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Research Article

Relative growth of Hadamard compositions of series in systems of functions

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Abstract

For an entire transcendental function f and a sequence (λ_n) of positive numbers increasing to $+\infty$, suppose that A and A_j, with $1 \le j \le p$, are entire functions represented by series in the system of functions $f(\lambda_n z)$ provided that the function A is a Hadamard composition of genus m of functions A_j. In terms of generalized orders, the connection between the growth of the function A with respect to the function f and the growth of the functions A_j with respect to f is studied.

Keywords: analytic function; regularly converging series; Hadamard composition; relative growth; generalized order.

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1. Introduction

Let f and g be entire transcendental functions and $M_f(r) = \max\{|f(z)| : |z| = r\}$. In order to study the relative growth of the functions f and g, Roy [15] used the order

$$\varrho_g[f] = \varlimsup_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$$

and the lower order

$$\lambda_g[f] = \varliminf_{r\to+\infty} \frac{\ln M_g^{-1}(M_f(r))}{\ln r}$$

of the function f with respect to the function g. Research on the relative growth of entire functions in terms of maximal terms, Nevanlinna characteristic function, and k-logarithmic orders was continued by several mathematicians; for example, see [5–8]. In [3], the relative growth of entire functions of two complex variables was considered. In [4], the relative growth of the entire Dirichlet series in terms of R-orders was studied. For the Dirichlet series $F(s) = \sum_{n=1}^{\infty} f_n \exp\{s\lambda_n\}$, the relative growth was studied in [11, 12, 14].

Let $f_j(z) = \sum_{n=0}^{\infty} a_{n,j} z^n$ $(j = 1, 2)$ be entire transcendental functions. The function $(f_1 * f_2)(z) = \sum_{n=0}^{\infty} a_n z^n$ is said [9] to be the Hadamard composition (product) of the functions f_j if $a_n = a_{n,1}a_{n,2}$ for every n. Hadamard [2, 10] obtained several properties of this composition and found some applications of this composition in the theory of the analytic continuation of the functions represented by power series. For the Dirichlet series, the usual Hadamard composition is defined in a similar way. For these series, the concept of Hadamard compositions of the genus m was introduced in [1, 13] and their relative growth was studied there.

In the present paper, we study the relative growth of Hadamard compositions of the genus m of series in systems of functions. Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \tag{1}$$

be an entire transcendental function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Let $\Lambda = (\lambda_n)$ be a sequence of positive numbers increasing to $+\infty$. Denote by $\mathfrak{S}(\Lambda, f, R)$ the class of the series

$$A(z) = \sum_{n=1}^{\infty} a_n f(\lambda_n z) \tag{2}$$

in the system $f(\lambda_n z)$ regularly convergent in $\mathbb{D}_R = \{z : |z| < R\}$; that is, for every $r \in [0, R)$,

$$\mathfrak{M}(r, A) := \sum_{n=1}^{\infty} |a_n| M_f(r\lambda_n) < +\infty. \tag{3}$$

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We remark that the function $\ln M_f(r)$ is logarithmically convex and therefore,

$$\Gamma_f(r) := \frac{d\ln M_f(r)}{d\ln r} \nearrow +\infty, \quad r\to+\infty.$$

(On the points where the derivative does not exist, under $\frac{d\ln M_f(r)}{d\ln r}$ means the right-hand derivative.)

If series (2) regularly converges in $\mathbb{C}$, then the function A is entire. Generalized orders are used to study its growth. Denote by L the class of continuous non-negative functions α on $(-\infty,+\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ and $\alpha(x)\uparrow+\infty$ as $x_0\leq x\to+\infty$. We say that $\alpha\in L^0$ if $\alpha\in L$ and $\alpha((1+o(1))x)=(1+o(1))\alpha(x)$ as $x\to+\infty$. Also, $\alpha\in L_{si}$ if $\alpha\in L$ and $\alpha(cx)=(1+o(1))\alpha(x)$ as $x\to+\infty$ for every $c\in(0,+\infty)$; that is, α is a slowly increasing function. Note that $L_{si}\subset L^0$. For $\alpha\in L$ and $\beta\in L$, the quantities

$$\varrho_{\alpha,\beta}[f]=\varlimsup_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(r)}$$

and

$$\lambda_{\alpha,\beta}[f]=\varliminf_{r\to+\infty}\frac{\alpha(\ln M_f(r))}{\beta(r)}$$

are called [17] generalized (α,β)-order and lower (α,β)-order of an entire function f, respectively. The relationship between the growth of functions $A\in\mathfrak{S}(\Lambda,f,+\infty)$ and f in terms of generalized orders was studied in [18]. Particularly, it has been proven that if $\alpha\in L_{si}$, $\beta(e^x)\in L^0$, $\frac{d\ln\beta^{-1}(c\alpha(x))}{d\ln x}=O(1)$ as $x\to+\infty$ for every $c\in(0,+\infty)$, $a_n\geq 0$ for all $n\geq 1$, $\ln\lambda_n=o\left(\ln\beta^{-1}\left(c\alpha\left(\frac{1}{\ln\lambda_n}\ln\frac{1}{a_n}\right)\right)\right)$ as $n\to\infty$ for every $c\in(0,+\infty)$, and $\ln n=O(\Gamma_f(\lambda_n))$ as $n\to\infty$, then $\varrho_{\alpha,\beta}[A]=\varrho_{\alpha,\beta}[f]$.

