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Research Article

Numerical analysis of a quasilinear parabolic problem with variable exponent

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Abstract

This paper deals with the numerical approximation of the mild solution of a quasilinear parabolic equation with variable exponent. Under some conditions, it is shown that the mild solution is a weak solution. Numerical tests are performed using the split Bregman method. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

Keywords: Leray-Lions operator with variable exponent; parabolic equation; numerical; iterative method; mild solution.

2020 Mathematics Subject Classification: 35K59.

1. Introduction

Let $\Omega \in \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with a smooth boundary $\partial\Omega$, and let T be a positive number. Our main goal in this paper is to make the numerical approximation of the mild solution of the following problem involving a quasilinear elliptic operator of Leray-Lions type with variable exponent:

$$\begin{cases} u_t - \displaystyle\sum_{j=1}^{d} \frac{\partial \mathbf{a}_j(x, \nabla u)}{\partial x_j} = f(x,u) & \text{in } Q :\equiv \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma :\equiv \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

In recent years, a growing number of researchers have focused on studying different mathematical problems with variable exponents. The applications of these problems are intriguing, and their use in electro-rheological fluids (also known as smart fluids) motivates their research. The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field (see [24] for details). Another noticeable application is related to image processing, where this kind of diffusion operator is used to identify the borders of a distorted image and to eliminate the noise (for example, see [8, 10, 17]). However, the study of these problems raises many difficult mathematical questions.

There is an abundant literature devoted to addressing questions on the existence and uniqueness of solutions to problem (1) (see [15]) and to stationary problems associated with (1) where $(a_j(x,\xi))_j = |\xi|^{p-2}\xi$ (for instance, see [2, 11, 12] and references cited therein). From a numerical analysis point of view, there is also an extensive amount of literature devoted to answering questions on the approximation of solutions to problem (1) when $(a_j(x,\xi))_j$ is either a $p(x)$-Laplacian or $p(x)$-Laplacian type operator and f does not depend on u (see [5–7, 10, 17, 20–22]).

Regarding the existing literature on quasilinear parabolic equations with variable exponent, we consider in the present paper a more general class of operators of Leray-Lions type than $p(x)$-Laplacian operator (see [14]) and we make a numerical approximation of the mild solution of problem (1). For this, we use the same technical tools as adopted in [19] while dealing with the case where $p(\cdot)$ is a constant exponent.

The remaining part of this paper is structured as follows. In Section 2, we recall the definitions of Lebesgue and Sobolev spaces with variable exponent and some of their properties; in addition, we formulate assumptions and give some results related to problem (1). We recall the notion of the mild solution in Section 3. We proceed to the numerical analysis in Section 4, where we demonstrate the existence and uniqueness of our numerical scheme's solution as well as its convergence; we then show that the mild solution is also a weak solution under a certain condition, and finally, we conclude this section with numerical tests by using the split Bregman algorithm.

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2. Mathematical preliminaries and assumptions

In what follows, we recall some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$, where $\Omega \subset \mathbb{R}^d (d \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$ and $p : \Omega \to (1, \infty)$ is a continuous bounded function, called a variable exponent on Ω. We refer the reader to [9, 18, 23] and references cited therein for details about variable exponent Lebesgue-Sobolev spaces.

Let $\mathcal{P}(\Omega)$ be the family of all measurable functions $q : \Omega \to (1, \infty)$ and set

$$\mathcal{P}^{\log} \stackrel{\text{def}}{=} \left\{ q \in \mathcal{P}(\Omega) : \ \frac{1}{q} \text{ globally Hölder continuous} \right\}.$$

In particular, for any $p \in \mathcal{P}^{\log}(\Omega)$, there exists a function ω such that

$$\forall (x, y) \in \Omega^2, \ \ |p(x) - p(y)| \leq \omega(|x - y|) \text{ and } \limsup_{t \to 0^+} (-\omega(t) \ln t) < \infty.$$

We define the variable exponent Lebesgue space and the corresponding Sobolev space as follows:

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable with } \int_\Omega |u(x)|^{p(\cdot)} dx < \infty \right\}$$

and

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

We recall that $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are normed linear spaces equipped respectively with the following norms:

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf \left\{ \mu > 0; \int_\Omega |\frac{u(x)}{\mu}|^{p(\cdot)} dx \leq 1 \right\}$$

and

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

We define $\mathbb{W} \stackrel{\text{def}}{=} W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. In the sequel, we assume that

$$p \in C(\overline{\Omega}) \cap \mathcal{P}^{\log}(\Omega) \text{ such that } 1 \leq \frac{2d}{d+2} < p^- \leq p^+ < d, \tag{2}$$

where

$$p^- := \inf_{x \in \overline{\Omega}} p(x) \text{ and } p^+ := \sup_{x \in \overline{\Omega}} p(x).$$

Due to (2), since Ω is a bounded domain, the Poincaré inequality holds and a natural norm of $\mathbb{W}$ is

$$\|u\|_{\mathbb{W}} = \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

For the vector field, we set

$$a_j(x, \xi) = \phi(x, |\xi|)\xi_j \text{ for all } \xi \in \mathbb{R}^d \text{ and } j = 1, \ldots, d,$$

such that ϕ is differentiable on $\Omega \times (0, \infty)$ and $\phi(x, s) > 0$ for $(x, s) \in \Omega \times (0, \infty)$. We define the function $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ as

$$\Phi(x, \kappa) = \int_0^\kappa \phi(x, |s|) s \, ds,$$

which is increasing on $\mathbb{R}^+$. Also, we define $A : \Omega \times \mathbb{R} \to \mathbb{R}$ by $A(x, \xi) = \Phi(x, |\xi|)$. We assume that a_j satisfies the following structural conditions:

$$a_j(x, \mathbf{0}) = 0 \quad \text{for almost every } x \in \Omega, \tag{3}$$

$$a_j(x, \xi) \in C^1(\Omega \times (\mathbb{R}^d \setminus \{\mathbf{0}\})) \cap C^0(\Omega \times \mathbb{R}^d), \tag{4}$$

$$\sum_{i,j=1}^d \frac{\partial a_j(x, \xi)}{\partial \xi_j} \eta_i \eta_j \geq \gamma |\xi|^{p(x)-2} |\eta|^2, \forall x \in \Omega, \forall \xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \forall \eta \in \mathbb{R}^d, \tag{5}$$

$$\sum_{i,j=1}^d \left| \frac{\partial a_j(x, \xi)}{\partial \xi_i} \right| \leq \Gamma |\xi|^{p(x)-2}, \forall x \in \Omega, \xi \in \mathbb{R}^d \setminus \{\mathbf{0}\}, \tag{6}$$

for some positive constants γ and Γ.

Using (6), we deduce that there exists a positive constant C_1 such that

$$|\mathbf{a}(x,\xi)| \le C_1|\xi|^{p(x)-1}, \quad \phi(x,|\xi|) \le C_1|\xi|^{p(x)-2}, \quad \forall x \in \Omega \text{ and } \xi \in \mathbb{R}^d, \tag{7}$$

where $\mathbf{a}(x,\xi) := (a_j(x,\xi))_j$, with $j = 1,\dots,d$. This notation will be used frequently in the rest of the paper. Finally, we assume that

$$(0,\infty) \ni \kappa \mapsto \Phi(x,\sqrt{\kappa}) \text{ is convex for a.e. } x \in \Omega^+ := \{s \in \Omega;\, p(s) \ge 2\}\,. \tag{8}$$

Remark 2.1. *From (3), (5), and (6), it follows that $\xi \mapsto A(x,\xi)$ is strictly convex and satisfies the following chain of inequalities for any fixed $x \in \Omega$:*

$$\frac{\gamma}{p^+-1}|\xi|^{p(x)} \le A(x,\xi) \le \frac{\Gamma}{p^--1}|\xi|^{p(x)}, \quad \forall \xi \in \mathbb{R}^d. \tag{9}$$

Remark 2.2. *Examples of the vector field satisfying the conditions (3)–(6) and (8) for problem (1), include the following:*

i) $\boldsymbol{a}\,(x,\xi) = |\xi|^{p(x)-2}\xi$.

ii) $\boldsymbol{a}\,(x,\xi) = (1+|\xi|^2)^{\frac{p(x)-2}{2}}\xi$.

For the reaction term $f(x,u)$, we assume the following:

(f_1) $f(x,s) \not\equiv 0$ is a Carathéodory function and $s \mapsto f(x,s)$ is locally Lipschitz uniformly in $x \in \Omega$.

(f_2) There exists $s_0 \in \mathbb{R}$ such that $x \mapsto f(x,s_0) \in L^2(\Omega) \cap L^q(\Omega)$ with $q > \frac{d}{p^-}$.

(f_3) f is nonincreasing with respect to the second variable and $x \mapsto f(x,0) \in L^\infty(\Omega)$.

(f_4) $\liminf\limits_{s\to 0} \dfrac{|f(x,s)|}{|s|^{p^--1}} > \Gamma\Lambda^{p^-}(p^-)_c$ for any $x \in \Omega$, where

$$(p^-)_c = \frac{p^-}{p^--1} \text{ and } \Lambda := \left(\sup_{\|u\|_{\mathbb{W}}=1} \|u\|_{L^{p^-}(\Omega)}\right)^{-1}.$$

(f_5) $\lim\limits_{|s|\to\infty} \sup \dfrac{|f(x,s)|}{|s|^{p^--1}} < \gamma\Lambda^{p^-}(p_c)^-$ for any $x \in \Omega$, where $(p_c^-) = \frac{p^-}{p^+-1}$.

Remark 2.3. *A prototype example for f satisfying all the conditions (f_1)–(f_5) is the following:*
$f(x,s) = -|s|^{r(x)-2}s$ with $r \in C(\Omega)$ such that $1 < r(x) < p^-$ for every $x \in \overline{\Omega}$.

We also recall the following Sobolev embedding theorem (see [9]):

Theorem 2.1. *Let $p \in \mathcal{P}^{\log}(\Omega)$ such that $1 \le p^- \le p^+ < d$. Then, $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ for any $\alpha \in L^\infty(\Omega)$ such that for every $x \in \Omega$, it holds that*

$$\alpha(x) \le p^*(x) = \frac{dp(x)}{d-p(x)}.$$

Also, the previous embedding is compact for $\alpha(x) < p^*(x) - \varepsilon$ a.e. in Ω, for any $\varepsilon > 0$.

We also need the following proposition (see [13, 14]) to prove the L^∞ uniform boundedness of our numerical solution:

Proposition 2.1 (Corollary A.2 in [14])**.** *Assume that conditions (3)–(6) hold. Let $p \in C(\overline{\Omega})$ such that $p^- < d$ and $u \in \mathbb{W}$ with*

$$\int_\Omega \boldsymbol{a}(x,\nabla u).\nabla\varphi\,dx = \int_\Omega (\tilde{f}(x,u)+\tilde{g})\varphi\,dx, \qquad \forall\ \varphi \in \mathbb{W},$$

where $\tilde{f}$ satisfies

$$|\tilde{f}(x,s)| \le C_1 + C_2|s|^{r(x)-1}$$

with $r \in C(\overline{\Omega})$ and $\forall\ x \in \overline{\Omega}$, $1 < r(x) < p^*(x)$, while $\tilde{g} \in L^q(\Omega)$ and $q > \frac{d}{p^-}$. Then, $u \in L^\infty(\Omega)$.

