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Research Article

Connected Ramsey numbers involving trees

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Abstract

A connected 2-coloring of a graph is a 2-coloring of its edges in which the subgraphs spanned by edges in each color are connected. First introduced by David Sumner in 1978, the connected Ramsey number $r_c(G_1, G_2)$ of two graphs G_1 and G_2 is the least positive integer p such that every connected 2-coloring of the complete graph of order p contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. In this paper, the connected Ramsey number is evaluated for stars and trees of order m with maximum degree equal to $m-2$. Some cases of paths versus cycles are also determined.

Keywords: Ramsey number; connectivity; graph factorization.

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1. Introduction

In 1978, David Sumner [16] introduced the concept of a "connected Ramsey number" by restricting to 2-colorings of complete graphs in which the subgraphs spanned by the edges in each color are connected. This topic was recently revitalized in [4], and the present paper continues this investigation. First, we focus on the relevant definitions and background.

Let K_p denote the complete graph of order p. A 2*-coloring* of K_p is a map

$$f : E(K_p) \longrightarrow \{\text{red}, \text{blue}\}.$$

For such a coloring, let G_R be the subgraph spanned by the red edges and G_B be the subgraph spanned by the blue edges. For graphs G_1 and G_2, the *Ramsey number* $r(G_1, G_2,)$ is the least $p \in \mathbb{N}$ such that for every 2-coloring of K_p, either G_R contains a subgraph isomorphic to G_1 or G_B contains a subgraph isomorphic to G_2. A 2-coloring of $K_{r(G_1,G_2)-1}$ that avoids a red subgraph isomorphic to G_1 and a blue subgraph isomorphic to G_2 is called a *critical coloring of* $r(G_1, G_2)$.

The *degree* of a vertex x in a graph G, denoted $\deg_G(x)$, is the number of edges incident with that vertex. If x is a vertex in a 2-colored complete graph, we may refer to its *red degree* $\deg_{G_R}(x)$ and its *blue degree* $\deg_{G_B}(x)$. The *maximum degree* of a graph G is given by

$$\Delta(G) := \max\{\deg_G(x) \mid x \in V(G)\}.$$

A 2-coloring f is called *connected* if both G_R and G_B are connected. Introduced in [16], the *connected Ramsey number* $r_c(G_1, G_2)$ is defined to be the least $p \in \mathbb{N}$ such that every connected 2-coloring of K_p contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. Since every connected 2-coloring of K_p is a 2-coloring, it follows that

$$r_c(G_1, G_2) \le r(G_1, G_2).$$

When equality holds, we say that (G_1, G_2) is *Ramsey-connected*.

One property that connected Ramsey numbers share with Ramsey numbers is that when $r_c(G_1, G_2) = p$, then every connected 2-coloring of K_n, where $n \ge p$, contains a red subgraph isomorphic to G_1 or a blue subgraph isomorphic to G_2. This property is fundamental to many proofs involving specific values of connected Ramsey numbers and follows from Theorem 11 of [3] and Theorem 2.1 of [4]. Recall that in a connected graph G, a *bridge* is an edge whose removal (while retaining all vertices) disconnects G. Sumner (Theorem 2.1 of [16]) proved the following theorem regarding Ramsey-connectedness.

Theorem 1.1 (see [16]). *Let G_1 and G_2 be connected graphs of order at least 4 that do not contain any bridges. Then* $r_c(G_1, G_2) = r(G_1, G_2)$.

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Thus, we focus our attention on the evaluation of connected Ramsey numbers when one of the arguments contains a bridge. In particular, we consider the cases when at least one of the graphs is a *tree* (i.e., a minimally connected graph).

The *path* P_m of order m is a sequence of m distinct vertices $x_1x_2\cdots x_m$ such that x_ix_{i+1} is an edge for every $i \in \{1, 2, \ldots, m-1\}$. If x_1x_m is also an edge, then the sequence $x_1x_2\cdots x_mx_1$ forms a *cycle* of order m, which we denote by C_m. A *spanning cycle* for a graph G is a subgraph of G that is a cycle that includes all of the vertices in G.

