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Research Article

On the existence of characteristic functions in bornological bispaces

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Abstract

Let $((S,\tau_1,\tau_2),\mathcal{B})$ be a bornological bispace, where τ_1 and τ_2 are collections of subsets of a set S such that (S,τ_1) and (S,τ_2) are spaces alongside a bornology $\mathcal{B}$. This paper extends boundedness concepts across bispaces. A property involving a (τ_1,τ_2)-characteristic function is introduced. It is shown that the existence of a (τ_1,τ_2)-characteristic function requires S to be weakly locally bounded, $\mathcal{B}$ to be τ_1-open and τ_2-closed, and to have a countable base.

Keywords: bispace; bornology; characteristic function; weakly locally bounded bornological bispace; countable base.

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1. Introduction

The generalization of topological spaces has evolved in various directions. One significant extension is the σ-space introduced by Alexandroff [1], where the requirement for arbitrary unions of open sets to be open was relaxed to include only countable unions. Other researchers further modified these conditions; see, for example, [3, 8]. In bitopological spaces, a set X is equipped with two topologies P and Q. Such spaces naturally arise from quasi-metrics, where two different topologies are induced by sets based on quasi-metrics. Kelly [5] expanded the study of bitopological spaces by introducing various separation properties and generalizing classical topological results. Subsequently, Pervin [9], Reilly [10], and others further investigated these properties. Levine [7] introduced the concept of semi-open sets in topological spaces. Later, Lahiri and Das [6] extended this concept to Alexandroff spaces; eventually, they generalized bitopological spaces to bispaces and studied their properties.

Bornological spaces are a key concept in functional analysis and topology, offering a generalized framework for studying bounded sets. Defined via a bornology — a collection of sets that represents the concept of boundedness — these spaces extend traditional ideas of boundedness beyond the scope of standard topological methods. They are particularly useful in the context of locally convex spaces, duality theory, and the study of continuous linear functions. Moreover, bornological spaces play a significant role in the theory of distributions, where they facilitate the handling of the intricacies associated with generalized functions. A well-articulated explanation of bornological concepts can be found in S.T. Hu's paper [4].

In the study of topology and functional analysis, the concept of bornological spaces plays a pivotal role, particularly when dealing with two topologies and bounded sets. A bornological bispace, denoted as $((S,\tau_1,\tau_2),\mathcal{B})$, introduces an additional layer of complexity alongside a bornology $\mathcal{B}$, where τ_1 and τ_2 are collections of subsets of a set S such that (S,τ_1) and (S,τ_2) are two spaces. This structure is instrumental in understanding the interplay between boundedness and topological properties across different spaces.

This paper investigates a bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ with a property $(*)$ and a (τ_1,τ_2)-characteristic function, where the property $(*)$ is stated as follows: given a subset P of S and a τ_1-open set Q such that τ_2-cl$(P)\subseteq Q$, there exist a τ_1-open set G and a set F such that

$$\tau_2\text{-cl}(P)\subseteq G\subseteq \tau_2\text{-cl}(F)\subseteq Q.$$

It is shown that the existence of such a characteristic function is equivalent to the following three conditions: (i) the space S must be weakly locally bounded, (ii) the bornology $\mathcal{B}$ must be τ_1-open and τ_2-closed, and (iii) $\mathcal{B}$ must possess a countable base.

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2. Preliminary definitions and results

We begin this section with a succinct review of the key terms and concepts related to spaces, bispaces, and boundedness.

Definition 2.1 (see [1]). *A set S is called an Alexandroff space, or σ-space, or simply a space, if a collection $\mathcal{F}$ of subsets of S is specified that satisfies the following axioms:*

(i) The intersection of a countable number of sets from $\mathcal{F}$ belongs to $\mathcal{F}$.

(ii) The union of a finite number of sets from $\mathcal{F}$ belongs to $\mathcal{F}$.

(iii) The void set $\emptyset$ belongs to $\mathcal{F}$.

(iv) The set S belongs to $\mathcal{F}$.

Sets of $\mathcal{F}$ are called closed sets. Their complementary sets are called open sets. It is clear that instead of closed sets in the definition of the space, one may put open sets, subject to the conditions of countable summability and finite intersectionality, and the condition that X and $\emptyset$ should be open. The collection of all such open sets is sometimes denoted by τ and the space by (S, τ). Note that a topological space is a space, but in general, τ is not a topology, as can be easily seen by taking $S = \mathbb{R}$ and τ as the collection of all F_σ sets in $\mathbb{R}$.

