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Research Article

N–pure and weak–pure elements in coherent quantales

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Abstract

The N-pure ideals of a commutative ring have been introduced and studied by Aghajani in the recent paper [*Bull. Korean Math. Soc.* **59** (2022) 1237–1246]. In the present paper, as an abstraction of the N-pure ideals, the N-pure elements of a quantale are defined. It is shown that N-pure elements of a coherent quantale A coincide with weak-pure elements of A (a notion introduced by the present author in [*Fuzzy Sets Syst.* **442** (2022) 196–221]). The primary objective of the current paper is to generalize a part of Aghajani's results to N-pure elements of a coherent quantale. Characterization theorems of coherent mp-quantales and coherent hyperarchimedean quantales in terms of N-pure elements are also proved. Moreover, mid quantales are defined as abstractions of mid rings, and a theorem for their characterization is proved.

Keywords: coherent quantales; N-pure elements; weak-pure elements; mp-quantales; mid quantales.

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1. Introduction

Abstract ideal theory is a branch of algebra that deals with the generalization of the notions and properties of ideals, filters, and congruences of concrete algebraic structures (rings, semirings, various types of lattices, etc.) to an abstract framework, represented mainly by various classes of multiplicative lattices (see e.g. [5,8,14]). Among them, quantales occupy a central place. Introduced by Mulvey in [17], quantales provide the most conducive abstract framework for abstracting notions and results from ring theory (see e.g. [7,19,21]).

The present paper belongs to abstract ideal theory and continues the research [10–12] on some special elements (pure elements, weak-pure elements, etc.) of quantales. The pure elements of a quantale have been introduced in several works as abstractions of the pure ideals of ring theory (see [10, 13, 18, 22]), and the weak-pure elements ($=$ w-pure elements) in [11, 12]. The pure and w-pure elements have been used in the works [10–12] to characterize the following important classes of quantales: normal quantales, mp-quantales, PF-quantales, purified quantales, and PP-quantales.

The N-pure ideals of a commutative ring have been introduced and studied by Aghajani [1,2]. In this paper, we define the N-pure elements of a quantale as an abstraction of the N-pure ideals. We prove that N-pure elements of a coherent quantale A coincide with weak-pure elements of A (a notion introduced by the present author in [12]). The primary objective of this paper is to generalize a part of Aghajani's results to the N-pure elements of a coherent quantale.

The rest of this paper is organized as follows. Section 2 provides some basic notions and results from quantale theory; after presenting some lemmas regarding the arithmetic of quantales, m-prime elements and m-prime spectrum $Spec(A)$ of the quantale A are defined, and then the notion of the radical of an element and its more important properties are mentioned. In Section 3, N-pure elements of a quantale A are defined, and it is proved that if A is coherent, then such elements coincide with w-elements of A (Proposition 3.1); this section also contains a theorem for characterizing N-pure elements and their other basic properties. In Section 4, some lemmas regarding the operator

$$O(\cdot) : Spec(A) \to A$$

are given, which are then used to obtain other results regarding N-pure elements. In Section 5, some characterization theorems of coherent mp-quantales and coherent hyperarchimedean quantales in terms of N-pure elements are proved. Section 6 is devoted to the mid quantales, which are the abstractions of the mid rings (introduced in [2]); a theorem for characterizing mid quantales is also proved in this section.

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2. Preliminaries on quantales

This section contains some basic definitions and results regarding the quantale theory (cf. [7,19,21]).

Let R be a unital commutative ring and $Id(R)$ be the set of its ideals. Then, $Id(R)$ can be endowed with a structure of a multiplicative complete lattice in which the join of a family of ideals is their sum and the multiplication is the ideal product. This concrete algebraic structure of $Id(R)$ is one of the sources of inspiration for defining the abstract notion of quantales and for developing the quantale theory. A *quantale* is a complete multiplicative lattice $(A, \bigvee, \wedge, \cdot, 0, 1)$, where the multiplication "$\cdot$" is associative, and for all $S \subseteq A$ and $a \in A$, the following infinite distributive law holds:

$$a \cdot \left(\bigvee S\right) = \bigvee\{a \cdot s \mid s \in S\} \quad \text{and} \quad \left(\bigvee S\right) \cdot a = \bigvee\{s \cdot a \mid s \in S\}.$$

Usually, we will write ab instead of $a \cdot b$ and denote by A the quantale $(A, \bigvee, \wedge, \cdot, 0, 1)$. The quantale A is said to be *integral* if the structure $(A, \cdot, 1)$ is a monoid. The quantale A is said to be *commutative* if the multiplication "$\cdot$" is commutative. A *frame* is a quantale in which the multiplication coincides with the meet (see [15,20]). Throughout this paper, quantales are assumed to be integral and commutative.

An element $c \in A$ is said to be *compact* if for any $S \subseteq A$ with $c \leq \bigvee S$, there exists a finite subset S_0 of S such that $c \leq \bigvee S_0$. The set of compact elements of A is denoted by $K(A)$. The finitely generated ideals of the commutative ring R are the compact elements of the quantale $Id(R)$.

The quantale A is *algebraic* if any element $a \in A$ has the form $a = \bigvee X$ for some subset X of $K(A)$. It follows that any element a of an algebraic quantale can be written as $a = \bigvee\{c \in K(A) \mid c \leq a\}$. An algebraic quantale A is said to be *coherent* if the top element 1 is a compact element and the set $K(A)$ of compact elements is closed under multiplication. An example of a coherent quantale is $Id(R)$.

Lemma 2.1 (see [5]). *Let A be a coherent quantale and $a, b, c \in A$.*

(i). *If $a \vee b = 1$, then $ab = a \wedge b$.*

(ii). *If $a \vee b = 1$, then $a^n \vee b^n = 1$. for any integer $n \geq 1$*

(iii). *If $a \vee b = 1$ and $a \vee c = 1$, then $a \vee (bc) = a \vee (b \wedge c) = 1$.*

Lemma 2.2. *Let A be a coherent quantale and $a, b, a_1, \ldots, a_k \in A$, where $k \geq 2$. Then, the following hold:*

(i). $(a \vee b)^n \leq a \vee b^n$ *for every integer $n \geq 1$;*

(ii). $(a \vee b)^{2n} \leq a^n \vee b^n$ *for every integer $n \geq 1$;*

(iiii). $(a_1 \vee \ldots \vee a_k)^{2^{k-1}n} \leq a_1^n \vee \ldots \vee a_k^n$ *for every integer $n \geq 1$.*

Proof. (i). The inequality is proved by induction on n.

(ii). By applying the inequality given in part **(i)** twice, we obtain

$$(a \vee b)^{2n} = ((a \vee b)^n)^2 \leq (a \vee b^n)^n \leq a^n \vee b^n.$$

(iii). We apply induction on k. For $k = 2$, by applying the inequality given in part **(ii)**, we obtain $(a_1 \vee a_2)^{2n} \leq a_1^n \vee a_2^n$. Assume that $(a_1 \vee \ldots \vee a_k)^{2^{k-1}n} \leq a_1^n \vee \ldots \vee a_k^n$. Using first the inequality given in part **(ii)** and then using the induction hypothesis, we have

$$(a_1 \vee \ldots \vee a_{k+1})^{2^k n} = (a_1 \vee \ldots \vee a_k \vee a_{k+1})^{2.2^{k-1}n} \leq (a_1 \vee \ldots \vee a_k)^{2^{k-1}n} \vee a_{k+1}^{2^{k-1}n} \leq a_1^n \vee \ldots a_k^n \vee a_{k+1}^n.$$

□

Recall from [21] that, on every quantale A, we can define:

- a binary operation $\rightarrow$, named *residuation* or *implication*: for all $a, b \in A$,

$$a \rightarrow b = \bigvee\{x \in A \mid ax \leq b\};$$

- a unary operation, named *annihilator operation*: for every $a \in A$,

$$a^\perp = a^{\perp_A} = a \rightarrow 0 = \bigvee\{x \in A \mid ax = 0\}.$$

Following a tradition in ring theory, some authors use the notation "$b : a$" for "$a \to b$". Recall from [21] that the implication $\to$ fulfills the following residuation rule: for all $a, b, c \in A$, $a \leq b \to c$ if and only if $ab \leq c$. Thus, the lattice $(A, \vee, \wedge, \cdot, \to, 0, 1)$ is a (commutative) residuated lattice. In this paper, without mentioning, we will use some elementary arithmetical properties of residuated lattices; the readers may consult the monograph [9], which is the standard text on residuated lattices.