Using the following lemma, we approximate fixed points of the non-expansive mapping that is defined in Section 3:

Lemma 2.1 (see [19]). *Let X be a Banach space and C be a convex subset of X, containing 0. Let $\tilde{T}$ be a non-expansive operator on C such that $\tilde{T}(C) \subseteq C$, admitting a unique fixed point x^* in C. Let λ_k be a sequence of $(0,1)$ such that*

$$\lim_{k\to\infty} \lambda_k = 1, \quad \prod_{k\geq 0} \lambda_k = 0, \quad \text{and} \quad \sum_{k\geq 0} |\lambda_{k+1} - \lambda_k| < \infty.$$

Let (x^k) be the sequence generated by the following iterative scheme:

$$x^0 \in C, \quad x^{k+1} = \lambda_{k+1} \tilde{T}(x^k). \tag{10}$$

Then, $\lim_{k\to\infty} \|x^k - \tilde{T}(x^k)\|_X = 0$.

3. Notion of a mild solution

Let A_0 be the operator in $L^\infty(\Omega)$ defined as $A_0 u := -\nabla \mathbf{a}(x, \nabla u)$ with domain $\mathcal{D}(A_0) = \{u \in \mathbb{W} \cap L^\infty(\Omega) \mid A_0 u \in L^\infty(\Omega)\}$. A classical method for solving problem (1) is approximating (1) for $\epsilon > 0$, by an implicit time discretization. Let $u_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$, and consider the time discretization of problem (1):

$$\begin{cases} \frac{u^\epsilon_{n+1} - u^\epsilon_n}{t_{n+1} - t_n} - \nabla.\mathbf{a}(x, \nabla u^\epsilon_{n+1}) - f(x, u^\epsilon_{n+1}) \ni 0 \text{ in } \mathcal{D}'(\Omega) \text{ for } n = 0, \ldots, N-1, \\ u^\epsilon_0 = u_0, \\ u^\epsilon_{n+1} \in \mathbb{W} \cap L^\infty(\Omega); \end{cases} \tag{11}$$

where

$$\begin{cases} 0 = t_0 < t_1 < \cdots < t_N \leq T \text{ is a partition of } [0,T], \text{ such that } t_n - t_{n-1} \leq \epsilon \text{ for } n = 1, \ldots, N, \\ u^\epsilon \text{ is the piecewise constant function defined by } u^\epsilon(t) = u^\epsilon_n \text{ on } (t_{n-1}, t_n] \text{ with } n = 1, \ldots, N; \;\; u^\epsilon(0) = u^\epsilon_0. \end{cases} \tag{12}$$

This method is called the method of nonlinear semigroups theory [4].

Definition 3.1. *A mild solution of (1) is a measurable function $u \in C([0,T]; \mathbb{W})$ such that, for every $\epsilon > 0$, there exists $(u^\epsilon_1, \ldots, u^\epsilon_N)$ verifying (11) provided that $\|u(t) - u^\epsilon(t)\|_{L^\infty(\Omega)} \leq \epsilon$ for every $t \in (t_{n-1}, t_n]$, $n = 1, \ldots, N$. This mild solution u is the uniform limit of the piecewise constant function u^ϵ.*

Remark 3.1. *In this paper, for the sake of simplicity and readability, we choose to present the constant step subdivision algorithm, where we set $t_{n+1} - t_n = h = \frac{T}{N}$ for all $n = 0, \ldots, N-1$. However, the techniques developed thereafter can be easily adapted to a different step subdivision.*

Note that using the theory of maximal accretive operators in Banach spaces [3,4], Giacomoni, Rădulescu, and Warnault [14] proved the existence and uniqueness of the mild solution of problem (1). In the next section, we approximate this mild solution.

4. Numerical study

4.1. Numerical scheme

We adopt the numerical approach to solve problem (11). Our idea is to approximate the mild solution of problem (1) using the fixed-point methods. Therefore, we must solve the implicit scheme (11). Consequently, we use the following iterative scheme (inspired by Maitre [19]) to obtain u^ϵ_{n+1} from u^ϵ_n:

$$\begin{cases} \text{Let } u^{\epsilon,0}_{n+1} = u^\epsilon_n \in \overline{\mathcal{D}(A_0)}^{L^\infty} \text{ and solve the following for } k = 0, 1, \ldots, \\ u^{\epsilon,k+1}_{n+1} - \rho \nabla.\mathbf{a}\left(x, \nabla u^{\epsilon,k+1}_{n+1}\right) = \left(1 - \frac{\rho}{h}\right) \lambda_k u^{\epsilon,k}_{n+1} + \rho f(x, \lambda_k u^{\epsilon,k}_{n+1}) + \frac{\rho}{h} u^\epsilon_n, \end{cases} \tag{13}$$

where $\rho > 0$ is a given parameter and $(\lambda_k)_{k\in\mathbb{N}}$ is a sequence of $(0,1)$ such that

$$\lim_{k\to\infty} \lambda_k = 1, \quad \prod_{k\geq 0} \lambda_k = 0, \quad \text{and} \quad \sum_{k\geq 0} |\lambda_{k+1} - \lambda_k| < \infty. \tag{14}$$

Remark 4.1. *It is obvious that*

$$\lambda_k = 1 - \frac{1}{k+1} \quad \textit{or} \quad \lambda_k = 1 - e^{-k} \quad \textit{satisfies (14).}$$

So, for the numerical tests, we will take $\lambda_k = 1 - \frac{1}{k+1}$.

Recall that the introduction of λ_k in the scheme is an application of the ideas of the Halpern algorithm (see [16]) for finding the fixed points of non-expansive mappings. Thus, the goal is to make the scheme sufficiently contractive to ensure convergence while allowing the constant to approach 1 so that the solution belongs to the non-contracting case.

4.2. Existence and uniqueness of the solution of problem (13)

First, we define the weak solution of problem (13).

Definition 4.1. *For any* $n = 0, \dots, N-1$, $\epsilon > 0$ *and* $u_n^\epsilon \in \overline{\mathcal{D}(A_0)}^{L^\infty}$*, a weak solution of* (13) *is a sequence* $\left(u_{n+1}^{\epsilon,k+1}\right)_{k\geq 0}$ *such that* $u_{n+1}^{\epsilon,k+1} \in \mathbb{W} \cap L^\infty(\Omega)$ *for all* $k = 0, 1, \dots,$ *and*

$$\int_\Omega u_{n+1}^{\epsilon,k+1} \varphi\, dx + \rho \int_\Omega \boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla\varphi\, dx = \int_\Omega g(x, u_{n+1}^{\epsilon,k})\varphi\, dx, \tag{15}$$

for every $\varphi \in \mathbb{W}$*, where*

$$g(x, u_{n+1}^{\epsilon,k}) := \left(1 - \frac{\rho}{h}\right)\lambda_k u_{n+1}^{\epsilon,k} + \rho f(x, \lambda_k u_{n+1}^{\epsilon,k}) + \frac{\rho}{h} u_n^\epsilon.$$

We now state the result concerning the existence and uniqueness of the mild solution of problem (13).

Theorem 4.1. *Assume that conditions* (3)–(6) *and* (9) *hold. Let* $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a function satisfying conditions* (f_1)–(f_2)*. For any* $n = 0, \dots, N-1$*, let* $\epsilon > 0$ *and* $u_{n+1}^{\epsilon,0} = u_n^\epsilon \in \overline{\mathcal{D}(A_0)}^{L^\infty}$*. Then, problem* (13) *admits a unique weak solution* $u_{n+1}^{\epsilon,k+1} \in \mathbb{W} \cap L^\infty(\Omega)$*, for all* $k = 0, 1, \dots$*.*

Proof. Fix n and let $\epsilon > 0$. For $k = 0$, we rewrite problem (13) as follows:

$$\begin{cases} u_{n+1}^{\epsilon,1} - \rho\, \nabla.\mathbf{a}\,(x, \nabla u_{n+1}^{\epsilon,1}) &= g(x, u_n^\epsilon) \quad \text{in } \Omega \\ u_{n+1}^{\epsilon,1} &= 0 \qquad\quad\;\; \text{on } \partial\Omega, \end{cases} \tag{16}$$

where

$$g(x, u_n^\epsilon) = \left(1 - \frac{\rho}{h}\right)\lambda_0 u_n^\epsilon + \rho f(x, \lambda_0 u_n^\epsilon) + \frac{\rho}{h} u_n^\epsilon.$$

Now, we define the energy functional J_ρ on $\mathbb{W}$ associated to (16) by

$$J_\rho(U) = \frac{1}{2}\int_\Omega U^2 dx + \rho \int_\Omega A\,(x, \nabla U)\,dx - \int_\Omega g(x, u_n^\epsilon) U dx.$$

We will establish that $J_\rho(U)$ has a minimizer $u_{n+1}^{\epsilon,1}$ in $\mathbb{W}$. The conditions (f_1) and (f_2) ensure that $f(\cdot, \lambda_0 u_n^\epsilon) \in L^q(\Omega)$ with $q > \frac{d}{p^-}$. Thus, $g(\cdot, u_n^\epsilon) \in L^q(\Omega)$. By (9), we note that J_ρ is well-defined and Gâteaux differentiable on $\mathbb{W}$. Moreover, $q > \frac{d}{p^-}$ and $1 < p^- \leq p^+ < d$ imply that $L^q \subset (L^{p^*(x)})'$. By Theorem 2.1 and (9), for $\|U\|_{\mathbb{W}} \geq 1$,

$$J_\rho(U) \geq \frac{\rho\gamma}{p^+(p^+-1)}\|U\|_{\mathbb{W}}^{p^-} - C\|U\|_{\mathbb{W}}.$$

Thus, J_ρ is coercive. Consequently, J_ρ admits a global minimizer $u_{n+1}^{\epsilon,1} \in \mathbb{W}$.

It remains to show that $u_{n+1}^{\epsilon,1} \in L^\infty(\Omega)$. For this, we set

$$\tilde{\mathbf{a}}\,(x, \nabla u_{n+1}^{\epsilon,1}) = \rho\, \mathbf{a}\,(x, \nabla u_{n+1}^{\epsilon,1}), \quad \tilde{f}(x, u_{n+1}^{\epsilon,1}) = -u_{n+1}^{\epsilon,1}, \quad \text{and} \quad \tilde{g}(\cdot, u_n^\epsilon) = g(\cdot, u_n^\epsilon).$$

It is clear that $\tilde{\mathbf{a}}$ verifies conditions (3)–(6), $\tilde{g}(\cdot, u_n^\epsilon) \in L^q(\Omega)$ with $q > \frac{d}{p^-}$ and $\tilde{f}$ satisfies for every $(x, \xi) \in \Omega \times \mathbb{R}$,

$$|\tilde{f}(x, \xi)| \leq C_2|\xi|^{r(x)-1}$$

with $r \in C(\overline{\Omega})$, and $\forall x \in \overline{\Omega}$, $1 < r(x) < p^*(x)$. Therefore, applying Proposition 2.1, we conclude that $u_{n+1}^{\epsilon,1} \in L^\infty(\Omega)$. Thus, $u_{n+1}^{\epsilon,1} \in \mathbb{W} \cap L^\infty(\Omega)$.

By (3), (5), and (6), J_ρ is strictly convex on $\mathbb{W}$, which guarantees the uniqueness of the critical point, and hence, the uniqueness of the weak solution of problem (16).

By induction, we deduce in the same manner as above that problem (13) has a unique weak solution $\left(u_{n+1}^{\epsilon,k+1}\right)_{k\geq 0}$ such that $u_{n+1}^{\epsilon,k+1} \in \mathbb{W} \cap L^\infty(\Omega)$ for every $k \in \mathbb{N}$. $\square$

4.3. Convergence of scheme (13)

In order to establish the convergence of the whole sequence $(u_{n+1}^{\epsilon,k+1})_{k\geq 0}$, we start with the following lemma, which provides an immediate recurrence and a crucial L^∞ uniform bound for this sequence.