Definition 2.2 (see [1]). *To every set M in a space (S, τ), we associate its closure $\overline{M}$, defined as the intersection of all closed sets containing M.*

In general, the closure of a set in a space is not a closed set.

Definition 2.3 (see [1]). *The interior of a set M in a space (S, τ) is defined as the union of all open sets contained in M and is denoted by $\tau - Int(M)$, or $Int(M)$ when there is no confusion about τ.*

In general, the interior of a set in a space is not an open set.

Definition 2.4. *Let (S, τ) be a space and $x \in S$. A subset N of S is said to be a τ-neighbourhood of x if there exists an open set G such that $x \in G \subseteq N$. The collection of all τ-neighbourhoods of x is called the τ-neighbourhood system at x.*

Definition 2.5. *Let (S, τ) be a space. A nonempty collection $\mathcal{B}(x)$ of τ -neighbourhoods of $x \in S$ is called a τ-neighbourhood base for the τ-neighbourhood system at x if and only if for every τ-neighbourhood N of x, there exists a $B \in \mathcal{B}(x)$ such that $B \subseteq N$.*

Definition 2.6 (see [6]). *A space (S, τ) is said to be bicompact if every open cover of it admits a finite subcover.*

Definition 2.7 (see [5]). *A set S equipped with two arbitrary topologies P and Q is called a bitopological space, and is denoted by (S, P, Q).*

Definition 2.8 (see [6]). *Let S be a nonempty set. If τ_1 and τ_2 are collections of subsets of S such that (S, τ_1) and (S, τ_2) are spaces, then S is called a bispace, and is denoted by (S, τ_1, τ_2).*

It is important to note that the bispace (S, τ_1, τ_2) becomes a bitopological space when τ_1 and τ_2 are topologies on S.

Example 2.1 (see [2]). *Let $S = [0, 2]$, and let $\{U_i\}$ be the collection of all countable subsets of irrational numbers in $[0, 1]$. Consider τ_1 as the collection of all sets of the form $U_i \cup \{\sqrt{3}\}$ together with S and $\emptyset$; and τ_2 as the collection of all sets U_i together with S and $\emptyset$. Since the uncountable union of countable sets is uncountable, (S, τ_1, τ_2) is a bispace but not a bitopological space.*

In order to apply the idea of boundedness to the case of a general topological space, Hu [4] developed the concepts of a bornology and a bornological space. A bornology on a set S is a collection $\mathcal{B}$ of subsets of a set S that satisfies all the following conditions:

(i) $\mathcal{B}$ covers S, that is $S = \bigcup\{B : B \in \mathcal{B}\}$;

(ii) $\mathcal{B}$ is stable under inclusions, that is, if $A \subseteq B$ and $B \in \mathcal{B}$ then $A \in \mathcal{B}$;

(iii) $\mathcal{B}$ is stable under finite unions; that is, if $B_1, \cdots, B_n \in \mathcal{B}$ then $B_1 \cup \cdots \cup B_n \in \mathcal{B}$.

The pair $(S,\mathcal{B})$ is referred to as a bornological space, and the sets belonging to $\mathcal{B}$ are considered bounded sets within this space. Given bornological spaces $(S,\mathcal{B}_S)$ and $(T,\mathcal{B}_T)$, a mapping $f:(S,\mathcal{B}_S)\to(T,\mathcal{B}_T)$ is called bounded if $f(A)\in\mathcal{B}_T$ for every $A\in\mathcal{B}_S$.

Key examples of bornological spaces $(S,\mathcal{B})$ include:

(i) A metric space, with the family of its bounded subsets;

(ii) A topological space, along with the family of its relatively compact subsets;

(iii) A uniform space, paired with the family of its totally bounded subsets.