The growth of the function $A\in\mathfrak{S}(\Lambda,f,+\infty)$ with respect to the function f is identified in [19] with the growth of the function $M_f^{-1}(\mathfrak{M}(r,A))$ as $r\to+\infty$. The generalized (α,β)-order $\varrho_{\alpha,\beta}[A]_f$ and the generalized lower (α,β)-order $\lambda_{\alpha,\beta}[A]_f$ of the function $A\in\mathfrak{S}(\Lambda,f,+\infty)$ with respect to a function f are defined [19] as

$$\varrho_{\alpha,\beta}[A]_f:=\varlimsup_{r\to+\infty}\frac{\alpha(M_f^{-1}(\mathfrak{M}(r,A)))}{\beta(r)} \quad\text{and}\quad \lambda_{\alpha,\beta}[A]_f:=\varliminf_{r\to+\infty}\frac{\alpha(M_f^{-1}(\mathfrak{M}(r,A)))}{\beta(r)}.$$

In [19], it was shown that if $A\in\mathfrak{S}(\Lambda,f,+\infty)$, $a_n\geq 0$ for every $n\geq 1$, $\ln n\leq q\Gamma_f(\lambda_n))$ for some $q>0$ and for every $n\geq n_0$, and $\varlimsup\limits_{x\to+\infty}\frac{\ln\mu(x)}{x\ln M_f^{-1}(e^x)}=\gamma$, where $\mu(x)=\max\{|a_n|\lambda_n^x : n\geq 1\}$, then

$$\lambda_{\alpha,\alpha}[A]_f=\varrho_{\alpha,\alpha}[A]_f=1$$

provided either $\gamma<1$ and $\alpha(e^x)\in L_{si}$ or $\gamma=0$ and $\alpha(e^x)\in L^0$.

Let $G_j(z)=\sum_{n=0}^\infty g_{n,j}z^n$ with $1\leq j\leq p$. Let $P(x_1,\dots,x_p)$ be a homogeneous polynomial of degree m; that is, $P(tx_1,\dots,tx_p)=t^mP(x_1,\dots,x_p)$ for every t from the above field on which a polynomial is defined. The function $G(z)=\sum_{n=0}^\infty g_nz^n$ is said [21] to be a Hadamard composition of genus $m\geq 1$ of functions G_j if $g_n=P(g_{n,1},\dots,g_{n,p})$, where

$$P(x_1,\dots,x_p)=\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}x_1^{k_1}\cdot\dots\cdot x_p^{k_p},\quad k_j\in\mathbb{Z}_+. \tag{4}$$

is a homogeneous polynomial of degree $m\geq 1$ with constant coefficients $c_{k_1\dots k_p}$. Denote by $(G_1*\dots*G_p)_m$ the Hadamard composition of genus $m\geq 1$ of functions f_j, that is

$$(G_1*\dots*G_p)_m(z)=\sum_{n=0}^\infty g_nz^n=\sum_{n=0}^\infty\left(\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}g_{n,1}^{k_1}\cdot\dots\cdot g_{n,p}^{k_p}\right)z^n.$$

Properties of Hadamard compositions of genus $m\geq 1$ of entire functions represented by series in a system of functions were studied in [21]. According to the definition, the function (2) is said to be a Hadamard composition of genus m of the functions

$$A_j(z)=\sum_{n=1}^\infty a_{n,j}f(\lambda_nz),\quad 1\leq j\leq p, \tag{5}$$

if $a_n=P(a_{n,1},\dots,a_{n,p})$, where P is defined by (4), that is,

$$A(z)=(A_1*\dots*A_p)_m(z)=\sum_{n=1}^\infty\left(\sum_{k_1+\dots+k_p=m}c_{k_1\dots k_p}a_{n,1}^{k_1}\cdot\dots\cdot a_{n,p}^{k_p}\right)f(\lambda_nz). \tag{6}$$

The function A_1 is said to be dominant if $|c_{m0\dots0}||a_{n,1}|^m\neq 0$ and $|a_{n,j}|=o(|a_{n,1}|)$ as $n\to\infty$ for $2\leq j\leq p$.

In [21], it was shown that if $\ln n = o(\Gamma_f(\lambda_n))$ as $n \to \infty$ and $A_j \in \mathfrak{S}(\Lambda, f, +\infty)$ for every j with $1 \le j \le p$, then $A(z) = (A_1 * \ldots * A_p)_m(z) \in \mathfrak{S}(\Lambda, f, +\infty)$. On the other hand, if $A \in \mathfrak{S}(\Lambda, f, +\infty)$ and the function A_1 is dominant, then $A_j \in \mathfrak{S}(\Lambda, f, +\infty)$ for every j with $1 \le j \le p$. If

$$\varrho_{\alpha,\beta}[A_j] = \overline{\lim_{r\to+\infty}} \frac{\alpha(\ln \mathfrak{M}(r, A_j))}{\beta(r)}$$

is the (α, β)-order of the function A_j, then under the conditions $\alpha \in L_{si}$, $\beta \in L_{si}$, $f_k \ge 0$ for every $k \ge 0$ and

$$\ln m < h \le \frac{d \ln \ln M_f(r)}{d \ln r} \le H < +\infty$$

for every $r \ge r_0$, the inequality $\varrho_{\alpha,\beta}[A] \le \max\{\varrho_{\alpha,\beta}[A_j] : 1 \le j \le p\}$ holds. Moreover, if the function A_1 is dominant, then $\varrho_{\alpha,\beta}[A_j] \le \varrho_{\alpha,\beta}[A_1] = \varrho_{\alpha,\beta}[A]$ for every j. In this article, the dependence of the growth of the function A with respect to the function f on the growth of functions A_j with respect to the function f is studied.