Lemma 4.1. *Let $\epsilon > 0$. For any $n = 0, \ldots, N$, set $M_n := M_0 + nh\|f(\cdot, 0)\|_\infty$, where $M_0 = \|u_0\|_\infty$ and $\|u_n^\epsilon\|_\infty \leq M_n$. Then, for every $k \in \mathbb{N}$,*

$$\|u_{n+1}^{\epsilon,k}\|_\infty \leq M_{n+1} := M_n + h\|f(\cdot, 0)\|_\infty \text{ if } \rho \leq \frac{2h}{2 + hL_{M_{n+1}}},$$

where

$$L_{M_{n+1}} := \sup_{|\xi|\leq M_{n+1}, x\in\Omega} \left| \frac{\partial}{\partial\xi} f(x, \xi) \right|$$

is the Lipschitz constant of f.

Remark 4.2. *Lemma 4.1 appears to require an adaptive step size; however, for a fixed final time T, one can choose a value of ρ independent of n, by taking $L = L_{M_0+T\|f(\cdot,0)\|_\infty}$. Indeed, by the construction, M_{n+1} is upper bounded by $M_0 + T\|f(\cdot, 0)\|_\infty$ for all n and h. Thus, one does not need any adaptive choice of ρ. This value can indeed be "pessimistic" compared to the values that work in practice.*

To prove Lemma 4.1, we first need the following result:

Lemma 4.2. *Let $\epsilon > 0$ and fix n. Let $u_{n+1}^{\epsilon,k+1} \in \mathbb{W} \cap L^\infty(\Omega)$ be a solution of problem (13). Then, $\|u_{n+1}^{\epsilon,k+1}\|_\infty \leq \|g(\cdot, u_{n+1}^{\epsilon,k})\|_\infty$ for all $k = 0, 1, \ldots$.*

Proof. Let $u_{n+1}^{\epsilon,k+1} \in \mathbb{W} \cap L^\infty(\Omega)$ be the weak solution of (13). We fix ψ such that

i) ψ is strictly increasing on $(0, +\infty)$

ii) $\psi(s) = 0$ for $s \in (-\infty, 0]$.

Let $\tau \in \mathbb{R}_+$. We shall show that $u \leq \|g(x, u_{n+1}^{\epsilon,k})\|_\infty$ on Ω. Note that $\psi(u_{n+1}^{\epsilon,k+1} - \tau) \in \mathbb{W}$. By plugging $\psi(u_{n+1}^{\epsilon,k+1} - \tau)$ into (15), we obtain

$$\int_\Omega u_{n+1}^{\epsilon,k+1}\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx + \rho\int_\Omega \mathbf{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx = \int_\Omega g(x, u_{n+1}^{\epsilon,k})\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx.$$

Since Ω is a bounded open domain, we have

$$\int_\Omega (u_{n+1}^{\epsilon,k+1} - \tau)\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx + \rho\int_\Omega \mathbf{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx = \int_\Omega (g(x, u_{n+1}^{\epsilon,k}) - \tau)\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx.$$

Note also that

$$I := \rho\int_\Omega \mathbf{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx \geq 0.$$

Indeed, we have

$$I = \rho\int_\Omega \mathbf{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla u_{n+1}^{\epsilon,k+1}\psi'(u_{n+1}^{\epsilon,k+1} - \tau)\, dx = \rho\sum_{j=1}^{d}\int_\Omega \phi(x, |\nabla u_{n+1}^{\epsilon,k+1}|)(\nabla u_{n+1}^{\epsilon,k+1})_j^2\psi'(u_{n+1}^{\epsilon,k+1} - \tau)\, dx \geq 0,$$

as $\varphi(x, s) > 0$ for $(x, s) \in \Omega \times (0, +\infty)$ and ψ is strictly increasing on $(0, +\infty)$. Thus, we have

$$\int_\Omega (u_{n+1}^{\epsilon,k+1} - \tau)\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx \leq \int_\Omega (g(x, u_{n+1}^{\epsilon,k}) - \tau)\psi(u_{n+1}^{\epsilon,k+1} - \tau)\, dx.$$

By choosing $\tau = \|g(x, u_{n+1}^{\epsilon,k})\|_\infty$, we obtain

$$g(x, u_{n+1}^{\epsilon,k}) - \|g(x, u_{n+1}^{\epsilon,k})\|_\infty \leq 0.$$

As $\psi(u_{n+1}^{\epsilon,k+1} - \|g(x, u_{n+1}^{\epsilon,k})\|_\infty) \geq 0$, it follows that

$$(u_{n+1}^{\epsilon,k+1} - \|g(x, u_{n+1}^{\epsilon,k+1})\|_\infty)\psi(u_{n+1}^{\epsilon,k} - \|g(x, u_{n+1}^{\epsilon,k})\|_\infty) \leq 0.$$

Since $s\psi(s) \geq 0$ for every $s \in \mathbb{R}$, the above inequality implies that

$$(u_{n+1}^{\epsilon,k+1} - \|g(x, u_{n+1}^{\epsilon,k})\|_\infty)\psi(u_{n+1}^{\epsilon,k+1} - \|g(x, u_{n+1}^{\epsilon,k})\|_\infty) = 0 \text{ a.e. in } \Omega.$$

Therefore, $u_{n+1}^{\epsilon,k+1} \leq \|g(x, u_{n+1}^{\epsilon,k})\|_\infty$ a.e. in Ω for all $k = 0, 1, 2, \ldots$. The lower bound for $u_{n+1}^{\epsilon,k+1}$ is obtained by applying this upper bound to $-u_{n+1}^{\epsilon,k+1}$. □

Proof of Lemma 4.1. Let $\epsilon > 0$ and fix n. For $k = 0$, we have $\|u_{n+1}^{\epsilon,0}\|_\infty = \|u_n^\epsilon\|_\infty \leq M_n \leq M_{n+1}$. Assume that $\|u_{n+1}^{\epsilon,k}\|_\infty \leq M_{n+1}$. We will show that $\|u_{n+1}^{\epsilon,k+1}\|_\infty \leq M_{n+1}$. By Lemma 4.2, we have

$$\left|u_{n+1}^{\epsilon,k+1}\right| \leq \left|\left(1 - \frac{\rho}{h}\right)\lambda_k u_{n+1}^{\epsilon,k} + \rho f(x, \lambda_k u_{n+1}^{\epsilon,k}) + \frac{\rho}{h} u_n^\epsilon\right|.$$

Note that Ω is the disjoint union of Ω^+ and Ω^-, where $\Omega^+ := \{s \in \Omega;\, p(s) \geq 2\}$ and $\Omega^- := \{s \in \Omega;\, p(s) < 2\}$.

First Step. Assume that $x \in \Omega^+$.

Case 1. Let $x \in \Omega^+$ such that $\lambda_k u_{n+1}^{\epsilon,k}(x) \geq 0$.
From the assumption (f_3), it follows that $f(x, \lambda_k u_{n+1}^{\epsilon,k}(x)) - f(x, 0) \leq 0$. By hypothesis (f_1), we have

$$-\rho L_{M_{n+1}}(\lambda_k u_{n+1}^{\epsilon,k}(x)) \leq \rho(f(x, \lambda_k u_{n+1}^{\epsilon,k}(x)) - f(x, 0)) \leq 0.$$

Therefore,

$$\left(1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x)) \leq g(x, u_{n+1}^{\epsilon,k}(x)) \leq \left(1 - \frac{\rho}{h}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x)).$$

So, we deduce that

$$|g(x, u_{n+1}^{\epsilon,k}(x))| \leq \max\left(\left|\left(1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x))\right|, \left|\left(1 - \frac{\rho}{h}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x))\right|\right)$$

If

$$|g(x, u_{n+1}^{\epsilon,k}(x))| \leq \left|\left(1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x))\right|,$$

then

$$\begin{aligned}
|u_{n+1}^{\epsilon,k+1}(x)| &\leq \left|1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right|\lambda_k |u_{n+1}^{\epsilon,k}(x)| + \frac{\rho}{h}|hf(x,0) + u_n^\epsilon(x)| \\
&\leq \left|1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right|\lambda_k \|u_{n+1}^{\epsilon,k}\|_\infty + \frac{\rho}{h}\left(h\|f(x,0)\|_\infty + \|u_n^\epsilon\|_\infty\right).
\end{aligned}$$

As $\rho \leq \dfrac{2h}{2 + hL_{M_{n+1}}}$, we have $\left|1 - \frac{\rho}{h} - \rho L_{M_{n+1}}\right| \leq 1 - \frac{\rho}{h}$. Therefore,

$$|u_{n+1}^{\epsilon,k+1}(x)| \leq \left(1 - \frac{\rho}{h}\right)\|u_{n+1}^{\epsilon,k}\|_\infty + \frac{\rho}{h}\left(h\|f(x,0)\|_\infty + \|u_n^\epsilon\|_\infty\right) \leq \left(1 - \frac{\rho}{h}\right)M_{n+1} + \frac{\rho}{h}M_{n+1} \leq M_{n+1}.$$

If $|g(x, u_{n+1}^{\epsilon,k}(x))| \leq \left|\left(1 - \frac{\rho}{h}\right)\lambda_k u_{n+1}^{\epsilon,k}(x) + \frac{\rho}{h}(hf(x,0) + u_n^\epsilon(x))\right|$, then

$$|u_{n+1}^{\epsilon,k+1}(x)| \leq \left|1 - \frac{\rho}{h}\right|\lambda_k |u_{n+1}^{\epsilon,k}(x)| + \frac{\rho}{h}\left(h|f(x,0)| + |u_n^\epsilon(x)|\right).$$

Since $\rho \leq \dfrac{2h}{2 + hL_{M_{n+1}}} < h$, we deduce that

$$|u_{n+1}^{\epsilon,k+1}(x)| \leq \left(1 - \frac{\rho}{h}\right)\|u_{n+1}^{\epsilon,k}\|_\infty + \frac{\rho}{h}\left(h\|f(x,0)\|_\infty + \|u_n^\epsilon\|_\infty\right) \leq \left(1 - \frac{\rho}{h}\right)M_{n+1} + \frac{\rho}{h}M_{n+1} \leq M_{n+1}.$$

Case 2. Let $x \in \Omega^+$ such that $\lambda_k u_{n+1}^{\epsilon,k}(x) < 0$.
The assumption (f_3) implies that $f(x, \lambda_k u_{n+1}^{\epsilon,k}(x)) - f(x, 0) > 0$. Following the same arguments as in Case 1, we obtain

$$|u_{n+1}^{\epsilon,k+1}(x)| \leq M_{n+1}.$$

Thus, from Cases 1 and 2, we conclude that for every $x \in \Omega^+$, it holds that $|u_{n+1}^{\epsilon,k+1}(x)| \leq M_{n+1}$.

Second Step. Assume that $x \in \Omega^-$.
Using the same way as in the first step, we deduce that for every $x \in \Omega^-$, it holds that $|u_{n+1}^{\epsilon,k+1}(x)| \leq M_{n+1}$.

