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Research Article

Extrinsic bounds for the spectrum of a Steklov problem associated with the (p, q)–Laplacian

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Abstract

A Steklov-like boundary problem associated with the (p, q)–Laplacian is considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary, where $p, q \in (1, +\infty)$ and $p \neq q$. It is known that the set of eigenvalues of this problem is $\{0\} \cup (\lambda_1, +\infty)$, where λ_1 is a positive constant. The aim of the present article is to give upper bounds for λ_1. These bounds are of two types: one based on the isoperimetric ratio and the other on the geometry of the boundary. Additionally, it is shown that if λ_1 is close to any of these upper bounds, then Ω is close to a ball.

Keywords: (p, q)–Laplacian; eigenvalues; mean curvature; isoperimetric ratio.

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1. Introduction

Let $p, q \in (1, +\infty)$ be two distinct real numbers. Let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. In this paper, the following eigenvalue problem of Steklov type in Ω is considered:

$$\begin{cases} \Delta_p u + \Delta_q u = 0 & \text{in } \Omega, \\ \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \frac{\partial u}{\partial \nu} = \lambda |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \tag{$S_{p,q}$}$$

where $\frac{\partial u}{\partial \nu}$ is the derivative of the function u with respect to the outward unit normal ν to the boundary $\partial\Omega$. Here, Δ_p and Δ_q denote, respectively, the p–Laplacian and the q–Laplacian on Ω defined, as usual, by $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$. The operator $\Delta_p + \Delta_q$ is called the (p, q)–Laplacian and appears in several branches of physics. For details, one may refer, for instance, to [4–6, 11] and the references therein.

From the mathematical point of view, the problem ($S_{p,q}$) has been widely studied during the last years. It appears to be a particular case (with $a \equiv 0$ and $b \equiv 1$) of the more general eigenvalue problem

$$\begin{cases} \Delta_p u + \Delta_q u = \lambda a(x) |u|^{q-2} u & \text{in } \Omega, \\ \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \frac{\partial u}{\partial \nu} = \lambda b(x) |u|^{q-2} u & \text{on } \partial\Omega, \end{cases} \tag{$S_{p,q}^{a,b}$}$$

where a and b are two nonnegative and bounded functions on Ω and $\partial\Omega$, respectively, satisfying

$$\int_\Omega a(x) dx + \int_{\partial\Omega} b(\sigma) d\sigma > 0.$$

The set of eigenvalues of this problem has been studied in a recent series of articles for different possible values of p and q. Solutions of ($S_{p,q}^{a,b}$) are understood here in the weak sense; that is, λ is an eigenvalue of the problem ($S_{p,q}^{a,b}$) if there exists a function u of the Sobolev space $W := W^{1,\max\{p,q\}}(\Omega)$ such that

$$\int_\Omega \left(|\nabla u|^{p-2} + |\nabla u|^{q-2}\right) \nabla u \cdot \nabla w dx = \lambda \left(\int_\Omega a |\nabla u|^{q-2} u v dx + \int_{\partial\Omega} b |\nabla u|^{q-2} u v d\sigma \right),$$

for any $w \in W$. The complete description of the set of eigenvalues for this problem has been obtained in [4] after partial results proved in previous papers (see [1, 3, 7, 12, 13]). Namely, in [4], Barbu and Moroşanu prove that the set of eigenvalues of the problem ($S_{p,q}^{a,b}$) is $\{0\} \cup (\lambda_1, +\infty)$, where λ_1 is given by

$$\lambda_1 = \inf_{w \in \mathcal{C} \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^q dx}{\displaystyle\int_\Omega a |w|^q dx + \int_{\partial\Omega} b |w|^q d\sigma} \tag{1}$$

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with

$$\mathcal{C} = \left\{ u \in W \mid \int_\Omega a|u|^{q-2}u dx + \int_{\partial\Omega} b|u|^{q-2}u d\sigma = 0 \right\}. \tag{2}$$

Note that, as shown in [4], λ is positive.