An ideal P of a commutative ring R is said to be a *prime ideal* if for all ideals I_1, I_2 of R, $I_1 I_2 \subseteq P$ implies $I_1 \subseteq P$ or $I_2 \subseteq P$ (see [4]). The notion of a prime ideal is generalized in quantale theory: an element $p < 1$ of a quantale A is *m-prime* if for all $a, b \in A$, $ab \leq p$ implies $a \leq p$ or $b \leq p$. If A is an algebraic quantale, then an element $p < 1$ is m-prime if and only if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq p$. An element $m < 1$ is said to be *maximal* if for any $x \in A$ such that $m \leq x < 1$ we have $m = x$. Recall from [21] that $Spec(A)$ denotes the set of m-prime elements of A and $Max(A)$ denotes the set of maximal elements of A. Keeping the usual terminology, we say that $Spec(A)$ is the m-prime spectrum of A and $Max(A)$ is the maximal spectrum of A.

If 1 is a compact element of the quantale A then for any $a < 1$ there exists $m \in Max(A)$ such that $a \leq m$. The same hypothesis $1 \in K(A)$ implies that $Max(A) \subseteq Spec(A)$. We remark that the set $Spec(R)$ of prime ideals in a commutative ring R is the prime spectrum of the quantale $Id(R)$, and the set $Spec_{Id}(L)$ of prime ideals in a bounded distributive lattice L is the prime spectrum of the frame $Id(L)$.

An m-prime element of the quantale A is *minimal* over $a \in A$ if for any m-prime element q, $a \leq q \leq p$ implies $q = p$. For any $a \in A$, there exists $p \in Spec(A)$, which is minimal over a. The minimal elements over 0 are called *minimal m-prime elements* of the quantale A. The set $Min(A)$ of minimal m-prime elements is called the minimal m-prime spectrum of A.

If R is a ring, then its *Jacobson radical* is the ideal $J(A) = \bigcap Max(A)$ (cf. [4]). This notion can be generalized to a quantale A: $r(A) = \bigwedge Max(A)$ is the *Jacobson radical* of A (cf. [6]).

Let $B(A)$ be the Boolean algebra of complemented elements of A. The quantale A is said to be *hyperarchimedean* if for any element $c \in K(A)$ there exists an integer $n \geq 1$ such that $c^n \in B(A)$ (see [6]).

Following [21], we recall that the *radical* $\rho(a)$ of an element a of the quantale A is defined by

$$\rho(a) = \rho_A(a) = \bigwedge\{p \in Spec(A) \mid a \leq p\}.$$

We remark that this notion is an abstraction of the radical of an ideal in a commutative ring. If $a = \rho(a)$ then a is said to be a *radical element* of A. The set of radical elements of A is denoted by $R(A)$. The quantale A is said to be *semiprime* if $\rho(0) = 0$.

We recall the following useful properties of the map $\rho : A \to A$. For all elements $a, b \in A$ the following hold:

(2.1) $a \leq \rho(a)$;

(2.2) $\rho(ab) = \rho(a) \wedge \rho(b)$;

(2.3) $\rho(\rho(a)) = \rho(a)$;

(2.4) $\rho(a \vee b) = \rho(\rho(a) \vee \rho(b))$;

(2.5) $\rho(a) = 1$ if and only if $a = 1$;

(2.6) $\rho(a) \vee \rho(b) = 1$ if and only if $a \vee b = 1$;

(2.7) for every integer $n \geq 1$, $\rho(a^n) = \rho(a)$.

(2.8) If S is a subset of A then the following equality holds: $\rho(\bigvee S) = \rho(\bigvee\{\rho(s) \mid s \in S\})$.

In the rest of the paper, we will often use the abovementioned properties, without mentioning them. It is obvious that the set $R(A)$ is closed under the arbitrary meets, so it is a complete lattice. For any subset S of $R(A)$, we denote $\dot{\bigvee} S = \rho(\bigvee S)$. Then $(R(A), \dot{\bigvee}, \wedge, 0, 1)$ is a frame (see [21]). According to Lemma 8 of [6], if A is a coherent quantale then $K(R(A)) = \rho(K(A))$ and $R(A)$ is a coherent frame.

Lemma 2.3. *Let A be a coherent quantale and $x \in A$. Then $x \to \rho(0) = \rho(x) \to \rho(0)$.*

Proof. Assume that $c \in K(A)$ and $c \leq x \to \rho(0)$, so $cx \leq \rho(0)$. Then, the following hold:

$$c\rho(x) \leq \rho(c) \wedge \rho(x) = \rho(cx) \leq \rho((\rho(0)) = \rho(0).$$

It follows that $c \leq \rho(x) \to \rho(0)$. Hence, using the fact that the quantale A is coherent, we obtain $x \to \rho(0) \leq \rho(x) \to \rho(0)$. From $x \leq \rho(x)$, we obtain the converse inequality $\rho(x) \to \rho(0) \leq x \to \rho(0)$. Therefore, $x \to \rho(0) = \rho(x) \to \rho(0)$. □

The following useful lemma is a quantale version of a well-known result in ring theory (see Proposition 1.8 of [4]).

Lemma 2.4 (see [16]). *Let A be a coherent quantale and $a \in A$. Then the following hold:*

(i). $\rho(a) = \bigvee\{c \in K(A) \mid c^k \leq a$ *for some integer* $k \geq 1\}$;

(ii). *For any $c \in K(A), c \leq \rho(a)$ if and only if $c^k \leq a$ for some integer $k \geq 1$;*

(iii). *A is semiprime if and only if for any integer $k \geq 1$, $c^k = 0$ implies $c = 0$.*

Let us consider two quantales A and B. Recall from [21] that a map $u : A \to B$ is a *quantale morphism* if it preserves the arbitrary joins and the multiplication (in particular, $u(0) = 0$); if $u(1) = 1$ then the quantale morphism u is an *integral quantale morphism*. We say that the quantale morphism u *preserves the compact elements* if $u(K(A)) \subseteq K(B)$.

Assume that A and B are coherent quantales. An integral quantale morphism $u : A \to B$ is said to be a *coherent quantale morphism* if it preserves the compact elements. The category of compact quantales and coherent quantale morphisms is denoted by CohQuant. Any coherent quantale morphism $u : A \to B$ has a right adjoint $u_\star : B \to A$, defined by $u_\star(b) = \bigvee\{a \in A \mid u(a) \leq b\}$, for any $b \in B$.

Let us fix an element a of a coherent quantale A. The interval $[a)_A = \{x \in A \mid a \leq x\}$ is closed under arbitrary joins of A. For all $x, y \in [a)_A$, we denote $x \cdot_a y = xy \vee a$. Then $[a)_A$ is closed under multiplication $\cdot_a$ and the algebraic structure $([a)_A, \bigvee, \wedge, \cdot_a, a, 1)$ is a quantale.