2. Hadamard compositions of entire functions

The following theorem is the first main result of the present paper:

Theorem 2.1. *Let $m \ge 2$, $\alpha \in L_{si}$, $\beta \in L_{si}$, $\ln n = O(\Gamma_f(\lambda_n))$ when $n \to \infty$, and $\Gamma_f(r) \asymp \ln M_f(r)$ when $r \to +\infty$. If $A_j \in \mathfrak{S}(\Lambda, f, +\infty)$ for every j with $1 \le j \le p$, then*

$$A(z) = (A_1 * \ldots * A_p)_m(z) \in \mathfrak{S}(\Lambda, f, +\infty)$$

and

$$\varrho_{\alpha,\beta}[A]_f \le \max\{\varrho_{\alpha,\beta}[A_j]_f : 1 \le j \le p\}. \tag{7}$$

Also, if the function A_1 is dominant, then

$$\varrho_{\alpha,\beta}[A]_f = \varrho_{\alpha,\beta}[A_1]_f \ge \varrho_{\alpha,\beta}[A_j]_f$$

and

$$\lambda_{\alpha,\beta}[A]_f = \lambda_{\alpha,\beta}[A_1]_f \ge \lambda_{\alpha,\beta}[A_j]_f$$

for every j with $1 \le j \le p$.

Proof. Let $\mu(r, A) = \max\{|a_n| M_f(r\lambda_n) : n \ge 1\}$ be the maximal term of series (3). Then, by [19], for $q > 1$ and $r \ge 1$, we have

$$\mu(r, A) \le \mathfrak{M}(r, A) \le \mu(qr, A) \sum_{n=1}^{\infty} \exp\{-\Gamma_f(\lambda_n) \ln q\}. \tag{8}$$

From the condition $\ln n = O(\Gamma_f(\lambda_n))$ when $n \to \infty$, it follows that $\ln n \le \eta \Gamma_f(\lambda_n)$ for some $\eta > 0$. For $q = e^{\eta+1}$, we have $\Gamma_f(\lambda_n) \ln q \ge ((\eta + 1)/\eta) \ln n$ and thus,

$$\sum_{n=1}^{\infty} \exp\{-\Gamma_f(\lambda_n) \ln q\} \le K < +\infty.$$

Therefore, (8) implies

$$\frac{\alpha(M_f^{-1}(\mu(r, A)))}{\beta(r)} \le \frac{\alpha(M_f^{-1}(\mathfrak{M}(r, A)))}{\beta(r)} \le \frac{\alpha(M_f^{-1}(K\mu(qr, A)))}{\beta(r)}. \tag{9}$$

For brevity, we put $\mu(r) = \mu(r, A)$ and $\mu_j(r) = \mu(r, A_j)$. Then

$$\varrho_{\alpha,\beta}[\mu]_f := \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(\mu(r)))}{\beta(r)}$$

and

$$\lambda_{\alpha,\beta}[\mu]_f := \underline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(\mu(r)))}{\beta(r)}$$

are the generalized (α, β)-order and lower (α, β)-order of the maximal term with respect to f. Since $M_f^{-1} \in L_{si}$, by the conditions $\alpha \in L^0$ and $\beta \in L_{si}$, we obtain $\varrho_{\alpha,\beta}[A]_f = \varrho_{\alpha,\beta}[\mu]_f$ and $\lambda_{\alpha,\beta}[A]_f = \lambda_{\alpha,\beta}[\mu]_f$.

By the condition $\Gamma_f(r) \asymp \ln M_f(r)$, we have

$$0 < h \le \frac{\Gamma_f(r)}{\ln M_f(r)} \le H < +\infty. \tag{10}$$

Take $B = 1 + (\ln m)/H > 1$. Then, from (10), we obtain

$$\ln \ln M_f(r) - \ln \ln M_f(r/B) = \int\limits_{r/B}^{r} \frac{\Gamma_f(t)}{\ln M_f(t)} d\ln t \le H \ln B = H \ln(1 + (\ln m)/H) \le \ln m,$$

On the other hand, for $C = m^{1/h} > 1$, we have

$$\ln \ln M_f(Cr) - \ln \ln M_f(r) = \int\limits_{r}^{Cr} \frac{\Gamma_f(t)}{\ln M_f(t)} d\ln t \ge h \ln C = \ln m.$$