From the conclusion made in the first and second steps, it follows that $\|u_{n+1}^{\epsilon,k+1}\|_\infty \leq M_{n+1}$, which completes the proof of Lemma 4.1. □

In what follows, we assume that M_{n+1} is the same number as defined in Lemma 4.1. Note that $L_{M_{n+1}}$ is the Lipschitz constant of f on $[-M_{n+1}, M_{n+1}]$. Now, we have the following convergence result:

Theorem 4.2. *Assume that conditions* (3)–(6) *and* (9) *hold. Let* $f : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ *be a function satisfying conditions* (f_1)–(f_5). *Then, for* $\rho \leq \dfrac{2h}{2+hL_{M_{n+1}}}$, *the iterative scheme* (13) *converges:*

$$\begin{aligned} u_{n+1}^{\epsilon,k} &\overset{*}{\rightharpoonup} u_{n+1}^{\epsilon} \quad \text{in} \quad L^{\infty}(\Omega), \text{ as } k\to+\infty, \\ u_{n+1}^{\epsilon,k+1} &\to u_{n+1}^{\epsilon} \quad \text{in} \quad L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega, \text{ as } k\to+\infty \end{aligned}$$

where u_{n+1}^{ϵ} *satisfies* (11). *Furthermore,*

$$\|u_{n+1}^{\epsilon}\|_{\infty} \leq M_{n+1} = \|u_0\|_{\infty} + (n+1)h\|f(\cdot,0)\|_{\infty}.$$

Proof. By Lemma 4.1, we can write (13) as

$$\frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\epsilon,k+1} - \rho\nabla.\mathbf{a}\left(x, \nabla\frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\epsilon,k+1}\right) = \left(1-\frac{\rho}{h}\right)\bar{u}_{n+1}^{\epsilon,k} + \rho f(x,\bar{u}_{n+1}^{\epsilon,k}) + \frac{\rho}{h}u_n^{\epsilon}, \tag{17}$$

where $\bar{u}_{n+1}^{\epsilon,k} = \lambda_k u_{n+1}^{\epsilon,k}$ and $\bar{u}_{n+1}^{\epsilon,k+1} = \lambda_{k+1}u_{n+1}^{\epsilon,k+1}$. We set $A_0 u = -\nabla.\mathbf{a}(x,\nabla u)$. The operator $A_0 : L^{\infty}(\Omega) \to L^{\infty}(\Omega)$ is m-accretive in $L^{\infty}(\Omega)$ with the following domain (for details, see [14, Proposition B.1]):

$$\mathcal{D}(A_0) = \{u \in \mathbb{W}\cap L^{\infty}(\Omega) \mid A_0 u \in L^{\infty}(\Omega)\}.$$

Hence, (17) yields

$$(I+\rho A_0)\left(\frac{1}{\lambda_{k+1}}\bar{u}_{n+1}^{\epsilon,k+1}\right) = \left(1-\frac{\rho}{h}\right)\bar{u}_{n+1}^{\epsilon,k} + \rho f(x,\bar{u}_{n+1}^{\epsilon,k}) + \frac{\rho}{h}u_n^{\epsilon}. \tag{18}$$

To complete the proof of Theorem 4.2, we use the following technical lemma:

Lemma 4.3. *Let* M_{n+1} *be the same number as defined in Lemma* 4.1. *Set* $C_{n+1} = \{u \in L^{\infty}(\Omega), \quad \|u\|_{\infty} \leq M_{n+1}\}$. *If*

$$\rho \leq \frac{2h}{2+hL_{M_{n+1}}},$$

then the iteration operator

$$\tilde{T}(\bar{u}) = (I+\rho A_0)^{-1}\left(\left(1-\frac{\rho}{h}\right)\bar{u} + \rho f(x,\bar{u}) + \frac{\rho}{h}u_n^{\epsilon}\right)$$

is an L^{∞}*-non-expanding operator from* C_{n+1} *to* C_{n+1}.

Proof. The fact that $\tilde{T}$ maps from C_{n+1} to C_{n+1} can be seen in the proof of Lemma 4.1, by replacing λ_k with 1. Now, let $(\bar{u},\bar{v}) \in C_{n+1}^2$. From the m-accretiveness of A_0 in L^{∞}, we conclude that $(I+\rho A_0)^{-1}$ is a contraction in L^{∞} (see [1, 4]). Therefore,

$$\begin{aligned} \|\tilde{T}(\bar{u})-\tilde{T}(\bar{v})\|_{\infty} &= \left\|(I+\rho A_0)^{-1}(\left(1-\frac{\rho}{h}\right)\bar{u} + \rho f(x,\bar{u}) + \frac{\rho}{h}u_n^{\epsilon}) - (I+\rho A_0)^{-1}(\left(1-\frac{\rho}{h}\right)\bar{v} + \rho f(x,\bar{v}) + \frac{\rho}{h}u_n^{\epsilon})\right\|_{\infty} \\ &\leq \left|1-\frac{\rho}{h}\right|\|\bar{u}-\bar{v}\|_{\infty} + \rho\|f(x,\bar{u})-f(x,\bar{v})\|_{\infty}. \end{aligned}$$

Since $\Omega = \Omega^+\cup\Omega^-$, we complete the proof of the lemma in two steps.

First Step. Let $x \in \Omega^+$.

Case 1. Assume that $\bar{u}(x)-\bar{v}(x) \geq 0$. Then, from (f_3), we have $f(x,\bar{u}(x))-f(x,\bar{v}(x)) \leq 0$. Since $f(x,.)$ is locally Lipschitz on $[-M_{n+1}, M_{n+1}]$ and $L_{M_{n+1}}$ is the Lipschitz constant, we have

$$\left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x)) - \rho L_{M_{n+1}}(\bar{u}(x)-\bar{v}(x)) \leq \left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x)) + \rho(f(x,\bar{u}(x))-f(x,\bar{v}(x))) \leq \left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x))$$

and

$$\left(1-\rho\frac{1+hL_{M_{n+1}}}{h}\right)(\bar{u}(x)-\bar{v}(x)) \leq \left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x)) + \rho(f(x,\bar{u}(x))-f(x,\bar{v}(x))) \leq \left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x)).$$

For $\rho \leq \dfrac{2h}{2+hL_{M_{n+1}}} < h$, we have

$$\left(1-\rho\frac{1+hL_M}{h}\right)(\bar{u}(x)-\bar{v}(x)) \leq \left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x)) + \rho(f(x,\bar{u}(x))-f(x,\bar{v}(x))) \leq (\bar{u}(x)-\bar{v}(x)).$$

Thus, for $\rho \leq \dfrac{2h}{2+hL_{M_{n+1}}}$, we obtain

$$\left|\left(1-\frac{\rho}{h}\right)(\bar{u}(x)-\bar{v}(x))+\rho(f(x,\bar{u}(x))-f(x,\bar{v}(x)))\right| \leq |(\bar{u}(x)-\bar{v}(x))|.$$

Consequently, we have

$$|\tilde{T}(\bar{u})(x)-\tilde{T}(\bar{v})(x)| \leq \|\bar{u}-\bar{v}\|_\infty.$$

Case 2. Assume that $\bar{u}(x)-\bar{v}(x)<0$. Then, from (f_3), we have $f(x,\bar{u}(x))-f(x,\bar{v}(x))>0$. Following the same arguments as in Case 1, we have $|\tilde{T}(\bar{u})(x)-\tilde{T}(\bar{v})(x)| \leq \|\bar{u}-\bar{v}\|_\infty$ for $\rho \leq \frac{2h}{2+hL_{M_{n+1}}}$.

From Cases 1 and 2, it follows that $|\tilde{T}(\bar{u})(x)-\tilde{T}(\bar{v})(x)| \leq \|\bar{u}-\bar{v}\|_\infty$ for every $x \in \Omega^+$.

Second Step. Let $x \in \Omega^-$.
Following the same arguments as in the first step, we deduce that $|\tilde{T}(\bar{u})(x)-\tilde{T}(\bar{v})(x)| \leq \|\bar{u}-\bar{v}\|_\infty$ for every $x \in \Omega^-$.

Thus, from the first and second steps, it follows that

$$\|\tilde{T}(\bar{u})-\tilde{T}(\bar{v})\|_\infty \leq \|\bar{u}-\bar{v}\|_\infty.$$

Hence, $\tilde{T}$ is a contraction map from C_{n+1} to C_{n+1}. This completes the proof of Lemma 4.3. □

From (18), one has the iteration $\bar{u}^{\epsilon,k+1}_{n+1} = \lambda_{k+1}\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})$, where $\tilde{T}$ is a non-expansive operator in $L^\infty(\Omega)$ defined in Lemma 4.3. To conclude the proof of Theorem 4.2, we set $X = L^\infty(\Omega)$ and $C = C_{n+1}$, which is clearly a convex subset of $L^\infty(\Omega)$ containing 0. Denote by u^* the fixed point of $\tilde{T}$. Then,

$$u^*-\rho\nabla.\mathbf{a}\,(x,\nabla u^*) = \left(1-\frac{\rho}{h}\right)u^*+\rho f(x,u^*)+\frac{\rho}{h}u^\epsilon_n.$$

Thus,

$$u^*-h\nabla.\mathbf{a}\,(x,\nabla u^*)-hf(x,u^*) = u^\epsilon_n \quad \text{in} \quad \mathcal{D}'(\Omega). \tag{19}$$

Equation (19) admits a unique weak solution $u^* \in \mathbb{W}\cap L^\infty(\Omega)$. Indeed, consider the energy functional J associated to (19) by

$$J_h(v) = \frac{1}{2}\int_\Omega v^2\,dx + h\int_\Omega A(x,\nabla v)\,dx - \int_\Omega F(x,v)\,dx - \int_\Omega u^\epsilon_n v\,dx, \tag{20}$$

where $F(x,\kappa) = \displaystyle\int_0^\kappa f(x,s)\,ds$. Under assumptions (f_4) and (f_5) (see [14]), for $\varepsilon>0$, there exists a constant $C_0 = C_0(\varepsilon,\alpha_\infty)$ large enough such that for any $C^{te} \geq C_0(\varepsilon,\alpha_\infty)$,

$$|f(x,s)| \leq C^{te} + (\alpha_\infty+\varepsilon)\,|s|^{p^--1} \quad \text{for any } (x,s)\in\Omega\times\mathbb{R},$$

with $\alpha_\infty+\varepsilon < \gamma\Lambda^{p^-}(p_c)^-$, where

$$\alpha_\infty := \sup_{x\in\Omega}\lim_{|s|\to+\infty}\sup\frac{|f(x,s)|}{|s|^{p^--1}}.$$

Thus, J_ρ is well-defined and Gâteau differentiable on $\mathbb{W}$, and

$$(J_h'(v),w) = \int_\Omega vw\,dx + h\int_\Omega \mathbf{a}\,(x,\nabla v).\nabla w\,dx - \int_\Omega f(x,v)w\,dx - \int_\Omega u^\epsilon_n w\,dx, \quad \forall v,w\in\mathbb{W}.$$

Using standard arguments, we prove that J_h admits a global minimizer $v\in\mathbb{W}$, which is a weak solution of problem (19). For $C^{te}>0$ large enough, C^{te} and $(-C^{te})$ are respectively supersolution and subsolution of problem (19), and hence, from the weak comparison principle, we have $u^*\in[-C^{te},C^{te}]$. Therefore, $u^*\in L^\infty(\Omega)$.