Lemma 2.5 (see [6]). *For any element a of a coherent quantale A, the following hold:*

(i). *$[a)_A$ is a coherent quantale.*

(ii). *$u_a^A : A \to [a)_A$ is a coherent quantale morphism.*

(iii). $K([a)_A) = \{a \vee c \mid c \in K(A)\}$.

(iv). *$\rho_{[a)_A}(x) = \rho_A(x)$, for any $x \in [a)_A$.*

3. N-pure and weak-pure elements

Let R be a unital commutative ring and $nil(R)$ be the nil-radical of R. According to Definition 2.1 of [2], an ideal I of R is *N-pure* if for any $a \in R$, there exists $b \in I$ such that $a(1-b) \in nil(R)$. Recall from [2], the following characterization of the N-pure ideals of R.

Theorem 3.1. *For any ideal I of R, the following are equivalent:*

(i). *I is an N-pure ideal.*

(ii). *For any $a \in I$, there exists an integer $n \geq 1$ such that $Ann(a^n) + I = R$.*

(iii). $\sqrt{I} = \{a \in R \mid \exists\, n \geq 1, Ann(a^n) + I = R\}$.

(iv). *$\sqrt{I}$ is an N-pure ideal.*

(v). *There exists a unique pure ideal J of R such that $\sqrt{I} = \sqrt{J}$.*

Let A be a quantale. Starting from the property (ii) of Theorem 3.1, we define the notion of N-pure element of A.

Definition 3.1. *An element a of A is said to be N-pure if for any compact element $c \in K(A)$ with $c \leq a$, there exists an integer $n \geq 1$ such that $(c^n)^\perp \vee a = 1$.*

Now, we recall from [12] the notion of weak-pure (= w-pure) elements of the quantale A.

Definition 3.2. *An element a of A is said to be w-pure if for any compact element $c \in K(A)$ such that $c \leq a$, we have $(c \to \rho(0)) \vee a = 1$.*

It is easy to verify that 0, 1, and $\rho(0)$ are w-pure elements.

Throughout the current section, A denotes a coherent quantale. The following result shows that the N-pure and w-pure elements of A coincide.

Proposition 3.1. *For any $a \in A$, the following statements are equivalent:*

(i). *a is N-pure.*

(ii). *a is w-pure.*

Proof. (i) $\Rightarrow$ (ii). Assume that a is N-pure. Let c be an element $c \in K(A)$ such that $c \leq a$. Hence, there exists an integer $n \geq 1$ such that $(c^n)^\perp \vee a = 1$. Since A is coherent, there exist two compact elements c, d, such that $e \leq a, f \leq (c^n)^\perp$, and $e \vee f = 1$. Using Lemma 2.4(ii) the following implications hold:

$$f \leq (c^n)^\perp \Rightarrow fc^n = 0 \Rightarrow f^n c^n = 0 \Rightarrow fc \leq \rho(0) \Rightarrow f \leq c \to \rho(0).$$

From $e \leq a$ and $f \leq c \to \rho(0)$, it follows that $1 = e \vee f \leq a \vee (c \to \rho(0))$. Hence, $a \vee (c \to \rho(0)) = 1$. Consequently, a is w-pure.

(ii) $\Rightarrow$ (i). Assume that a is w-pure. Let c be an element $c \in K(A)$ such that $c \leq a$. So, $a \vee (c \to \rho(0)) = 1$. Then, there exist two compact elements c, d, such that $e \leq a$, $f \leq c \to \rho(0)$, and $e \vee f = 1$. Thus, $fc \leq \rho(0)$. Thus, using Lemma 2.4(ii), we obtain $f^n c^n = 0$ for some integer $n \geq 1$. We obtain the inequality $f^n \leq (c^n)^\perp$. By Lemma 2.1(ii), we have $e^n \vee f^n = 1$. So, $e^n \leq e \leq a$ and $f^n \leq (c^n)^\perp$ imply $(c^n)^\perp \vee a = 1$. Hence, a is N-pure. □

According to Proposition 3.1, we will identify the N-pure and w-pure elements of the coherent quantale A. Recall that an element $a \in A$ is said to be *pure* if for any compact element $c \leq a$, we have $a \vee c^\perp = 1$ (for example, see [12]). It is obvious that any pure element of A is w-pure, but the converse assertion does not hold (cf. [2]). If the quantale A is semiprime then an element $a \in A$ is pure if and only it is N-pure.

Theorem 3.2. *For any $a \in A$, the following statements are equivalent:*

(i). *a is N-pure.*

(ii). *a is w-pure.*

(iii). $\rho(a) = \bigvee\{c \in K(A) \mid \exists\, n \geq 1, (c^n)^\perp \vee a = 1\}$.

(iv). *$\rho(a)$ is N-pure.*

Proof. (i) $\Leftrightarrow$ (ii). The desired conclusion follows from Proposition 3.1.

(i) $\Rightarrow$ (iii). Assume that a is N-pure. Let c be a compact element of A such that $(c^n)^\perp \vee a = 1$ for some integer $n \geq 1$. Then $c^n = c^n((c^n)^\perp \vee a) = c^n a \leq a$. By virtue of Lemma 2.4(ii), we have $c \leq \rho(a)$. Therefore, we obtain the inequality

$$\bigvee\{c \in K(A) \mid \exists\, n \geq 1, (c^n)^\perp \vee a = 1\} \leq \rho(a).$$

Suppose now that c is a compact element of A such that $c \leq \rho(a)$. Hence, there exists an integer $n \geq 1$ such that $c^n \leq a$ (cf. Lemma 2.4(ii)). Since $c^n \in K(A)$ and a is N-pure, we have $((c^n)^k)^\perp \vee a = 1$ for some integer $k \geq 1$. Then, $(c^{nk})^\perp \vee a = 1$, and hence

$$\rho(a) \leq \bigvee\{c \in K(A) \mid \exists\, n \geq 1, (c^n)^\perp \vee a = 1\}.$$

(iii) $\Rightarrow$ (iv). Assume that c is a compact element of A such that $c \leq \rho(a)$, so $c \leq \bigvee\{c \in K(A) \mid \exists\, n \geq 1, (c^n)^\perp \vee a = 1\}$. Thus, there exist the integers $m, n_1, \ldots, n_m \geq 1$ and $d_1, \ldots, c_m \in K(A)$ such that $(d_i^{n_i})^\perp \vee a = 1$, for $i = 1, \ldots, m$, and $c \leq d_1 \vee \ldots \vee d_m$. Let us denote $n = max\{n_i\}$ and $d = d_1 \vee \ldots \vee d_m$, so $(d_i^{n_i})^\perp \leq (d_i^n)^\perp$ for $i = 1, \ldots, m$. Then, $d \in K(A)$ and $(d_i^n)^\perp \vee a = 1$ for $i = 1, \ldots, m$. Using Lemma 2.1(iii), we obtain

$$\left(\bigvee_{i=1}^{m} d_i^n\right)^\perp \vee a = \bigwedge_{i=1}^{m} ((d_i^n)^\perp \vee a) = 1.$$

Take $k = 2^{k-1}n$. By virtue of Lemma 2.2(iii), we have

$$d^k = \left(\bigvee_{i=1}^{m} d_i\right)^{2^{m-1}n} \leq \bigvee_{i=1}^{m} d_i^n.$$

Therefore, $(\bigvee_{i=1}^{m} d_i^n)^\perp \leq (d^k)^\perp$. Consequently, we have $(d^k)^\perp \vee a = 1$ and hence $(d^k)^\perp \vee \rho(a) = 1$ (because $a \leq \rho(a)$). So, we conclude that $\rho(a)$ is N-pure.