In the present paper, we focus on the case $a = 0$ and $b = 1$, for which λ_1 becomes

$$\lambda_1 = \inf_{w \in \mathcal{C}\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^q dx}{\displaystyle\int_{\partial\Omega} |w|^q d\sigma} \tag{3}$$

with

$$\mathcal{C} = \left\{ u \in W \mid \int_{\partial\Omega} |u|^{q-2}u d\sigma = 0 \right\}. \tag{4}$$

The first aim is to give upper bounds for λ_1. Specifically, we provide two types of upper bounds. The first bound is of Reilly type (see [14]) and is expressed in terms of the mean curvature H of the boundary $\partial\Omega$, whereas the second one is of isoperimetric type; that is, it depends on both volume of Ω and $\partial\Omega$.

The first inequality, involving the isoperimetric ratio, is given in the following theorem.

Theorem 1.1. *Let $n \geqslant 2$ an integer, $p, q \in (1, +\infty)$ two distinct real numbers, and Ω a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Then,*

$$\lambda_1 \leqslant \begin{cases} \dfrac{1}{n}\left(\dfrac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)}\right)^{q-1} & \text{if } q \geqslant 2, \\[2ex] \dfrac{1}{n^{q-1}}\left(\dfrac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)}\right)^{q-1} & \text{if } 1 < q \leqslant 2. \end{cases}$$

Moreover, equality occurs if and only if $q = 2$ and Ω is a ball.

The next inequality involves mean curvature-type quantities.

Theorem 1.2. *Let $n \geqslant 2$ be an integer, and let $p, q \in (1, +\infty)$ be two distinct real numbers. Let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ of mean curvature H. Then,*

$$\lambda_1 \left| \int_{\partial\Omega} \mathrm{tr}\,(T) d\sigma \right|^q \leqslant \begin{cases} n^{q-1}\left(\displaystyle\int_{\partial\Omega} \|H_T\|^{\frac{q}{q-1}} d\sigma\right)^{q-1} \mathrm{Vol}(\Omega) & \text{if } q \geqslant 2, \\[2ex] n\left(\displaystyle\int_{\partial\Omega} \|H_T\|^{\frac{q}{q-1}} d\sigma\right)^{q-1} \mathrm{Vol}(\Omega) & \text{if } 1 < q \leqslant 2. \end{cases}$$

Moreover, equality occurs if and only if $q = 2$, $\mathrm{tr}\,(T)$ is constant and Ω is a ball.

In particular, in the case where T is the tensor T_r associated with the r-th mean curvature, we obtain the following result.

Corollary 1.1. *Let $n \geqslant 2$ be an integer, and let $p, q \in (1, +\infty)$ be two distinct real numbers. Let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ such that the integral of its r-th mean does not vanish. Then,*

$$\lambda_1 \left| \int_{\partial\Omega} H_r d\sigma \right|^q \leqslant \begin{cases} n^{q-1}\left(\displaystyle\int_{\partial\Omega} |H_{r+1}|^{\frac{q}{q-1}} d\sigma\right)^{q-1} \mathrm{Vol}(\Omega) & \text{if } q \geqslant 2, \\[2ex] n\left(\displaystyle\int_{\partial\Omega} |H_{r+1}|^{\frac{q}{q-1}} d\sigma\right)^{q-1} \mathrm{Vol}(\Omega) & \text{if } 1 < q \leqslant 2. \end{cases}$$

Moreover, equality occurs if and only if $q = 2$ and Ω is a ball.

In the case where $r = 0$, we obtain the following estimates involving the mean curvature of the boundary.

Corollary 1.2. *Let $n \geqslant 2$ be an integer, and let $p, q \in (1, +\infty)$ be two distinct real numbers. Let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ of mean curvature H. Then,*

$$\lambda_1 \leqslant \begin{cases} n^{q-1} \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \|H\|^q_{\frac{q}{q-1}} & \text{if } q \geqslant 2, \\ n \dfrac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \|H\|^q_{\frac{q}{q-1}} & \text{if } 1 < q \leqslant 2. \end{cases}$$

Moreover, equality occurs if and only if $q = 2$ and Ω is a ball.

