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Research Article

Bounds of the forgotten topological index and some Hamiltonian properties of graphs

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Abstract

The forgotten topological index of a graph is defined as the sum of the cubes of the degrees of vertices in the graph. In this article, an upper bound and a lower bound for the forgotten topological index of graphs are established. Using the ideas of obtaining these bounds, sufficient conditions (based on the forgotten topological index) for some Hamiltonian properties of graphs are also presented.

Keywords: forgotten topological index; Hamiltonian graph; traceable graph.

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1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [2]. Let $G = (V(G), E(G))$ be a graph with n vertices and e edges. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ (or $d(v)$ when there is no confusion regarding the graph under consideration). We use δ and Δ to denote the minimum degree and maximum degree of G, respectively. A set of vertices in a graph G is independent if the vertices in the set are pairwise nonadjacent. A maximum independent set in a graph G is an independent set of the largest possible size. The independence number of a graph G is denoted by $\beta(G)$ and is defined as the cardinality of a maximum independent set in G. For disjoint vertex subsets X and Y of $V(G)$, we use $E(X, Y)$ to denote the set of all those edges in $E(G)$ whose one end vertex belongs to X and the end vertex belongs to Y; namely, $E(X, Y) := \{ f : f = xy \in E(G), x \in X, y \in Y \}$. A bipartite graph G is balanced if the two partite sets of G have the same size. A bipartite graph G is said to be a balanced regular bipartite graph if G is balanced and all the vertices of G have the same degree. A cycle C in a graph G is said to be a Hamiltonian cycle of G if C contains all the vertices of G. A graph containing a Hamiltonian cycle is called a Hamiltonian graph. A path P in a graph G is said to be a Hamiltonian path of G if P contains all the vertices of G. A graph containing a Hamiltonian path is known as a traceable graph.

The forgotten topological index of a graph was introduced by Furtula and Gutman in [4]. For a graph G, its forgotten topological index is denoted by $F(G)$ and is defined as $\sum_{u \in V(G)} d^3(u)$. Recently, several sufficient conditions (based on different topological indices) for the Hamiltonian properties of graphs have been obtained; for example, see [1, 5–14]. Motivated by the results reported in the aforementioned references, we in the present article present an upper bound and a lower bound for the forgotten topological index of a graph. Using the ideas of obtaining these bounds, we then present sufficient conditions (based on the forgotten topological index) for Hamiltonian and traceable graphs. The main result of the current article is as follows.

Theorem 1.1. *Let G be a graph with n vertices, e edges, and the independence number β.*

(i). *If the maximum degree of G is Δ, then*

$$F(G) \leq \Delta^2(4n\Delta - 3\Delta\beta - 3e)$$

with equality if and only if G is a balanced regular bipartite graph.

(ii). *If the minimum degree of G is δ, then*

$$F(G) \geq \delta^2(n\delta + 3\delta\beta - 3e)$$

with equality if and only if G is a balanced regular bipartite graph.

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Using the ideas of proving Theorem 1.1, we prove the following two theorems:

Theorem 1.2. *Let G be a k-connected graph with n vertices and e edges, where $k \geq 2$ and $n \geq 3$.*

(i). *If the maximum degree of G is Δ and*
$$F(G) \geq \Delta^2(4n\Delta - 3\Delta(k+1) - 3e),$$
then G is Hamiltonian.

(ii). *If the minimum degree of G is δ and*
$$F(G) \leq \delta^2(n\delta + 3\delta(k+1) - 3e),$$
then G is Hamiltonian.

Theorem 1.3. *Let G be a k-connected graph with n vertices and e edges, where $n \geq 9$ and $k \geq 1$.*

(i). *If the maximum degree of G is Δ and*
$$F(G) \geq \Delta^2(4n\Delta - 3\Delta(k+2) - 3e),$$
then G is traceable.

(ii). *If the minimum degree of G is δ and*
$$F(G) \leq \delta^2(n\delta + 3\delta(k+2) - 3e),$$
then G is traceable.

2. Lemmas

In this section, we recall some known results, which we use to prove our results.

Lemma 2.1 (see [3]). *Let G be a k-connected graph of order n (with $n \geq 3$) and the independence number β. If $\beta \leq k$, then G is Hamiltonian.*

Lemma 2.2 (see [3]). *Let G be a k-connected graph of order n and the independence number β. If $\beta \leq k+1$, then G is traceable.*

Lemma 2.3 (see [15]). *Let G be a balanced bipartite graph of order $2n$ with bipartition (A, B). If*
$$d(x) + d(y) \geq n + 1$$
for any $x \in A$ and any $y \in B$ with $xy \notin E(G)$, then G is Hamiltonian.

