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Research Article

A new product of monoids based on bicrossed and Schützenberger products

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Abstract

Let E and F be two monoids. A new product of E and F, using the bicrossed product $E \bowtie F$ and the Schützenberger product $E\Diamond F$, is defined. This new product is denoted by $E \Diamond_{\bowtie} F$ and is referred to as the bicrossed-Schützenberger product. Under specific conditions, it is demonstrated that this new product is a monoid. A presentation of the bicrossed-Schützenberger product of any two monoids is also obtained. Furthermore, necessary and sufficient conditions for the regularity of $E \Diamond_{\bowtie} F$ are provided.

Keywords: bicrossed product; Schützenberger product; regularity.

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1. Introduction and preliminaries

The classification of groups has long been a topic of great interest. In the present study, a new concept related to the classification of groups is discussed. Particularly, a novel product of monoids is proposed. It may have the advantage of obtaining some new monoids by using the existing ones. It may also be used to simplify some complex monoids and to form new groups that inherit some of the characteristics of old groups of monoids.

The appearance of the bicrossed product, first introduced by Zappa [20] and later published in Takeuchi's article [19], was a major turning point in the investigation of monoids' classification. This product is also referred to as the Zappa-Szep product [3] and the knit product [15]. For numerous structures, including Lie groups, locally compact quantum groups, algebras, Lie algebras, Hopf algebras, and groupoids, the bicrossed products were considered and studied in the literature. The bicrossed product for finite groups was initially examined in [1]. The essential components that are employed to obtain the bicrossed product of groups are the so-called matched pairings of groups. A group G is said to be the internal bicrossed product of its subgroups E and F if $EF = G$ and their intersection is trivial. When neither of the factors is normal, this product generalizes the semidirect product.

In order to solve automata theory problems and analyze the syntactic properties of the concatenation product using formal language theory, the Schützenberger product of monoids was proposed in [17]. Initially, Schützenberger defined the Schützenberger product for two monoids. Later, this product was generalized by Straubing [18] for any finite number of monoids. The presentation of the Schützenberger product for two monoids was demonstrated in [10], where details on the normal form of the main components were also reported.

In the literature, there are various studies on the products that are obtained by combining the Schützenberger product with other products. For instance, a new product using the semi-direct product and Schützenberger product for any two monoids was defined in [2], where the regularity of this new product was also investigated. A new monoid construction under crossed products for specified monoids was proposed in [9], where the authors not only presented a generating set and a relator set for this product but also provided sufficient and necessary conditions for its regularity. As an extension of this work, the present author [8] studied the n-generalized Schützenberger-crossed product and proposed a novel version of this product.

The primary purpose of the current article is to define a new product for two monoids using the bicrossed product and the Schützenberger product. The newly defined product is referred to as the bicrossed-Schützenberger product. Under certain conditions, it is demonstrated that this new product is a monoid. A presentation of the bicrossed-Schützenberger product of any two monoids is also obtained. Moreover, necessary and sufficient conditions for the regularity of the bicrossed-Schützenberger product are provided.

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In what follows, the basic definitions used in the rest of the article are provided. These definitions can be found in [1,4,7,10,11,13,14,17].

Let E and F be two monoids. Consider the mappings $\alpha : F \times E \to E$ and $\beta : F \times E \to F$ defined as follows:

$$\alpha(f,e) = f \triangleright e \qquad \text{and} \qquad \beta(f,e) = f \triangleleft e$$

for all $e \in E$ and $f \in F$. If $\alpha : F \times E \to E$ is an action of F on E as group automorphisms, then we denote by $E \rtimes_\alpha F$ the semidirect product of E on F: $E \rtimes_\alpha F = E \times F$, as a group using the multiplication given by

$$(e_1, f_1)(e_2, f_2) = (e_1(f_1 \triangleright e_2), f_1 f_2),$$

for all $e_1, e_2 \in E$ and $f_1, f_2 \in F$.