In the case where $q \geqslant 2$, we obtain another upper bound involving two higher-order mean curvatures, namely H_r and H_{r+2}, in contrast to Corollary 1.1, which involves H_r and H_{r+1}. Precisely, the result is stated as follows.

Theorem 1.3. *Let $n \geqslant 2$ be an integer. Let $p \in (1, +\infty)$ and $q \in [2, +\infty)$ be two distinct real numbers, and let Ω be a bounded domain of the Euclidean space $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ with positive $(r + 2)$-th mean curvature H_{r+2}. Then,*

$$\lambda_1 \leqslant n^{q-1} \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \left(\frac{\|H_{r+2}\|_\infty}{\|H_r\|_1} \right)^{\frac{q}{2}}. \tag{5}$$

Moreover, equality occurs if and only if $q = 2$ and Ω is a ball.

Inequality (5) is comparable to similar eigenvalue estimates obtained in [18] for other operators of boundary problems.

2. Preliminaries

2.1. L_T operators and generalized Hsiung-Minkowski formula

Let (M^n, g) be a connected, oriented and closed Riemannian manifold isometrically immersed into $\mathbb{R}^{n+1}$. We denote by X its vector position and by ν its unit normal vector. We consider a divergence-free symmetric $(1,1)$-tensor T on M. We associate to T, the second-order differential operator L_T defined by

$$L_T u := -\mathrm{div}(T\nabla u),$$

for any C^2-function u on M. We also associate with T the following generalized mean curvature vectors:

$$H_T = T^{ij} X_{ij}, \tag{6}$$

where X_{ij} is the second covariant derivative of the immersion vector X. Then, we have

$$L_T X = -H_T \tag{7}$$

and easily deduce the following identity:

$$\frac{1}{2} L_T |X|^2 = -\langle X, H_T \rangle - \mathrm{tr}\,(T). \tag{8}$$

After integrating, this gives the so-called generalized Hsiung-Minkowski formula

$$\int_M \Big(\langle X, H_T \rangle + \mathrm{tr}\,(T) \Big) dv_g = 0. \tag{9}$$

The classical Hsiung-Minkowski formula (see [9]) is, in fact, this formula for the particular case where T is the identity, and more generally, where T is the tensor T_r associated to the r-th mean curvature. In the next subsection, we recall the definition of T_r and the associated operator L_r.

2.2. Higher-order mean curvatures

The higher-order mean curvatures are extrinsic quantities defined from the second fundamental form and generalizing the notion of mean curvature. Up to a normalization constant, the mean curvature H is the trace of the second fundamental form B:

$$H = \frac{1}{n} \mathrm{tr}\,(B). \tag{10}$$

In other words, the mean curvature is

$$H = \frac{1}{n} S_1(\kappa_1, \dots, \kappa_n), \tag{11}$$

where S_1 is the first elementary symmetric polynomial and $\kappa_1, \dots, \kappa_n$ are the principal curvatures. Higher-order mean curvatures are defined, in a similar way for $r \in \{1, \dots, n\}$, by

$$H_r = \frac{1}{\binom{n}{r}} S_r(\kappa_1, \cdots, \kappa_n), \tag{12}$$

where S_r is the r-th elementary symmetric polynomial, that is, for any n-tuple $(x_1, \cdots, x_n)$,

$$S_r(x_1, \dots, x_n) = \sum_{1 \leqslant i_1 < \cdots < i_r \leqslant n} x_{i_1} \cdots x_{i_r}. \tag{13}$$

To each H_r, we associate a symmetric $(2,0)$-tensor, which is in coordinates given by

$$T_r = (T_r^{ij}) = \left(\frac{\partial S_{r+1}}{\partial B_{ij}} \right), \tag{14}$$

where S_{r+1} is now understood to depend on the second fundamental form and the metric. These tensors T_r are divergence-free and satisfy the following relations:

$$\mathrm{tr}\,(T_r) = c(r) H_r \quad \text{and} \quad H_{T_r} = -c(r) H_{r+1} \nu, \tag{15}$$

where

$$c(r) = (n-r)\binom{n}{r}$$

and H_{T_r} is given by the relation (6).