3. Proofs of Theorems 1.1, 1.2, and 1.3

Proof of Theorem 1.1. Let G be a graph with n vertices and e edges. Let $I := \{ u_1, u_2, \ldots, u_\beta \}$ be a maximum independent set in G. Then
$$\sum_{u \in I} d(u) = |E(I, V-I)| \leq \sum_{v \in V-I} d(v).$$
Since
$$\sum_{u \in I} d(u) + \sum_{v \in V-I} d(v) = 2e,$$
we have that
$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V-I} d(v).$$
(i). Let v be any vertex in $V - I$. From the inequality $(\Delta - d(v))^3 \geq 0$, it follows that
$$\Delta^3 - 3\Delta^2 d(v) + 3\Delta d^2(v) - d^3(v) \geq 0$$
and
$$\Delta^3 + 3\Delta d^2(v) \geq 3\Delta^2 d(v) + d^3(v).$$

Thus, we obtain

$$\sum_{v\in V-I}(\Delta^3+3\Delta d^2(v)) \geq \sum_{v\in V-I}(3\Delta^2 d(v)+d^3(v))$$

and hence

$$(n-\beta)\Delta^3+3(n-\beta)\Delta\,\Delta^2 \geq 3\Delta^2 e+\sum_{v\in V-I} d^3(v).$$

Consequently, we have

$$\sum_{v\in V-I} d^3(v) \leq 4(n-\beta)\Delta^3-3\Delta^2 e.$$

Therefore,

$$\begin{aligned} F(G) &= \sum_{w\in V} d^3(w) \\ &= \sum_{u\in I} d^3(u)+\sum_{v\in V-I} d^3(v) \\ &\leq \beta\Delta^3+4(n-\beta)\Delta^3-3\Delta^2 e \\ &= \Delta^2(4n\Delta-3\Delta\beta-3e). \end{aligned}$$

If $F(G)=\Delta^2(4n\Delta-3\Delta\beta-3e)$, then by the above discussion, we obtain

$$\sum_{v\in V-I} d(v)=e,$$

which implies that

$$\sum_{u\in I} d(u)=e$$

and G is a bipartite graph with partite sets I and $V-I$ such that $|I|=\beta$. Furthermore, we have $d(u)=\Delta$ for every $u\in I$ and $d(v)=\Delta$ for every $v\in V-I$. Thus,

$$\Delta\beta=|E(I,V-I)|=(n-\beta)\Delta \quad \text{and} \quad \beta=\frac{n}{2}$$

Hence, G is a balanced regular bipartite graph.

On the other hand, if G is a balanced regular bipartite graph, then simple computations yield

$$F(G)=\Delta^2(4n\Delta-3\Delta\beta-3e).$$

This completes the proof of part **(i)** of Theorem 1.1.

(ii). Let u be any vertex in I. From the inequality $(d(u)-\delta)^3\geq 0$, we obtain

$$d^3(u)-3d^2(u)\delta+3\delta^2 d(u)-\delta^3\geq 0$$

and

$$\delta^3+3d^2(u)\delta \leq 3\delta^2 d(u)+d^3(u).$$

Thus, it holds that

$$\sum_{u\in I}(\delta^3+3d^2(u)\delta) \leq \sum_{u\in I}(3\delta^2 d(u)+d^3(u)).$$

Therefore, we obtain

$$\beta\delta^3+3\beta\delta^2\,\delta \leq 3\delta^2 e+\sum_{u\in I} d^3(u)$$

and hence

$$\sum_{u\in I} d^3(u) \geq 4\beta\delta^3-3\delta^2 e.$$

Consequently, we have

$$\begin{aligned} F(G) &= \sum_{w\in V} d^3(w) \\ &= \sum_{u\in I} d^3(u) + \sum_{v\in V-I} d^3(v) \\ &\geq 4\beta\delta^3 - 3\delta^2 e + (n-\beta)\delta^3 \\ &= \delta^2(n\delta + 3\delta\beta - 3e). \end{aligned}$$

If $F(G) = \delta^2(n\delta + 3\delta\beta - 3e)$, then from the proof of the above inequality, we have

$$\sum_{u\in I} d(u) = e,$$

which implies that

$$\sum_{v\in V-I} d(v) = e$$

and G is a bipartite graph with partite sets I and $V-I$ such that $|I| = \beta$. Furthermore, we have $d(u) = \delta$ for every $u \in I$ and $d(v) = \delta$ for every $v \in V - I$. Thus,

$$\delta\beta = |E(I, V-I)| = (n-\beta)\delta \quad \text{and} \quad \beta = \frac{n}{2}.$$

Hence, G is a balanced regular bipartite graph.