Definition 1.1. *A quadruple (E, F, α, f) is a matched pair of monoids, where E and F are groups, $\alpha : F \times E \to E$ is a left action of the group F on the set E, $\beta : F \times E \to E$ is a right action of the group E on the set F, provided that the following compatibility requirements are satisfied:*

$$\left.\begin{aligned} f \triangleright (e_1 e_2) &= (f \triangleright e_1)((f \triangleleft e_1) \triangleright e_2), \\ (f_1 f_2) \triangleright e &= f_1 \triangleright (f_2 \triangleright e), \\ (f_1 f_2) \triangleleft e &= (f_1 \triangleleft (f_2 \triangleright e))(f_2 \triangleleft e), \\ f \triangleleft e_1 e_2 &= f \triangleleft (e_1 \triangleleft e_2), \end{aligned}\right\} \tag{1}$$

for all $e, e_1, e_2 \in E$ and $f, f_1, f_2 \in F$. The quadruple (E, F, α, f) is said to be the matched pair if $f \triangleright 1 = 1$ and $1 \triangleleft e = 1$ for all $e \in E$ and $f \in F$.

Consider two monoids E and F with the mappings $\alpha : F \times E \to E$ and $\beta : F \times E \to F$. The bicrossed product of E and F, represented by $E \bowtie_\beta F = E \bowtie F$, is the set $E \times F$ with the multiplication

$$(e_1, f_1)(e_2, f_2) = (e_1(f_1 \triangleright e_2), (f_1 \triangleleft e_2) f_2),$$

for all $e_1, e_2 \in E$ and $f_1, f_2 \in F$. The bicrossed product $E \bowtie F$ is a monoid with the unit element $(1_E, 1_F)$ if and only if (E, F, α, f) is a matched pair [1].

Definition 1.2. *Let $E = \langle X_1 \mid R_1 \rangle$ and $F = \langle X_2 \mid R_2 \rangle$ be monoids. For $P \subseteq E \times F$, $e \in E$, and $f \in F$, define*

$$eP = \{(ec, d) \mid (c, d) \in P\} \quad \textit{and} \quad Pf = \{(c, df) \mid (c, d) \in P\}.$$

The Schützenberger product of E and F, denoted by $E \Diamond F$, is the set $E \times \mathcal{P}(E \times F) \times F$ with the multiplication

$$(e_1, P_1, f_1)(e_2, P_2, f_2) = (e_1 e_2, P_1 f_2 \cup e_1 P_2, f_1 f_2).$$

It is known that $E \Diamond F$ is a monoid with the identity $(1_E, \emptyset, 1_F)$, where $\emptyset$ is an empty set. The Schützenberger product of E and F is presented by

$$\wp_{E \Diamond F} = \Big\langle Z \mid R_1,\ R_2,\ z_{e_1,f_1}^2 = z_{e_1,f_1},\ z_{e_1,f_1} z_{e_1',f_1'} = z_{e_1',f_1'} z_{e_1,f_1},$$

$$x_1 z_{e_1,f_1} = z_{x_1 e_1, f_1} x_1,\quad z_{e_1,f_1} x_2 = x_2 z_{e_1, f_1 x_2},\ x_1 x_2 = x_2 x_1 \Big\rangle,$$

where $x_i \in X_i$ $(i \in \{1,2\})$, $e_1, e_1' \in E$, $f_1, f_1' \in F$, and $Z = X_1 \cup X_2 \cup \{z_{e_1,f_1} \mid e_1 \in E,\ f_1 \in F\}$; see [10].

2. Main results

In this section, a new product of two monoids is defined by combining the concepts of the Schützenberger product and bicrossed product. This new product is referred to as the bicrossed-Schützenberger product. Under specific conditions, it is proved that the bicrossed-Schützenberger product of two monoids is also a monoid. Additionally, the presentation for the bicrossed-Schützenberger product for any two monoids is provided in the current section.