We conclude this subsection by stating some classical and well-known inequalities involving H_r. From [2], we know that if $H_{r+1} > 0$, then for any $s \in \{1, \cdots, r\}$, T_s is positive definite, so H_s is positive everywhere. Moreover, in this case, we have the following (see, for instance, [8, p. 104]):

$$H_{r+1}^{\frac{1}{r+1}} \leqslant H_r^{\frac{1}{r}} \leqslant \cdots \leqslant H_2^{\frac{1}{2}} \leqslant H. \tag{16}$$

If equality occurs in one inequality of this sequence, then M is totally umbilical, and all inequalities are in fact equalities.

We remark here that in the case where $T = T_r$, the Hsiung-Minkowski formula (9) becomes, using (15),

$$\int_M \Big(H_{r+1} \langle X, \nu \rangle + H_r \Big) dv_g = 0. \tag{17}$$

2.3. Two lemmas

The strategy of the proof of all inequalities is to use coordinates $\{X_i\}_{1 \leqslant i \leqslant n}$ as test functions in order to estimate from above λ_1. For this, we recall the following elementary lemma. One can find this lemma in [17], for instance, but for the sake of completeness, we recall its proof here.

Lemma 2.1. *There exists a point $y = (y_1, \cdots, y_n) \in \mathbb{R}^n$ such that*

$$\int_{\partial\Omega} |(X^i - y_i)|^{q-2} (X^i - y_i) d\sigma = 0$$

for every $i \in \{1, \cdots, n\}$.

Proof. We consider the function $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ defined by

$$f(y_1, \cdots, y_n) = \frac{1}{p} \int_{\partial M} \sum_{i=1}^{N} |X^i - y_i|^p dv_h,$$

where $(X^1, \cdots, X^N)$ are the Euclidean coordinates centered at the origin. This function is nonnegative, and we denote by $\alpha > 0$ its infimum. By compactness of ∂M, there exists $R > 0$ such that ∂M is contained in the ball $B(0, R)$. Moreover, we set

$$\rho = \left(\frac{2\alpha p}{\mathrm{Vol}(\partial M)} \right)^{\frac{1}{p}}.$$

Now, let $(y_1, \cdots, y_n) \in \mathbb{R}^N$, outside the hypercube $[-R-\rho, R+\rho]^N$ such that there exists $i \in \{1, \cdots, N\}$ so that $|y_i| > R+\rho$. Hence, for any $(X^1, \cdots, X^n) \in \partial M$, we have $\sum_{i=1}^N |X^i - y_i|^p \geqslant \rho^p$. From this, we deduce that

$$f(y_1, \cdots, y_N) \geqslant \frac{1}{p}\int_M \rho^p dv_h = \frac{1}{p}\rho^p \mathrm{Vol}(\partial M) = 2\alpha.$$

By the continuity of f and compactness of $[-R-\rho, R+\rho]^N$, the infimum α of f is necessarily attained inside $[-R-\rho, R+\rho]^N$. Let $y_0 = (y_1, \cdots, y_n)$ be a point where the minimum is attained. At this point, $(\nabla f)_{y_0} = 0$, and we deduce that for each $i \in \{1, \cdots, N\}$, we have

$$\langle \nabla f, \partial_i \rangle_{y_0} = \int_{\partial M} |(X^i - y_i)|^{p-2}(X^i - y_i) dv_h = 0,$$

where $\{\partial_1, \cdots, \partial_N\}$ is the canonical basis of $\mathbb{R}^N$. □

We now present the second elementary lemma, which is useful in the proof of Theorem 1.3.