(iv) ⇒ (i). Suppose that $\rho(a)$ is N-pure. Let c be a compact element of A such that $c \leq a$. Then $c \leq \rho(a)$, so

$$(c^n)^\perp \vee \rho(a) = 1$$

for some integer $n \geq 1$. Using (2.6) (see the properties listed before Lemma 2.3), we obtain immediately $(c^n)^\perp \vee a = 1$, and hence a is N-pure. □

Theorem 3.2 generalizes some parts of Theorem 2.6 of [2] (more precisely, the equivalence of the conditions (iii)-(v)). Theorem 2.6(vi) of [2] suggests to us to ask if an element $a \in A$ is N-pure if and only if there exists a unique element $b \in A$ such that $\rho(a) = \rho(b)$.

Corollary 3.1. *For any $a \in A$, the following statements are equivalent:*

(i). *a is N-pure.*

(ii). *a^n is N-pure for any integer $n \geq 1$.*

Proof. We recall from property (2.7) that $\rho(a^n) = \rho(a)$ for every integer $n \geq 1$. Therefore, by Theorem 3.2(iv), a is N-pure if and only if $\rho(a)$ is N-pure, if and only if $\rho(a^n)$ is N-pure for any integer $n \geq 1$, if and only if a^n is N-pure for any integer $n \geq 1$. □

Keeping the notation of [10], we denote by $Vir(A)$ the set of pure elements of A and by $Vir_w(A)$ the set of w-pure elements of A. By Proposition 3.1, the set $Vir_N(A)$ of N-pure elements of A coincides with $Vir_w(A)$.

Proposition 3.2. *The set $Vir_N(A)$ of N-pure elements of A is closed under arbitrary joins, finite meets, and multiplication.*

Proof. The result follows from Lemma 4.7 of [12] and Proposition 3.1. □

Theorem 3.3. *The following statements hold:*

(i). *If $a, b \in Vir_N(A)$ then $ab = a \wedge b$.*

(ii). *$Vir_N(A)$ is a $\bigvee$-sublattice of A which is a frame.*

Proof. (i). Assume that $a, b \in Vir_N(A)$ and c is a compact element such that $c \leq a \wedge b$. Then, $c \leq a$, $c \leq b$, and $a, b \in Vir_w(A)$. So, we have $a \vee (c \to \rho(0)) = b \vee (c \to \rho(0)) = 1$. Using Lemma 2.1(iii), we obtain $ab \vee (c \to \rho(0)) = 1$. Hence,

$$c = c(ab \vee (c \to \rho(0))) = cab \vee c(c \to \rho(0)) \leq cab \leq ab.$$

Now, it follows that $a \wedge b \leq ab$, and hence $a \wedge b = ab$.

(ii). This part follows from Proposition 3.2 and part (i). □

Proposition 3.3. *For any $a \in A$, the following statements are equivalent:*

(i). *a is an N-pure element of A.*

(ii). *$a \vee \rho(0)$ is an N-pure element of $[\rho(0))_A$.*

Proof. Firstly, we recall that $[\rho(0))_A$ is a coherent quantale. So, the N-pure elements and the w-pure elements of $[\rho(0))_A$ coincide (cf. Proposition 3.1). According to Lemma 2.5(iv), we have $\rho_{[\rho(0))_A}(b) = \rho_A(b)$ for any $b \in [\rho(0))_A$ and hence $\rho_{[\rho(0))_A}(\rho_A(0)) = \rho_A(0)$. Now, we prove the equivalence of (i) and (ii).

(i) ⇒ (ii). Assume that a is an N-pure element of A. Consider a compact element x of $[\rho(0))_A$ such that $x \leq a \vee \rho(0)$. By Lemma 2.5(iii), there exists $c \in K(A)$ such that $x = c \vee \rho(0)$. Since A is a coherent quantale and $c \leq a \vee \rho(0)$, there exist $e, f \in K(A)$ such that $e \leq a$, $f \leq \rho(0)$, and $c \leq e \vee f$. By Proposition 3.1, a is w-pure, and so $e \leq a$ implies that $a \vee (e \to \rho(0)) = 1$. Since $c \leq e \vee f$ and $f \leq \rho(0)$, we have

$$1 = a \vee (e \to \rho(0)) = a \vee ((e \vee f) \to \rho(0)) \leq a \vee (c \to \rho(0)).$$

Hence, $a \vee (c \rightarrow \rho(0)) = 1$. But, on the other hand, we have $x \rightarrow \rho(0) = (c \vee \rho(0)) \rightarrow \rho(0) = c \rightarrow \rho(0)$, and hence $a \vee (x \rightarrow \rho(0)) = a \vee (c \rightarrow \rho(0)) = 1$. Since $\rho(0) \leq x \rightarrow \rho(0)$, we have $a \vee \rho(0) \vee (x \rightarrow \rho(0)) = a \vee (x \rightarrow \rho(0)) = 1$. Thus, $a \vee \rho(0)$ is a w-pure element of $[\rho(0))_A$. Therefore, by Proposition 3.1, $a \vee \rho(0)$ is an N-pure element of $[\rho(0))_A$.

(ii) ⇒(i). Assume that $a \vee \rho(0)$ is an N-pure element of $[\rho(0))_A$. In order to prove that a is an N-pure element of A, let us consider a compact element c of A such that $c \leq a$. We have to prove that $a \vee (c \rightarrow \rho(0)) = 1$. By Lemma 2.5(iii), $x = c \vee \rho(0)$ is a compact element of $[\rho(0))_A$ and $x = c \vee \rho(0) \leq a \vee \rho(0)$. Then, $a \vee \rho(0) \vee (x \rightarrow \rho(0)) = 1$ (because $a \vee \rho(0)$ is w-pure in $[\rho(0))_A$). On the other hand, since $\rho(0) \leq c \rightarrow \rho(0)$, following equalities hold:

$$a \vee (c \rightarrow \rho(0)) = a \vee \rho(0) \vee (c \rightarrow \rho(0)) = a \vee \rho(0) \vee ((c \vee \rho(0)) \rightarrow \rho(0)) = a \vee \rho(0) \vee (x \rightarrow \rho(0)).$$

Therefore, $a \vee (c \rightarrow \rho(0)) = 1$, and hence a is an w-pure element of A. □

Proposition 3.4. *For any $a \in A$, the following statements are equivalent:*

(i). *a is an N-pure element of A.*

(ii). *For any $b \in A$ such that $a \leq b$, b is N-pure in A if and only if b is N-pure in $[a)_A$.*

Proof. **(i) ⇒ (ii)**. Let b be an element of A such that $a \leq b$. According to Lemma 2.5, the following statements are equivalent:

- b is N-pure in $[a)_A$.
- For any $x \in K([a)_A)$, $x \leq b$ implies $(x \rightarrow \rho_{[a)_A}(a)) \vee b = 1$.
- For any $c \in K(A)$, $a \vee c \leq b$ implies $((a \vee c) \rightarrow \rho_A(a)) \vee b = 1$.
- For any $c \in K(A)$, $c \leq b$ implies $(c \rightarrow \rho_A(a)) \vee b = 1$.

Firstly, assume that b is an N-pure element of A. In order to prove that b is N-pure in $[a)_A$, suppose that $c \in K(A)$ and $c \leq b$. Since b is N-pure in A, $c \leq b$ implies that $(c \rightarrow \rho(0)) \vee b = 1$. But, $\rho(0) \leq \rho(a)$ implies $c \rightarrow \rho(a) \leq c \rightarrow \rho(a)$. So, $(c \rightarrow \rho(a)) \vee b = 1$. Then, b is N-pure in $[a)_A$.