Recall that for $k \geq 1$, a k-*factor* of a graph G is a spanning k-regular subgraph. A 1-factor of a graph is often called a *perfect matching* while a 2-factor of a graph is a spanning cycle. A k-*factorization* of a graph G is a factorization of G into k-factors. When a graph has a k-factorization, we say that it is k-*factorable*. The following results are well known (e.g., see Theorems 9.6 and 9.7 of Harary's text [9]).

Theorem 1.2 (see [9, 14]). *For every $k \in \mathbb{N}$, the complete graph K_{2k+1} factors into k spanning cycles.*

Theorem 1.3 (see [9]). *For every $k \in \mathbb{N}$, the complete graph K_{2k} factors into $k-1$ spanning cycles and a 1-factor.*

A graph G is called *totally connected* if its complement $\overline{G}$ is also connected. Note that at least one of G and $\overline{G}$ must be connected (Theorem 1.1 of [2]). Many results concerning connected Ramsey numbers can be rephrased in terms of totally connected graphs (see Corollaries 3.2, 3.4, and 3.6 of [4]).

In the case of paths, Sumner proved the following theorem (Theorem 3.1 of [16]).

Theorem 1.4 (see [16]). *If $n \geq m \geq 4$, then*

$$r_c(P_m, P_n) = \begin{cases} 4 & \text{if } m = 4 \\ n + \left\lfloor \dfrac{m-3}{2} \right\rfloor - 1 & \text{if } m \geq 5. \end{cases}$$

The main results proved in [4] include the following:

$$r_c(P_m, K_3) = m \quad \text{for all } m \geq 4,$$

$$r_c(P_5, K_n) = n + 2 \quad \text{for al } n \geq 3, \text{ and}$$

$$r_c(K_{1,3}, K_n) = 2n \quad \text{for all } n \geq 3.$$

In this paper, we consider additional cases of trees versus trees as well as some cases of paths versus cycles. For $m, n \geq 3$, it is shown in Section 2 that $(K_{1,m}, K_{1,n})$ is Ramsey-connected:

$$r_c(K_{1,m}, K_{1,n}) = \begin{cases} m + n - 1 & \text{if } m \text{ and } n \text{ are even} \\ m + n & \text{otherwise.} \end{cases}$$

If T_m^* denotes a tree of order m such that $\Delta(T_m^*) = m - 2$, we prove that

$$r_c(T_m^*, T_n^*) = \begin{cases} m + n - 5 & \text{if } m \text{ and } n \text{ are even} \\ m + n - 4 & \text{otherwise,} \end{cases}$$

whenever $m, n \geq 5$. The section is then concluded with a proof that for all $m \geq 4$ and $n \geq 3$,

$$r_c(T_m^*, K_{1,n}) = \begin{cases} m + n - 3 & \text{if } m \text{ and } n \text{ are even} \\ m + n - 2 & \text{otherwise.} \end{cases}$$

In Section 3, we turn our attention to the cases of paths versus cycles. We prove that

$$r_c(P_m, C_m) \geq m + 1 \quad \text{for all } m \geq 5,$$

$$r_c(P_m, C_5) \leq m + 1 \quad \text{for all } m \geq 4, \text{ and}$$

$$r_c(P_5, C_n) = n + 1 \quad \text{for all } n \geq 4.$$

Section 4 concludes with some directions for future work.

2. Trees versus trees

In 1972, Harary [10] proved that

$$r(K_{1,m}, K_{1,n}) = \begin{cases} m+n-1 & \text{if } m \text{ and } n \text{ are even} \\ m+n & \text{otherwise.} \end{cases}$$

The corresponding connected Ramsey number is considered in the following theorem. Note that the assumption $m, n \geq 3$ restricts our focus to stars that are not paths, which were already considered by Sumner [16].