Definition 2.9 (see [1])**.** *If $\mathcal{B}$ is a bornology on S, then a collection $\mathcal{B}_0$ is called a base for $\mathcal{B}$ if $\mathcal{B}_0\subseteq\mathcal{B}$ and every set of $\mathcal{B}$ is a subset of a member of $\mathcal{B}_0$.*

Definition 2.10. *A bornology $\mathcal{B}$ is said to be a second countable bornology if it has a countable base.*

Definition 2.11. *A bornological bispace is an ordered pair $((S,\tau_1,\tau_2),\mathcal{B})$, where (S,τ_1,τ_2) is a bispace and $\mathcal{B}$ is a bornology on S .*

Definition 2.12. *A bounded set B' of a bornology $\mathcal{B}$ on a set S is said to be maximal if every $B\in\mathcal{B}$ is a subset of B'.*

Definition 2.13. *A bornology $\mathcal{B}$ in a bispace (S,τ_1,τ_2) is said to be τ_i-open if every bounded set is contained in some τ_i-open bounded set, for $i=1,2$.*

Definition 2.14. *A bornology $\mathcal{B}$ in a bispace (S,τ_1,τ_2) is said to be τ_i-closed if the τ_i-closure of every bounded set is bounded, for $i=1,2$.*

Proposition 2.1. *Let $\mathcal{P}$ be a countable base for a bornology $\mathcal{B}$ on a set S such that $\mathcal{B}$ does not have a maximal bounded set. Then there exists a strictly increasing sequence $\{Q_n\}$ of members of $\mathcal{P}$ such that the collection $\{Q_n:n\in\mathbb{N}\}$ is a base for $\mathcal{B}$.*

Proof. By the countability of $\mathcal{P}$, we can write $\mathcal{P}=\{P_n:n\in\mathbb{N}\}$. Let $Q_1=P_1$. Since $\mathcal{B}$ does not contain any maximal bounded set, there exists an element $B\in\mathcal{B}$ such that B is not a subset of $Q_1\cup P_2$, and there exists an element $P\in\mathcal{P}$ such that $Q_1\cup B\cup P_2\subset P$. This establishes the existence of an element $P\in\mathcal{P}$ such that $Q_1\cup P_2\neq P$ and $Q_1\cup P_2\subset P$. Let $n_1=\min\{n\in\mathbb{N}:Q_1\cup P_2\subset P_n\}$ and $Q_2=P_{n_1}$. Here, we use the symbol $\subset$ for strict inclusion. Suppose that, for $m\in\mathbb{N}$, we have already defined the set $Q_m\in\mathcal{P}$; then, analogous to the aforementioned procedure, we take

$$n_{m+1}=\min\{n\in\mathbb{N}:Q_m\cup P_{m+1}\subset P_n\}$$

and subsequently define $Q_{m+2}=P_{n_{m+1}}$. This demonstrates that the sequence $\{Q_n\}$ is a strictly increasing sequence of members of $\mathcal{P}$.

To demonstrate that the sequence $\{Q_n\}$ constitutes a base of $\mathcal{B}$, let us consider an arbitrary element B of $\mathcal{B}$. Given that $\mathcal{P}$ is a base of $\mathcal{B}$, it follows that there exists a positive integer n such that $B\subseteq P_n\subseteq Q_{n+1}$. Therefore, it can be concluded that $\{Q_n\}$ is a base of $\mathcal{B}$. □

Proposition 2.2. *Let (S,τ_1,τ_2) be a bispace. Let $\mathcal{B}$ be a second countable, τ_1-open and τ_2-closed bornology on S such that $\mathcal{B}$ does not have a maximal bounded set. Then there exists a strictly increasing sequence $\{Q_n\}$ of τ_1-open sets such that $\mathcal{Q}=\{Q_n:n\in\mathbb{N}\}$ is a base for $\mathcal{B}$ provided that τ_2-cl$(Q_n)\subset Q_{n+1}$ for each $n\in\mathbb{N}$.*

Proof. By Proposition 2.1, $\mathcal{B}$ possesses a strictly increasing countable base $\mathcal{P}=\{P_n:n\in\mathbb{N}\}$. Let Q_1 be an arbitrary τ_1-open bounded set. Given that $\mathcal{B}$ is τ_1-open, it guarantees the existence of a τ_1-open bounded set. Here τ_2- cl$(Q_1\cup P_2)$ is a bounded set, as $\mathcal{B}$ is τ_2-closed. Since $\mathcal{B}$ does not contain any maximal bounded set, there exists an element $B\in\mathcal{B}$ such that B is not a subset of τ_2- cl$(Q_1\cup P_2)$ and there also exists an element $P\in\mathcal{P}$ such that τ_2-cl$(Q_1\cup P_2)\cup B\subset P$. This establishes the existence of an element $P\in\mathcal{P}$ satisfying τ_2-cl$(Q_1\cup P_2)\neq P$ and τ_2-cl$(Q_1\cup P_2)\subset P$.