Secondly, assume that b is an N-pure element of $[a)_A$. In order to prove that b is N-pure in A, suppose that $c \in K(A)$ and $c \leq b$. So, b is w-pure in $[a)_A$. From $c \leq b$, we infer that $(c \rightarrow \rho_A(a)) \vee b = 1$. Since $1 \in K(A)$, there exist $d, e \in K(A)$ such that $d \leq c \rightarrow \rho_A(a)$, $e \leq b$, and $d \vee e = 1$. From $d \leq c \rightarrow \rho_A(a)$, we obtain $dc \leq \rho_A(a)$, and hence $d^n c^n \leq a$ for some integer $n \geq 1$ (cf. Lemma 2.4(ii)). Therefore, using $d^n c^n \in K(A)$, we obtain $(d^n c^n \rightarrow \rho(0)) \vee a = 1$ (because a is N-pure in A). Thus, there exist $x, y \in K(A)$ such that $x \leq d^n c^n \rightarrow \rho(0)$, $y \leq a$, and $x \vee y = 1$. The inequality $x \leq d^n c^n \rightarrow \rho(0)$ implies that $x d^n c^n \leq \rho(0)$, and hence $x^k d^{nk} c^{nk} = 0$ for some integer $k \geq 1$ (cf. Lemma 2.4(ii)).

Thus, $(xdc)^{nk} = 0$. So, we obtain $xdc \leq \rho(0)$, and therefore $xd \leq c \rightarrow \rho(0)$. We remark that $yd \leq y \leq a$, $xe \leq e \leq b$, and $ye \leq y \leq a$. So, the following hold:

$$1 = (x \vee y)(d \vee c) = xd \vee yd \vee xe \vee ye \leq (c \rightarrow \rho(0)) \vee a \vee b \vee a = (c \rightarrow \rho(0)) \vee b.$$

It follows that $(c \rightarrow \rho(0)) \vee b = 1$. Hence, b is N-pure in A.

(ii) ⇒ (i). If one takes $b = a$ in the statement (ii), one obtains the statement (i). □

4. Some properties of the N-pure elements

We fix a coherent quantale A. Following [12], for any $p \in Spec(A)$, we denote $O(p) = \bigvee\{c \in K(A) \mid c^{\perp} \not\leq p\}$.

Lemma 4.1 (see [12]). *For any $p \in Spec(A)$, the following hold:*

(i). *For any $c \in K(A)$, $c \leq O(p)$ if and only if $c^{\perp} \not\leq p$.*

(ii). $O(p) \leq p$.

(iii). $\rho(O(p)) \leq p$.

Lemma 4.2 (see [12]). *Assume that $p \in Spec(A)$. Then $p \in Min(A)$ if and only if for each $c \in K(A)$, $c \leq p$ implies $c \rightarrow \rho(0) \not\leq p$.*

Lemma 4.3 (see [11]). *For any $p \in Spec(A)$, $p \in Min(A)$ if and only if $\rho(O(p)) = p$.*

Lemma 4.4. *If $m \in Max(A)$ and $c \in K(A)$, then $c \leq \rho(O(m))$ if and only if $(c^n)^\perp \vee m = 1$ for some integer $n \geq 1$.*

Proof. By Lemmas 2.4(ii) and 4.1(i), we obtain the following equivalences:
- $c \leq \rho(O(m))$.
- $c^n \leq O(m)$ for some integer $n \geq 1$.
- $(c^n)^\perp \not\leq m$ for some integer $n \geq 1$.
- $(c^n)^\perp \vee m = 1$ for some integer $n \geq 1$. □

Theorem 4.1. *If $a \in A$ is N-pure, then*

$$\rho(a) = \bigwedge\{\rho(O(m)) \mid m \in V(a) \cap Max(A)\}.$$

Proof. Assume that c is a compact element of A such that $c \leq \rho(a)$ and $m \in V(a) \cap Max(A)$. We want to show that $c \leq \rho(O(m))$. By Lemma 2.4(ii), we have $c^n \leq a$ for some integer $n \geq 1$. Since a is N-pure and $c^n \in K(A)$, there exists an integer $k \geq$ such that $((c^n)^k)^\perp \vee a = 1$. So, $(c^{nk})^\perp \vee a = 1$. But, on the other hand, $a \leq m$. Hence, $((c^n)^k)^\perp \vee m = 1$. Using Lemma 4.4, we obtain $c \leq \rho(O(m))$. It follows that

$$\rho(a) \leq \bigwedge\{\rho(O(m)) \mid m \in V(a) \cap Max(A)\}.$$

Now, we prove the converse inequality

$$\bigwedge\{\rho(O(m)) \mid m \in V(a) \cap Max(A)\} \leq \rho(a).$$

Let c be a compact element of A satisfying the inequality $c \leq \rho(O(m))$ for all $m \in V(a) \cap Max(A)$. Since a is N-pure, it holds that $\rho(a) = \bigvee\{c \in K(A) \mid \exists\, n \geq 1, (c^n)^\perp \vee a = 1\}$ (cf. Theorem 3.2). Assume to the contrary that $c \not\leq \rho(a)$. Hence, $(c^n)^\perp \vee a \neq 1$ for every integer $n \geq 1$. Since

$$t \leq s \Rightarrow c^s \leq c^t \Rightarrow (c^t)^\perp \leq (c^s)^\perp \Rightarrow (c^t)^\perp \vee a \leq (c^s)^\perp \vee a,$$

the sequence $((c^t)^\perp \vee a)_{t\geq 1}$ is increasing. Due to the compactness of 1, we have $\bigvee_{t\geq 1}((c^t)^\perp \vee a) < 1$. So, there exists $m_0 \in Max(A)$ such that $\bigvee_{t\geq 1}((c^t)^\perp \vee a) \leq m_0$. Hence, $(c^t)^\perp \vee a \leq m_0$ for all $t \geq 1$. Then, $a \leq m_0$, and so we obtain $m_0 \in V(a) \cap Max(A)$. By virtue of the hypothesis, we obtain $c \leq \rho(O(m_0))$. By Lemma 4.4, we have $(c^k)^\perp \vee m_0 = 1$ for some integer $k \geq 1$, contradicting that $(c^t)^\perp \vee m_0 = m_0 < 1$ for all integers $t \geq 1$. Hence, the converse inequality follows, and so the proof is completed. □

Lemma 4.5. *If p is an N-pure m-prime element of A, then $\rho(O(p)) = p$.*

Proof. We know that $\rho(O(p)) \leq p$ (by Lemma 4.1(iii)). In order to prove the converse inequality $p \leq \rho(O(p))$, assume that $c \in K(A)$ and $c \leq p$. Since p is N-pure, there exists an integer $n \geq 1$ such that $(c^n)^\perp \vee p = 1$. So, $c^n \not\leq p$, and hence we obtain $c^n \leq O(p)$ (by Lemma 4.1(i)). Therefore, using Lemma 2.4(ii), we obtain $c \leq \rho(O(p))$. □

Recall that the quantale generalization of the Jacobson radical of a commutative ring is $r(A) = \bigwedge Max(A)$.