Theorem 2.1. *For all $m, n \geq 3$, $(K_{1,m}, K_{1,n})$ is Ramsey-connected.*

Proof. We consider cases based upon the parities of m and n, and provide connected critical colorings in each case.

Case 1. Suppose that m and n have different parities. Without loss of generality, suppose that m is odd and n is even. Then $(m+n)-1$ is even and we let $m+n-1 = 2k$, for some $k \in \mathbb{N}$. By Theorem 1.3, K_{m+n-1} is the sum of $k-1$ spanning cycles and a single 1-factor. The spanning cycles can be split into $\frac{m-1}{2}$ red spanning cycles and $\frac{n-2}{2}$ blue spanning cycles. Coloring the 1-factor blue, each vertex will have red degree $m-1$ and blue degree $n-1$. It follows that

$$r_c(K_{1,m}, K_{1,n}) \geq m+n = r(K_{1,m}, K_{1,n})$$

in this case.

Case 2. Suppose that m and n are both odd. Then $m+n-1 = 2k+1$ for some $k \in \mathbb{N}$. By Theorem 1.2, K_{m+n-1} factors into k spanning cycles. Color $\frac{m-1}{2}$ of the spanning cycles red and the other $\frac{n-1}{2}$ of them blue. Each vertex then has red degree $m-1$ and blue degree $n-1$ and it follows that

$$r_c(K_{1,m}, K_{1,n}) \geq m+n = r(K_{1,m}, K_{1,n})$$

in the considered case.

Case 3. Finally, suppose that m and n are both even. Then $m+n-2 = 2k$ for some $k \in \mathbb{N}$, and by Theorem 1.3, K_{m+n-2} factors into $k-1$ spanning cycles and a single 1-factor. Color $\frac{m-2}{2}$ spanning cycles red and $\frac{n-2}{2}$ spanning cycles blue. Coloring the 1-factor red, the red degree of each vertex is $m-1$ and the blue degree of each vertex is $n-2$. It follows that

$$r_c(K_{1,m}, K_{1,n}) \geq m+n-1 = r(K_{1,m}, K_{1,n})$$

in this case.

In all cases, we find that $r_c(K_{1,m}, K_{1,n}) = r(K_{1,m}, K_{1,n})$. It follows that $(K_{1,m}, K_{1,n})$ is Ramsey Connected. □

In 1995, Guo and Volkmann [8] considered Ramsey numbers involving a tree T_m^* of order m that satisfies $\Delta(T_m^*) = m-2$. Such a tree is necessarily a *broom*; that is, it can be formed by constructing a single edge between one vertex in the path P_2 and the center vertex in the star $K_{1,m-3}$. Specifically, Guo and Volkmann proved that

$$r(T_m^*, T_n^*) = \begin{cases} m+n-3 & \text{if } (m-1)|(n-3) \text{ or } (n-1)|(m-3) \\ m+n-5 & \text{if } m \text{ is even and } m = n \\ m+n-4 & \text{otherwise.} \end{cases}$$

Before we consider the corresponding connected Ramsey number, we prove the following lemma.

Lemma 2.1. *Let G be a totally connected graph of order $n \geq m-1 \geq 2$. If $\Delta(G) \geq m-2$, then G contains a subgraph that is isomorphic to T_m^*.*

Proof. Identify the graph G with the red induced subgraph of a connected 2-coloring of K_n in which $\Delta(G) \geq m-2$. Let x be a vertex of degree at least $m-2$. Partition the neighbors of x into two sets:

$$R := \{y \in V(G) \mid xy \text{ is red}\} \quad \text{and} \quad B := \{y \in V(G) \mid xy \text{ is blue}\}.$$

