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Research Article

Bounds on the Euler Sombor index of maximal outerplanar graphs

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Abstract

In the recent paper [*Int. J. Quantum Chem.* **124** (2024) #e27387], Tang, Li, and Deng proposed a novel vertex-degree-based topological index, namely the Euler Sombor index. This index for a graph G is denoted by $ES(G)$ and is defined as $ES(G) = \sum_{uv\in E(G)} \sqrt{d^2(u) + d^2(v) + d(u)d(v)}$, where $d(u)$ is the degree of the vertex u in G and $E(G)$ is the edge set of G. In the present paper, we obtain sharp lower and upper bounds on the Euler Sombor index of maximal outerplanar graphs with a fixed number of vertices. We also characterize the graphs attaining these bounds.

Keywords: maximal outerplanar graph; vertex-degree-based topological index; Euler Sombor index.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C92.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of G are called its order and size, respectively. The set consisting of all neighbors of v in G is denoted by $N_G(v)$ (or $N(v)$). The cardinality of $N_G(v)$ is called the degree of v and is denoted by $d_G(v)$ (or $d(v)$); i.e., $d_G(v) = |N_G(v)|$. The subgraph of G obtained by deleting its vertex v and the edges incident to v is denoted by $G - v$.

An outerplanar graph is a planar graph that has a planar embedding such that all of its vertices lie on the boundary of the unbounded face. An outerplanar graph is said to be maximal if its outerplanar property is affected by adding an edge between two nonadjacent vertices. Consequently, it follows that a maximal outerplanar graph of order n has $2n - 3$ edges and at least one vertex with degree 2. The structure and properties of maximal outerplanar graphs have been studied in several publications, see for example [6, 7, 9, 10, 16, 17].

In 2021, Gutman [1] introduced a new topological index, namely the Sombor index, from a new perspective of geometry, and obtained its basic properties. The Sombor index of a graph G is defined as

$$SO(G) = \sum_{uv\in E(G)} \sqrt{d^2(u) + d^2(v)}.$$

Based on this index, many vertex-degree-based topological indices defined from a geometric perspective have been proposed and extensively studied by scholars, see for example [2, 4, 5]. From the perspective of the elliptical area, Gutman, Furtula, and Oz [4] recently proposed the elliptic Sombor index, which is defined as follows:

$$ESO(G) = \sum_{uv\in E(G)} (d(u) + d(v))\sqrt{d^2(u) + d^2(v)}.$$

In [4], the authors established some bounds on the elliptic Sombor index and determined the correlation between this index and other topological indices. The extremal value problems of the elliptic Sombor index have attracted considerable attention from scholars and many results on this topic have been obtained [8, 11–13, 15].

Recently, the Euler Sombor index based on the perimeter of the ellipse was proposed [3, 14], which is defined as

$$ES(G) = \sum_{uv\in E(G)} \sqrt{d^2(u) + d^2(v) + d(u)d(v)}.$$

In [14], the chemical applicability and mathematical properties of the Euler Sombor index were studied. The extremal values for the Euler Sombor index and the corresponding extremal graphs among all (molecular) trees were also determined in [14]. Gutman [3] established the relationship between the Euler Sombor index and the Sombor index.

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In this paper, we continue to study the extremal value problems of the Euler Sombor index. Particularly, we determine the graphs attaining the extremal values of the Euler Sombor index over the class of maximal outerplanar graphs with a fixed order.

2. Preliminaries

Denote by P_n^2 the graph obtained from the paths $P_a = v_1v_2\cdots v_a$ and $P_b = u_1u_2\cdots u_b$, where $a+b=n$ and $0 \leq b-a \leq 1$, by adding new edges v_iu_i and v_ju_{j+1} for

- $i = 1,2,\ldots,a$ and $j = 1,2,\ldots,a-1$ when $a=b$,
- $i = 1,2,\ldots,a$ and $j = 1,2,\ldots,a$ when $a=b-1$.

The graph P_n^2 is shown in Figure 2.1. Let Q_n be the set of maximal outerplanar graphs of order n whose every member G satisfies the following two properties:

(i) there is a vertex v in $V(G)$ such that $d(v)=2$ and $N(v)=\{u,w\}$,

(ii) for every vertex $x \in V(G)\backslash\{u,w\}$, either $x \in N(u)$ or $x \in N(w)$.