Definition 2.1. *Consider a quadruple (E, F, α, f) as a matched pair of two monoids E and F. For $P \subseteq E \times F$, $e \in E$, and $f \in F$, let*

$$eP = \{(ec, d) \mid (c, d) \in P\} \quad \textit{and} \quad Pf = \{(c, df) \mid (c, d) \in P\}.$$

Considering the above-mentioned concepts, the bicrossed-Schützenberger product of E and F is the set $E \times F$ equipped with the operation

$$(e_1, P_1, f_1)(e_2, P_2, f_2) = (e_1(f_1 \triangleright e_2),\ P_1 f_2 \cup e_1 P_2,\ (f_1 \triangleleft e_2) f_2)$$

for all $e_1, e_2 \in E$ and $f_1, f_2 \in F$. The bicrossed-Schützenberger product of E and F is denoted by $E \lozenge_{\bowtie} F$.

The first main result of this paper is the next theorem, which confirms that the newly defined product is a monoid when certain conditions are satisfied.

Theorem 2.1. *Let E and F be any monoids. For all $e_1, e_2, e_3 \in E$ and $f_1, f_2, f_3 \in F$, assume that the conditions listed in (1) with the following features are satisfied:*

$$f_1 \triangleright e_2 = e_2 \qquad \textit{and} \qquad f_2 \triangleleft e_3 = f_2. \tag{2}$$

Then, the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ of E and F specifies a monoid.

Proof. We examine the monoid characteristics of the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$. We start by demonstrating the associative property. To accomplish it, for any $e_1, e_2, e_3 \in E$ and $f_1, f_2, f_3 \in F$, let

$$(e_1, P_1, f_1), (e_2, P_2, f_2), (e_3, P_3, f_3) \in E \lozenge_{\bowtie} F.$$

Hence, the left-hand side, say $[(e_1, P_1, f_1)(e_2, P_2, f_2)](e_3, P_3, f_3)$, of the desired equation is

$$\begin{aligned}
&= (e_1(f_1 \triangleright e_2),\ P_1 f_2 \cup e_1 P_2,\ (f_1 \triangleleft e_2) f_2)(e_3, P_3, f_3) \\
&= (e_1(f_1 \triangleright e_2)(((f_1 \triangleleft e_2) f_2) \triangleright e_3),\ (P_1 f_2 \cup e_1 P_2) f_3 \cup (e_1(f_1 \triangleright e_2)) P_3,\ (((f_1 \triangleleft e_2) f_2) \triangleleft e_3) f_3) \\
&= (e_1(f_1 \triangleright e_2)(((f_1 \triangleleft e_2) f_2) \triangleright e_3),\ P_1 f_2 f_3 \cup e_1 P_2 f_3 \cup e_1(f_1 \triangleright e_2)) P_3,\ f_1 \triangleleft (e_2 \triangleleft (f_2 \triangleright e_3))(f_2 \triangleleft e_3) f_3) \\
&= (e_1 e_2(((f_1 \triangleleft e_2) f_2) \triangleright e_3),\ P_1 f_2 f_3 \cup e_1 P_2 f_3 \cup e_1 e_2 P_3,\ f_1 \triangleleft (e_2 \triangleleft (f_2 \triangleright e_3)) f_2 f_3) \\
&= (e_1 e_2((f_1 \triangleleft e_2) \triangleright (f_2 \triangleright e_3)),\ P_1 f_2 f_3 \cup e_1 P_2 f_3 \cup e_1 e_2 P_3,\ f_1 \triangleleft (e_2 \triangleleft (f_2 \triangleright e_3)) f_2 f_3),
\end{aligned}$$

and the right-hand side $(e_1, P_1, f_1)[(e_2, P_2, f_2)(e_3, P_3, f_3)]$ of the equation regarding the associative property is