Then

$$m-2 \leq |R| \leq n-1 \quad \text{and} \quad |B| \geq 1.$$

Since G is connected, there exists some red edge yz such that $y \in R$ and $z \in B$. Then the subgraph induced by the set containing x, y, z, and $m-3$ vertices in the red subgraph contains a copy of T_m^* with x being the degree $m-2$ vertex. □

Theorem 2.2. *Let $m, n \geq 5$. Then*

$$r_c(T_m^*, T_n^*) = \begin{cases} m+n-5 & \text{if } m \text{ and } n \text{ are even} \\ m+n-4 & \text{otherwise.} \end{cases}$$

Proof. We start by showing that for all $m, n \geq 5$, $r_c(T_m^*, T_n^*) \leq m+n-4$. Consider a connected 2-coloring of K_{m+n-4} Then each vertex has degree $m+n-5$. If $\Delta(G_R) \geq m-2$, then by Lemma 2.1, there exists a red T_m^*. Otherwise, $\Delta(G_R) \leq m-3$, from which it follows that

$$\Delta(G_B) \geq m+n-5-(m-3) = n-2.$$

Lemma 2.1 then implies that there exists a blue T_n^*. Hence, $r_c(T_m^*, T_n^*) \leq m+n-4$.

Next, we show that this upper bound can be improved to

$$r_c(T_m^*, T_n^*) \leq m+n-5$$

when m and n are even. With this assumption, consider a connected 2-coloring of K_{m+n-5} and note that $m+n-5$ is odd. As with the general case, $\Delta(G_R) \geq m-2$ implies there exists a red T_m^* and $\Delta(G_B) \geq n-3$ implies there exists a blue T_n^*, by Lemma 2.1. The only other possibility is that all vertices have red degree equal to $m-3$ and blue degree equal to $n-3$ (since the vertices in K_{m+n-5} all have total degree equal to $m+n-6$. Observe that such a coloring cannot occur as G_R and G_B would both be graphs that have an odd number of odd degree vertices. Hence, $r_c(T_m^*, T_n^*) \leq m+n-5$ when m and n are both even.

For the lower bounds, we consider cases, based on the parities of m and n.

Case 1. Suppose that m and n are both odd. Then $m+n-5$ is odd and letting $m+n-5 = 2k+1$, Theorem 1.2 implies that K_{m+n-5} can be factored into $k = \frac{m+n-6}{2}$ spanning cycles. Coloring $\frac{m-3}{2}$ spanning cycles red and $\frac{n-3}{2}$ spanning cycles blue results in a connected 2-coloring that avoids a red T_m^* and a blue T_n^*. Thus,

$$r_c(T_m^*, T_n^*) \geq m+n-4$$

in this case.

Case 2. Suppose that one of m and n is odd and the other is even. Without loss of generality, assume that m is odd and n is even. Then $m+n-5$ is even and letting $m+n-5 = 2k$, Theorem 1.3 implies that K_{m+n-5} can be factored into $k-1$ spanning cycles and a single 1-factor. Color $\frac{m-3}{2}$ spanning cycles red, and $\frac{n-4}{2}$ spanning cycles and the 1-factor blue. This results in a connected 2-coloring of K_{m+n-5} that avoids a red T_m^* and a blue T_n^*. Thus,

$$r_c(T_m^*, T_n^*) \geq m+n-4$$

in the considered case.

Case 3. Suppose that m and n are both even. Then $m+n-6$ is even and letting $m+n-6 = 2k$, Theorem 1.3 implies that K_{m+n-6} can be factored into $\frac{m+n-8}{2}$ spanning cycles and a single 1-factor. Coloring $\frac{m-4}{2}$ spanning cycles red, $\frac{n-4}{2}$ spanning cycles blue, and the 1-factor red results in a connected 2-coloring that avoids a red T_m^* and a blue T_n^*. Thus,

$$r_c(T_m^*, T_n^*) \geq m+n-5$$

in this case. □

Theorem 2.3. *For all $m \geq 4$ and $n \geq 3$,*

$$r_c(T_m^*, K_{1,n}) = \begin{cases} m+n-3 & \text{if } m \text{ and } n \text{ are even} \\ m+n-2 & \text{otherwise.} \end{cases}$$