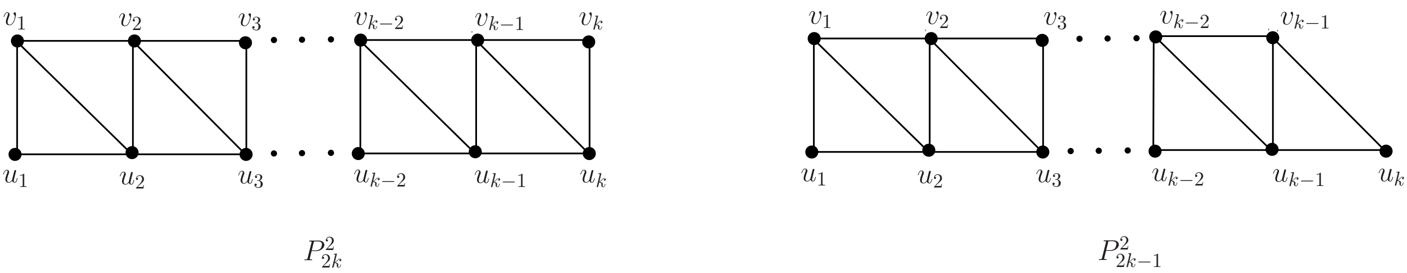

Figure 2.1: The graphs P_{2k}^2 and P_{2k-1}^2.

For two vertex disjoint graphs G_1 and G_2, the join of G_1 and G_2, denoted by $G_1 \vee G_2$, refers to the graph formed by adding edges between every vertex of G_1 and every vertex of G_2. Clearly, $K_1 \vee P_{n-1} \in Q_n$.

Lemma 2.1 (see [7]). *Let G be a maximal outerplanar graph and v be a vertex in G with $d(v)=2$ and $N(v)=\{u,w\}$. Then $uv \in E(G)$.*

Lemma 2.2 (see [7]). *Let G be a maximal outerplanar graph with order $n \geq 4$ and v be a vertex in G with $d(v)=2$ and $N(v)=\{u,w\}$. Then $|N(u)\cap N(w)|=2$.*

Lemma 2.3 (see [7]). *Let G be a maximal outerplanar graph with order $n \geq 5$ and v be a vertex in G with $d(v)=2$ and $N(v)=\{u,w\}$. Then $7 \leq d(u)+d(w) \leq n+2$, where the left and right equalities hold if and only if $G[N(u)\cup N(w)] \cong P_5^2$ and $G \in Q_n$, respectively.*

Lemma 2.4. *Let $f(x,y) = \sqrt{4+x^2+2x} + \sqrt{4+y^2+2y} + \sqrt{x^2+y^2+xy} - (x+y-2)\sqrt{(x-1)^2+(y-1)^2+(x-1)(y-1)}$. Then $f(x,y)$ is strictly increasing for $x,y \geq 2$.*

Proof. Since $x^2+2x+xy-4[(x-1)^2+(y-1)^2+(x-1)(y-1)] \leq 0$, we have

$$\begin{aligned} f_x(x,y) &= \frac{x+1}{\sqrt{x^2+2x+4}} - \frac{2x+y-3}{2\sqrt{(x-1)^2+(y-1)^2+(x-1)(y-1)}} + \frac{2x+y}{2\sqrt{x^2+y^2+xy}} \\ &\geq \frac{2x+y}{\sqrt{x^2+y^2+xy}} - \frac{2x+y-3}{2\sqrt{(x-1)^2+(y-1)^2+(x-1)(y-1)}}, \\ &> 0. \end{aligned}$$

Thus, the function $f(x,y)$ is strictly increasing for $x \geq 2$. Since $f(x,y)$ is a symmetric function, the function $f(x,y)$ is also strictly increasing for $y \geq 2$. □

Lemma 2.5. *Let* $g(x,y) = \sqrt{x^2+y^2+xy} - \sqrt{(x-1)^2+y^2+(x-1)y}$. *Then* $g(x,y)$ *is strictly increasing for* $x \geq 2$ *and strictly decreasing for* $y \geq 2$.