$$\begin{aligned}
&= (e_1, P_1, f_1)(e_2(f_2 \triangleright e_3),\ P_2 f_3 \cup e_2 P_3,\ (f_2 \triangleleft e_3) f_3) \\
&= (e_1(f_1 \triangleright (e_2(f_2 \triangleright e_3)),\ P_1((f_2 \triangleleft e_3) f_3) \cup e_1(P_2 f_3 \cup e_2 P_3),\ (f_1 \triangleleft (e_2(f_2 \triangleright e_3))(f_2 \triangleleft e_3) f_3)) \\
&= (e_1(f_1 \triangleright e_2)((f_1 \triangleleft e_2) \triangleright (f_2 \triangleright e_3)),\ P_1(f_2 \triangleleft e_3) f_3 \cup e_1 P_2 f_3 \cup e_1 e_2 P_3),\ (f_1 \triangleleft (e_2(f_2 \triangleright e_3))(f_2 \triangleleft e_3) f_3)) \\
&= (e_1 e_2((f_1 \triangleleft e_2) \triangleright (f_2 \triangleright e_3)),\ P_1 f_2 f_3 \cup e_1 P_2 f_3 \cup e_1 e_2 P_3,\ (f_1 \triangleleft (e_2(f_2 \triangleright e_3)) f_2 f_3)) \\
&= (e_1 e_2((f_1 \triangleleft e_2) \triangleright (f_2 \triangleright e_3)),\ P_1 f_2 f_3 \cup e_1 P_2 f_3 \cup e_1 e_2 P_3,\ f_1 \triangleleft (e_2 \triangleleft (f_2 \triangleright e_3)) f_2 f_3).
\end{aligned}$$

Thus, the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ has the associative property. Now, for the identity element $(1_E, \emptyset, 1_F)$ of $E \lozenge_{\bowtie} F$, we have

$$(e, P, f)(1_E, \emptyset, 1_F) = (e(f \triangleright 1_E), \emptyset \cup P, (f \triangleleft 1_E) 1_F) = (e 1_E, P, f 1_F) = (e, P, f)$$

and

$$(1_E, \emptyset, 1_F)(e, P, f) = (1_E(1_F \triangleright e), P \cup \emptyset, 1_F(1_E \triangleleft f)) = (1_E e, P, 1_F f) = (e, P, f).$$

Therefore, the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ is a monoid with the identity $(1_E, \emptyset, 1_F)$. □

Next, for any two monoids E and F, we obtain a presentation for the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$.

Theorem 2.2. *Let $E = \langle X \mid R \rangle$ and $F = \langle Y \mid S \rangle$ be monoids. Then the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ is determined by the generator*

$$Z = X \cup Y \cup \{z_{e,f} \,;\, e \in E,\ f \in F\}$$

and the relations

$$R = 1, \qquad S = 1, \tag{3}$$

$$yx = (y \triangleright x)(y \triangleleft x) \qquad (x \in X,\ y \in Y), \tag{4}$$

$$z_{e,f}^2 = z_{e,f}, \qquad z_{e,f} z_{e',f'} = z_{e',f'} z_{e,f}, \tag{5}$$

$$x z_{e,f} = z_{xe,f} x, \qquad z_{e,f} y = y z_{e,fy} \qquad (e \in E,\ f \in F,\ x \in X,\ y \in Y). \tag{6}$$

Proof. To symbolize the full word list in W, we use the notation W^*. Let θ be a form of the homomorphism specified by

$$\begin{aligned} \theta : W^* &\to E \lozenge_{\bowtie} F, \\ e &\mapsto \theta(e) = (e, \emptyset, 1_F), \\ f &\mapsto \theta(f) = (1_E, \emptyset, f), \\ z_{e,f} &\mapsto \theta(z_{e,f}) = (1_E, \{(e,f)\}, 1_F). \end{aligned}$$

Note that

$$(e_1, \emptyset, 1_F)(e_2, \emptyset, 1_F) = (e_1 e_2, \emptyset, 1_F), \tag{7}$$

$$(1_E, \emptyset, f_1)(1_E, \emptyset, f_2) = (1_E, \emptyset, f_1 f_2), \tag{8}$$

$$(1_E, P_1, 1_F)(1_E, P_2, 1_F) = (1_E, P_1 \cup P_2, 1_F), \tag{9}$$

$$(e, \emptyset, 1_F)(1_E, P, 1_F)(1_E, \emptyset, f) = (e, ePf, f).$$

for $e, e_1, e_2 \in E$, $f, f_1, f_2 \in F$, and $P, P_1, P_2 \subseteq E \times F$. Taking into account the above-mentioned actions, we conclude that θ is an onto mapping.