Proof. We begin by proving that $r_c(T_m^*, K_{1,n}) \leq m+n-2$ for all $m \geq 4$ and $n \geq 3$. Consider a connected red/blue coloring of K_{m+n-2} that lacks a blue $K_{1,n}$. Then the blue degree of each vertex is at most $n-1$. It follows that the red degree of every vertex is at least $m+n-3-(n-1) = m-2$. By Lemma 2.1, there exists a red T_m^*. It follows that $r_c(T_m^*, K_{1,n}) \leq m+n-2$.

In the case where m and n are both even, avoiding a red T_m^* and a blue $K_{1,n}$ forces each vertex in a connected red/blue coloring of K_{m+n-3} to have red degree at most $m-3$ and blue degree at most $n-1$. If these degrees are actually achieved, then the subgraph spanned by red edges has order $m+n-3$ (which is odd) with every vertex having degree $m-3$ (which is also odd). No such graph exists since it is not possible for a graph to have an odd number of vertices of odd degree. It follows that

$$r_c(T_m^*, K_{1,n}) \leq m+n-3$$

when m and n are both even.

For the lower bounds, we again consider cases, based on the parities of m and n.

Case 1. Assume that m and n are both even. Then $m+n-4$ is even and by Theorem 1.3, K_{m+n-4} can be factored into $k-1$ spanning cycles and a 1-factor, where $m+n-4=2k$. Color $\frac{m-4}{2}$ spanning cycles red, $\frac{n-2}{2}$ spanning cycles blue, and the 1-factor red. Each vertex has red degree $m-3$ and blue degree $n-2$, avoiding a red T_m^* and a blue $K_{1,n}$. It follows that

$$r_c(T_m^*, K-1, n) \geq m+n-3.$$

Case 2. Assume that m and n are both odd. Then $m+n-3$ is odd and by Theorem 1.2, K_{m+n-3} can be factored into k spanning cycles, where $m+n-3=2k+1$. Color $\frac{m-3}{2}$ spanning cycles red and $\frac{n-1}{2}$ spanning cycles blue. Then each vertex has red degree $m-3$ and blue degree $n-1$, avoiding a red T_m^* and a blue $K_{1,n}$. It follows that

$$r_c(T_m^*, K-1, n) \geq m+n-2.$$

Case 3. Assume that m is even and n is odd. Then $m+n-3$ is even and by Theorem 1.3,K_{m+n-3} can be factored into $k-1$ spanning cycles and a 1-factor, where $m+n-3=2k$. Color $\frac{m-4}{2}$ spanning cycles and the 1-factor red, and $\frac{n-1}{2}$ spanning cycles blue. Then each vertex has red degree $m-3$ and blue degree $n-1$, avoiding a red T_m^* and a blue $K_{1,n}$. It follow that

$$r_c(T_m^*, K-1, n) \geq m+n-2.$$

Case 4. Assume that m is odd and n is even. Then $m+n-3$ is even and by Theorem 1.3, K_{m+n-3} can be factored into $k-1$ spanning cycles and a 1-factor, where $m+n-3=2k$. Color $\frac{m-3}{2}$ spanning cycles red, and $\frac{n-2}{2}$ spanning cycles and the 1-factor blue. Then each vertex has red degree $m-3$ and blue degree $n-1$, avoiding a red T_m^* and a blue $K_{1,n}$. It follow that $r_c(T_m^*, K-1, n) \geq m+n-2$. □

3. Paths versus cycles

At the conclusion of his paper, Sumner [16] recommended the next step in the evaluation of connected Ramsey numbers be the case of paths versus cycles. In this section, we begin this investigation. First, note the following fact about connected 2-colorings.