Hence, for some $B > 1$ and $C > 1$, we have

$$M_f(r/C)^m \le M_f(r) \le M_f(r/B)^m. \tag{11}$$

Therefore,

$$\begin{aligned}
\mu(r) &= \max\left\{\left|\sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p} a_{n,1}^{k_1} \cdot \ldots \cdot a_{n,p}^{k_p}\right| M_f(r\lambda_n) : n \ge 1\right\} \\
&\le \max\left\{\sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}| a_{n,1}|^{k_1} \cdot \ldots \cdot |a_{n,p}|^{k_p} M_f(r\lambda_n/B)^m : n \ge 1\right\} \\
&\le \max\left\{\sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|(|a_{n,1}| M_f(r\lambda_n/B))^{k_1} \cdot \ldots \cdot (|a_{n,p}| M_f(r\lambda_n/B))^{k_p} : n \ge 1\right\} \\
&\le \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}| \mu_1(r/B)^{k_1} \cdot \ldots \cdot \mu_p(r/B)^{k_p}.
\end{aligned} \tag{12}$$

Let $\varrho^* = \max\{\varrho_{\alpha,\beta}[\mu_j]_f : 1 \le j \le p\} < +\infty$. Since $\beta \in L_{si}$ for every $\varrho > \varrho^*$ and for every $r \ge r_0 = r_0(\varrho)$, we have

$$\mu_j(r/B) \le M_f(\alpha^{-1}(\varrho\beta(r)))$$

for $r \ge r_0$. Therefore, (12) implies

$$\mu(r) \le K_0 M_f(\alpha^{-1}(\varrho\beta(r)))^m \quad \text{and} \quad K_0 = \sum_{k_1+\dots+k_p=m} |c_{k_1\dots k_p}|.$$

Since $M_f^{-1} \in L_{si}$ and $\alpha \in L_{si}$, we obtain

$$\begin{aligned}
\varrho_{\alpha,\beta}[\mu]_f &\le \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(K_0 M_f^m(\alpha^{-1}(\varrho\beta(r)))))}{\beta(r)} \\
&= \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(M_f^m(\alpha^{-1}(\varrho\beta(r)))))}{\beta(r)} \\
&= \varrho \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(M_f^m(r)))}{\alpha(r)} \\
&\le \varrho \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(M_f(Cr)))}{\alpha(r)} \\
&= \varrho \overline{\lim_{r\to+\infty}} \frac{\alpha(M_f^{-1}(M_f(Cr)))}{\alpha(Cr)} \varrho.
\end{aligned}$$

In view of the arbitrariness of ϱ, we obtain the inequality $\varrho_{\alpha,\beta}[\mu]_f \le \varrho^* = \max\{\varrho_{\alpha,\beta}[\mu_j]_f : 1 \le j \le p\}$ that is obvious for $\varrho^* = +\infty$. Since $\varrho_{\alpha,\beta}[\mu]_f = \varrho_{\alpha,\beta}[A]_f$ and $\varrho_{\alpha,\beta}[\mu_j]_f = \varrho_{\alpha,\beta}[A_j]_f$, inequality (7) is proved.

Now, suppose that the function A_1 is dominant and $m \geq 2$. We put

$$\Sigma'_n = \sum_{k_1+\dots+k_p=m, k_1\neq m} c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\dots\cdot(a_{n,p})^{k_p}$$
$$= \sum_{k_1+\dots+k_p=m} c_{k_1\dots k_p}(a_{n,1})^{k_1}\cdot\dots\cdot(a_{n,p})^{k_p} - c_{m0\dots 0}(a_{n,1})^m.$$

Note that for each monomial of the polynomial Σ'_n, the sum of the exponents is equal to m. Hence, we have

$$\frac{|a_{n,1}|^{k_1}\cdot\dots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^m} = \frac{|a_{n,2}|^{k_2}\cdot\dots\cdot|a_{n,p}|^{k_p}}{|a_{n,1}|^{m-k_1}} \to 0, \quad n\to\infty$$

and, thus $\Sigma'_n = o(|a_{n,1}|^m)$ as $n\to\infty$. Therefore, $|a_n| = (1+o(1))|c_{m0\dots 0}||a_{n,1}|^m$ as $n\to\infty$, and in view of (11), we obtain

$$(1+o(1))|c_{m0\dots 0}||a_{n,1}|^m M_f(r\lambda_n/C)^m \leq |a_n|M_f(r\lambda_n) \leq (1+o(1))|c_{m0\dots 0}||a_{n,1}|^m M_f(r\lambda_n/B)^m$$

as $n\to\infty$. Using these inequalities, it is easy to show that $c_1\mu_1^m(r/C) \leq \mu(r) \leq c_2\mu_1^m(r/B)$ for $r\geq r_0$, where c_1 and c_2 are positive constants. Since $\alpha\in L_{si}$, $\beta\in L_{si}$, and $M_f^{-1}\in L_{si}$, we obtain

$$\frac{\alpha(M_f^{-1}(\mu(r)))}{\beta(r)} = (1+o(1))\frac{\alpha(M_f^{-1}(\mu_1^m(r)))}{\beta(r)}, \quad r\to+\infty. \tag{13}$$

Using (11) again, we have $M_f(r)^m \leq M_f(Cr)$. Hence, $M_f^{-1}(x^m) \leq CM_f^{-1}(x)$ and thus, $M_f^{-1}(\mu_1^m(r)) \leq CM_f^{-1}(\mu_1(r))$. Therefore, (13) implies

$$\frac{\alpha(M_f^{-1}(\mu(r)))}{\beta(r)} = (1+o(1))\frac{\alpha(M_f^{-1}(\mu_1(r)))}{\beta(r)}, \quad r\to+\infty,$$

that is, $\varrho_{\alpha,\beta}[A]_f = \varrho_{\alpha,\beta}[A_1]_f$ and $\lambda_{\alpha,\beta}[A]_f = \lambda_{\alpha,\beta}[A_1]_f$. □