Next, we show that $E \lozenge_{\bowtie} F$ satisfies relations (3)–(6). Let $R = e_1 e_2 \cdots e_s$ and $S = f_1 f_2 \cdots f_k$, where $e_1, e_2, \cdots, e_s \in E$, $f_1, f_2, \cdots, f_k \in F$. The relation (3) follows from

$$(e_1, \emptyset, 1_F)(e_2, \emptyset, 1_F) \cdots (e_s, \emptyset, 1_F) = (e_1 e_2 \cdots e_s, \emptyset, 1_F)$$

and

$$(1_E, \emptyset, f_1)(1_E, \emptyset, f_2) \cdots (1_E, \emptyset, f_k) = (1_E, \emptyset, f_1 f_2 \cdots f_k).$$

For the relation (4), observe the following fact:

$$(1_E, \emptyset, y)(x, \emptyset, 1_F) = (y \triangleright x, \emptyset, y \triangleleft x) = (y \triangleright x, \emptyset, 1_F)(1_E, \emptyset, y \triangleleft x).$$

The relationships listed in (5) follow from (7), (8), and (9). Finally, the relations listed in (6) hold because

$$(x, \emptyset, 1_F)(1_E, \{(e,f)\}, 1_F) = (x, \{(xe,f)\}, 1_F) = (1_E, \{(xe,f)\}, 1_F)(x, \emptyset, 1_F)$$

and

$$(1_E, \{(e,f)\}, 1_F)(1_E, \emptyset, y) = (1_E, \{(e,fy)\}, y) = (1_E, \emptyset, y)(1_E, \{(e,fy)\}, 1_F).$$

Therefore, θ induces an epimorphism $\overline{\theta}$ from the monoid defined by (3)–(6), say M, to $E \lozenge_{\bowtie} F$.

Let $w \in W^*$ be any nonempty word. Observe that, due to relations (4)–(6), there exist words $w_x \in X^*$, $w_y \in Y^*$, and $w_{e,f} \in \{z_{e,f}\ ;\ e \in E, f \in F\}$ such that the relation $w = w_x w_{e,f} w_y$ holds in M. Moreover, because of the relation (5), there exists a set $P(w) \subseteq E \times F$ such that

$$w_{e,f} = \prod_{(e,f)\in P(w)} z_{e,f}.$$

Therefore, for any word $w \in W^*$, we have

$$\begin{aligned}\overline{\theta}(w) &= \theta(w)\\ &= \theta(w_x w_{e,f} w_y)\\ &= \theta(w_x)\theta(w_{e,f})\theta(w_y)\\ &= (w_x, \emptyset, 1_F)(1_E, \{(e,f)\}, 1_F)(1_E, \emptyset, w_y)\\ &= (w_x, \{(w_x e, f w_y)\}, w_y)\end{aligned}$$

for any word $w \in W^*$.

Now, we take $w' = w'_x w'_y w'_{e,f}$ and $w'' = w''_x w''_y w''_{e,f}$ for some $w', w'' \in W^*$, If $\theta(w') = \theta(w'')$, then by the equality of these components, we deduce that $w'_x = w''_x$ in E and $w'_y = w''_y$ in F. Relation (3) implies that $w'_x = w''_x$ and $w'_y = w''_y$ hold in M. Hence, $w' = w''$ holds as well. Therefore, $\overline{\theta}$ is injective. □

Next, we discuss two cyclic monoids and provide an example, which supports Theorem 2.2.

Example 2.1. *Let $E = \mathbb{Z}_2 =< e;\ e^2 = 1 >$ and $F = \mathbb{Z}_3 =< f;\ f^3 = 1 >$ be two monoids. Let $\alpha : \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ and $\beta : \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_3$ be two maps. Considering the notation*

$$\alpha(f,e) = f \triangleright e \qquad \text{and} \qquad \beta(f,e) = f \triangleleft e$$

for all $e \in E$ and $f \in F$, we write

$$\begin{aligned}\alpha(1,1) &= 1, & \beta(1,1) &= 1,\\ \alpha(1,e) &= e, & \beta(1,e) &= 1,\\ \alpha(f,1) &= 1, & \beta(f,1) &= f,\\ \alpha(f,e) &= f \triangleright e, & \beta(f,e) &= f \triangleleft e,\\ \alpha(f^2,1) &= 1, & \beta(f^2,1) &= f^2,\\ \alpha(f^2,e) &= f^2 \triangleright e, & \beta(f^2,e) &= f^2 \triangleleft e.\end{aligned}$$