Lemma 2.2. *If H_{k+1} is positive everywhere, then, for any $l \in \mathbb{N}$, the following inequality holds:*

$$\int_M H_k \|X\|^l dv_g \leqslant \int_M H_{k+1}\|X\|^{l+1} dv_g.$$

Proof. First, if we take $T = T_k$, then (8) becomes

$$\frac{1}{2} L_k \|X\|^2 = -c(k)\left(H_{k+1}\langle X, \nu\rangle + H_k\right), \tag{18}$$

where L_k denotes the operator $L_{T_k} = -\mathrm{div}\,(T_k \nabla \cdot)$. Thus, multiplying by $\|X\|^l$, we obtain

$$\begin{aligned} \frac{1}{2}\|X\|^l L_k \|X\|^2 &= -c(k)\left(H_{k+1}\langle X, \nu\rangle \|X\|^l + H_k\|X\|^l\right) \\ &\leqslant c(r)\left(H_{k+1}\|X\|^{l+1} - H_k\|X\|^l\right). \end{aligned} \tag{19}$$

Note that, in the last line, we have used the fact that H_{k+1} is positive by assumption.

Now, we integrate (18) over M to obtain

$$\begin{aligned} c(k)\int_M \left(H_{k+1}\|X\|^{l+1} - H_k\|X\|^l\right) dv_g &\geqslant \frac{1}{2}\int_M \|X\|^l L_k \|X\|^2 dv_g \\ &\geqslant l\int_M \|X\|^l \langle T_k \nabla \|X\|, \nabla \|X\|\rangle dv_g \end{aligned} \tag{20}$$

where the second line comes from integration by parts with the fact that L_k is defined by $L_k = -\mathrm{div}\,(T_k \nabla \cdot)$. Moreover, as we have already mentioned, H_{k+1} is positive everywhere, which implies that T_k is positive definite. Hence

$$\int_M \|X\| \langle T_k \nabla \|X\|, \nabla \|X\|\rangle dv_g \geqslant 0$$

and therefore, we obtain

$$\int_M H_k \|X\|^l dv_g \leqslant \int_M H_{k+1}\|X\|^{l+1} dv_g. \tag{21}$$

□

3. Proofs of the inequalities

With Lemma 2.1, up to the appropriate translation, the coordinate functions belong to $\mathcal{C}$ and can be used as test functions, without loss of generality. Hence, we obtain

$$\lambda_1 \int_{\partial\Omega} \sum_{i=1}^n |X^i|^q \leqslant \int_\Omega \sum_{i=1}^n \|\nabla X^i\|^q. \tag{22}$$

We discuss the cases $q \geqslant 2$ and $q \in (1, 2]$ separately.

3.1. The case $q \geqslant 2$

First, since $q \geqslant 2$ and $\sum_{i=1}^{n} \|\nabla X^i\|^2 = n$ (see [15, Lemma 2.1] for instance), we have

$$\sum_{i=1}^{n} \|\nabla X^i\|^q \leqslant \left(\sum_{i=1}^{n} \|\nabla X^i\|^2 \right)^{\frac{q}{2}} = n^{\frac{q}{2}}. \tag{23}$$

Hence, we obtain

$$\lambda_1 \int_{\partial\Omega} \sum_{i=1}^{n} |X^i|^q \sigma \leqslant n^{\frac{q}{2}} V(M).$$

Moreover, by the Hölder inequality, we have

$$\|X\|^2 = \sum_{i=1}^{n} |X^i|^2 \leqslant \left(\sum_{i=1}^{n} |X^i|^q \right)^{\frac{2}{q}} n^{\frac{q-2}{q}}, \tag{24}$$

which gives

$$\sum_{i=1}^{n} |X^i|^q \geqslant \frac{1}{n^{\frac{q-2}{2}}} \|X\|^q$$

and hence

$$\lambda_1 \int_{\partial\Omega} \|X\|^q d\sigma \leqslant n^{q-1} V(\Omega). \tag{25}$$

From now on, we differentiate the proofs of the inequalities given in Theorems 1.1, 1.2 and 1.3.