Now, let $m = 1$. Then

$$(A_1 * \dots * A_p)_1(z) = \sum_{n=1}^\infty (c_1a_{n,1}+\cdot\dots\cdot+c_pa_{n,p})\,f(\lambda_nz) = .\sum_{j=1}^p c_j\sum_{n=1}^\infty a_{n,j}f(\lambda_nz) = \sum_{j=1}^p c_jA_j(z).$$

Also, for $A = (A_1 * \dots * A_p)_1$, we have

$$\mathfrak{M}(r,A) \leq \sum_{j=1}^p |c_j|\mathfrak{M}(r,A_j).$$

It is not difficult to show that this implies (7) provided $\alpha\in L^0$ and $\beta\in L$. If, in addition, the function A_1 is dominant, then

$$|c_1a_{n,1}+\cdot\dots\cdot+c_pa_{n,p}| = (1+o(1))|c_1||a_{n,1}|$$

as $n\to\infty$, that is

$$q_1|a_{n,1}| \leq |c_1a_{n,1}+\cdot\dots\cdot+c_pa_{n,p}| \leq q_2|a_{n,1}|$$

for some $0<q_1<q_2<+\infty$. Therefore,

$$q_1\mathfrak{M}(r,A_1) \leq \mathfrak{M}(r,A) \leq q_2\mathfrak{M}(r,A_1).$$

Therefore, if $\alpha\in L^0$ and $\beta\in L$, then $\varrho_{\alpha,\beta}[A]_f = \varrho_{\alpha,\beta}[A_1]_f$ and $\lambda_{\alpha,\beta}[A]_f = \lambda_{\alpha,\beta}[A_1]_f$. Consequently, we have the following result:

Proposition 2.1. *Let $m=1$, $\alpha\in L^0$, $\beta\in L$, and $A_j\in\mathfrak{S}(\Lambda,f,+\infty)$ for every j with $1\leq j\leq p$. Then*

$$A(z) = (A_1 * \dots * A_p)_m(z) \in \mathfrak{S}(\Lambda,f,+\infty)$$

and (7) holds. Also, if the function A_1 is dominant, then $\varrho_{\alpha,\beta}[A]_f = \varrho_{\alpha,\beta}[A_1]_f$ and $\lambda_{\alpha,\beta}[A]_f = \lambda_{\alpha,\beta}[A_1]_f$.

3. Hadamard compositions of functions analytic in a disk

Now, we consider the class $\mathfrak{S}(\Lambda, f, 1)$; that is, we suppose that (3) holds for $r<1$, and it does not hold for $r>1$. In [20], it was shown that if $\ln n = o(\Gamma_f(\lambda_n))$ when $n\to\infty$ and $\Gamma_f(cr)\asymp\Gamma_f(r)$ for every $c\in(0,+\infty)$, then

$$\varliminf_{n\to\infty}\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_n|}\right)=1. \tag{14}$$

Let $A_j\in\mathfrak{S}(\Lambda, f, 1)$ for $1\le j\le p$. Then, in view of (14), it holds that

$$\frac{1}{\lambda_n}M_f^{-1}\left(\frac{1}{|a_{n,j}|}\right)\ge 1-\varepsilon,$$

that is,

$$|a_{n,j}|\le\frac{1}{M_f((1-\varepsilon)\lambda_n)}$$

for all ε, n, and j, with $\varepsilon\in(0,1)$, $n\ge n_0(\varepsilon)$, and $1\le j\le p$. Therefore, if $A(z)=(A_1*\ldots*A_p)_m(z)$, then for $n\ge n_0(\varepsilon)$, we have

$$|a_n|\le\sum_{k_1+\cdots+k_p=m}|c_{k_1\ldots k_p}||a_{n,1}|^{k_1}\cdot\ldots\cdot|a_{n,p}|^{k_p}\le\frac{K}{M_f((1-\varepsilon)\lambda_n)^m}\le\frac{K}{M_f((1-\varepsilon)\lambda_n)},$$

where

$$K=\sum_{k_1+\cdots+k_p=m}|c_{k_1\ldots k_p}|.$$

Therefore,

$$\frac{1}{\lambda_n}M_f^{-1}\left(\frac{K}{|a_n|}\right)\ge 1-\varepsilon$$

and, since $M_f^{-1}\in L_{si}$, in view of the arbitrariness of ε, the following result is true.