Let $n_1=\min\{n\in\mathbb{N}:\tau_2\text{-cl}(Q_1\cup P_2)\subset P_n\}$ and $Q_2=P_{n_1}$. Here, we employ the notation $\subset$ to denote the strict inclusion. Assume that, for $m\in\mathbb{N}$, we have previously established a τ_1-open set $Q_m\in\mathcal{P}$; then, in a manner analogous to the preceding discussion, we define $n_{m+1}=\min\{n\in\mathbb{N}:\tau_2\text{-cl}(Q_m\cup P_{m+1})\subset P_n\}$ and $Q_{m+2}=P_{n_{m+1}}$. The sequence $\{Q_n\}$ possesses the requisite properties.

It is essential to demonstrate that $\mathcal{Q}$ constitutes a base for $\mathcal{B}$. For this purpose, let B denote any member of $\mathcal{B}$. Since $\mathcal{P}$ is a base of $\mathcal{B}$, there exists a positive integer n satisfying $B\subseteq P_n\subseteq Q_{n+1}$. Consequently, $\mathcal{Q}$ is a base of $\mathcal{B}$. □

Remark 2.1. *In Proposition 2.2, it is important to note that τ_1 and τ_2 are interchangeable.*

Definition 2.15. *A point s of a bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ is said to be a τ_i-finite point if it has a bounded τ_i-neighbourhood in S, for $i = 1, 2$.*

Definition 2.16. *A bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ is said to be weakly locally bounded if every point of S is either τ_1-finite or τ_2-finite.*

Definition 2.17. *A bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ is said to be locally bounded if every point of S is τ_1-finite and τ_2-finite.*

Definition 2.18. *A bispace (S,τ_1,τ_2) is said to be bicompact if every cover $\mathcal{P} \subseteq \tau_1 \cup \tau_2$ has a finite subcover.*

We remark that if (S,τ_1,τ_2) is bicompact, then both the spaces (S,τ_1) and (S,τ_2) are bicompact.

Definition 2.19 (see [6])**.** *A cover $\mathcal{P}$ of a bispace (S,τ_1,τ_2) is said to be pairwise open if $\mathcal{P} \subseteq \tau_1 \cup \tau_2$ and if $\mathcal{P}$ contains at least one nonempty member of τ_1 and atleast one nonempty member of τ_2.*

Definition 2.20. *A bispace (S,τ_1,τ_2) is said to be pairwise compact if every pairwise open cover of S has a finite subcover.*

Proposition 2.3. *Let (S,τ_1,τ_2) be a bispace. Any τ_1-closed or τ_2-closed subset of a pairwise compact space S is pairwise compact.*

Proof. Let K be a τ_1-closed subset of S. Let $\mathcal{P}$ be a pairwise open cover of K by open sets of S. Since K is τ_1-closed, the complement $S \setminus K$ is τ_1-open. Hence, the members of $\mathcal{P}$ together with the τ_1-open set $S \setminus K$ form a pairwise open cover of S. Since S is pairwise compact, the pairwise open cover of S contains a finite subcover of S. In other words, there is a finite number of open sets $P_1, P_2, \cdots, P_n$, where $P_i \in \tau_1 \cup \tau_2, i = 1, 2, \cdots, n$ in $\mathcal{P}$ such that

$$P_1 \cup \cdots \cup P_n \cup (S \setminus K) = S.$$

Hence, $\{P_1, \cdots, P_n\}$ covers K, and K is pairwise compact. □

Proposition 2.4. *Every bicompact subset of a weakly locally bounded bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ is bounded.*