Proof of Theorem 1.1, $q \geqslant 2$. We start from (25), which can be rewritten as

$$\lambda_1 \|X\|_q^q \mathrm{Vol}(\partial\Omega) \leqslant n^{q-1} \mathrm{Vol}(\Omega).$$

From the fact that the L^s-norms are nondecreasing for the parameter s and $q \geqslant 2$, we have $\|X\|_2 \leqslant \|X\|_q$ and so

$$\lambda_1 \|X\|_2^q \mathrm{Vol}(\partial\Omega) \leqslant n^{q-1} \mathrm{Vol}(\Omega). \tag{26}$$

On the other hand, we have $\mathrm{div}_{\,\Omega}(X) = n$, which gives after integration over Ω and using the divergence theorem

$$n\mathrm{Vol}(\Omega) = \int_{\Omega} \mathrm{div}_{\,\Omega}(X) dx = \left| \int_{\partial\Omega} \langle X, \nu \rangle d\sigma \right|,$$

where ν a unit normal vector field to $\partial\Omega$. Moreover, we have

$$\begin{aligned} \left| \int_{\partial\Omega} \langle X, \nu \rangle d\sigma \right| &\leqslant \int_{\partial\Omega} \|X\| d\sigma \\ &= \|X\|_1 \mathrm{Vol}(\partial\Omega) \\ &\leqslant \|X\|_2 \mathrm{Vol}(\partial\Omega), \end{aligned}$$

and so

$$\|X\|_2 \geqslant \frac{n\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)}.$$

Substituting this into (26), we obtain

$$\lambda_1 \leqslant \frac{1}{n} \left(\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)} \right)^{q-1}.$$

Hence, the inequality of Theorem 1.2 is proved for $q \geqslant 2$. If equality occurs in this inequality, then equality occurs in all the above inequalities. First, we have

$$\left| \int_{\partial\Omega} \langle X, \nu \rangle d\sigma \right| = \int_{\partial\Omega} \|X\| d\sigma,$$

which implies that X is normal everywhere $\partial\Omega$, and so $\partial\Omega$ is a sphere, or equivalently, Ω is a ball. Moreover, if $q \neq 2$ and equality occurs in (24), which means that equality occurs at any point in the pointwise Hölder inequality; that is, X is colinear to the vector $(1, 1, \dots, 1)$ at any point of $\partial\Omega$. This is a contradiction. Hence, if equality occurs, then $q = 2$.

Conversely, assume that $q = 2$ and Ω is ball. Since $q = 2$, from its definition, λ_1 is equal to the first positive eigenvalue σ_1 of the usual Steklov problem. We know that, on the ball,

$$\sigma_1 = \frac{1}{n}\frac{\mathrm{Vol}(\partial\Omega)}{\mathrm{Vol}(\Omega)}$$

(see [16] for instance), and therefore, equality occurs. $\square$

Proof of Theorem 1.2, $q \geqslant 2$**.** We again begin with inequality (25), multiply it by $\left(\int_{\partial\Omega} \|H_T\|^{\frac{q}{q-1}} d\sigma\right)^{q-1}$, and apply the integral Hölder inequality to obtain

$$\lambda_1 \left|\int_{\partial\Omega} \langle X, H_T\rangle d\sigma\right|^p \leqslant n^{q-1}\left(\int_{\partial\Omega} \|H_T\|^{\frac{p}{p-1}} d\sigma\right)^{p-1} \mathrm{Vol}(\Omega). \tag{27}$$

Now, we apply the Hsiung-Minkowski formula (9) to the integral of the left-hand side to get

$$\lambda_1 \left|\int_{\partial\Omega} \mathrm{tr}\,(T) d\sigma\right|^q \leqslant n^{q-1}\left(\int_{\partial\Omega} \|H_T\|^{\frac{q}{q-1}} d\sigma\right)^{q-1} \mathrm{Vol}(\Omega),$$