On the other hand, if G is a balanced regular bipartite graph, then after simple computations, we have

$$F(G) = \delta^2(n\delta + 3\delta\beta - 3e).$$

This completes the proof of part **(ii)** of Theorem 1.1. □

Proof of Theorem 1.2. Let G be a k-connected graph with n vertices and e edges, where $n \geq 3$ and $k \geq 2$. Suppose to the contrary that G is not Hamiltonian. Then, Lemma 2.1 implies that $\beta \geq k+1$. Let I be an independent set of size $(k+1)$ in G. Following the proof of Theorem 1.1, we have

$$\sum_{u\in I} d(u) + \sum_{v\in V-I} d(v) = 2e$$

and

$$\sum_{u\in I} d(u) \leq e \leq \sum_{v\in V-I} d(v).$$

(i). Following the proof of Theorem 1.1(i), we obtain

$$F(G) \leq \Delta^2(4n\Delta - 3\Delta(k+1) - 3e).$$

Since

$$F(G) \geq \Delta^2(4n\Delta - 3\Delta(k+1) - 3e),$$

we have that

$$F(G) = \Delta^2(4n\Delta - 3\Delta(k+1) - 3e).$$

Following again the proof of Theorem 1.1(i), we conclude that G is a bipartite graph with partite sets I and $V-I$ such that $|I| = k+1$, $d(u) = \Delta$ for every $u \in I$, and $d(v) = \Delta$ for every vertex v in $V-I$. Since

$$\Delta|I| = |E(I, V-I)| = \Delta(|V| - |I|),$$

we have $|I| = \frac{n}{2}$. Now, by Lemma 2.3, the graph G is Hamiltonian, which is a contradiction. This completes the proof of part bf (i) of Theorem 1.2.

(ii). Following the proof of Theorem 1.1(ii), we obtain

$$F(G) \leq \delta^2(n\delta + 3\delta(k+1) - 3e).$$

Since

$$F(G) \geq \delta^2(n\delta + 3\delta(k+1) - 3e),$$

it holds that

$$F(G) = \delta^2(n\delta + 3\delta(k+1) - 3e).$$

Following again the proof of Theorem 1.1(ii), we conclude that G is a bipartite graph with partite sets I and $V - I$ such that $|I| = k + 1$, $d(u) = \delta$ for every $u \in I$, and $d(v) = \delta$ for every vertex v in $V - I$. Since

$$\delta|I| = |E(I, V - I)| = \delta(|V| - |I|),$$

we have $|I| = \frac{n}{2}$. Hence, by Lemma 2.3, the graph G is Hamiltonian, which is a contradiction. This completes the proof of part **(ii)** of Theorem 1.2. □

The proof of Theorem 1.3 is similar to the proof of Theorem 1.2. However, for the sake of completeness, we present the complete proof of Theorem 1.3.

Proof of Theorem 1.3. Let G be a k-connected graph with n vertices and e edges, where $k \geq 1$ and $n \geq 9$. Suppose to the contrary that G is not traceable. Then, Lemma 2.2 implies that $\beta \geq k + 2$. Let I be an independent set of size $(k + 2)$ in G. Following the proof of Theorem 1.1, we have

$$\sum_{u \in I} d(u) + \sum_{v \in V - I} d(v) = 2e$$

and

$$\sum_{u \in I} d(u) \leq e \leq \sum_{v \in V - I} d(v).$$

(i). Following the proof of Theorem 1.1(i), we derive the following inequality:

$$F(G) \leq \Delta^2(4n\Delta - 3\Delta(k+2) - 3e).$$

Since

$$F(G) \geq \Delta^2(4n\Delta - 3\Delta(k+2) - 3e),$$

it holds that

$$F(G) = \Delta^2(4n\Delta - 3\Delta(k+2) - 3e).$$

Following again the proof of Theorem 1.1(i), we conclude that G is a bipartite graph with partite sets I and $V - I$ such that $|I| = k + 2$, $d(u) = \Delta$ for every $u \in I$, and $d(v) = \Delta$ for every vertex v in $V - I$. Since $\Delta|I| = |E(I, V - I)| = \Delta(|V| - |I|)$, we have $|I| = \frac{n}{2}$. Since $n \geq 9$, we have $k \geq 3$. Now, Lemma 2.3 confirms that G is Hamiltonian and thereby G is traceable, which is a contradiction. This completes the proof of part **(i)** of Theorem 1.3.

(ii). Following the proof of Theorem 1.1(ii), obtain

$$F(G) \leq \delta^2(n\delta + 3\delta(k+2) - 3e).$$

Since

$$F(G) \geq \delta^2(n\delta + 3\delta(k+2) - 3e),$$

it holds that

$$F(G) = \delta^2(n\delta + 3\delta(k+2) - 3e).$$

Following again the proof of Theorem 1.1(ii), we conclude that G is a bipartite graph with partite sets I and $V - I$ such that $|I| = k + 2$, $d(u) = \delta$ for every $u \in I$, and $d(v) = \delta$ for every vertex v in $V - I$. Since

$$\delta|I| = |E(I, V - I)| = \delta(|V| - |I|),$$

we have $|I| = \frac{n}{2}$. Also, since $n \geq 9$, we have $k \geq 3$. Now, by Lemma 2.3, the graph G is Hamiltonian. Hence, G is traceable, which is a contradiction. This completes the proof of part **(ii)** of Theorem 1.3. □

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