Proposition 3.1. *If $A_j\in\mathfrak{S}(\Lambda, f, 1)$ for $1\le j\le p$ and $A(z)=(A_1*\ldots*A_p)_m(z)$ then $A\in\mathfrak{S}(\Lambda, f, 1)$.*

As above, let $\mu(r, A)$ be the maximal term of series (3) regularly converging in $\mathbb{D}_1$. Suppose that $\mu(r, A)\uparrow K<+\infty$ as $r\uparrow 1$. Then $|a_n|M_f(r\lambda_n)\le K$ for all $n\ge 1$ and $r\in(0,1)$. Letting $r\uparrow 1$, we obtain

$$|a_n|M_f(\lambda_n)\le K$$

for every $n\ge 1$, that is

$$\varlimsup_{n\to\infty}|a_n|M_f(\lambda_n)\le K.$$

Conversely, if $|a_n|M_f(\lambda_n)\le K$, then

$$|a_n|M_f(r\lambda_n)\le K$$

for all $n\ge 1$ and $r\in(0,1)$, and hence

$$\mu(r, A)\le K<+\infty$$

for every $r<1$. Therefore, the following result is true:

Proposition 3.2. *In order that $\mu(r, A)\uparrow+\infty$ as $r\uparrow 1$, it is necessary and sufficient that*

$$\varlimsup_{n\to\infty}|a_n|M_f(\lambda_n)=+\infty.$$

In what follows, we assume that the condition given in Proposition 3.2 is satisfied.

For $\alpha\in L$, $\beta\in L$, and for an analytic function $\varphi(z)=\sum\limits_{k=0}^{\infty}\varphi_k z^k$ in $\mathbb{D}_1$, the generalized (α,β)-order and lower generalized (α,β)-order are defined [16] as

$$\varrho^1_{\alpha,\beta}[\varphi]=\varlimsup_{r\uparrow 1}\frac{\alpha(\ln M_\varphi(r))}{\beta(1/(1-r))}$$

and

$$\lambda^1_{\alpha,\beta}[\varphi]=\varlimsup_{r\uparrow 1}\frac{\alpha(\ln M_\varphi(r))}{\beta(1/(1-r))}.$$

By analogy, we define

$$\varrho^1_{\alpha,\beta}[A]_f := \varlimsup_{r\uparrow 1}\frac{\alpha(M_f^{-1}(\mathfrak{M}(r,A)))}{\beta(1/(1-r))},$$

$$\lambda^1_{\alpha,\beta}[A]_f := \varliminf_{r\uparrow 1}\frac{\alpha(M_f^{-1}(\mathfrak{M}(r,A)))}{\beta(1/(1-r))},$$

and similar characteristic for $\mu(r)=\mu(r,A)$. Let $n(r)=\sum\limits_{\lambda_n\le r}1$ be the counting function of the sequence Λ.

Proposition 3.3. *Let $\alpha\in L_{si}$, $\beta\in L_{si}$,*

$$\varliminf_{x\to+\infty}\frac{\alpha(M_f^{-1}(x))}{\beta(x/\ln x)}=\eta>0, \tag{15}$$

and

$$\varliminf_{x\to+\infty}\frac{\alpha(M_f^{-1}(x))}{\beta(x/\ln n(x))}=0. \tag{16}$$

If $\Gamma_f(r)\asymp\ln M_f(r)$ as $r\to+\infty$ and $\Gamma_f(r)\ge hr$ for some $h>0$ and every $r\in[0,+\infty)$, then

$$\varrho^1_{\alpha,\beta}[A]_f=\varrho^1_{\alpha,\beta}[\mu]_f \quad \textit{and} \quad \lambda^1_{\alpha,\beta}[A]_f=\lambda^1_{\alpha,\beta}[\mu]_f.$$

Proof. Since $\mu(r,A)\le\mathfrak{M}(r,A)$, we have $\varrho^1_{\alpha,\beta}[A]_f\ge\varrho^1_{\alpha,\beta}[\mu]_f$ and $\lambda^1_{\alpha,\beta}[A]_f\ge\lambda^1_{\alpha,\beta}[\mu]_f$. On the other hand, for all $q>1$ and $r\in[0,1)$, we have

$$\begin{aligned}\mathfrak{M}(r,A)&=\sum_{n=1}^{\infty}|a_n|M_f\left(\left(r+\frac{1-r}{q}\right)\lambda_n\right)\frac{M_f(r\lambda_n)}{M_f((r+(1-r)/q)\lambda_n)}\\ &\le\mu\left(r+\frac{1-r}{q}\right)\sum_{n=1}^{\infty}\exp\left\{-\int\limits_{r\lambda_n}^{(r+(1-r)/q)\lambda_n}\Gamma_f(t)d\ln t\right\}\\ &\le\mu\left(r+\frac{1-r}{q}\right)\sum_{n=1}^{\infty}\exp\left\{-\Gamma_f(r\lambda_n)\ln\left(1+\frac{1-r}{qr}\right)\right\}.\end{aligned} \tag{17}$$

Since $\beta\in L_{si}$, by the definition of $\varrho^1_{\alpha,\beta}[\mu]_f$, we have

$$\begin{aligned}\varrho^1_{\alpha,\beta}[\mu]_f&=\varlimsup_{r\uparrow 1}\frac{\alpha(M_f^{-1}(\mu\left(r+(1-r)/q\right)))}{\beta\left(\dfrac{1}{1-r-(1-r)/q}\right)}\\ &=\varlimsup_{r\uparrow 1}\frac{\alpha(M_f^{-1}(\mu\left(r+(1-r)/q\right)))}{\beta\left(\dfrac{q}{(q-1)(1-r)}\right)}\\ &=\varlimsup_{r\uparrow 1}\frac{\alpha(M_f^{-1}(\mu\left(r+(1-r)/q\right)))}{\beta(1/(1-r))},\end{aligned}$$

that is, if $\varrho^1_{\alpha,\beta}[\mu]_f<+\infty$, then for all $\varrho>\varrho^1_{\alpha,\beta}[\mu]_f$ and $r\ge r_0(\varrho)$, it holds that

$$\mu\left(r+\frac{1-r}{q}\right)\le M_f\left(\alpha^{-1}\left(\varrho\beta\left(\frac{1}{1-r}\right)\right)\right). \tag{18}$$