Lemma 4.6. *If a is an N-pure element of A such that $a \leq r(A)$ then $a \leq \rho(0)$.*

Proof. Suppose to the contrary that $a \not\leq \rho(0)$. Then, there exists $c \in K(A)$ such that $c \leq a$ and $a \not\leq \rho(0)$. By Lemma 2.4(ii), $c^n \neq 0$ for every integer $n \geq 1$. Then, $(c^n)^\perp \neq 1$ for every integer $n \geq 1$ (because $c^n = 1$ implies that $c^n \leq c^{\perp\perp} = 0$). Since a is N-pure, $c \leq a$ implies that $(c^k)^\perp \vee a = 1$ for some integer $k \geq 1$. Thus, $(c^k)^\perp < 1$, and so $(c^k)^\perp \leq m_0$ for some maximal element m_0. By the hypothesis, we have $a \leq r(A) \leq m_0$, and so we obtain $1 = (c^k)^\perp \vee a \leq m_0$, contradicting that $m_0 \in Max(A)$. Hence, we have $a \leq \rho(0)$. □

Proposition 4.1. *For any $a \in A$, the following statements are equivalent:*

(i). *a is an N-pure element of A.*

(ii). *$\rho(a)$ is a pure element of the frame $R(A)$.*

Proof. (i) ⇒ (ii). Assume that a is an N-pure element of A. Let x be a compact element of the frame $R(A)$ such that $x \leq \rho(a)$. In accordance with Theorem 3.2, $\rho(a)$ is w-pure, and hence $\rho(a) \vee (x \to \rho(0)) = 1$. Therefore, using the property (2.6) (given in Section 2), we obtain $\rho(a)\dot\vee(x \to \rho(0)) = 1$. Noticing that $x^{\perp_{R(A)}} = x \to \rho(0)$, it follows that $x^{\perp_{R(A)}}\dot\vee\rho(a) = 1$, and hence $\rho(a)$ is a pure element of the frame A.

(ii) ⇒ (i). Assume that $\rho(a)$ is a pure element of the frame A. Let c be a compact element of A such that $c \leq a$. Then, $\rho(c) \in K(R(A))$ (by Lemma 4.8 of [6]) and $\rho(c) \leq \rho(a)$, and hence

$$\rho(a)\dot{\vee}(\rho(c) \to \rho(0)) = (\rho(c))^{\perp_{R(A)}}\dot{\vee}\rho(a) = 1.$$

By virtue of Lemma 2.3, we have $\rho(c) \to \rho(0) = c \to \rho(0)$, and so we obtain $\rho(a)\dot{\vee}(c \to \rho(0)) = 1$. Using the property (2.6), we obtain $a \vee (c \to \rho(0)) = 1$, and hence a is w-pure. Now, from Proposition 3.1, it follows that a is N-pure. □

Proposition 4.2. *$r(A) = \rho(0)$ if and only if $r(A)$ is N-pure.*

Proof. We know that $\rho(0)$ is N-pure, and so $r(A) = \rho(0)$ implies that $r(A)$ is N-pure. Conversely, assume that $r(A)$ is N-pure. We want to show that $r(a) \leq \rho(0)$. Assume that c is a compact element of A such that $c \leq r(a)$. So, $r(a) \vee (c^k)^{\perp} = 1$ for some integer $k \geq 1$.

Assume to the contrary that $c \not\leq \rho(0)$. Then, $c^n \neq 0$ for every integer $n \geq 1$ (by Lemma 2.4(ii)). So, $(c^n)^{\perp} \neq 1$ for every $n \geq 1$. Note that the sequence $((c^n)^{\perp})_{n\geq 1}$ is increasing. Therefore, by using the compactness of 1, we have $\bigvee_{n\geq 1}(c^n)^{\perp} < 1$. Then, $\bigvee_{n\geq 1}(c^n)^{\perp} \leq m_0$ for some maximal element m_0 of A. Thus, $r(a) \leq m_0$ and $(c^k)^{\perp} \leq m_0$. Hence, $r(a) \vee (c^k)^{\perp} \leq m_0$, contradicting that $r(a) \vee (c^k)^{\perp} = 1$. Therefore, $c \leq \rho(0)$, and so we obtain the inequality $r(a) \leq \rho(0)$. The converse inequality $\rho(0) \leq r(A)$ is obvious, and hence $r(A) = \rho(0)$. □

We say that the quantale A is *semisimple* if $r(A) = 0$.

Proposition 4.3. *The quantale A is semisimple if and only if $r(A)$ is pure.*

Proof. We start with the hypothesis that $r(A)$ is pure. Assume to the contrary that $r(A) \neq 0$. Then, there exists a non-zero compact element c such that $c \leq r(A)$. So, it follows that $c^{\perp} < 1$. Hence, $c^{\perp} \leq m_0$ for some $m_0 \in Max(A)$. The inequalities $r(A) \leq m_0$ and $c^{\perp} \leq m_0$ imply that $r(A) \vee c^{\perp} \leq m_0$. On the other hand, from $c \leq r(A)$, we obtain $r(A) \vee c^{\perp} = 1$ (because $r(A)$ is pure). We have arrived at a contradiction. Therefore, $r(A) \neq 0$. The converse implication is obvious. □

Remark 4.1. Let R be a commutative ring. If we particularize the previous two results to the quantale $Id(R)$, then Proposition 3.2 of [2] is obtained. Keeping the terminology of [2], the quantales that fulfill the equivalent conditions of Proposition 4.2 are called *NJ-quantales*.

5. Characterizing some classes of coherent quantales

The results of this section highlight how the N-elements can be used in obtaining characterizations of coherent mp-quantales and coherent hyperarchimedean quantales.

From [10], we recall that a quantale A is an *mp-quantale* if for every m-prime element p of A there exists a unique minimal m-prime element q such that $q \leq p$. We note that a commutative ring A is an mp-ring [3] if and only if the quantale $Id(R)$ is an mp-quantale. The papers [10–12] contain several conditions that characterize the coherent mp-quantale. We start recalling some of these conditions.

Theorem 5.1 (see [10]). *For a coherent quantale A, the following statements are equivalent:*

(i). *A is an mp-quantale.*

(ii). *For all distinct elements $p, q \in Min(A)$, it holds that $p \vee q = 1$.*

(iii). *$R(A)$ is an mp-frame.*

(iv). *$[\rho(0))_A$ is an mp-quantale.*

Theorem 5.2 (see [11]). *For a coherent quantale A, the following statements are equivalent:*

(i). *A is an mp-quantale.*

(ii). *For all distinct elements $p, q \in Min(A)$, it holds that $O(p) \vee O(q) = 1$.*

(iii). *For any $p \in Spec(A)$, $\rho(O(p))$ is an m-prime element.*

(iv). *For any $m \in Max(A)$, $\rho(O(m))$ is an m-prime element.*

(v). *For all $p, q \in Spec(A)$, $q \leq p$ implies $\rho(O(p)) = \rho(O(q))$.*

The next theorem contains some old and new characterizations of coherent mp-quantales.

Theorem 5.3. *For a coherent quantale A, the following statements are equivalent:*

(i). *A is an mp-quantale.*

(ii). *Any $p \in Min(A)$ is w-pure.*

(iii). *For all $c, d \in K(A)$, $cd \leq \rho(0)$ implies $(c \to \rho(0)) \vee (d \to \rho(0)) = 1$.*

(iv). *For all $c, d \in K(A)$, $cd \to \rho(0) = (c \to \rho(0)) \vee (d \to \rho(0))$.*

(v). *For all $c, d \in K(A)$, $cd = 0$ implies $(c^n)^\perp \vee (d^n)^\perp = 1$ for some integer $n \geq 1$.*

(vi). *Any $p \in Min(A)$ is N-pure.*

(vii). *For any $p \in Min(A)$, $O(p)$ is N-pure.*

(viii). *For any $p \in Spec(A)$, $\rho(O(p))$ is N-pure.*

(ix). *For any $p \in Spec(A)$, $O(p)$ is N-pure.*

Proof. The equivalence of the statements **(i)–(v)** follows from Theorem 5.4 of [11].

(ii) ⇔ (vi). This is true because of Proposition 3.1.

(vi) ⇔ (vii). Assume that $p \in Min(A)$. In accordance with Lemma 4.3, we obtain $p = \rho(O(p))$. Therefore, by virtue of the equivalence $(i) \Leftrightarrow (iv)$ of Theorem 3.2, we conclude that p is N-pure if and only if $\rho(O(p))$ is N-pure, if and only if $O(p)$ is N-pure.

