Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **8** (2024) 19–30
DOI: 10.47443/ejm.2024.035

Research Article

Frullani formula and distributional integration

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(Received: 29 June 2024. Received in revised form: 1 August 2024. Accepted: 1 August 2024. Published online: 6 August 2024.)

Abstract

The Frullani integral formula is considered for the distributional integral. The existence of the Frullani integral is proven to be equivalent to the existence of the distributional point value at zero and Cesàro limit at infinity. Connections to finite parts and Cesàro summability are drawn. Applications and illustrations are also given.

Keywords: Frullani integrals; distributional integral; distributional point values.

2020 Mathematics Subject Classification: 26A03, 40A10, 46F10.

1. Introduction

A simple but interesting result for the evaluation of integrals is Frullani's formula [5, 6]:

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\, dx = C \ln\left(\frac{a}{b}\right) , \tag{1}$$

where $a > 0$ and $b > 0$. Formula (1) holds for the continuous functions in $(0, \infty)$ that have limits at both 0 and ∞. Here

$$C = f(\infty) - f\left(0^+\right) . \tag{2}$$

We shall refer to the integral in this formula as the *Frullani Integral* and the formula itself as the *Frullani integral formula*. This formula was communicated by Frullani to Plana in 1821 and was eventually published in 1828 [14]. Cauchy also published a version in 1823. For the formula to hold true, it is not necessary for the function to have a limit at infinity or at zero. With an appropriate interpretation of the value of the function at infinity and at zero, one can show that it holds for a larger class of functions.

In 1940, Iyengar [15] obtained the necessary and sufficient conditions for the existence of the Frullani integral in (1), under the assumption that $f : (0, \infty) \to \mathbb{R}$ is Lebesgue integrable over every finite closed interval. He proved that if the integral exists for all values of a/b in an interval $[\rho_1, \rho_2]$ with $0 < \rho_1 < \rho_2$, then it exists for every $a, b > 0$ and the equality follows. He proved, albeit erroneously [22], that the existence of the Frullani integral is equivalent to the existence of the following four expressions:

$$\int_{0^+}^1 f(x)\, dx = \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 f(x)\, dx , \qquad m_* = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{0^+}^\varepsilon f(x)\, dx , \tag{3}$$

$$\int_1^\infty \frac{f(x)}{x^2}\, dx = \lim_{\lambda \to \infty} \int_1^\lambda \frac{f(x)}{x^2}\, dx , \qquad M_* = \lim_{\lambda \to \infty} \lambda \int_\lambda^\infty \frac{f(x)}{x^2}\, dx . \tag{4}$$

In this case

$$C = M_* - m_* . \tag{5}$$

A correct proof was later given by Agnew [1, 2] and then by Ostrowski [21, 22], who showed that the improper Frullani integrals exist for all $a, b > 0$ if and only if the two mean values

$$m = \lim_{\varepsilon \to 0^+} \varepsilon \int_\varepsilon^1 \frac{f(x)}{x^2}\, dx , \tag{6}$$

$$M = \lim_{\lambda \to \infty} \frac{1}{\lambda} \int_1^\lambda f(x)\, dx , \tag{7}$$

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both exist, in which case $m_* = m$ and $M_* = M$. Actually if the Frullani integrals exist for all $a, b > 0$ for *any* integral, not just for the Lebesgue integral, Arias de Reyna [4] shows that (1) holds for some constant C; for the Denjoy-Perron-Henstock-Kurzweil integral he proves that C is a distributional point value; actually the arguments of Ostrowski show that $C = M - m$ for this integral as well. It should be clear that the Lebesgue integral is not the right framework to study these problems since the integrals in (1) are not absolutely convergent, in general; integrals that allow one to integrate highly oscillatory functions, like the Denjoy-Perron-Henstock-Kurzweil integral or the more powerful distributional integral introduced recently [12] seem better suited to understand the situation.

Recently, Ortner and Wagner [20] employed a distributional framework to rigorously obtain several integrals that arise in Mathematical Physics, and placed many Frullani integral formulas and finite part evaluations in this context.