The uniqueness of the solution follows from the strict convexity of J_h. By the definition of the mild solution, the fixed point is u^ϵ_{n+1}. We also have

$$\begin{aligned}
\|u^{\epsilon,k}_{n+1}-u^{\epsilon,k+1}_{n+1}\|_\infty &= \|\frac{\bar{u}^{\epsilon,k}_{n+1}}{\lambda_k}-\frac{\bar{u}^{\epsilon,k+1}_{n+1}}{\lambda_{k+1}}-\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})+\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty \\
&\leq \frac{1}{\lambda_k}\|\bar{u}^{\epsilon,k}_{n+1}-\lambda_k\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty + \frac{1}{\lambda_{k+1}}\|\bar{u}^{\epsilon,k+1}_{n+1}-\lambda_{k+1}\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty \\
&\leq \frac{1}{\lambda_k}\left(\|\bar{u}^{\epsilon,k}_{n+1}-\lambda_k\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})-\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})+\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty\right) \\
&\leq \frac{1}{\lambda_k}\|\bar{u}^{\epsilon,k}_{n+1}-\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty + \left(\frac{1}{\lambda_k}-1\right)\|\tilde{T}(\bar{u}^{\epsilon,k}_{n+1})\|_\infty.
\end{aligned}$$

By Lemma 2.1, we deduce that

$$\|u_{n+1}^{\epsilon,k} - u_{n+1}^{\epsilon,k+1}\|_\infty \to 0 \text{ as } k \to \infty. \tag{21}$$

By the uniform boundedness of $(u_{n+1}^{\epsilon,k+1})_k$, there exists a subsequence denoted by $(u_{n+1}^{\epsilon,k+1})_k$ such that

$$u_{n+1}^{\epsilon,k+1} \overset{*}{\rightharpoonup} \tilde{u}_{n+1}^{\epsilon} \text{ in } L^\infty(\Omega) \text{ as } k \to +\infty. \tag{22}$$

We choose $\varphi = u_{n+1}^{\epsilon,k+1}$ as a test function in (15). Then, we have

$$\int_\Omega \left(u_{n+1}^{\epsilon,k+1}\right)^2 dx + \rho \int_\Omega \boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla u_{n+1}^{\epsilon,k+1}\, dx = \int_\Omega g(x, u_{n+1}^{\epsilon,k}) u_{n+1}^{\epsilon,k+1}\, dx.$$

By keeping in mind (2) and Theorem 2.1, we deduce that $W_0^{1,p(x)}(\Omega) \hookrightarrow L^2(\Omega)$ with compact embedding. Since $f(., \lambda_k u_{n+1}^{\epsilon,k})$ belongs to $L^q(\Omega)$, by conditions (f_1) and (f_2), we have $g(., u_{n+1}^{\epsilon,k}) \in L^q(\Omega)$. As $q > \frac{d}{p^-}$ and $1 < p^- \le p^+ \le d$, we conclude that $L^q(\Omega) \subset \left(L^{p^\star(.)}\right)'(\Omega)$. By Young's inequality and the uniform boundedness of $u_{n+1}^{\epsilon,k+1}$, we obtain

$$\int_\Omega \boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla u_{n+1}^{\epsilon,k+1}\, dx \le C'(n, \Omega).$$

From Remark 9, we deduce that

$$\begin{aligned} C'(n,\Omega) &\ge \int_\Omega \boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right).\nabla u_{n+1}^{\epsilon,k+1}\, dx \\ &\ge \int_\Omega A(x, \nabla u_{n+1}^{\epsilon,k+1})\, dx \\ &\ge \frac{\gamma}{p^+ - 1} \int_\Omega |\nabla u_{n+1}^{\epsilon,k+1}|^{p(x)}\, dx. \end{aligned}$$

Therefore, $(u_{n+1}^{\epsilon,k+1})_k$ is bounded in $W_0^{1,p(x)}(\Omega)$. There exists a subsequence $(u_{n+1}^{\epsilon,k+1})_k$ such that

$$\begin{aligned} u_{n+1}^{\epsilon,k+1} &\rightharpoonup \tilde{u}_{n+1}^{\epsilon} \quad \text{in} \quad W_0^{1,p(\cdot)}(\Omega) \text{ as } k \to +\infty, \\ u_{n+1}^{\epsilon,k+1} &\to \tilde{u}_{n+1}^{\epsilon} \quad \text{in} \quad L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega, \text{ as } k \to +\infty. \end{aligned}$$

By choosing $\varphi = u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon}$ as a test function in (15), we deduce that

$$\begin{aligned} &\int_\Omega \left(\boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right) - \boldsymbol{a}\left(x, \nabla \tilde{u}_{n+1}^{\epsilon}\right)\right).\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\, dx \\ &= \int_\Omega \left(g(x, u_{n+1}^{\epsilon,k}) - g(x, \tilde{u}_{n+1}^{\epsilon})\right)(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\, dx - \int_\Omega (u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})^2\, dx. \end{aligned}$$

Because of the above convergences, we have

$$\int_\Omega \left(\boldsymbol{a}\left(x, \nabla u_{n+1}^{\epsilon,k+1}\right) - \boldsymbol{a}\left(x, \nabla \tilde{u}_{n+1}^{\epsilon}\right)\right).\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\, dx \overset{k\to+\infty}{\longrightarrow} 0\ . \tag{23}$$

Now, we show that

$$\int_\Omega \left|\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\right|^{p(x)} dx \to 0 \text{ as } k \to +\infty.$$

To do this, we distinguish two cases.

Case 1. $x \in \Omega$ such that $p(x) < 2$.
We set $q(x) = \frac{p(x)(2-p(x))}{2}$ and define Ω^- as $\Omega^- := \{x \in \Omega : p(x) < 2\}$. Since $u_{n+1}^{\epsilon,k+1}, \tilde{u}_{n+1}^{\epsilon} \in W_0^{1,p(x)}(\Omega)$, by using Hölder inequality we obtain

$$\begin{aligned} \int_{\Omega^-} \left|\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\right|^{p(x)} dx &\le \tilde{C} \left\| \frac{\left|\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}}{\left(|\nabla u_{n+1}^{\epsilon,k+1}| + |\nabla \tilde{u}_{n+1}^{\epsilon}|\right)^{q(x)}} \right\|_{L^{\frac{2}{p(x)}}(\Omega^-)} \times \left\| \left(|\nabla u_{n+1}^{\epsilon,k+1}| + |\nabla \tilde{u}_{n+1}^{\epsilon}|\right)^{q(x)} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega^-)} \\ &\le \tilde{C}_1 \left\| \frac{\left|\nabla(u_{n+1}^{\epsilon,k+1} - \tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}}{\left(|\nabla u_{n+1}^{\epsilon,k+1}| + |\nabla \tilde{u}_{n+1}^{\epsilon}|\right)^{q(x)}} \right\|_{L^{\frac{2}{p(x)}}(\Omega^-)}. \end{aligned}$$

Next, we set

$$\Pi := \left\| \frac{\left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{p(x)}}{\left(|\nabla u^{\epsilon,k+1}_{n+1}| + |\nabla \tilde{u}^{\epsilon}_{n+1}|\right)^{q(x)}} \right\|_{L^{\frac{2}{p(x)}}(\Omega^-)}.$$

If $\Pi < 1$, then we have

$$\int_{\Omega^-} \left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{p(x)} dx \leq \tilde{C}_1 \left(\int_{\Omega^-} \frac{\left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{2}}{\left(|\nabla u^{\epsilon,k+1}_{n+1}| + |\nabla \tilde{u}^{\epsilon}_{n+1}|\right)^{2-p(x)}} \, dx \right)^{\frac{1}{2}\sup\limits_{\Omega^-} p(x)}. \tag{24}$$

Using (5), we deduce that

$$\gamma \int_{\Omega^-} \frac{\left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{2}}{\left(|\nabla u^{\epsilon,k+1}_{n+1}| + |\nabla \tilde{u}^{\epsilon}_{n+1}|\right)^{2-p(x)}} dx \leq \int_{\Omega} \left(\boldsymbol{a}\left(x, \nabla u^{\epsilon,k+1}_{n+1}\right) - \boldsymbol{a}\left(x, \nabla \tilde{u}^{\epsilon}_{n+1}\right)\right) . \nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1}) \, dx. \tag{25}$$

Thus, by combining (24) and (25), we obtain

$$\int_{\Omega^-} \left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{p(x)} dx \to 0 \text{ as } k \to +\infty.$$

If $\Pi \geq 1$, then we have

$$\int_{\Omega^-} \left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{p(x)} dx \leq \tilde{C}_1 \left(\int_{\Omega^-} \frac{\left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{2}}{\left(|\nabla u^{\epsilon,k+1}_{n+1}| + |\nabla \tilde{u}^{\epsilon}_{n+1}|\right)^{2-p(x)}} \, dx \right)^{\frac{p^-}{2}}. \tag{26}$$

Using the same arguments as in the case when $\Pi < 1$, we obtain

$$\int_{\Omega^-} \left|\nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1})\right|^{p(x)} dx \to 0 \text{ as } k \to +\infty.$$

Case 2. $x \in \Omega$ such that $p(x) \geq 2$.
By the convexity of Φ, we have

$$\int_{\Omega^+} \Phi(x, |\nabla \tilde{u}^{\epsilon}_{n+1}|) \, dx \leq \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(\tilde{u}^{\epsilon}_{n+1} + u^{\epsilon,k+1}_{n+1})|}{2}\right) dx + \frac{1}{2} \int_{\Omega^+} \boldsymbol{a}\left(x, \nabla \tilde{u}^{\epsilon}_{n+1}\right) . \nabla(\tilde{u}^{\epsilon}_{n+1} - u^{\epsilon,k+1}_{n+1}) \, dx \tag{27}$$

and

$$\int_{\Omega^+} \Phi(x, |\nabla u^{\epsilon,k+1}_{n+1}|) \, dx \leq \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(\tilde{u}^{\epsilon}_{n+1} + u^{\epsilon,k+1}_{n+1})|}{2}\right) dx + \frac{1}{2} \int_{\Omega^+} \boldsymbol{a}\left(x, \nabla u^{\epsilon,k+1}_{n+1}\right) . \nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1}) \, dx. \tag{28}$$

By adding (27) and (28), we obtain

$$\begin{aligned} &\int_{\Omega} \left(\boldsymbol{a}\left(x, \nabla u^{\epsilon,k+1}_{n+1}\right) - \boldsymbol{a}\left(x, \nabla \tilde{u}^{\epsilon}_{n+1}\right)\right) . \nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1}) \, dx \\ &\quad \geq 2 \int_{\Omega^+} \Phi(x, |\nabla u^{\epsilon,k+1}_{n+1}|) \, dx + 2 \int_{\Omega^+} \Phi(x, |\nabla \tilde{u}^{\epsilon}_{n+1}|) \, dx - 4 \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(\tilde{u}^{\epsilon}_{n+1} + u^{\epsilon,k+1}_{n+1})|}{2}\right) dx. \end{aligned} \tag{29}$$

By applying Clarkson's type inequality, i.e. $v, w \in W^{1,p(x)}_0(\Omega)$,

$$\int_{\Omega^+} \Phi(x, |\nabla v|) \, dx + \int_{\Omega^+} \Phi(x, |\nabla w|) \, dx \geq 2 \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(v+w)|}{2}\right) dx + 2 \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(v-w)|}{2}\right) dx,$$

to (29), we obtain

$$\int_{\Omega} \left(\boldsymbol{a}\left(x, \nabla u^{\epsilon,k+1}_{n+1}\right) - \boldsymbol{a}\left(x, \nabla \tilde{u}^{\epsilon}_{n+1}\right)\right) . \nabla(u^{\epsilon,k+1}_{n+1} - \tilde{u}^{\epsilon}_{n+1}) \, dx \geq 4 \int_{\Omega^+} \Phi\left(x, \frac{|\nabla(\tilde{u}^{\epsilon}_{n+1} - u^{\epsilon,k+1}_{n+1})|}{2}\right) dx. \tag{30}$$

From (30) and Remark 9, it follows that

$$\int_{\Omega}\left(\boldsymbol{a}\left(x,\nabla u_{n+1}^{\epsilon,k+1}\right)-\boldsymbol{a}\left(x,\nabla \tilde{u}_{n+1}^{\epsilon}\right)\right).\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\,dx \geq \frac{4\gamma}{2^{p^+}(p^+-1)}\int_{\Omega+}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx. \tag{31}$$