Proof. Since

$$(2x+y)^2[(x-1)^2+y^2+(x-1)y] - (2x+y-2)^2[x^2+y^2+xy] = 3y^2(2x+y-1) > 0,$$

we have

$$g_x(x,y) = \frac{2x+y}{2\sqrt{x^2+y^2+xy}} - \frac{2x+y-2}{2\sqrt{(x-1)^2+y^2+(x-1)y}} > 0.$$

Thus, $g(x,y)$ is strictly increasing for $x \geq 2$. Since

$$(x+2y)^2[(x-1)^2+y^2+(x-1)y] - (2x+y-1)^2[x^2+y^2+xy] = (1-x)(3y^2+3xy) - 3xy^2 < 0,$$

we have

$$g_y(x,y) = \frac{x+2y}{2\sqrt{x^2+y^2+xy}} - \frac{2x+y-1}{2\sqrt{(x-1)^2+y^2+(x-1)y}} < 0.$$

Thus, $g(x,y)$ is strictly decreasing for $y \geq 2$. □

Lemma 2.6. *Let*

$$h(x) = \sqrt{4+x^2+2x} + \frac{1}{2}(\sqrt{x^2+a^2+ax} + \sqrt{(x-1)^2+(a-1)^2+(a-1)(x-1)}) + (x-4)g(x,3) + g(x,2) + g(x,4),$$

where a *is a constant. Then* $h(x)$ *is strictly convex for* $x \geq 4$.

Proof. We note that

$$\begin{aligned} h''(x) = {} & \frac{5}{(x^2+2x+4)^{\frac{3}{2}}} + \frac{0.375a^2}{(x^2+ax+a^2)^{\frac{3}{2}}} + \frac{0.375(a-1)^2}{[(x-1)^2+(a-1)(x-1)+(a-1)^2]^{\frac{3}{2}}} \\ & + \frac{2x^3+9x^2+33.75x}{(x^2+3x+9)^{\frac{3}{2}}} - \frac{2x^3+x^2+19.75x-34}{(x^2+x+7)^{\frac{3}{2}}} \\ & - \frac{4}{(x^2+3)^{\frac{3}{2}}} - \frac{12}{(x^2+2x+13)^{\frac{3}{2}}} - \frac{15}{(x^2+2x+16)^{\frac{3}{2}}} > 0. \end{aligned}$$

Thus, $h(x)$ is a strictly convex function for $x \geq 4$. □

Lemma 2.7. *Let* $p(x)$ *be a strictly convex symmetric continuous function on* $[a,b]$ *and* $P(x,y) = p(x) + p(y)$. *Then*

$$P(x,y) \leq P(a, s-a)$$

for $x + y = s$.

Proof. Let $x, y \in [a,b]$, $a \leq x' < x \leq y < y' \leq b$ and $x - x' = y' - y$. Then there exists, $\lambda = \frac{x-x'}{y-x'} = \frac{y'-y}{y'-x} \in (0,1]$, such that

$$y = \lambda x + (1-\lambda)y' \quad \text{and} \quad x = \lambda y + (1-\lambda)x'.$$

Since $p(x)$ is strictly convex on $[a,b]$, we have

$$p(y) = p(\lambda x + (1-\lambda)y') < \lambda p(x) + (1-\lambda)p(y') \tag{1}$$

and

$$p(x) = p(\lambda y + (1-\lambda)x') < \lambda p(y) + (1-\lambda)p(x'). \tag{2}$$

By adding (1) and (2), we have

$$p(x) + p(y) < \lambda[p(x) + p(y)] + (1 - \lambda)[p(x') + p(y')],$$

that is,

$$p(x) + p(y) < p(x') + p(y'),$$

which implies that

$$P(x, y) < P(x', y').$$

Thus, we have

$$P(x, y) \le P(a, s - a)$$

for $x + y = s$. □

By Lemmas 2.6 and 2.7, we obtain the following corollary:

Corollary 2.1. *Let $\phi(x, y) = h(x) + h(y)$ and $x + y \in [4, n + 2]$. Then*

$$\phi(x, y) \le \phi(4, n - 2)$$

with equality if and only if $x = 4, y = n - 2$.