Put $b=h/(2q)$ and

$$T(r)=M_f(\alpha^{-1}(\varepsilon\beta(2/(b(1-r))).$$

Then, from (16), for all $\varepsilon\in(0,\varrho)$ and $t\ge t_0(\varepsilon)$, we obtain

$$\ln n(t)\le\frac{t}{\beta^{-1}(\alpha(M_f^{-1}(t))/\varepsilon)}=o(t),\quad t\to+\infty,$$

and $T(r)>t_0(\varepsilon)$ for all $r\ge r_0(\varepsilon)$.

Therefore, in view of the condition $\Gamma_f(r) \geq hr$, for $b = h/q$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \exp\left\{-\Gamma_f(r\lambda_n)\ln\left(1+\frac{1-r}{qr}\right)\right\} &\leq \sum_{n=1}^{\infty} \exp\{-b\lambda_n(1-r)\} \\
&= \int_0^{\infty} \exp\{-bt(1-r)\}dn(t) \\
&= b(1-r)\int_0^{\infty} n(t)\exp\{-bt(1-r)\}dt \\
&= b(1-r)\left(\int_0^{T(r)} n(t)\exp\{-bt(1-r)\}dt + \int_{T(r)}^{\infty} \exp\{-bt(1-r)+\ln\, n(t)\}dt\right) \\
&\leq b(1-r)\left(\int_0^{T(r)} n(t)dt + \int_{T(r)}^{\infty} \exp\left\{-t\left(b(1-r) - \frac{1}{\beta^{-1}(\alpha(M_f^{-1}(t))/\varepsilon)}\right)\right\}dt\right) \\
&\leq b(1-r)\left(T(r)n(T(r)) + \int_{T(r)}^{\infty} \exp\left\{-t\left(b(1-r) - \frac{1}{\beta^{-1}(\alpha(M_f^{-1}(T(r)))/\varepsilon)}\right)\right\}dt\right) \\
&= b(1-r)\left(T(r)n(T(r)) + \int_{T(r)}^{\infty} \exp\left\{-t\left(b(1-r) - \frac{b(1-r)}{2}\right)\right\}dt\right) \\
&\leq b(1-r)T(r)\exp\left\{\frac{T(r)}{\beta^{-1}(\alpha(M_f^{-1}(T(r)))/\varepsilon)}\right\} + 2. \qquad (19)
\end{aligned}$$

From (15), it follows that

$$\alpha(M_f^{-1}(x)) \geq \varepsilon\beta(x/\ln\, x)$$

for all $\varepsilon \in (0, \eta)$ and $x \geq x_0$; that is,

$$\beta^{-1}(\alpha(M_f^{-1}(T(r)))/\varepsilon) \geq T(r)/\ln\, T(r).$$

Hence,

$$\exp\left\{\frac{T(r)}{\beta^{-1}(\alpha(M_f^{-1}(T(r)))/\varepsilon)}\right\} \leq T(r), \quad r \in [r_0(\varepsilon), 1).$$

Therefore, (19) implies that

$$\begin{aligned}
\sum_{n=1}^{\infty} \exp\left\{-\Gamma_f(r\lambda_n)\ln\left(1+\frac{1-r}{qr}\right)\right\} &\leq b(1-r)T(r)^2 + 2 \\
&\leq T(r)^2 + 2 \\
&= M_f^2(\alpha^{-1}(\varepsilon\beta(2/(b(1-r))) + 2, \quad r \in [r_0^*(\varepsilon), 1). \qquad (20)
\end{aligned}$$

Since $\beta \in L_{si}$ and $\varepsilon < \varrho$, from (17), (18), and (20), it follows that

$$\mathfrak{M}(r, A) \leq M_f^4\left(\alpha^{-1}\left(\varrho\beta\left(\frac{1}{1-r}\right)\right)\right) \qquad (21)$$

for every $r \in [0, 1)$ sufficiently close to 1. As above, the condition $\Gamma_f(r) \asymp \ln\, M_f(r)$ implies (11), and hence $M_f(r)^4 \leq M_f(Cr)$ for some $C > 1$. Therefore, (21) gives

$$\mathfrak{M}(r, A) \leq M_f\left(C\alpha^{-1}\left(\varrho\beta\left(\frac{1}{1-r}\right)\right)\right).$$

Hence, in view of the condition $\alpha \in L_{si}$, we obtain $\varrho^1_{\alpha,\beta}[A]_f \le \varrho$, and in view of the arbitrariness of ϱ, we have

$$\varrho^1_{\alpha,\beta}[A]_f \le \varrho^1_{\alpha,\beta}[\mu]_f.$$

If $\varrho^1_{\alpha,\beta}[\mu]_f = +\infty$, then this inequality is obvious. Thus, the equality $\varrho^1_{\alpha,\beta}[A]_f = \varrho^1_{\alpha,\beta}[\mu]_f$ is proved.