Thus, by using (23), we have

$$\int_{\Omega+}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx \to 0 \text{ as } k\to+\infty.$$

From Cases 1 and 2, it follows that

$$\int_{\Omega}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx = \int_{\Omega-}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx + \int_{\Omega+}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx.$$

Hence, we conclude that

$$\int_{\Omega}\left|\nabla(u_{n+1}^{\epsilon,k+1}-\tilde{u}_{n+1}^{\epsilon})\right|^{p(x)}dx \to 0 \text{ as } k\to+\infty. \tag{32}$$

Therefore,

$$\begin{aligned}\nabla u_{n+1}^{\epsilon,k+1} &\to \nabla\tilde{u}_{n+1}^{\epsilon} \quad \text{in} \quad L^{p(\cdot)}(\Omega),\ \text{as } k\to+\infty,\\ u_{n+1}^{\epsilon,k+1} &\to \tilde{u}_{n+1}^{\epsilon} \quad \text{in} \quad W_0^{1,p(\cdot)}(\Omega),\ \text{as } k\to+\infty.\end{aligned}$$

Finally, we have to show that

$$\boldsymbol{a}\left(x,\nabla u_{n+1}^{\epsilon,k+1}\right)\to\boldsymbol{a}\left(x,\nabla\tilde{u}_{n+1}^{\epsilon}\right) \text{ in } \left(L^{p_c(x)}(\Omega)\right)^d \text{ as } k\to+\infty.$$

From (32), we deduce that

$$|\nabla u_{n+1}^{\epsilon,k+1}|^{p(x)} \to |\nabla\tilde{u}_{n+1}^{\epsilon}|^{p(x)} \text{ as } k\to+\infty.$$

Hence, there exists a subsequence $(u_{n+1}^{\epsilon,k+1})_k$ such that

$$\nabla u_{n+1}^{\epsilon,k+1}\to\nabla\tilde{u}_{n+1}^{\epsilon} \text{ a.e. in } \Omega \text{ and } |\nabla u_{n+1}^{\epsilon,k+1}|^{p(x)}\leq b\in L^1(\Omega).$$

From (7), we deduce that

$$|\boldsymbol{a}\left(x,\nabla u_{n+1}^{\epsilon,k+1}\right)|\leq C_1|\nabla u_{n+1}^{\epsilon,k+1}|^{p(x)-1}\leq b^{\frac{1}{p_c(x)}}\in L^{p_c(x)}.$$

Using the dominated convergence theorem, we conclude that

$$\boldsymbol{a}\left(x,\nabla u_{n+1}^{\epsilon,k+1}\right)\to\boldsymbol{a}\left(x,\nabla\tilde{u}_{n+1}^{\epsilon}\right) \text{ in } \left(L^{p_c(x)}(\Omega)\right)^d \text{ as } k\to+\infty.$$

Since $\lambda_k\to1$ as $k\to\infty$ and according to (21), we pass to the limit, in the distribution sense, in (13), to conclude that $\tilde{u}_{n+1}^{\epsilon}$ is a weak solution of (11). By the uniqueness of the weak solution of (11), we deduce that

$$u_{n+1}^{\epsilon}=\tilde{u}_{n+1}^{\epsilon}.$$

Therefore,

$$u_{n+1}^{\epsilon,k+1}\rightharpoonup u_{n+1}^{\epsilon} \text{ weakly} * \text{ in } L^{\infty}(\Omega).$$

Since

$$\|u_{n+1}^{\epsilon,k}\|_{\infty}\leq M_{n+1}:=\|u_0\|_{\infty}+(n+1)h\|f(\cdot,0)\|_{\infty} \text{ for every } k, \tag{33}$$

passing to the limit in (33) as $k\to\infty$, we complete the proof of Theorem 4.2. □

Remark 4.3. *For numerical tests, the simple choice $\rho=h$ works, even if it may not satisfy the theoretical (sufficient) assumptions of Lemma 4.1.*

4.4. Convergence when $\epsilon\to0$ toward a solution of problem (1)

We recall that, for a mild solution, convergence in time need not be shown explicitly, as it is inherent in the definition: once convergence in k is achieved for u_{n+1}^{ϵ}, then by the definition of the mild solution, it follows that u_{n+1}^{ϵ} approaches $u^{\epsilon}(t)$ on $(t_n,t_{n+1}]$ up to ϵ. Consequently, our scheme converges to the mild solution when ϵ goes to zero (since $h\leq\epsilon$).

By choosing ϵ in such a way that $h = \epsilon$ (i.e. $\epsilon = \min(\epsilon, 1/(2L_M))$), we obtain the next result.

Proposition 4.1. *Assume that conditions (3)–(6) and (8)–(9) hold. Let f be a function satisfying hypotheses (f_1)–(f_3). Let $u_0 \in \mathbb{W} \cap \overline{\mathcal{D}(A_0)}^{L^\infty}$ and u be the unique mild solution of (1). Then, u is a weak solution of (1). A weak solution is understood to be a measurable function $u \in L^\infty(0,T;\mathbb{W})$ such that $u_t \in L^2(Q)$, $f(\cdot,u) \in L^\infty(0,T,L^2(\Omega))$ and for any $\varphi \in C_0^\infty(Q)$,*

$$\int_0^T \int_\Omega u_t \varphi \, dx\, dt + \int_0^T \int_\Omega \boldsymbol{a}(x,\nabla u).\nabla\varphi \, dx\, dt = \int_0^T \int_\Omega f(x,u)\varphi \, dx\, dt,$$

and $u(0,.) = u_0$ a.e. in Ω.

Proof. Let u be the mild solution of (1). For $n = 0,\dots,N-1$, u^ϵ_{n+1} is the unique weak solution of (11). We have

$$\int_\Omega \frac{u^\epsilon_{n+1} - u^\epsilon_n}{h}\varphi \, dx + \int_\Omega \boldsymbol{a}(x,\nabla u^\epsilon_{n+1}).\nabla\varphi \, dx = \int_\Omega f(x,u^\epsilon_{n+1})\varphi, \quad \forall \varphi \in \mathbb{W} \cap L^\infty(\Omega)$$

and

$$\begin{cases} 0 = t_0 < t_1 < \cdots < t_N \leq T, \text{ such that } t_n - t_{n-1} = h = \epsilon \text{ for } n = 1,\dots,N, \\ \|u_0 - u^\epsilon_0\|_{L^\infty(\Omega)} \leq h. \end{cases} \tag{34}$$

We set $u^\epsilon(t) = u^\epsilon_{n+1}\ \forall\, t \in (t_n,t_{n+1}]$, where $n = 0,\dots,N-1$ and $u^\epsilon(0) = u^\epsilon_0$. By Lemma 4.1, Theorem 4.2, and (34), we have

$$\begin{aligned}\|u^\epsilon(t)\|_{L^\infty(\Omega)} &\leq \|u_0\|_{L^\infty(\Omega)} + Nh\|f(\cdot,0)\|_{L^\infty(\Omega)} \\ &\leq C' := \|u_0\|_{L^\infty(\Omega)} + T\|f(\cdot,0)\|_{L^\infty(\Omega)}.\end{aligned}$$

By using the uniform boundedness of u^ϵ and conditions (f_1)–(f_3), we obtain

$$|f(x,u^\epsilon(t))| \leq \max\left(|f(x,-C)|, |f(x,C)|\right).$$

Therefore,

$$\|f(x,u^\epsilon(t))\|_{L^2(\Omega)} \leq C_1.$$

Now, we define for $n = 0,\dots,N-1$ and $t \in [t_n,t_{n+1})$ the functions

$$u_h(t) = u^\epsilon_n \quad \text{and} \quad \tilde{u}_h(t) = \frac{(t-t_n)}{h}(u^\epsilon_{n+1} - u^\epsilon_n) + u^\epsilon_n, \tag{35}$$

which satisfy

$$\frac{\partial \tilde{u}_h}{\partial t} - \nabla.\mathbf{a}(x,\nabla u_h) = f(x,u_h) \quad \text{in } Q. \tag{36}$$

Thus, by the uniform boundedness of u^ϵ,

$$(u_h) \text{ and } (\tilde{u}_h) \text{ are bounded in } L^\infty(Q), \text{ uniformly with respect to } h. \tag{37}$$

By taking $(u^\epsilon_{n+1} - u^\epsilon_n)$ as a test function in (11) and summing up for $n = 0$ to $N' \leq N-1$, we obtain

$$\sum_{n=0}^{N'} h \int_\Omega \left(\frac{u^\epsilon_{n+1} - u^\epsilon_n}{h}\right)^2 dx + \sum_{n=0}^{N'} \int_\Omega \mathbf{a}(x,\nabla u^\epsilon_{n+1}).\nabla(u^\epsilon_{n+1} - u^\epsilon_n)\, dx = \sum_{n=0}^{N'} \int_\Omega f(x,u^\epsilon_{n+1})(u^\epsilon_{n+1} - u^\epsilon_n)\, dx.$$

Hence, by using Young's inequality and the convexity of $u \mapsto \displaystyle\int_\Omega A(x,\nabla u)\, dx$, we obtain

$$\frac{1}{2}\sum_{n=0}^{N'} h \int_\Omega \left(\frac{u^\epsilon_{n+1} - u^\epsilon_n}{h}\right)^2 dx + \sum_{n=0}^{N'} \int_\Omega A(x,\nabla u^\epsilon_{n+1}) - A(x,\nabla u^\epsilon_n)\, dx \leq \frac{1}{2}\sum_{n=0}^{N'} h \int_\Omega f(x,u^\epsilon_{n+1})^2\, dx,$$

$$\frac{1}{2}\sum_{n=0}^{N'} h \int_\Omega \left(\frac{u^\epsilon_{n+1} - u^\epsilon_n}{h}\right)^2 dx + \frac{\gamma}{p^+ - 1}\int_\Omega |\nabla u^\epsilon_{N'+1}|^{p(x)} - \int_\Omega A(x,\nabla u^\epsilon_0)\, dx \leq \frac{1}{2}\|f(x,u_\epsilon)\|^2_{L^2(Q)} \leq C_1.$$

Therefore,

$$\left(\frac{\partial \tilde{u}_h}{\partial t}\right)_h \text{ is bounded in } L^2(Q), \text{ uniformly with respect to } h. \tag{38}$$

Consequently, $(\tilde{u}_h)$ is equicontinuous in $C([0,T];L^r(\Omega))$ for $1 \leq r \leq 2$. Hence, we conclude that

$$(u_h) \text{ and } (\tilde{u}_h) \text{ are bounded in } L^\infty(0,T,\mathbb{W}), \text{ uniformly with respect to } h. \tag{39}$$

Thus, for $h \to 0^+$, there exists $u, v \in L^\infty(0,T,\mathbb{W})$ such that (up to a subsequence)

$$\frac{\partial \tilde{u}_h}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } L^2(Q), \tag{40}$$

$$\tilde{u}_h \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,T,\mathbb{W}), \quad u_h \overset{*}{\rightharpoonup} v \text{ in } L^\infty(0,T,\mathbb{W}). \tag{41}$$

From (38), it follows that

$$\sup_{[0,T]} \|u_h - \tilde{u}_h\|_{L^2(\Omega)} \leq \max_{n=0,\ldots,N-1} \|u^\epsilon_{n+1} - u^\epsilon_n\|_{L^2(\Omega)} \leq C\sqrt{h}. \tag{42}$$

We conclude from (42) that $u \equiv v$. From (41), it follows that

$$\tilde{u}_h, u_h \rightharpoonup u \text{ in } L^r(0,T,\mathbb{W}) \text{ for any } r \geq 1.$$