3. Main results

Theorem 3.1. *Let G be a maximal outerplanar graph with order $n \ge 6$. Then*

$$ES(G) \ge 4(2n - 11)\sqrt{3} + 4\sqrt{7} + 2\sqrt{19} + 4\sqrt{37}$$

with equality if and only if $G \cong P_n^2$.

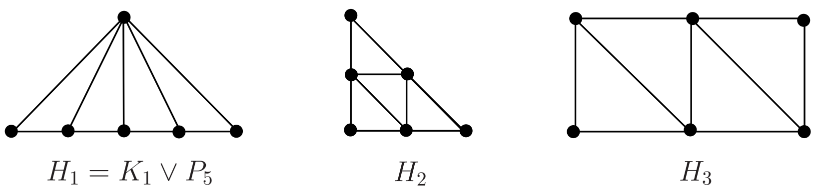

Figure 3.1: Non-isomorphic maximal outerplanar graphs with order 6.

Proof. We prove $ES(G) \ge ES(P_n^2)$ by induction on $n \ge 6$. If $n = 6$, then G is one of the graphs H_1, H_2, and H_3, shown in Figure 3.1. By direct calculations, we have

$$ES(H_1) = 6\sqrt{3} + 2\sqrt{19} + 2\sqrt{39} + 21 \approx 52.60,$$

$$ES(H_2) = 12\sqrt{3} + 12\sqrt{7} \approx 52.53, \text{ and}$$

$$ES(H_3) = ES(P_6^2) = 4\sqrt{3} + 4\sqrt{7} + 2\sqrt{19} + 4\sqrt{37} \approx 50.56.$$

Clearly,

$$ES(G) \ge ES(P_6^2).$$

Assume that the result holds for all maximal outerplanar graphs with order $n - 1 \ge 6$. Let G be a maximal outerplanar graph of order n with the minimum Euler Sombor index. There must be a vertex v with degree 2 in G. Let $N_G(v) = \{u, w\}$ and $d(u) \le d(w)$. By Lemmas 2.1 and 2.2, $uw \in E(G)$ and there is a vertex $p \in N(u) \cap N(w)$. By Lemma 2.3, we have $d(u) \ge 3, d(w) \ge 4$ and there is a vertex $q \in N(w) \backslash \{N(u) \cap N(w)\}$.

Let $G' = G - v$. Then $d_{G'}(u) = d(u) - 1, d_{G'}(w) = d(w) - 1$, and the degree of the other vertices remains unchanged. Let $N_0(u) = N(u)\backslash\{v, w\}$ and $N_0(w) = N(w)\backslash\{v, u\}$. For $d(u) \geq 3$ and $d(w) \geq 4$, by Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
ES(G) = ES(G') &+ \sqrt{d^2(v) + d^2(u) + d(v)d(u)} + \sqrt{d^2(v) + d^2(w) + d(v)d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{d^2_{G'}(u) + d^2_{G'}(w) + d^2_{G'}(u)d^2_{G'}(w)} \\
&+ \sum_{u_i \in N_0(u)} \left(\sqrt{d^2(u) + d^2(u_i) + d(u)d(u_i)} - \sqrt{d^2_{G'}(u) + d^2_{G'}(u_i) + d_{G'}(u)d_{G'}(u_i)} \right) \\
&+ \sum_{w_j \in N_0(w)} \left(\sqrt{d^2(w) + d^2(w_j) + d(w)d(w_j)} - \sqrt{d^2_{G'}(w) + d^2_{G'}(w_j) + d_{G'}(w)d_{G'}(w_j)} \right) \\
= ES(G') &+ \sqrt{2^2 + d^2(u) + 2d(u)} + \sqrt{2^2 + d^2(w) + 2d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{(d(u)-1)^2 + (d(w)-1)^2 + (d(u)-1)(d(w)-1)} \\
&+ \sum_{u_i \in N_0(u)} \left(\sqrt{d^2(u) + d^2(u_i) + d(u)d(u_i)} - \sqrt{(d(u)-1)^2 + d^2(u_i) + (d(u)-1)d(u_i)} \right) \\
&+ \sum_{w_j \in N_0(w)} \left(\sqrt{d^2(w) + d^2(w_j) + d(w)d(w_j)} - \sqrt{(d(w)-1)^2 + d^2(w_j) + (d(w)-1)d(w_j)} \right) \\
\geq ES(G') &+ \sqrt{4 + d^2(u) + 2d(u)} + \sqrt{4 + d^2(w) + 2d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{(d(u)-1)^2 + (d(w)-1)^2 + (d(u)-1)(d(w)-1)} \\
&+ \sqrt{d^2(u) + d^2(p) + d(u)d(p)} - \sqrt{(d(u)-1)^2 + d^2(p) + (d(u)-1)d(p)} \\
&+ \sqrt{d^2(w) + d^2(p) + d(w)d(p)} - \sqrt{(d(w)-1)^2 + d^2(p) + (d(w)-1)d(p)} \\
&+ \sqrt{d^2(w) + d^2(q) + d(w)d(q)} - \sqrt{(d(w)-1)^2 + d^2(q) + (d(w)-1)d(q)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
ES(G) - ES(G') &\geq f(d(u), d(w)) + g(d(u), d(p)) + g(d(w), d(p)) + g(d(w), d(q)) \\
&\geq f(3, 4) + g(3, d(p)) + g(4, d(p)) + g(4, d(q))
\end{aligned}$$