which proves the inequality given in Theorem 1.2 for the case $q \geqslant 2$. If equality occurs in this inequality, then equality occurs in (27). Hence, we have the first inequality in the pointwise Cauchy-Schwarz inequality $\langle X, H_T\rangle \leqslant |H_T|$ so that X is normal everywhere on $\partial\Omega$, and thus, Ω is a ball. Secondly, equality occurs in the integral Hölder inequality, which implies that $\|X\|$ and $|H_T|$ are colinear. Since $\|X\|$ is constant, H_T is also constant. Finally, since $\partial\Omega$ is a sphere, it is totally umbilical, that is, $A = \frac{1}{R} Id_{\partial\Omega}$, where R is the radius of Ω, so that $H_T = \mathrm{tr}\,(A \circ T) = \frac{1}{R}\mathrm{tr}\,(T)$. Since $|H_T|$ is constant, it follows that $\mathrm{tr}(T)$ is also constant. To prove that $q = 2$, the argument is the same as in the proof of Theorem 1.1.

Conversely, if $q = 2$, then, as in Theorem 1.1, λ_1 coincides with σ_1. Moreover, if Ω is a ball and $\mathrm{tr}(T)$ is constant, then the inequality stated in the theorem becomes

$$\lambda_1 \mathrm{tr}\,(T)^2 \mathrm{Vol}(\partial\Omega) \leqslant n|H_T|^2 \mathrm{Vol}(\Omega),$$

that is,

$$\lambda_1 \leqslant n\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} H^2.$$

On the other hand, in this case (see [10]), we have

$$\sigma_1 = n\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} H^2,$$

and therefore, equality occurs. $\square$

Proof of Theorem 1.3. Once again, from (22) and the fact that the L^s-norms are nondecreasing for the parameter s and $q \geqslant 2$, we have (26); that is

$$\lambda_1 \left(\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} \|X\|^2 d\sigma\right)^{\frac{q}{2}} \leqslant n^{q-1}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)}.$$

We multiply it by $\|H_{r+2}\|_\infty^{\frac{q}{2}}$, use Lemma 2.2 and the Hsiung-Minkowski formula (17) to obtain

$$\begin{aligned}
n^{q-1}\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)}\|H_{r+2}\|_\infty^{\frac{q}{2}} &\geqslant \lambda_1 \|H_{r+2}\|_\infty^{\frac{q}{2}} \left(\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} \|X\|^2 d\sigma\right)^{\frac{q}{2}} \\
&\geqslant \lambda_1 \left(\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} H_{r+2}\|X\|^2 d\sigma\right)^{\frac{q}{2}} \\
&\geqslant \lambda_1 \left(\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} H_{r+1}\|X\| d\sigma\right)^{\frac{q}{2}} \\
&\geqslant \lambda_1 \left|\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} H_{r+1}\langle X, \nu\rangle d\sigma\right|^{\frac{q}{2}} \\
&\geqslant \lambda_1 \left(\frac{1}{\mathrm{Vol}(\partial\Omega)}\int_{\partial\Omega} H_r d\sigma\right).
\end{aligned}$$

Note that we also have used the fact that $H_{r+2} > 0$, which implies that $H_{r+1} > 0$ and $H_r > 0$. Hence, we obtain the desired inequality

$$\lambda_1 \leqslant n^{q-1} \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \left(\frac{\|H_{r+2}\|_\infty}{\|H_r\|_1} \right)^{\frac{q}{2}}.$$

If equality occurs, then once again, we have $\langle X, \nu \rangle = \|X\|$ on $\partial\Omega$, which implies, as previously shown, that Ω is a ball. The condition $q = 2$ comes here again from the equality case in (24), similar to the equality case in Theorems 1.1 and 1.2.

Conversely, if $q = 2$ and Ω is a ball, then

$$\lambda_1 = n \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} H^2 = n \frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \left(\frac{\|H_{r+2}\|_\infty}{\|H_r\|_1} \right)^{\frac{q}{2}},$$

and hence, equality occurs. □

3.2. The case $q \in (1, 2]$

Here again, we start from (22), that is,

$$\lambda_1 \int_{\partial\Omega} \sum_{i=1}^{n} |X^i|^q \leqslant \int_{\Omega} \sum_{i=1}^{n} \|\nabla X^i\|^q.$$