The inequality $\lambda^1_{\alpha,\beta}[A]_f \le \lambda^1_{\alpha,\beta}[\mu]_f$ is similarly proved. Indeed, as before, we have

$$\lambda^1_{\alpha,\beta}[\mu]_f = \varliminf_{r\uparrow 1} \frac{\alpha(M_f^{-1}(\mu\,(r+(1-r)/q)))}{\beta(1/(1-r)}.$$

Therefore, if $\lambda^1_{\alpha,\beta}[\mu]_f < +\infty$ then for every $\lambda > \lambda^1_{\alpha,\beta}[\mu]_f$ there exists a sequence $(r_k)\uparrow 1$ such that

$$\mu\left(r_k + \frac{1-r_k}{q}\right) \le M_f\left(\alpha^{-1}\left(\lambda\beta\left(\frac{1}{1-r_k}\right)\right)\right). \tag{22}$$

Since (20) holds for every $r \in [r_0^*(\varepsilon), 1)$, from (17), (22), and (20) it follows (as before) that

$$\mathfrak{M}(r_k, A) \le M_f\left(C\alpha^{-1}\left(\lambda\beta\left(\frac{1}{1-r_k}\right)\right)\right)$$

for $r_0^*(\varepsilon) \le r_k \uparrow 1$. Hence, the inequality

$$\lambda^1_{\alpha,\beta}[A]_f \le \lambda^1_{\alpha,\beta}[\mu]_f$$

follows. Therefore, $\lambda^1_{\alpha,\beta}[A]_f = \lambda^1_{\alpha,\beta}[\mu]_f$, which completes the proof of the proposition. □

Using Propositions 3.1 and 3.3, we prove the next result.

Theorem 3.1. *Let $m \ge 2$, $M_f(0) \ge 1$, and the conditions of Proposition 3.3 be satisfied. If $A_j \in \mathfrak{S}(\Lambda, f, 1)$ for every j with $1 \le j \le p$, then*

$$A(z) = (A_1 * \ldots * A_p)_m(z) \in \mathfrak{S}(\Lambda, f, 1)$$

and

$$\varrho^1_{\alpha,\beta}[A]_f \le \max\{\varrho^1_{\alpha,\beta}[A_j]_f : 1 \le j \le p\}. \tag{23}$$

Proof. Since $M_f(r) \ge M_f(0) \ge 1$, we have $M_f(r\lambda_n) \le M_f^m(r\lambda_n)$ and instead of (12), we obtain

$$\mu(r) := \mu(r, A) \le \sum_{k_1+\cdots+k_p=m} |c_{k_1\ldots k_p}|\mu(r, A_1)^{k_1}\cdot\ldots\cdot\mu(r, A_p)^{k_p}. \tag{24}$$

Let $\mu_j(r) = \mu(r, A_j)$ and $\varrho^* = \max\{\varrho_{\alpha,\beta}[\mu_j]_f : 1 \le j \le p\} < +\infty$. Then,

$$\mu_j(r) \le M_f(\alpha^{-1}(\varrho\beta(1/(1-r)))$$

for all $\varrho > \varrho^*$ and $r \ge r_0$. Therefore, (24) yields

$$\mu(r) \le K_0 M_f(\alpha^{-1}(\varrho\beta(1/(1-r)))^m.$$

Hence, as in the proof of Theorem 2.1, we obtain

$$\varrho^1_{\alpha,\beta}[\mu]_f \le \varlimsup_{r\uparrow 1} \frac{\alpha(M_f^{-1}(K_0 M_f^m(\alpha^{-1}(\varrho\beta(1/(1-r))))}{\beta(1/(1-r))} = \varrho.$$

In view of the arbitrariness of ϱ, we obtain the inequality

$$\varrho^1_{\alpha,\beta}[\mu]_f \le \varrho^* = \max\{\varrho^1_{\alpha,\beta}[\mu_j]_f : 1 \le j \le p\},$$

which is obvious for $\varrho^* = +\infty$. Since the equalities $\varrho^1_{\alpha,\beta}[\mu]_f = \varrho^1_{\alpha,\beta}[A]_f$ and $\varrho^1_{\alpha,\beta}[\mu_j]_f = \varrho^1_{\alpha,\beta}[A_j]_f$ hold (by Proposition 3.3), inequality (23) also holds. □

Remark 3.1. *Theorem 3.1 is an analogue of only one part of Theorem 2.1. An analogue of the second part (which is concerned with the dominant) could not be obtained. However, as seen from the proof of Proposition 2.1, the next proposition, being an analogue of this statement, is correct.*

Proposition 3.4. *Let $m = 1$, $\alpha \in L^0$, $\beta \in L$, and $A_j \in \mathfrak{S}(\Lambda, f, 1)$ for every j with $1 \le j \le p$. Then*

$$A(z) = (A_1 * \ldots * A_p)_m(z) \in \mathfrak{S}(\Lambda, f, 1)$$

and (23) *holds. Moreover, if the function A_1 is dominant, then $\varrho^1_{\alpha,\beta}[A]_f = \varrho^1_{\alpha,\beta}[A_1]_f$ and $\lambda^1_{\alpha,\beta}[A]_f = \lambda^1_{\alpha,\beta}[A_1]_f$.*

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