Lemma 3.1. *Every connected red/blue coloring of K_4 factors into a red P_4 and a blue P_4.*

Proof. Let $\{a,b,c,d\}$ be the vertices of a connected red/blue coloring of K_4. Then each vertex must be incident with at least one red edge, at least one blue edge, and exactly two edges of the same color. Without loss of generality, suppose that ab and ac are red and ad is blue. Vertex d must also be incident with a red edge, so without loss of generality, assume that bd is red. Then bc must be blue so that b is incident with a blue edge. In order for the blue subgraph to be connected, cd must also be blue. It follows that $adcb$ is a blue P_4 and $cabd$ is a red P_4. □

It was shown in [4] that $r(P_m, C_3) = m$, for all $m \geq 4$. Since every connected 2-coloring of K_4 contains a red P_4 by Lemma 3.1, it follows that $r_c(P_4, C_n) = 4$ for all $n \geq 3$.

Theorem 3.1. *For all $m \geq 5$, $r_c(P_m, C_m) \geq m+1$.*

Proof. Consider the connected red/blue coloring of K_m in which the subgraph spanned by the red edges is isomorphic to T_m^* (e.g., Figure 3.1 shows a red T_6^* and the corresponding connected red/blue coloring of K_6). In this broom, denote the vertex of degree $m-2$ by x. No red P_m exists since the longest path contained in T_m^* has order 4. Also, no blue C_m exists since such a cycle would have to include all of the vertices and x has blue degree equal to 1. It follows that $r_c(P_m, C_m) \geq m+1$. □

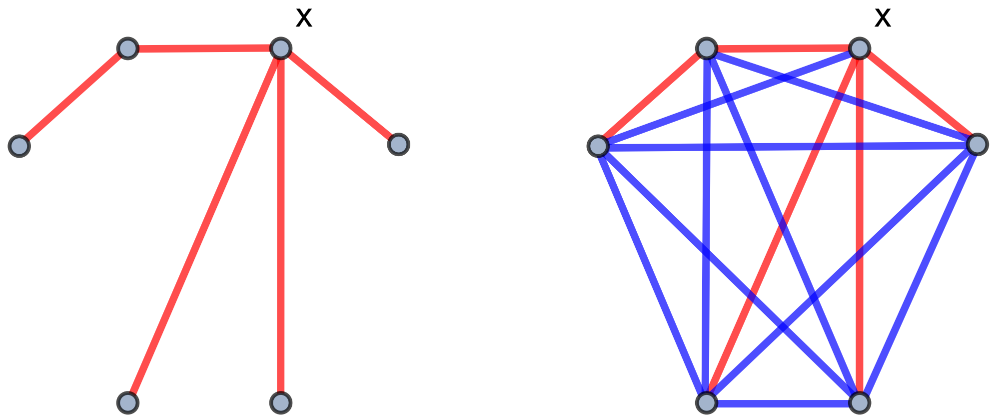

Figure 3.1: A red T_6^* and the corresponding connected 2-coloring of K_6. This coloring avoids a red P_6 and a blue C_6.

Theorem 3.2. *For all* $m \geq 4$, $r_c(P_m, C_5) \leq m+1$.