(i) ⇒ (viii). Assume that A is an mp-quantale. From the equivalence (i) ⇔ (iii) of Theorem 5.2, it follows that $\rho(O(p))$ is an m-prime element of A. Let q be a minimal m-prime element of A such that $q \leq p$. Using Lemma 4.3 and Theorem 5.2(5), we obtain $q = \rho(O(q))) = \rho(O(p))$. Hence, $\rho(O(p))$ is a minimal m-prime element. From the equivalence $(ii) \Leftrightarrow (vi)$, $\rho(O(p))$ is w-pure, and hence by Proposition 3.1, it is also N-pure.

(viii) ⇔ (ix). By Theorem 3.2, $\rho(O(p))$ is N-pure if and only if $O(p)$ is N-pure.

(ix) ⇒ (vii). This is obvious. □

If $p_0 < p_1 < \ldots < p_n$ is a chain of m-prime elements of the quantale A, then the integer n is *the length* of the chain. *The dimension* of A is denoted by $dim(A)$ and is defined as the supremum of the lengths of chains of m-prime elements of A. It is clear that $dim(A) = 0$ if and only if $Spec(A) = Max(A)$, if and only if $Max(A) = Min(A)$.

The next result contains several equivalent conditions that characterize the zero-dimensional coherent quantales in terms of N-elements.

Theorem 5.4. *For a coherent quantale A, the following statements are equivalent:*

(i). *$dim(A) = 0$.*

(ii). *A is hyperarchimedean.*

(iii). *Any $a \in A$ is N-pure.*

(iv). *Any $c \in K(A)$ is N-pure.*

(v). *Any maximal element of A is N-pure.*

(vi). *$\rho(O(m)) = m$ for any $m \in Max(A)$.*

Proof. (i) ⇔ (ii). It is true due to Proposition 6.2 of [10].

(ii) ⇒ (iii). Let a be an arbitrary element of A and c be a compact element of A such that $c \leq a$. Since A is hyperarchimedean, there exists an integer $n \geq 1$ such that $c^n \in B(A)$. So, $c^n \vee (c^n)^\perp = 1$. But, $c^n \leq a$. So, $a \vee (c^n)^\perp = 1$, and hence a is N-pure.

(iii) ⇒ (iv). It is obvious.

(iv) ⇒ (v). Assume that any $c \in K(A)$ is N-pure. Let m be a maximal element of A and c be a compact element of A such that $c \leq m$. Since c is N-pure and $c \leq c$, one can find an integer $k \geq 1$ such that $c \vee (c^k)^\perp = 1$, so $m \vee (c^k)^\perp = 1$. Hence, m is N-pure.

(v) ⇒ (vi). Let m be a maximal element of A. By Lemma 4.1(iii), we have $\rho(O(m)) \leq m$. In order to prove the converse inequality $m \leq \rho(O(m))$, assume that c is a compact element of A such that $c \leq m$. Then, $m \vee (c^k)^\perp = 1$ for some integer $k \geq 1$. Therefore, using Lemma 4.4, we obtain $c \leq \rho(O(m))$, and hence $\rho(O(m)) = m$.

(vi) ⇒ (i). Let m be a maximal element of A. By the hypothesis, we have $\rho(O(m)) = m$. Therefore, from Lemma 4.3, it follows that m is a minimal m-prime element. Thus, $Max(A) = Min(A)$, and hence $dim(A) = 0$. □

Corollary 5.1. *For a coherent quantale A, the following statements are equivalent:*

(i). *A is semiprime and $dim(A) = 0$.*

(ii). *A is semiprime and hyperarchimedean.*

(iii). *Any $a \in A$ is pure.*

(iv). *Any $c \in K(A)$ is pure.*

(v). *Any maximal element of A is pure.*

(vi). *$O(m) = m$ for any $m \in Max(A)$.*

Proof. (i) ⇔ (ii). It holds by Theorem 5.4.

(ii) ⇒ (iii). By Theorem 5.4, any $a \in A$ is N-pure. Since A is semiprime, any N-pure element of A is pure.

(iii) ⇒ (iv). It holds obviously.

(iv) ⇒ (v). Let m be a maximal element of A and c be a compact element with property that $c \leq m$. By the hypothesis, c is pure. So, $c \leq c$ implies $c \vee c^\perp = 1$. Thus, $m \vee c^\perp = 1$, and hence m is a pure element of A.

(v) ⇒ (vi). Let m be a maximal element of A. In order to show that $O(m) = m$, it suffices to check that $m \leq O(m)$. Assume that c is a compact element such that $c \leq m$. Then, $m \vee c^\perp = 1$ (because m is pure). So, $c^\perp \not\leq m$. Therefore, using Lemma 4.1(i), we obtain $c \leq O(m)$. Hence, the inequality $m \leq O(m)$ is proven.

(vi) ⇒ (i). By the given hypothesis, we have $\rho(O(m)) = \rho(m) = m$ for any $m \in Max(A)$. Therefore, from the equivalence (i) ⇔ (vi) of Theorem 5.4, it follows that the quantale A is zero-dimensional; that is,

$$Max(A) = Spec(A) = Min(A).$$

So, we obtain $\rho(0) = r(A)$. Using the hypothesis that $O(m) = m$ for any $m \in Max(A)$, we obtain

$$\rho(0) = r(A) = \bigwedge\{m \mid m \in Max(A)\} = \bigwedge\{O(m) \mid m \in Max(A)\}.$$

Assume to the contrary that $\rho(0) \neq 0$. Then, there exists $c \in K(A)$ such that

$$0 < c \leq \bigwedge\{O(m) \mid m \in Max(A)\}.$$

So, $c \leq O(m)$ for any $m \in Max(A)$. From Lemma 4.1(i), it follows that $c^\perp \not\leq m$ for any $m \in Max(A)$. On the other hand, $0 < c$ implies $c^\perp \neq 1$. Hence, $c^\perp \leq m_0$ for some $m_0 \in Max(A)$. We have obtained a contradiction. So, $\rho(0) = 0$. Therefore, we conclude that A is semiprime and $dim(A) = 0$. □

Remark 5.1. Let R be a commutative ring. If we apply Theorem 5.4 to the particular quantale $Id(R)$, then we obtain Theorem 3.3 of [2]. Meanwhile, our Corollary 5.1 generalizes Corollary 3.4 of [2] (this last result contains some conditions that describe the commutative von Neumann regular rings).

6. Mid coherent quantales

Let R be a unital commutative ring. Recall from [2] that R is said to be a *mid ring* if for any $a \in A$, the annihilator $Ann(a)$ of a is an N-pure ideal.

Lemma 6.1. *The following statements are equivalent:*

(i). *R is a mid ring.*

(ii). *For any finitely generated ideal I of R, $Ann(I)$ is an N-pure ideal of R.*

Proof. **(i) ⇒ (ii).** Assume that the ideal I is generated by the set $\{a_1, \dots, a_n\}$. So, $I = \sum_{i=1}^{n} < a_i >$, where $< a_i >$ is the ideal generated by $\{a_i\}$ for $i = 1, \dots, n$. Then,

$$Ann(I) = Ann\left(\sum_{i=1}^{n} < a_i >\right) = \bigcap_{i=1}^{n} Ann(a_i)$$

and $Ann(a_i)$ is N-pure for every $i = 1, \dots, n$. According to Theorem 2.8(ii) of [2], $Ann(I)$ is an N-pure ideal.

(ii) ⇒ (i). It holds obviously. □

Taking into account Lemma 6.1, we define the notion of mid quantale.