By Theorem 2.2, the bicrossed-Schützenberger product $E \diamondsuit_{\bowtie} F$ has generators

$$e,\ f,\ z_{1,1},\ z_{1,f},\ z_{1,f^2},\ z_{e,1},\ z_{e,f},\ z_{e,f^2},$$

and relations

$$e^2 = 1,\ f^3 = 1,\ \Big\}\ \text{by } (3)$$

$$fe = (f \triangleright e)(f \triangleleft e),\ f^2 e = (f^2 \triangleright e)(f^2 \triangleleft e),\ \Big\}\ \text{by } (4)$$

$$\left.\begin{array}{c} z_{1,1}^2 = z_{1,1},\quad z_{1,f}^2 = z_{1,f},\quad z_{1,f^2}^2 = z_{1,f^2},\quad z_{e,1}^2 = z_{e,1},\quad z_{e,f}^2 = z_{e,f},\\ z_{e,f^2}^2 = z_{e,f^2},\quad z_{1,1}z_{1,f} = z_{1,f}z_{1,1},\quad z_{1,1}z_{1,f^2} = z_{1,f^2}z_{1,1},\quad z_{1,1}z_{e,1} = z_{e,1}z_{1,1},\\ z_{1,1}z_{e,f} = z_{e,f}z_{1,1},\quad z_{1,1}z_{e,f^2} = z_{e,f^2}z_{1,1},\ z_{1,f}z_{1,f^2} = z_{1,f^2}z_{1,f},\ z_{1,f}z_{e,1} = z_{e,1}z_{1,f},\\ z_{1,f}z_{e,f} = z_{e,f}z_{1,f},\ z_{1,f^2}z_{e,f} = z_{e,f^2}z_{1,f},\ z_{1,f^2}z_{e,1} = z_{e,1}z_{1,f^2},\ z_{1,f^2}z_{e,f} = z_{e,f}z_{1,f^2},\\ z_{1,f^2}z_{e,f^2} = z_{e,f^2}z_{1,f^2},\ z_{e,1}z_{e,f} = z_{e,f}z_{e,1},\ z_{e,1}z_{e,f^2} = z_{e,f^2}z_{e,1},\ z_{e,f}z_{e,f^2} = z_{e,f^2}z_{e,f},\end{array}\right\}\ \text{by } (5)$$

$$\left.\begin{array}{c} ez_{1,1} = z_{e,1}e,\quad ez_{1,f} = z_{e,f}e,\quad ez_{1,f^2} = z_{e,f^2}e,\quad ez_{e,1} = z_{1,1}e,\quad ez_{e,f} = z_{1,f}e,\\ ez_{e,f^2} = z_{1,f^2}e,\quad z_{1,1}f = fz_{1,f},\quad z_{1,f}f = fz_{1,f^2},\quad z_{1,f^2}f = fz_{1,1},\\ z_{e,1}f = fz_{e,f},\quad z_{e,f}f = fz_{e,f^2},\quad z_{e,f^2}f = fz_{e,1}.\end{array}\right\}\ \text{by } (6)$$

3. Regularity of $E \lozenge_{\bowtie} F$

The purpose of this section is to demonstrate the regularity of the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ of any two monoids E and F. Particularly, necessary and sufficient conditions for the regularity of $E \lozenge_{\bowtie} F$ are specified in this section.

A monoid M is said to be *regular* if, for every $e \in M$, there is $f \in M$ such that $efe = e$ and $fef = e$. Equivalently, for the set $e^{-1} = \{f \in M : efe = e \text{ and } fef = f\}$ of inverses of e in M, the monoid M is regular if and only if, for every $e \in M$, the set e^{-1} is nonempty. For details about the regularity of monoids, the reader is referred to [5, 6, 12, 16].