Proof. Let K be a bicompact subset of a weakly locally bounded bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$. Let $x \in K \subseteq S$. As S is weakly locally bounded, so x is either τ_1-finite or τ_2-finite. Let x be a τ_1-finite point. Then, x has a bounded τ_1-neighbourhood, say $B_x \in \mathcal{B}$ in S. So, there exists $P_x \in \tau_1$ such that $x \in P_x \subseteq \mathcal{B}_x$. Now, $\mathcal{P} = \{P_x : x \in K\} \subseteq \tau_1 \cup \tau_2$ is a cover of K. By bicompactness of K, there exists a finite subcollection say, $\{P_1, P_2, \cdots, P_n\}$ of $\mathcal{P}$ such that

$$K \subseteq \bigcap_{i=1}^{n} P_i.$$

Hence,

$$K \subseteq \bigcap_{i=1}^{n} P_i \subseteq \bigcap_{i=1}^{n} B_i.$$

Now, since $\bigcap_{i=1}^{n} B_i$ is bounded, K must also be bounded. □

Proposition 2.5. *Every pairwise compact subset of a locally bounded bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ is bounded.*

Proof. The proof is similar to that of Proposition 2.4. □

3. Main result

Before stating and proving the primary result of this paper, some additional preparation is needed.

Definition 3.1. *Let (S,τ) be a σ-space and $f : S \to \mathbb{R}$ be a function. Then*

(i) f is called upper semicontinuous on S if for each $y \in \mathbb{R}$, the set $f^{-1}((-\infty, y)) = \{x \in S : f(x) < y\}$ is open in S; equivalently, for each $y \in \mathbb{R}$, the set $f^{-1}([y, -\infty)) = \{x \in X : f(x) \geq y\}$ is closed in S.

(ii) f is called lower semicontinuous on S if for each $y \in \mathbb{R}$, the set $f^{-1}((y, \infty)) = \{x \in S : f(x) > y\}$ is open in S; equivalently, for each $y \in \mathbb{R}$, the set $f^{-1}((-\infty, y]) = \{x \in S : f(x) \leq y\}$ is closed in S.

The next result follows directly from Definition 3.1.

Proposition 3.1. *(i) If $f : S \to \mathbb{R}$ is an upper semicontinuous function on S, then for each point $x_0 \in S$ and for every $\epsilon > 0$, there exists a neighbourhood U of x_0 such that $f(x) < f(x_0) + \epsilon$ for all $x \in U$.*

(ii) If $f : S \to \mathbb{R}$ is a lower semicontinuous function on S, then for each point $x_0 \in S$ and for every $\epsilon > 0$, there exists a neighbourhood U of x_0 such that $f(x) > f(x_0) - \epsilon$ for all $x \in U$.

Definition 3.2. *Let (S, τ_1, τ_2) be a bispace and $f : S \to \mathbb{R}$ be a function. Then f is said to be τ_i-upper (lower) semicontinuous if $f : (S, \tau_i) \to \mathbb{R}$ is upper (lower) semicontinuous, for $i = 1, 2$.*

Definition 3.3. *Let (S, τ_1, τ_2) be a bispace. Then a (τ_i, τ_j)-characteristic function for a bornology $\mathcal{B}$ on S is a real-valued τ_i-upper semicontinuous and τ_j-lower semicontinuous function f such that*

$$\mathcal{B} = \{A \subseteq S : \sup\{f(x) : x \in A\} < +\infty\}$$

for $i, j \in \{1, 2\}$.

Definition 3.4. *A bispace (S, τ_1, τ_2) is said to have property $(*)$ if, given a set P and a τ_1-open set Q such that τ_2-cl$(P) \subseteq Q$, there exist a τ_1-open set G and a set F such that*

$$\tau_2\text{-}cl(P) \subseteq G \subseteq \tau_2\text{-}cl(F) \subseteq Q.$$

Lemma 3.1. *If (S, τ_1, τ_2) satisfies property $(*)$, then given a set G and a τ_1-closed set H such that τ_2-cl$(G) \cap H = \emptyset$, there exists a real-valued function α on S such that*

(i) $\alpha(x) = 0$ for all $x \in \tau_2$-cl(G), $\alpha(x) = 1$ for all $x \in H$, and $0 \le \alpha(x) \le 1$ for all $x \in S$;

(ii) α is τ_1-upper semicontinuous and τ_2-lower semicontinuous.