First, since $q \leqslant 2$, we have

$$\|X\|^q = \left(\sum_{i=1}^{n} |X^i|^2 \right)^{\frac{q}{2}} \leqslant \sum_{i=1}^{n} |X^i|^q. \tag{28}$$

On the other hand, by the Hölder inequality, we have

$$\sum_{i=1}^{n} \|\nabla X^i\|^q \leqslant n^{\frac{2-q}{2}} \left(\sum_{i=1}^{n} \|\nabla X^i\|^2 \right)^{\frac{q}{2}} = n^{\frac{2-q}{2}} n^{\frac{q}{2}} = n.$$

Hence, using the last two inequalities in (22), we obtain

$$\lambda_1 \int_{\partial\Omega} \|X\|^q d\sigma \leqslant n V(\partial\Omega). \tag{29}$$

We provide the proofs of Theorems 1.1 and 1.2 separately.

Proof of Theorem 1.1, $q \in (1, 2]$**.** The proof is similar to the case $q \geqslant 2$. Here, we have

$$\begin{aligned} n\mathrm{Vol}(\Omega) &= \left| \int_{\partial\Omega} \langle X, \nu \rangle d\sigma \right| \\ &\leqslant \left| \int_{\partial\Omega} \|X\| d\sigma \right| \\ &= \|X\|_1 \mathrm{Vol}(\partial\Omega) \\ &\leqslant \|X\|_q \mathrm{Vol}(\partial\Omega), \end{aligned}$$

and so

$$\|X\|_q \geqslant \frac{n\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)}.$$

Substituting this into (29), we obtain

$$\lambda_1 \leqslant \frac{1}{n^{q-1}} \left(\frac{\mathrm{Vol}(\Omega)}{\mathrm{Vol}(\partial\Omega)} \right)^{q-1}.$$

Hence, the inequality given in Theorem 1.1 is also proved for the case $q \in (1, 2]$. If equality occurs in this inequality, then all previous inequalities are equalities. In particular, again here, we have, on $\partial\Omega$,

$$\langle X, \nu \rangle = \|X\|,$$

which implies that Ω is a ball.

On the other hand, we have equality in (28), which implies that

$$\left(\sum_{i=1}^{n}|X^i|^2\right)^{\frac{q}{2}} \leqslant \sum_{i=1}^{n}|X^i|^q$$

at any point of $\partial\Omega$. Assume that $q \neq 2$. This is possible at a point x only if $X(x)$ is one of the vectors ∂_i of the canonical frame of $\mathbb{R}^n$. By continuity, this implies that X is constant on $\partial\Omega$, which leads to a contradiction. Hence, if equality occurs, then $q = 2$.

Conversely, if $q = 2$ and Ω is a ball, then equality holds, since the bound coincides with λ_1 in this case. □

Proof of Theorem 1.2, $q \in (1, 2]$**.** Here also, the proof is similar to the case $q \geqslant 2$. We multiply (29) by

$$\left(\int_{\partial\Omega}\|H_T\|^{\frac{q}{q-1}}\,d\sigma\right)^{q-1}$$

and then apply the integral Hölder inequality to obtain

$$\lambda_1\left|\int_{\partial\Omega}\langle X, H_T\rangle d\sigma\right|^q \leqslant n\left(\int_{\partial\Omega}\|H_T\|^{\frac{q}{q-1}}d\sigma\right)^{q-1}\mathrm{Vol}(\Omega).$$

We apply the Hsiung–Minkowski formula (9) to get

$$\lambda_1\left|\int_{\partial\Omega}\mathrm{tr}\,(T)d\sigma\right|^q \leqslant n\left(\int_{\partial\Omega}\|H\|^{\frac{q}{q-1}}d\sigma\right)^{q-1}\mathrm{Vol}(\Omega),$$

which proves Theorem 1.1 for the case $q \in (1, 2]$. The proof of the equivalence in the case of equality is the same as previously presented. □

We end the paper with the observation that Corollary 1.1 is an immediate consequence of Theorem 1.2 due to relations (15), and Corollary 1.2 corresponds to the particular case $r = 0$.

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