Proof. We proceed by induction on $m \geq 4$, with the base case corresponding to $r_c(P_4, C_5) = 4 \leq 5$. Assume that for $m \geq 4$, $r_c(P_m, C_5) \leq m+1$ and consider a connected red/blue coloring of K_{m+2}. By the inductive hypothesis, there exists a red P_m or a blue C_5. Assume the former case and denote the red P_m by $x_1x_2\cdots x_m$. Denote the other two vertices by y and z. If a red P_{m+1} is avoided, then x_1y, x_1z, x_my, and x_mz must be blue (see the first image in Figure 3.2). In order for this coloring to be connected, some x_i, with $2 \leq i \leq m-1$, must join to $\{y, z\}$ via a red edge. Without loss of generality, assume that x_iy is red. Since $m \geq 4$, at least one of x_{i-1} and x_{i+1} must be distinct from x_1 and x_m, respectively. Without loss of generality, assume that $x_{i+1} \neq x_m$. If $x_{i+1}y$ is red, then $x_1x_2\cdots x_iyx_{i+1}\cdots x_m$ is a red P_{m+1}. So, assume $x_{i+1}y$ is blue and now consider the edge x_1x_{i+1} (see the second image in Figure 3.2). If x_1x_{i+1} is red, then $yx_ix_{i-1}\cdots x_1x_{i+1}\cdots x_m$ is a red P_{m+1}. If x_1x_{i+1} is blue, then $x_1x_{i+1}yx_mzx_1$ is a blue C_5. It follows that $r_c(P_{m+1}, C_5) \leq m+2$. □

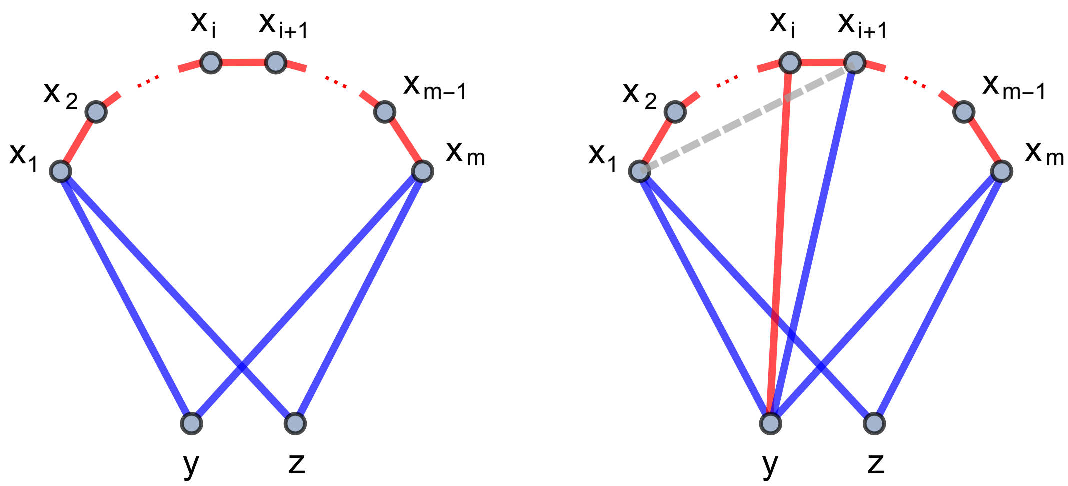

Figure 3.2: Avoiding a red P_{m+1} and a blue C_5 in a connected 2-coloring of K_{m+1}.

Combining the results of Theorems 3.1 and 3.2, we find that $r_c(P_5, C_5) = 6$. The following theorem goes a step further, evaluating all of the cases of P_5 versus a cycle of order at least 4.

Theorem 3.3. *For all* $n \geq 4$, $r_c(P_5, C_n) = n+1$.

Proof. If $n = 4$, then the connected 2-coloring of K_4 described in Lemma 3.1 avoids a red P_5 and a blue C_4. It follows that $r_c(P_5, C_4) \geq 5$. If $n \geq 5$, consider a connected 2-coloring of a K_n consisting of a red T_n^* and all other edges colored blue. Then one vertex has red degree $n-2$ and blue degree 1, preventing the existence of a blue C_n. The longest red path has order 4, so a red P_5 is also avoided. This implies that $r_c(P_5, C_n) \geq n+1$ when $n \geq 5$.