with equality if and only if $d(u) = 3$ and $d(w) = 4$. By inductive hypothesis, we have

$$ES(G') \geq 4(2n - 13)\sqrt{3} + 4\sqrt{7} + 2\sqrt{19} + 4\sqrt{37}$$

with equality holds if and only if $G' \cong P^2_{n-1}$, $G[N(u) \cup N(w)] \cong P^2_5$, and $d(p) = d(q) = 4$. Consequently, we have

$$\begin{aligned}
ES(G) &\geq ES(P^2_{n-1}) + f(3, 4) + g(3, 4) + g(4, 4) + g(4, 4) \\
&= 4(2n - 11)\sqrt{3} + 4\sqrt{7} + 2\sqrt{19} + 4\sqrt{37}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let G be a maximal outerplanar graph with order $n \geq 4$. Then*

$$ES(G) \leq 3(n - 4)\sqrt{3} + 2\sqrt{19} + 2\sqrt{n^2 + 3} + (n - 3)\sqrt{n^2 + n + 7}$$

with equality if and only if $G \cong K_1 \vee P_{n-1}$.

Proof. We prove $ES(G) \leq ES(K_1 \vee P_{n-1})$ by induction on $n \geq 4$. If $n = 4$, then there is only one maximum outerplanar graph $K_1 \vee P_3$. Thus, the result holds for $n = 4$. Assume that the result holds for all maximal outerplanar graphs with order $n - 1 \geq 4$. Let G be a maximal outerplanar graph of order n with the maximum Euler Sombor index. There must be a vertex v with degree 2 in G. Let $N_G(v) = \{u, w\}$ and $d(u) \leq d(w)$. By Lemmas 2.1 and 2.2, $uw \in E(G)$ and there is a vertex $p \in N(u) \cap N(w)$.

Let $G^* = G - v$. Let $N_0(u) = N(u) \backslash \{v, w\} = \{u_1, u_2, \ldots, u_s, p\}$ and $N_0(w) = N(w) \backslash \{v, u\} = \{w_1, w_2, \ldots, w_t, p\}$. Also, let $s = d(u) - 3$ and $t = d(w) - 3$. For $d(u) \geq 3$ and $d(w) \geq 4$, by Lemma 2.5, we have