Theorem 3.1. *Let E and F be any monoids. Consider the conditions given in (1) and (2) with the features*

$$\left.\begin{aligned} P_1 = e_1e_2P_1 \subseteq E \times F, \\ P_2 = e_2e_1P_1 \subseteq E \times F, \\ f_2 \triangleright e_1 = e_1, \\ f_1 \triangleleft e_2 = f_1, \end{aligned}\right\} \tag{10}$$

where $e_1, e_2 \in E$ and $f_1, f_2 \in F$. Then, the bicrossed-Schützenberger product $E \lozenge_{\bowtie} F$ is regular if and only if E is a regular monoid and F is a group.

Proof. First, suppose that $E \lozenge_{\bowtie} F$ is regular. Then, for $(e, \{(1_E, 1_F)\}, 1_F) \in E \lozenge_{\bowtie} F$, there exists $(x, P, y) \in E \lozenge_{\bowtie} F$ such that

$$\begin{aligned} (e, \{(1_E, 1_F)\}, 1_F) &= (e, \{(1_E, 1_F)\}, 1_F)(x, P, y)(e, \{(1_E, 1_F)\}, 1_F) \\ &= (e(1_F \triangleright x),\ \{(1_E, 1_F)\}\, y \cup eP,\ (1_F \triangleleft x)y)(e, \{(1_E, 1_F)\}, 1_F) \\ &= (ex,\ \{(1_E, 1_F)\}\, y \cup eP,\ y)(e, \{(e, f)\}, 1_F) \\ &= (ex(y \triangleright e),\ \{(1_E, 1_F)\}\, y \cup eP \cup ex\, \{(1_E, 1_F)\},\ y \triangleleft e) \end{aligned}$$

and

$$\begin{aligned} (x, P, y) &= (x, P, y)(e, \{(1_E, 1_F)\}, 1_F)(x, P, y) \\ &= (x(y \triangleright e), P \cup x\, \{(1_E, 1_F)\}, y \triangleleft e)(x, P, y) \\ &= (x(y \triangleright e)((y \triangleleft e) \triangleright x),\ Py \cup x\, \{(1_E, 1_F)\}\, y \cup x(y \triangleright e)P,\ ((y \triangleleft e) \triangleleft x)y). \end{aligned}$$

Hence, we have $y = 1_F$. This means that $exe = e$ and $xex = x$. Thus, E is regular. Also, for $(1_E, \{(1_E, 1_F)\}, f) \in E \lozenge_{\bowtie} F$, there exists $(x, P, y) \in E \lozenge_{\bowtie} F$ such that

$$\begin{aligned} (1_E, \{(1_E, 1_F)\}, f) &= (1_E, \{(1_E, 1_F)\}, f)(x, P, y)(1_E, \{(1_E, 1_F)\}, f) \\ &= (1_E(f \triangleright x),\ \{(1_E, 1_F)\}\, y \cup P,\ (f \triangleleft x)y)(1_E, \{(1_E, 1_F)\}, f) \\ &= ((f \triangleright x)((f \triangleleft x)y \triangleright 1_E), \{(1_E, 1_F)\}\, yf \cup Pf \cup (f \triangleleft x)\, \{(1_E, 1_F)\}, ((f \triangleleft x)y \triangleleft 1_E)f) \\ &= (f \triangleright x, \{(1_E, 1_F)\}\, yf \cup Pf \cup (f \triangleleft x)\, \{(1_E, 1_F)\}, (f \triangleleft x)yf) \end{aligned}$$

and

$$\begin{aligned} (x, P, y) &= (x, P, y)(1_E, \{(1_E, 1_F)\}, f)(x, P, y) \\ &= (x(y \triangleright 1_E),\ Pf \cup x\, \{(1_E, 1_F)\},\ (y \triangleleft 1_E)f)(x, P, y) \\ &= (x,\ Pf \cup x\, \{(1_E, 1_F)\},\ yf)(x, P, y) \\ &= (x(yf \triangleright x),\ Pfy \cup x\, \{(1_E, 1_F)\}\, y \cup xP,\ (yf \triangleleft x)y). \end{aligned}$$

Here, we have

$$\{(1_E, 1_F)\} = \{(1_E, 1_F)\}\, yf \cup Pf \cup (f \triangleleft x)\, \{(1_E, 1_F)\}$$

$$P = Pfy \cup x\, \{(1_E, 1_F)\}\, y \cup xP$$

and $(1_E, yf) = (1_E, 1_F)$ with $Pfy = P$. Hence, we obtain $yf = fy \in F$. Therefore, F is a group.