To prove the reverse inequality, consider a connected 2-colored K_{n+1}. By Lemma 3.1 there exists a red P_4. Label the vertices of such a path by $abcd$. If $n = 4$, then label the other vertex x and note that ax and dx must be blue if a red P_5 is to be avoided. Since the 2-coloring is connected, one of bx and cx must be red. Without loss of generality, assume that bx is red. Then cx must be blue or $abxcd$ would be a red P_5. Also, ad must be blue, otherwise $xbcda$ is a red P_5. Now consider edge ac. If it is red, then $xbacd$ is a red P_5. If it is blue, then $acxda$ is a blue C_4. It follows that $r_c(P_5, C_4) \leq 5$.

If $n = 5$, label the vertices of a red P_4 by $abcd$ and label the other vertices x and y. If a red P_5 is avoided, then all edges joining $\{a, d\}$ to $\{x, y\}$ are blue. Since the 2-coloring is connected, some red edge must join $\{b, c\}$ to $\{x, y\}$. Without loss of generality, assume that bx is red. Then cx must be blue or $abxcd$ would be a red P_5. If ad is red, then $xbcda$ is a red P_5, so assume that ad is blue. If ac is red, then $dcabx$ is a red P_5. So, ac must be blue and $xdyacx$ is a blue C_5. It follows that $r_c(P_5, C_5) \leq 6$.

For the cases where $n \geq 6$, let $abcd$ be a red P_4 and label the other vertices $\{x_1, x_2, \ldots, x_{n-3}\}$. Avoiding a red P_5 forces ax_i and dx_i to be blue for all $i \in \{1, 2, \ldots, n-3\}$. Since we are considering a connected 2-coloring, at least one of b or c must join to $\{x_1, x_2, \ldots, x_{n-3}\}$ via a red edge. Without loss of generality, assume that bx_1 is red. Then cx_1 must be blue, otherwise abx_1cd is a red P_5. If any edge joining x_1 to $\{x_2, x_3, \ldots, x_{n-3}\}$ is red, say x_1x_2, then x_2x_1bcd is a red P_5. So, assume that al such edges are blue. Now, we proceed by induction, assuming that exactly one of b or c joins to each of $x_1, x_2, \ldots, x_i$ via a red edge, the subgraph induced by $\{x_1, x_2, \ldots, x_i\}$ is a blue K_i, and x_i joins to $\{x_{i+1}, x_{i+2}, \ldots, x_{n-3}\}$ via only blue edges. Since our 2-coloring is connected, exactly one of b and c must join to $\{x_{i+1}, x_{i+2}, \ldots, x_{n-3}\}$ via a red edge. If any edge joining x_{i+1} to $\{x_{i+2}, x_{i+3}, \ldots, x_{n-3}\}$ is red, then a red P_5 is formed. So, assume all such edges are blue. The result of this inductive process is that the subgraph induced by $\{x_1, x_2, \ldots, x_{n-3}\}$ is a blue K_{n-3}. Also, note that if ad is red, then x_1bcda is a red P_5. So, ad must be blue and the subgraph induced by $\{a, d, x_1, x_2, \ldots, x_{n-3}\}$ is a blue K_{n-1}. At least one of b or c joins to at least two vertices in $\{x_1, x_2, \ldots, x_{n-3}\}$ via blue edges (since $n \geq 6$), and this vertex, along with the blue K_{n-1} forms a blue C_n. It follows that $r_c(P_5, C_n) \leq n + 1$. □

4. Conclusion

Besides the cases of trees versus trees considered here, there are still many trees T_m of order m such that $\Delta(T_m) \leq m - 3$ that have not yet been considered. Much work is also still open on the general problem of determining the connected Ramsey number for paths versus cycles. Other cases worth considering include paths versus stars (see [5] and [13]), path versus books (see [15]), and paths versus wheels (see [1], [6], and [12]). Other variations of connected Ramsey numbers one may investigate include a weakened version (similar to [7]) or a star-critical version (similar to [11]). At present, no multicolor analogue of the connected Ramsey number has been studied.

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