$$\begin{aligned}
ES(G) = ES(G^*) &+ \sqrt{d^2(v) + d^2(u) + d(v)d(u)} + \sqrt{d^2(v) + d^2(w) + d(v)d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{d^2_{G^*}(u) + d^2_{G^*}(w) + d^2_{G^*}(u)d^2_{G^*}(w)} \\
&+ \sum_{u_i \in N_0(u)} \left(\sqrt{d^2(u) + d^2(u_i) + d(u)d(u_i)} - \sqrt{d^2_{G^*}(u) + d^2_{G^*}(u_i) + d_{G^*}(u)d_{G^*}(u_i)} \right) \\
&+ \sum_{w_j \in N_0(w)} \left(\sqrt{d^2(w) + d^2(w_j) + d(w)d(w_j)} - \sqrt{d^2_{G^*}(w) + d^2_{G^*}(w_j) + d_{G^*}(w)d_{G^*}(w_j)} \right) \\
= ES(G^*) &+ \sqrt{2^2 + d^2(u) + 2d(u)} + \sqrt{2^2 + d^2(w) + 2d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{(d(u)-1)^2 + (d(w)-1)^2 + (d(u)-1)(d(w)-1)} \\
&+ \sum_{u_i \in N_0(u)} \left(\sqrt{d^2(u) + d^2(u_i) + d(u)d(u_i)} - \sqrt{(d(u)-1)^2 + d^2(u_i) + (d(u)-1)d(u_i)} \right) \\
&+ \sum_{w_j \in N_0(w)} \left(\sqrt{d^2(w) + d^2(w_j) + d(w)d(w_j)} - \sqrt{(d(w)-1)^2 + d^2(w_j) + (d(w)-1)d(w_j)} \right) \\
= ES(G^*) &+ \sqrt{2^2 + d^2(u) + 2d(u)} + \sqrt{2^2 + d^2(w) + 2d(w)} \\
&+ \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{(d(u)-1)^2 + (d(w)-1)^2 + (d(u)-1)(d(w)-1)} \\
&+ \sum_{u_i \in N_0(u)} g(d(u), d(u_i)) + \sum_{w_j \in N_0(w)} g(d(w), d(w_i)).
\end{aligned}$$

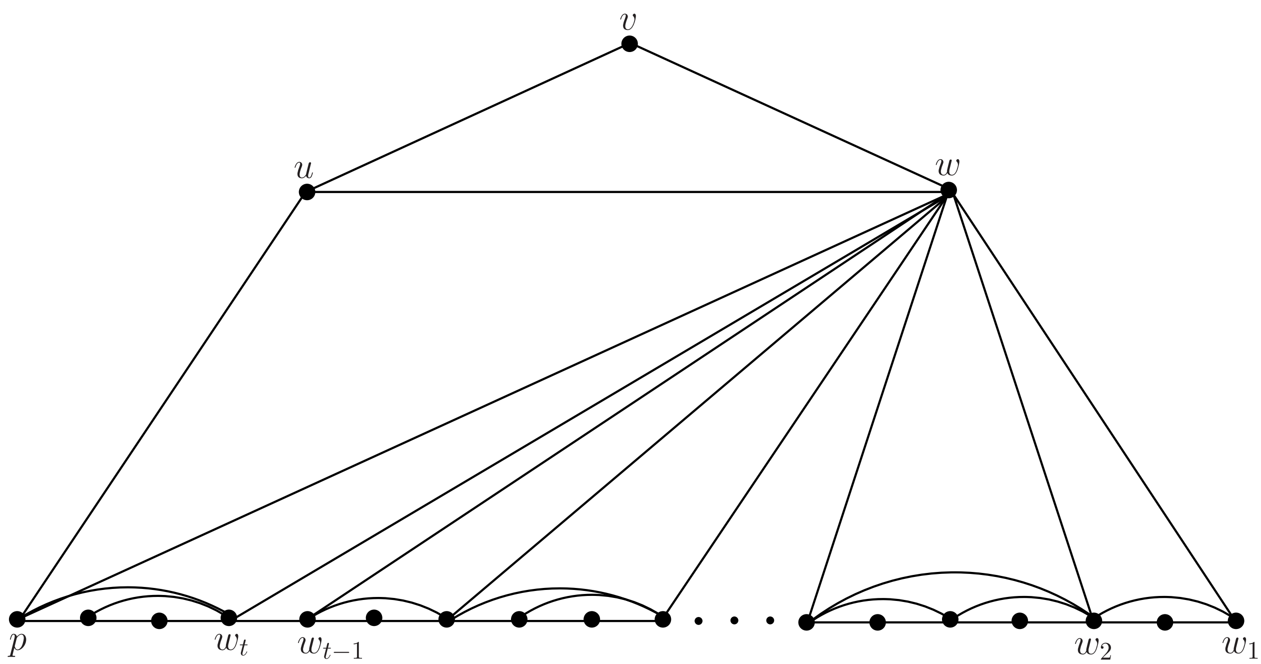

Figure 3.2: The maximal outerplanar graph with order n and $d(u) = 3$.

Next, we consider two cases in terms of the degree $d(u)$.