Proof. Let G and H be subsets of S such that H is τ_1-closed and τ_2-cl$(G) \cap H = \emptyset$. Let $G_0 = \tau_2$-cl(G) and let $K_1 = S \setminus H$. Then K_1 is τ_1-open and $G_0 \subseteq K_1$. Since (S, τ_1, τ_2) satisfies the property $(*)$, there exists a τ_1-open set $K_{\frac{1}{2}}$ and a set $G_{\frac{1}{2}}$ such that

$$G_0 \subseteq K_{\frac{1}{2}} \subseteq \tau_2\text{-}\mathbf{cl}\left(G_{\frac{1}{2}}\right) \subseteq K_1.$$

By applying the hypothesis of (S, τ_1, τ_2) to each pair of sets G_0, $K_{\frac{1}{2}}$ and τ_2-cl$(G_{\frac{1}{2}})$, K_1, we obtain τ_1-open sets $K_{\frac{1}{4}}$, $K_{\frac{3}{4}}$ and two sets $G_{\frac{1}{4}}$, $G_{\frac{3}{4}}$ such that

$$\tau_2\text{-cl}(G_0) \subseteq K_{\frac{1}{4}} \subseteq \tau_2\text{-}\mathbf{cl}\left(G_{\frac{1}{4}}\right) \subseteq K_{\frac{1}{2}} \subseteq \tau_2\text{-}\mathbf{cl}\left(G_{\frac{1}{2}}\right) \subseteq K_{\frac{3}{4}} \subseteq \tau_2\text{-}\mathbf{cl}\left(G_{\frac{3}{4}}\right) \subseteq K_1.$$

Continuing this process, we generate two families $\{G_s\}$ and $\{K_s\}$, where $s = \frac{p}{2^q}$, $p = 1, 2, \cdots, 2^q - 1$, and $q = 1, 2, \cdots$. For any other dyadic rational s, set $K_s = \emptyset$ for $s \le 0$, $K_s = X$ for $s > 1$, and $G_s = \emptyset$ for $s < 0$, $G_s = X$ for $s \ge 1$. Then

$$K_r \subseteq K_s \subseteq \tau_2\text{-}\mathbf{cl}(G_s) \subseteq \tau_2\text{-}\mathbf{cl}(G_t)$$

for $r \le s \le t$, and τ_2-cl$(G_s) \subseteq K_t$ for $s < t$. Let α be the function from S to $[0, 1]$ defined by

$$\alpha(x) = \inf\{t : x \in K_t\} \ for \ x \in S.$$

Then

$$\alpha(x) = \inf\{t : x \in \tau_2\text{-}\mathbf{cl}(g_t)\} \ for \ x \in S.$$

Clearly, $0 \le \alpha(x) \le 1$ for $x \in S$, $\alpha(x) = 0$ for $x \in G_0$, and $\alpha(x) = 1$ for $x \in S/K_1 = H$. As in the proof of the classical Urysohn lemma, we can show, using the sets K_s, that α is τ_1-upper semicontinuous, and using the sets τ_2-cl(G_s), that α is τ_2-lower semicontinuous. This completes the proof. □

Now, we are in a position to prove the main result.

Theorem 3.1. *A bornological bispace $((S, \tau_1, \tau_2), \mathcal{B})$ with property $(*)$ admits a (τ_1, τ_2)-characteristic function α if and only if S is weakly locally bounded, the bornology $\mathcal{B}$ is τ_1-open and τ_2-closed, and S has a countable base.*

Proof. Assume that the bornological bispace $((S,\tau_1,\tau_2),\mathcal{B})$ admits a (τ_1,τ_2)-characteristic function α. Let $p \in S$ and denote $q = f(p)$. Let us choose a real number $b > q$. Let B_b denote the subset of S that consists of all points $x \in S$ satisfying $\alpha(x) < b$, i.e., $B_b = \{x \in S : \alpha(x) < b\}$. From the τ_1-upper semicontinuity of α, it follows that B_b is an τ_1-open set. According to the definition of a (τ_1,τ_2)-characteristic function, B_b is bounded. Since $p \in B_b$, p is a τ_1-finite point in S. Thus, S is weakly locally bounded.

Next, let B denote an arbitrary bounded set. We choose a real number t such that

$$t > \sup\{\alpha(x) : x \in B\}.$$

Here $\sup\{\alpha(x) : x \in B\}$ is finite, as B is bounded. Now, we define $B_t = \{x \in S : \alpha(x) < t\}$. From the τ_1-upper semicontinuity of α, it follows that B_t is an τ_1-open set. Also, B_t is bounded. So, B_t is a bounded τ_1-open set containing B. Thus, $\mathcal{B}$ is τ_1-open boundedness.