By using the interpolation inequality and (37), we deduce that $(\tilde{u}_h)$ is equicontinuous in $C([0,T];L^r(\Omega))$ for any $r > 1$. By using (39) and Theorem 2.1, and applying the Ascoli-Arzela theorem, we deduce that (up to a subsequence)

$$\tilde{u}_h \to u \text{ in } C([0,T];L^r(\Omega)) \text{ for any } r > 1. \tag{43}$$

Since (u_h) is uniformly bounded in $L^\infty(Q)$, condition (f_1) implies the following:

$$\|f(\cdot,u_h(t)) - f(\cdot,u(t))\|_{L^2(\Omega)} \leq C_f \|u_h(t) - u(t)\|_{L^2(\Omega)}.$$

Hence, we deduce that

$$f(\cdot,u_h(t)) \to f(\cdot,u(t)) \text{ in } L^\infty(0,T,L^2(\Omega)). \tag{44}$$

Following the same arguments as given in Step 3 of the proof of Theorem 1.1 in [14], we obtain

$$\mathbf{a}(x,\nabla u_h) \to \mathbf{a}(x,\nabla u) \text{ in } \left(L^{p_c(\cdot)}(Q)\right)^d, \text{ where } p_c(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}. \tag{45}$$

Additionally, by (34) and the convergence of the time discretization functions u_h and $\tilde{u}_h$ to u in $C([0,T];L^r(\Omega))$ for $r > 1$, the limit function u satisfies $u(0,\cdot) = u_0$ in the weak sense. Finally, (40), (45), and (44) allow us to pass to the limit as $h \to 0^+$, in the distribution sense, in (36) to conclude that u is a weak solution of (1). □

4.5. Numerical tests

4.5.1. Implementation

Note that solving (13) is equivalent to solve the following minimization problem for $n = 0,1,\ldots,N-1$ and $k = 0,1,\ldots$:

$$u^{\epsilon,k+1}_{n+1} = \text{argmin}_{v \in \mathbb{W}} J(v), \tag{46}$$

where,

$$\mathbb{W} := \left\{ v \in W^{1,p(\cdot)}_0(\Omega) \cap L^\infty(\Omega) \right\}$$

and the functional J is given as

$$J(v) = \frac{1}{2}\int_\Omega v^2\,dx + \rho \int_\Omega A(x,\nabla v)\,dx - \left(1 - \frac{\rho}{h}\right)\lambda_k \int_\Omega u^{\epsilon,k}_{n+1} v - \frac{\rho}{h}\int_\Omega u^\epsilon_n v\,dx - \rho \int_\Omega f(x,\lambda_k u^{\epsilon,k}_{n+1}) v\,dx.$$

We formulate a basic procedure for solving problem (46) by following the split Bregman technique (see [10]). We solve the minimization problem by introducing an auxiliary variable b. We have

$$\min_v \left\{ \frac{1}{2}\int_\Omega v^2\,dx + \rho\int_\Omega A(x,b)\,dx - \left(1-\frac{\rho}{h}\right)\lambda_k \int_\Omega u^{\epsilon,k}_{n+1} v\,dx - \frac{\rho}{h}\int_\Omega u^\epsilon_n v\,dx - \rho\int_\Omega f(x,\lambda_k u^{\epsilon,k}_{n+1}) v\,dx \text{ subject to } b = \nabla v \right\}. \tag{47}$$

By adding one quadratic penalty function term, we convert (47) to an unconstrained splitting formulation as follows:

$$\min_{v,b}\left\{\frac{1}{2}\int_\Omega v^2dx+\rho\int_\Omega A(x,b)dx+\frac{\gamma}{2}\int_\Omega |b-\nabla v|^2dx-\left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u^{\epsilon,k}_{n+1}vdx-\frac{\rho}{h}\int_\Omega u^\epsilon_n vdx-\rho\int_\Omega f(x,\lambda_k u^{\epsilon,k}_{n+1})vdx\right\}, \tag{48}$$

where γ is a positive parameter, which controls the weight of the penalty term. Similar to the split Bregman iteration, we propose the following scheme:

$$\left\{\begin{array}{rcl}(v^{l+1},b^{l+1}) &=& \text{argmin}_{v,b}\{\frac{1}{2}\int_\Omega v^2\,dx+\rho\int_\Omega A(x,b)\,dx+\frac{\gamma}{2}\int_\Omega |b-\nabla v-\delta^l|^2\,dx\\ &-& \left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u^{\epsilon,k}_{n+1}v\,dx-\frac{\rho}{h}\int_\Omega u^\epsilon_n v\,dx-\rho\int_\Omega f(x,\lambda_k u^{\epsilon,k}_{n+1})v\,dx\},\\ \delta^{l+1} &=& \delta^l+\nabla v^{l+1}-b^{l+1}.\end{array}\right. \tag{49}$$

Alternatively, this joint minimization problem can be solved by decomposing it into several subproblems.

4.5.2. Subproblem v with fixed b and δ

Given the fixed variables b^l and δ^l, our aim here is to find the solution to the following problem:

$$v^{l+1}=\text{argmin}_v\left\{\frac{1}{2}\int_\Omega v^2dx+\frac{\gamma}{2}\int_\Omega |b^l-\nabla v-\delta^l|^2dx-\left(1-\frac{\rho}{h}\right)\lambda_k\int_\Omega u^{\epsilon,k}_{n+1}vdx-\frac{\rho}{h}\int_\Omega u^\epsilon_n vdx-\rho\int_\Omega f(x,\lambda_k u^{\epsilon,k}_{n+1})v\,dx\right\}. \tag{50}$$

Note that solving (50) is equivalent to solving the following optimality condition:

$$v-\gamma\Delta v=\gamma\nabla.(\delta^l-b^l)+\left(1-\frac{\rho}{h}\right)\lambda_k u^{\epsilon,k}_{n+1}+\frac{\rho}{h}u^\epsilon_n+\rho f(x,\lambda_k u^{\epsilon,k}_{n+1}). \tag{51}$$

Since the discrete system is strictly diagonally dominant with Neumann boundary conditions, the most natural choice is the Gauss-Seidel method. Considering that $\delta=(\delta_x,\delta_y)$, $b=(b_x,b_y)$, and using κ as discretization step for x and y, the Gauss-Seidel solution to the subproblem (51) can be written componentwise as

$$\begin{aligned}v^{l+1}_{i,j}&=\frac{-\gamma\kappa}{\kappa^2+4\gamma}\left(\delta^l_{x,i-1,j}+\delta^l_{y,i,j-1}-\delta^l_{x,i,j}-\delta^l_{y,i,j}+b^l_{x,i-1,j}+b^l_{y,i,j-1}-b^l_{x,i,j}-b^l_{y,i,j}\right)\\&+\frac{\gamma}{\kappa^2+4\gamma}\left(v^l_{i+1,j}+v^l_{i-1,j}+v^l_{i,j+1}+v^l_{i,j-1}\right)+\frac{\kappa^2}{\kappa^2+4\gamma}g_{i,j}\left(u^{\epsilon,k}_{n+1,i,j}\right),\end{aligned}$$

where

$$g_{i,j}\left(u^{\epsilon,k}_{n+1,i,j}\right):=\left(1-\frac{\rho}{h}\right)\lambda_k u^{\epsilon,k}_{n+1,i,j}+\frac{\rho}{h}u^\epsilon_{n,i,j}+\rho f\left(i,j,\lambda_k u^{\epsilon,k}_{n+1,i,j}\right).$$

4.5.3. Subproblem b with fixed v and δ

Similar to the previous section, we solve

$$b^{l+1}=\text{argmin}_b\left\{\rho\int_\Omega A(x,b)\,dx+\frac{\gamma}{2}\int_\Omega |b-\nabla v^{l+1}-\delta^l|^2\,dx\right\}. \tag{52}$$

Solving (52) is equivalent to solving the following optimality condition:

$$\rho\,\mathbf{a}\,(x,b)+\gamma(b-\nabla v^{l+1}-\delta^l)=0. \tag{53}$$

4.5.4. Applications

We take $\mathbf{a}\,(x,y,\nabla u)=|\nabla u(x,y,t)|^{p(x,y)-2}\nabla u(x,y,t)$. We set $b=(b_x,b_y)$ and $\delta=(\delta_x,\delta_y)$. Then, the resolution of (52) is equivalent to solve the following optimality condition:

$$\left\{\begin{array}{ll}\rho|b|^{p(x,y)-2}b_x+\gamma(b_x-\nabla_x v^{l+1}-\delta^l_x) &=0\\ \rho|b|^{p(x,y)-2}b_y+\gamma(b_y-\nabla_y v^{l+1}-\delta^l_y) &=0,\end{array}\right. \tag{54}$$

where $\nabla v=(\nabla_x v,\nabla_y v)$. If b_x and b_y are not zero, then

$$b_x=\frac{\nabla_x v^{l+1}+\delta^l_x}{\nabla_y v^{l+1}+\delta^l_y}b_y. \tag{55}$$

By substituting (55) in (54), we obtain

$$\mathrm{sign}(b_y)T|b_y|^{p(x,y)-1} + \gamma(b_y - \nabla_y v^{l+1} - \delta_y^l) = 0, \tag{56}$$

where

$$T = \rho\left(\left(\frac{\nabla_x v^{l+1} + \delta_x^l}{\nabla_y v^{l+1} + \delta_y^l}\right)^2 + 1\right)^{\frac{p(x,y)-2}{2}}$$

and sign is defined as follows:

$$\mathrm{sign}(\omega) := \begin{cases} 1 & \text{if} \quad \omega > 0, \\ 0 & \text{if} \quad \omega = 0, \\ -1 & \text{if} \quad \omega < 0. \end{cases}$$

Note that

$$\mathrm{sign}(b_x) = \mathrm{sign}(\nabla_x v^{l+1} + \delta_x^l) \tag{57}$$

and

$$\mathrm{sign}(b_y) = \mathrm{sign}(\nabla_y v^{l+1} + \delta_y^l). \tag{58}$$

So, (56) can be expressed as

$$\mathrm{sign}(\nabla_y v^{l+1} + \delta_y^l)T|b_y|^{p(x,y)-1} + \gamma(b_y - \nabla_y v^{l+1} - \delta_y^l) = 0. \tag{59}$$

Unfortunately, we cannot obtain the explicit solution of (59). However, we can use Newton's method to obtain an approximate solution. If r_y is solved, then r_x can be easily determined using (55) and (57).

In the following numerical simulation, the iteration process stops when the following condition is satisfied:

$$\frac{\|u_{n+1}^{k+1} - u_{n+1}^k\|_2}{\|u_{n+1}^{k+1}\|_2} \le \text{stop} := 10^{-4}, \tag{60}$$

where $\|.\|_2$ is the Euclidean norm and u_{n+1}^k is the vector approaching, at iteration k, the space-discretization of u_{n+1}. After stopping the iterations at $k = k_{last}$, we denote $u_{n+1} = u_{n+1}^{k_{last}}$ and switch to the next time step.

Note that for implementation, the finite difference method is used to approximate the partial derivatives. Moreover, for the sake of simplicity, the domain Ω will be a square. The domain Ω will be subdivided into N_x^2 uniform squares.

For numerical simulation, we will use the following parameters:

$$N_x = 80 \quad \text{and} \quad h = 0.002.$$

We recall that h is the time step. The space step is easily computed because of N_x and Ω.