Conversely, suppose that E is a regular monoid and F is a group. Take $(e_1, P_1, f_1),\ (e_2, P_2, f_2) \in E \lozenge_{\bowtie} F$. Since F is a group, there exists $f_2 \in F$ such that $f_1 f_2 = f_2 f_1$. Consider the actions given in (1), (2), and (10). By choosing $P_2 = e_2 P_1 f_2 \subseteq E \times F$ and $P_1 = e_1 P_2 f_1 \subseteq E \times F$, we obtain

$$\begin{aligned} P_1 f_2 f_1 \cup e_1 P_2 f_1 \cup e_1 (f_1 \triangleright e_2) P_1 &= P_1 \cup e_1 e_2 P_1 f_2 f_1 \cup e_1 e_2 P_1 \\ &= P_1 \cup e_1 e_2 P_1 \cup e_1 e_2 P_1 \\ &= P_1 \cup P_1 \cup P_1 \\ &= P_1 \end{aligned}$$

and

$$\begin{aligned} P_2 f_1 f_2 \cup e_2 P_1 f_2 \cup e_2 (f_2 \triangleright e_1) P_2 &= P_2 \cup e_2 e_1 P_2 f_1 f_2 \cup e_2 e_1 P_2 \\ &= P_2 \cup e_2 e_1 P_2 \cup e_2 e_1 P_2 \\ &= P_2 \cup P_2 \cup P_2 \\ &= P_2. \end{aligned}$$

Also, by using the above conditions, obtain

$$\begin{aligned} e_1 (f_1 \triangleright e_2)((f_1 \triangleleft e_2) f_2 \triangleleft e_1) &= e_1, \\ ((f_1 \triangleleft e_2) f_2 \triangleleft e_1) f_1 &= f_1, \\ e_2 (f_2 \triangleright e_1)((f_2 \triangleleft e_1) f_1 \triangleleft e_2) &= e_2, \\ ((f_2 \triangleleft e_1) f_1 \triangleleft e_2) f_2 &= f_2. \end{aligned}$$

Finally, we have

$$\begin{aligned} (e_1, P_1, f_1)(e_2, P_2, f_2)(e_1, P_1, f_1) &= (e_1 (f_1 \triangleright e_2),\ P_1 f_2 \cup e_1 P_2, (f_1 \triangleleft e_2) f_2)(e_1, P_1, f_1) \\ &= (e_1 (f_1 \triangleright e_2)((f_1 \triangleleft e_2) f_2 \triangleleft e_1),\ P_1 f_2 f_1 \cup e_1 P_2 f_1 \cup e_1 (f_1 \triangleright e_2) P_1,\ ((f_1 \triangleleft e_2) f_2 \triangleleft e_1) f_1) \\ &= (e_1, P_1, f_1) \end{aligned}$$

and

$$\begin{aligned} (e_2, P_2, f_2)(e_1, P_1, f_1)(e_2, P_2, f_2) &= (e_2 (f_2 \triangleright e_1),\ P_2 f_1 \cup e_2 P_1, (f_2 \triangleleft e_1) f_1)(e_2, P_2, f_2) \\ &= (e_2 (f_2 \triangleright e_1)((f_2 \triangleleft e_1) f_1 \triangleleft e_2),\ P_2 f_1 f_2 \cup e_2 P_1 f_2 \cup e_2 (f_2 \triangleright e_1) P_2,\ ((f_2 \triangleleft e_1) f_1 \triangleleft e_2) f_2) \\ &= (e_2, P_2, f_2). \end{aligned}$$

Therefore, $E \lozenge_{\bowtie} F$ is regular. □

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