By τ_2-lower semicontinuity of α, the set $\{x \in S : \alpha(x) \le t\}$ is τ_2-closed. So

$$\tau_2\text{- }\mathbf{cl}(B_t) \subseteq \{x \in S : \alpha(x) \le t\}.$$

By the definition of a (τ_1,τ_2)-characteristic function, the set $\{x \in S : \alpha(x) \le t\}$ is bounded, and a subset of a bounded set is bounded. So, τ_2- $\mathbf{cl}(B_t)$ is a bounded set. Also, $B \subseteq B_t$ implies that

$$\tau_2\text{-}\mathbf{cl}(B) \subseteq \tau_2\text{-}\mathbf{cl}(B_t).$$

Since τ_2-$\mathbf{cl}(B_t)$ is a bounded set, τ_2-$\mathbf{cl}(B)$ is also bounded. Hence, $\mathcal{B}$ is τ_2-closed boundedness.

Now, for every positive integer n, let $B_n = \{x \in S : \alpha(x) < n\}$. Then, $\mathcal{C} = \{B_n : n \in \mathbb{N}\}$ is a sequence of bounded τ_1-open sets. Let B denote an arbitrary bounded set of S. Choose an integer n satisfying

$$n > \sup\{\alpha(x) : x \in B\}.$$

Then, we have $B \subseteq B_n$. Hence, $\mathcal{C}$ is a countable base of the boundedness $\mathcal{B}$.

Conversely, assume that $((S,\tau_1,\tau_2),\mathcal{B})$ is a bornological bispace with property $(*)$ satisfying the conditions of the theorem. If S is bounded, then the constant zero function on S is a characteristic function of S. Hereafter, we assume that S is not bounded. Since S is weakly locally bounded, S cannot have a maximal bounded set. It follows from Proposition 2.2 that the boundedness of S has a base $\mathcal{C}$, which consists of a strictly increasing sequence of bounded τ_1-open sets $G_1, G_2, \cdots, G_n, \cdots$, satisfying τ_2-$\mathbf{cl}(G_n) \subseteq G_{n+1}$ for every $n = 1, 2, \cdots$. From Lemma 3.1, it follows that there exists a real-valued funtion ϕ_n defined on S such that

(i) $\phi_n(x) = 0$ for every $x \in \tau_2$-$\mathbf{cl}(G_n)$, $\phi_n(x) = 1$ for every $x \in S \setminus G_{n+1}$, and $0 \le \phi_n(x) \le 1$ for every $x \in S$;

(ii) ϕ_n is τ_1-upper semicontinuous and τ_2-lower semicontinuous.

Now, we define a function $\alpha : S \to \mathbb{R}$ as

$$\alpha(x) = n - 1 + \phi_n(x)$$

for every $x \in G_{n+1} \setminus G_n$ and for every n. Next, we examine the τ_1-upper semicontinuity and τ_2-lower semicontinuity of α. For $x \in G_{n+1} \setminus G_n$, $\alpha(x) = n - 1 + \phi_n(x)$, where ϕ_n is τ_1-upper semicontinuous. This means that for any $c \in \mathbb{R}$, the set $\{x \in S : \phi_n(x) < c - (n-1)\}$ is τ_1-open. Thus, α is τ_1-upper semicontinuous on S. Also, the fact that ϕ_n is τ_2-lower semicontinuous on S implies that α is τ_2-lower semicontinuous on S.

Now, let E denote an arbitrary subset of S. If E is bounded, then there exists a positive integer n satisfying $E \subseteq G_{n+1}$. This leads to the conclusion that $\alpha(x) \le n$ for every $x \in E$. That is, $\sup\{\alpha(x) : x \in E\}$ is finite. Conversely, assume that $\sup\{\alpha(x) : x \in E\}$ is finite. We choose a sufficiently large positive integer n such that $E \subseteq G_{n+1}$. In fact, if for some $x \in E$, we have $x \notin G_{n+1}$, then $x \in G_{n+2} \setminus G_{n+1}$ (say), and in that case, we have

$$\alpha(x) = (n+1) - 1 + \phi_{n+1}(x) = n + \phi_{n+1}(x).$$

As $0 \le \phi_{n+1}(x) \le 1$, we obtain $\alpha(x) \ge n$, which is a contradiction. So, $E \subseteq G_{n+1}$, and E is bounded. Therefore, α is a (τ_1,τ_2)-characteristic function of S. $\square$

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