Case 1. $d(u) = 3$.
In this case, we have $N_0(u) = \{p\}$, see Figure 3.2. By Lemma 2.5, we obtain

$$\begin{aligned} ES(G) &= ES(G^*) + \sqrt{19} + \sqrt{4 + d^2(w) + 2d(w)} + \sqrt{9 + d^2(w) + 3d(w)} \\ &\quad - \sqrt{4 + (d(w) - 1)^2 + 2(d(w) - 1)} + g(3, d(p)) + \sum_{w_j \in N_0(w)} g(d(w), d(w_i)) \\ &\leq ES(G^*) + \sqrt{19} + \sqrt{4 + d^2(w) + 2d(w)} + \sqrt{9 + d^2(w) + 3d(w)} \\ &\quad - \sqrt{4 + (d(w) - 1)^2 + 2(d(w) - 1)} + g(3, 3) + g(d(w), 2) + (d(w) - 3)g(d(w), 3) \\ &= ES(G^*) + \sqrt{19} + 3\sqrt{3} - 5 + \sqrt{9 + d^2(w) + 3d(w)} + 2g(d(w), 2) + (d(w) - 3)g(d(w), 3), \end{aligned}$$

with equality holds if and only if $d(p) = 3$, $d(w_1) = 2$, and $d(w_j) = 3$, where $j \in \{2, 3, \ldots, t\}$. Let

$$\varphi(x) = \sqrt{9 + x^2 + 3x} + 2g(x, 2) + (x - 3)g(x, 3).$$

Then $\varphi(x)$ is strictly increasing for $x \geq 2$. By the inductive hypothesis and $4 \leq d(w) \leq n - 1$, we have

$$\begin{aligned} ES(G) &\leq ES(G^*) + \varphi(n - 1) + \sqrt{19} + 3\sqrt{3} - 5 \\ &\leq ES(K_1 \vee P_{n-2}) + \sqrt{19} + 3\sqrt{3} - 5 + \sqrt{9 + (n - 1)^2 + 3(n - 1)} \\ &\quad + 2g(n - 1, 2) + (n - 1 - 3)g(n - 1, 3) \\ &= 3(n - 4)\sqrt{3} + 2\sqrt{19} + 2\sqrt{n^2 + 3} + (n - 3)\sqrt{n^2 + n + 7}, \end{aligned}$$

the equality holds if and only if $G^* \cong K_1 \vee P_{n-2}$ and $d(u) = 3, d(w) = n - 1$, which means $G \cong K_1 \vee P_{n-1}$.