Definition 6.1. *A quantale A is said to be a mid quantale if for any $c \in K(A)$, $c^\perp$ is an N-pure element of A.*

Remark 6.1. A quantale A is a mid quantale if and only if for all $c, d \in K(A)$, $d \leq c^\perp$ implies $c^\perp \vee (d^n)^\perp = 1$ for some integer $n \geq 1$, if and only if for all $c, d \in K(A)$, $d \leq c^\perp$ implies $c^\perp \vee (d \to \rho(0)) = 1$.

According to [12], a quantale A is a *PF-quantale* if $c^\perp$ is a pure element for any $c \in K(A)$. We know from [10] that a coherent A is a PF-quantale if and only if it is a semiprime mp-quantale. It is clear that any PF-quantale is a mid quantale.

Recall from [14] that an element $p < 1$ of a quantale A is a *primary element* of A if for all $c, d \in K(A)$, $cd \leq p$ implies $c \leq p$ or $d \leq \rho(p)$.

Lemma 6.2. *If p is a minimal m-prime element of a coherent quantale A, then $O(p)$ is a primary element.*

Proof. Assume that p is a minimal m-prime element of the coherent quantale A. Let c, d, be two compact elements of A such that $cd \leq O(p)$ and $c \not\leq O(p)$. In accordance with Lemma 4.1(i), we have $(cd)^\perp \not\leq p$ and $c^\perp \leq p$. Then, there exists $e \in K(A)$ such that $ecd = 0$ and $e \not\leq p$. Hence, $ed \leq c^\perp \leq p$. So, we obtain $d \leq p$ (because $p \in Spec(A)$ and $e \not\leq p$). By the hypothesis, p is a minimal m-prime element, so $p = \rho(O(m))$ (cf. Lemma 4.3). Therefore, $d \leq \rho(O(p))$, and so $O(p)$ is a primary element. □

Theorem 6.1. *For a coherent quantale A, the following statements are equivalent:*

(i). *A is a mid quantale.*

(ii). *For all $c, d \in K(A)$ such that $cd = 0$ there exists an integer $n \geq 1$ such that $c^\perp \vee (d^n)^\perp = 1$.*

(iii). *For any $p \in Spec(A)$, $O(p)$ is pure.*

(iv). *For any $p \in Min(A)$, $O(p)$ is pure.*

(v). *For all $p, q \in Spec(A)$, $p \leq q$ implies $O(p) = O(q)$.*

(vi). *For any $p \in Spec(A)$, $O(p)$ is a primary element.*

(vii). *For any $m \in Max(A)$, $O(m)$ is a primary element.*

Proof. **(i) ⇔ (ii).** It holds by Remark 6.1.

(ii) ⇒ (iii). Let p be an m-prime element of A. We must prove that the following statement is true:

$$\forall\, c \in K(A)\ [c \leq O(p) \Rightarrow c^\perp \vee O(p) = 1].$$

Assume that $c \in K(A)$ and $c \leq O(p)$. Then, $c^\perp \not\leq p$ (cf. Lemma 4.1(i)). So, there exists $e \in K(A)$ such that $e \leq c^\perp$ and $e \not\leq p$. It follows that $ec = 0$, and hence there exists an integer $n \geq 1$ such that $c^\perp \vee (e^n)^\perp = 1$ (by the given hypothesis).

We have to show that $(e^n)^\perp \leq O(p)$. Let d be a compact element of A such that $d \leq (e^n)^\perp$. Then, $de^n = 0$ and so we obtain $e^n \leq d^\perp$. From $e \not\leq p$ we obtain $e^n \not\leq p$, and hence $d^\perp \not\leq p$. From Lemma 4.1(i), it follows that $d \leq O(p)$. Hence, the inequality $(e^n)^\perp \leq O(p)$ follows.

Therefore, $1 = c^\perp \vee (e^n)^\perp \leq c^\perp \vee O(p)$, and hence $c^\perp \vee O(p) = 1$. Consequently, we conclude that $O(p)$ is pure.

(iii) ⇒ (iv). It holds obviously.

(iv) ⇒ (v). Assume that $p, q \in Spec(A)$ and $p \leq q$. Then, $r \leq p \leq q \leq m$ for some $r \in Min(A)$ and $m \in Max(A)$. Note that $r \leq m$ implies $O(m) \leq O(r)$. In order to prove that $O(r) \leq O(m)$, assume that $d \in K(A)$ and $d \leq O(r)$. Since $O(r)$ is assumed to be pure, we obtain $d^{\perp} \vee O(r) = 1$. But, on the other hand, we have $O(r) \leq r \leq m$, and hence $d^{\perp} \vee m = 1$. Since $m \in Max(A)$, we have $d^{\perp} \nleq m$, and so $d \leq O(m)$ (by Lemma 4.1(i)). We have proved that $O(r) \leq O(m)$, and hence $O(r) = O(m)$. Therefore, we conclude that $O(p) = O(q)$.

(v) ⇒ (vi). Let p be an m-prime element of A. One can find an $r \in Min(A)$ such that $r \leq p$. By Lemma 6.2, $O(r)$ is a primary element of A. According to the given hypothesis, we have $O(r) = O(p)$, and hence $O(p)$ is a primary element of A.

(vi) ⇒ (vii). It holds obviously.

(vii) ⇒ (ii). Let c, d, be two compact elements of A such that $cd = 0$. Assume to the contrary that $c^{\perp} \vee (d^n)^{\perp} < 1$ for every integer $n \geq 1$. Note that the sequence $(c^{\perp} \vee (d^n)^{\perp})_{n \geq 1}$ is increasing. Hence, using the fact that $1 \in K(A)$, we obtain

$$\bigvee_{n \geq 1} (c^{\perp} \vee (d^n)^{\perp}) < 1.$$

Then, we have

$$\bigvee_{n \geq 1} (c^{\perp} \vee (d^n)^{\perp}) \leq m$$

for some maximal element m. Hence, $c^{\perp} \leq m$ and $(d^n)^{\perp} \leq m$ for every integer $n \geq 1$. By virtue of Lemma 4.1(i), we obtain $c \nleq O(m)$ and $d^n \nleq O(m)$ for every integer $n \geq 1$. Hence, $c \nleq O(m)$ and $d \nleq \rho(O(m))$, contradicting that $O(m)$ is a primary element. So, it follows that there exists an integer $n \geq 1$ such that $c^{\perp} \vee (d^n)^{\perp} = 1$. □

A quantale A is said to be a *primary quantale* if 0 is a primary element. Hence, A is primary if and only if for all $c, d \in K(A)$, $cd = 0$ implies $c = 0$ or $d \leq \rho(0)$.

Corollary 6.1. *Any primary coherent quantale A is a mid quantale.*

Proof. Let c, d, be two compact elements of A such that $cd = 0$. So, $c = 0$ or $d \leq \rho(0)$. If $c = 0$ then $c^{\perp} = 1$, and hence $c^{\perp} \vee d^{\perp} = 1$. If $d \leq \rho(0)$ then $d^n = 0$ for some integer $n \geq 1$. Thus, $(d^n)^{\perp} = 1$, and hence $c^{\perp} \vee (d^n)^{\perp} = 1$. Consequently, the property (ii) of Theorem 6.1 is fulfilled, and hence A is a mid quantale. □

Corollary 6.2. *Any mid coherent quantale A is an mp-quantale.*

Proof. Assume that $p \in Min(A)$. By Lemma 4.3, we have $\rho(O(p)) = p$. According to Theorem 6.1(iv), $O(p)$ is a pure element and hence $O(p)$ is N-pure. Therefore, $O(p)$ is N-pure for any minimal m-prime element of A. By virtue of the equivalence (i) ⇔ (vi) of Theorem 5.3, A is an mp-quantale. □

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