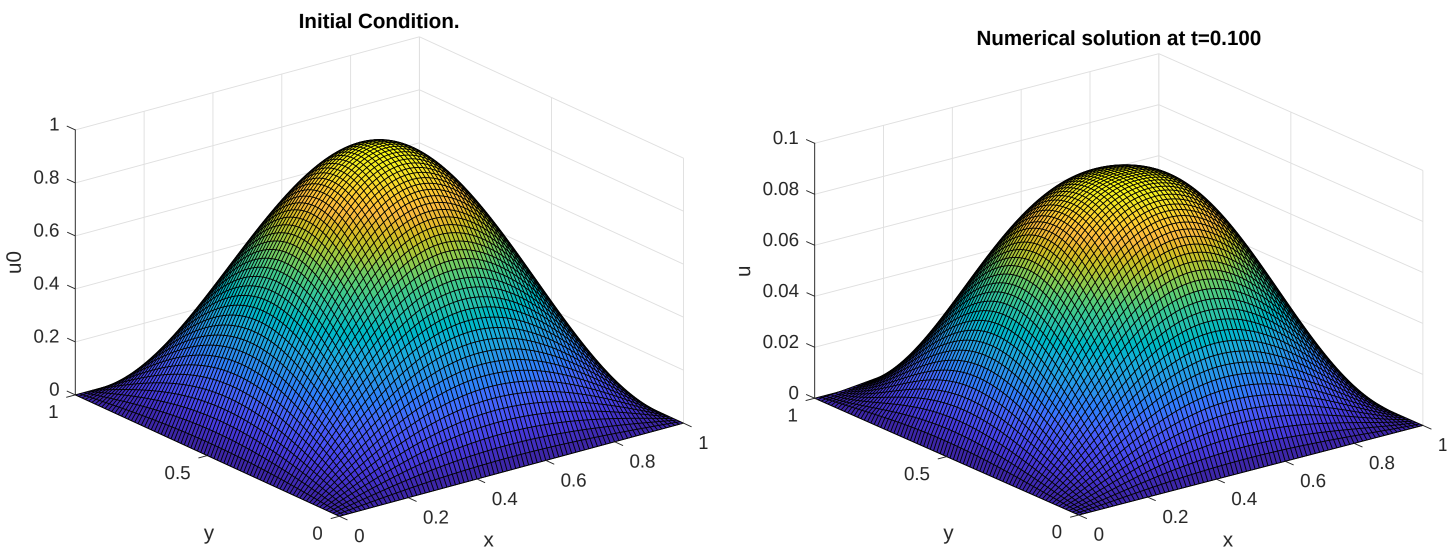

Figure 1: Left: initial solution. Right: numerical solution at $t = 0.1$.

Example 4.1. *We take* $\Omega = (0,1) \times (0,1)$, $T = 0.1$, $p(x,y) = 1.95 - \frac{x}{2}$, *and* $f(x,u) = -|u|^{r(x,y)-2}u$, *where* $r(x,y) = 1.15 - x^2$. *As an initial condition, we set*

$$u_0(x,y) = \sin(\pi x)\sin(\pi y).$$

As parameters, we take $\gamma = 0.003$ *and* $\rho = 0.002$. *We obtain Figure 1.*

Example 4.2. *In this example, we present the numerical solution in the following case (see Figure 2). We set* $\Omega = (0,1)\times(0,1)$*,* $T = 0.1$*,* $p(x,y) = 1.01$*, and* $f(x,y,u) = \frac{1}{1+u}$*. As an initial condition, we set*

$$u_0(x,y) = \sin(\pi x)\sin(\pi y).$$

As parameters, we take $\gamma = 0.003$ *and* $\rho = 0.002$.

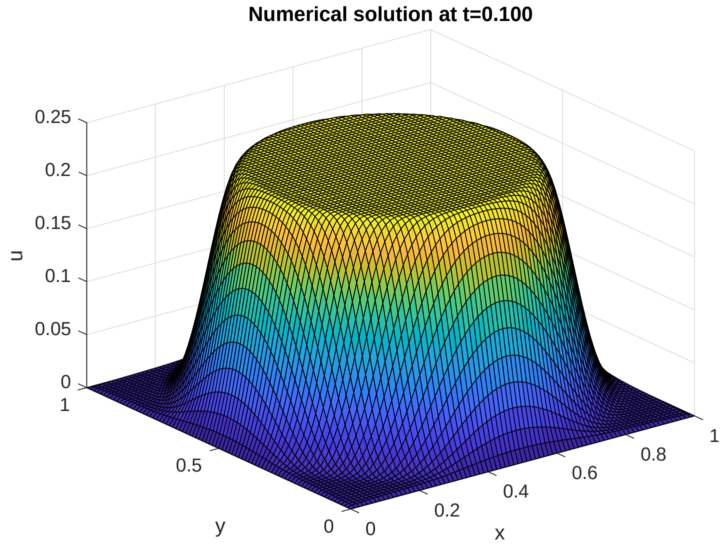

Figure 2: The numerical solution for the case considered in Example 4.2.

Example 4.3. *Take* $\Omega = (-1,1)\times(-1,1)$*,* $T = 0.1$*,* $p(x,y) = 1.3 + \frac{|x+y|}{4}$*, and* $f(x,y,u) = 1$*. As an initial condition, set*

$$u_0(x,y) = e^{(1-x^2)(1-y^2)} - 1.$$

As parameters, take $\gamma = 0.003$ *and* $\rho = 0.002$*. We obtain Figure 3.*

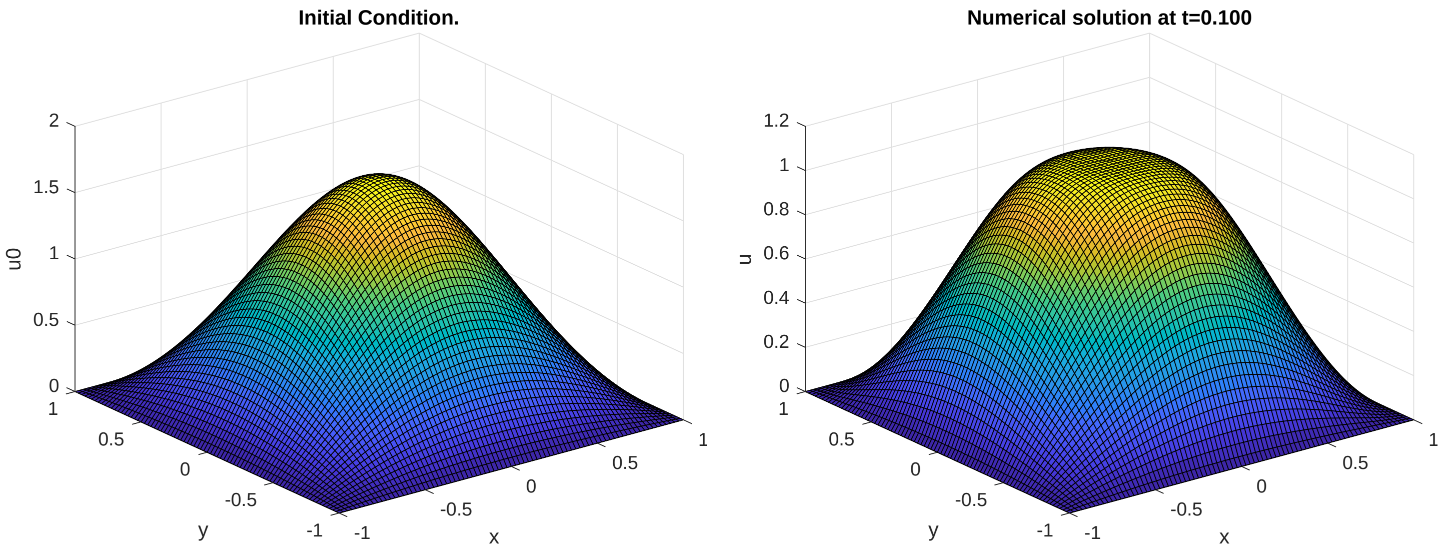

Figure 3: The solutions for the case considered in Example 4.3.

Example 4.4. *We take* $\Omega = (0,1)\times(0,1)$*,* $T = 1$*,* $p(x,y) = 2$*, and* $f = xy(1-x)(1-y) + 2t((1-y)y + (1-x)x)$*. As an initial condition, we set*

$$u_0(x,y) = 0.$$

Then, the exact solution is $u(x,y,t) = txy(1-x)(1-y)$*. Figure 4 shows the numerical solution for* $\gamma = 0.003$ *and* $\rho = h$*, and the exact solution.*

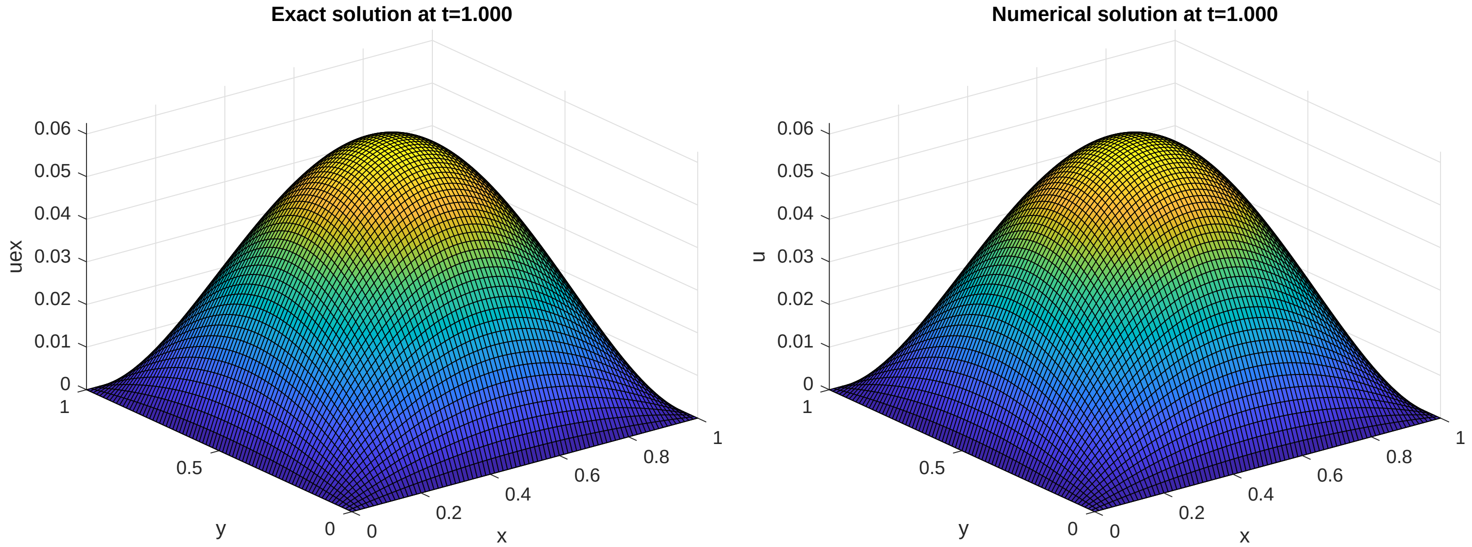

Figure 4: The numerical solution for $\gamma = 0.003$ and $\rho = h$, and the exact solution for the case considered in Example 4.4.

Denote by u_h and u the numerical solution and the exact solution, respectively, of the problem considered in Example 4.4. We obtain the following table concerning the error approximation:

t	0.1	0.2	0.3	0.4	0.5
$\|u_h - u\|_1$	$2.5090.10^{-5}$	$5.6931.10^{-5}$	$7.9781.10^{-5}$	$1.0704.10^{-4}$	$1.331.10^{-4}$

t	0.6	0.7	0.8	0.9	1
$\|u_h - u\|_1$	$1.6182.10^{-4}$	$1.8924.10^{-4}$	$2.1658.10^{-4}$	$2.4383.10^{-4}$	$2.7136.10^{-4}$

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