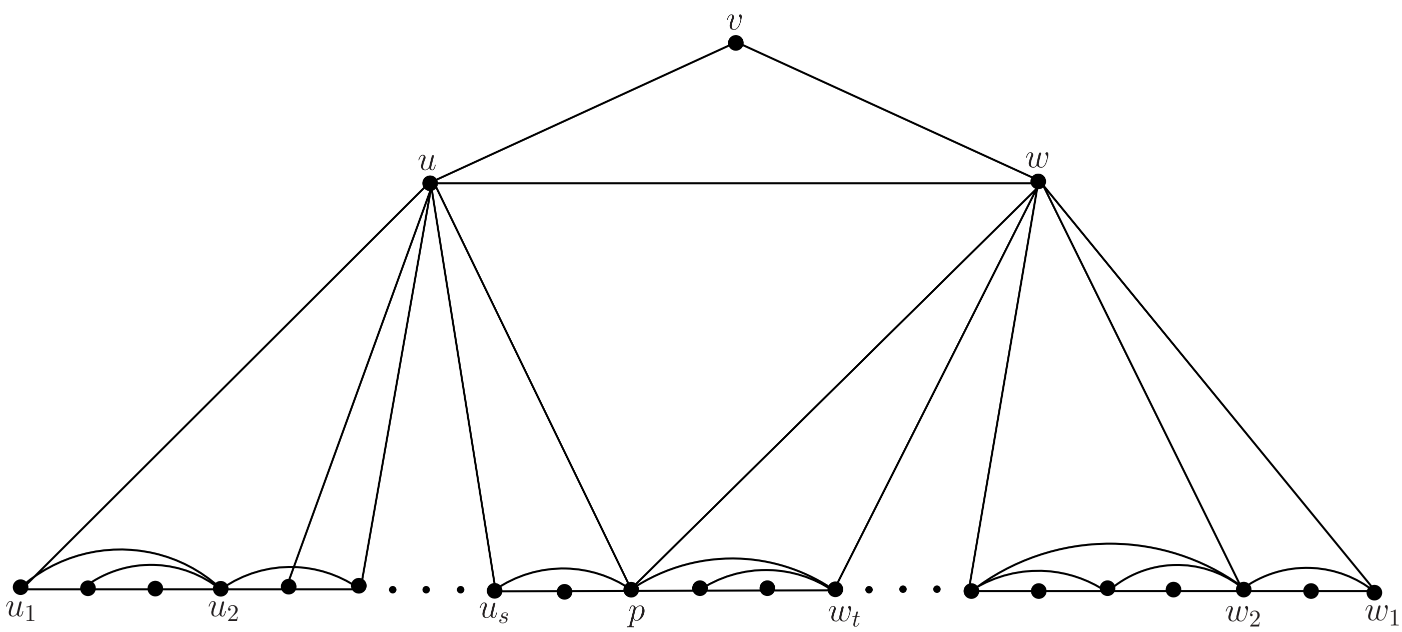

Figure 3.3: The maximal outerplanar graph with order n and $d(u) \geq 4$.

Case 2. $d(u) \geq 4$.
In this case, the graph G is illustrated in Figure 3.3. By Lemma 2.5, we have

$$\begin{aligned} ES(G) &\leq ES(G^*) + \sqrt{4 + d^2(u) + 2d(u)} + \sqrt{4 + d^2(w) + 2d(w)} \\ &\quad + \sqrt{d^2(u) + d^2(w) + d(u)d(w)} - \sqrt{(d(u)-1)^2 + (d(w)-1)^2 + (d(u)-1)(d(w)-1)} \\ &\quad + (d(u)-4)g(d(u),3) + g(d(u),2) + g(d(u),4) \\ &\quad + (d(w)-4)g(d(w),3) + g(d(w),2) + g(d(w),4) \\ &= ES(G^*) + h(d(u)) + h(d(w)) \end{aligned}$$

with equality if and only if $d(p) = 4$, $d(u_1) = d(w_1) = 2$, and $d(u_i) = d(w_j) = 3$, where $i \in 2, 3, \ldots, s$ and $j \in \{2, 3, \ldots, t\}$. By Lemma 2.3 and Corollary 2.1, we have

$$ES(G) \leq ES(G^*) + \phi(d(u), d(w)) \leq ES(G^*) + \phi(4, n-2),$$

where the equality holds if and only if $d(u) = 4$ and $d(w) = n - 2$. Since

$$\begin{aligned} \varphi(n-1) - \phi(4, n-2) &= \sqrt{9 + (n-1)^2 + 3(n-1)} + 2g(n-1, 2) + (n-1-3)g(n-1, 3) \\ &\quad - 2\sqrt{7} - \sqrt{4 + (n-2)^2 - 2(n-2)} + \sqrt{16 + (n-2)^2 + 4(n-2)} \\ &\quad - \sqrt{9 + (n-3)^2 + 3(n-3)} \\ &\geq \sqrt{9 + (n-1)^2 + 3(n-1)} + \sqrt{16 + (n-2)^2 + 4(n-2)} \\ &\quad - \sqrt{4 + (n-2)^2 - 2(n-2)} - \sqrt{9 + (n-3)^2 + 3(n-3)} \\ &> 0, \end{aligned}$$

by the inductive hypothesis, we have

$$\begin{aligned} ES(G) &\leq ES(G^*) + \phi(4, n-2) \\ &\leq ES(K_1 \vee P_{n-2}) + \phi(4, n-2) \\ &\leq ES(K_1 \vee P_{n-2}) + \varphi(n-1) \\ &\leq ES(K_1 \vee P_{n-1}). \end{aligned}$$

By combining the above arguments, we have

$$ES(G) \leq 3(n-4)\sqrt{3} + 2\sqrt{19} + 2\sqrt{n^2+3} + (n-3)\sqrt{n^2+n+7}$$

with equality if and only if $G \cong K_1 \vee P_{n-1}$. This completes the proof of the theorem. □

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