The aim of the present article is to establish that for the *distributional integral* there is a direct equivalence between the existence of the Frullani integrals and the existence of distributional limits at the origin and Cesàro limits at infinity. If f is distributionally integrable in $[0, \infty)$, then the distributional integral

$$\int_0^1 \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x \, , \tag{8}$$

exists for all $a, b > 0$ if and only if the distributional limit

$$f(0^+) \quad (\mathfrak{dist}) \, , \tag{9}$$

exists. We prove a corresponding result at infinity for Cesàro integrals, namely

$$\int_1^\infty \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x \quad (\mathrm{C}) \, , \tag{10}$$

the Cesàro limit of the distributional integrals $\int_1^\lambda (f(ax) - f(bx))/x \, \mathrm{d}x$ as $\lambda \to \infty$, exists for every $a, b > 0$ if and only if the Cesàro limit

$$f(\infty) = \lim_{\lambda \to \infty} f(x) \quad (\mathrm{C}) \, , \tag{11}$$

exists. If both (9) and (11) exist then Frullani's formula (1) holds with $C = f(\infty) - f(0^+)$ as the original formula (2), except that now $f(0^+)$ is a *distributional limit* and $f(\infty)$ is a *Cesàro limit*. A quick review of the distributional integral, distributional limits and Cesàro integrals and limits is given in Section 2.

Actually, we are able to prove a very general form of this equivalence, valid for distributions in $(0, \infty)$ whether those distributions are "regular" in the sense that they are evaluations with respect to some integral or not. We show that if f is a distribution of $\mathcal{D}'(0, \infty)$ then all primitives of

$$\frac{\mathsf{f}(ax) - \mathsf{f}(bx)}{x} \, , \tag{12}$$

have a distributional limit as $x \to 0^+$ for every $a, b > 0$ if and only if $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists. We also show that all primitives have a Cesàro limit at infinity for every $a, b > 0$ if and only if the Cesàro limit at infinity $\mathsf{f}(\infty)$ (C) exists. If the limits exist at both 0^+ and at ∞ then for any primitive $\mathsf{F}_{a,b}$ formula (1) becomes

$$\lim_{x \to 0^+} \mathsf{F}_{a,b}(x) - \lim_{x \to \infty} \mathsf{F}_{a,b}(x) = \left(\mathsf{f}(\infty) - \mathsf{f}\left(0^+\right)\right) \ln\left(\frac{a}{b}\right) \, . \tag{13}$$

Here $\lim_{x \to 0^+} \mathsf{F}_{a,b}(x)$ is a distributional limit and $\lim_{x \to \infty} \mathsf{F}_{a,b}(x)$ is a Cesàro limit.

The plan of the article is as follows. Section 2 gives a summary of ideas from the theory of distributions needed in our analysis, including the concept of "regular" distributions. In Section 3 we give a general equivalence result, that holds for all distributions –whether they are regular or not– for the existence of distributional Frullani-type limits. Next, in Section 4 we apply the general results to the distributional integral, proving the announced equivalence of the existence of the Frullani integrals and the existence of the distributional limit at zero and the corresponding results in the Cesàro sense at infinity. We show that the ideas are related to finite parts in Section 5 and then give applications to sequences in Section 6. The question of whether it is enough to assume the existence of the integrals for some pairs (a, b) is treated in Section 7 while the final section gives several examples.

2. Distributions, functions, and distributional integration

Basic ideas from the theory of distributions and details about the topological vector space structure of the spaces of test functions and distributions can be found in the textbooks [3, 16, 25, 29]; for the local behavior of distributions we refer to [11, 13, 23, 24, 30]. Our distributions will be from the space $\mathcal{D}'(0, \infty)$, dual of the space of test functions $\mathcal{D}(0, \infty)$. The space $\mathcal{D}(0,\infty)$ consists of those smooth functions defined in $(0,\infty)$ with compact support, that is, whose support is contained in an interval of the form $[a,b]$ where $0<a<b$. The space $\mathcal{E}(0,\infty)$ consists of all smooth functions in $(0,\infty)$ and its dual $\mathcal{E}'(0,\infty)$ is the subspace of $\mathcal{D}'(0,\infty)$ formed by those distributions with compact support, that is, with support contained in a closed interval $[c,d]$ for some $0<c<d$.

Since we would like to separate the analysis at 0 and at infinity, it is sometimes convenient to work with distributions whose support is bounded on the right. In other cases, we would need to consider distributions $f\in\mathcal{D}'(0,\infty)$ whose support is bounded on the left, that is, is contained in an interval of the form $[c,\infty)$ for some $c>0$.[†]

Distributions will be denoted as $\mathsf{f},\mathsf{g},\mathsf{h}$, and so on, while functions will be denoted as f,g,h, etc. Several distributions f are *regular* [9] in the sense that they correspond to a measurable function f, $\mathsf{f}\leftrightarrow f$, such that

$$\langle \mathsf{f}(x),\phi(x)\rangle=\int_0^\infty f(x)\,\phi(x)\;\mathrm{d}x\,, \tag{14}$$

when $\phi\in\mathcal{D}(0,\infty)$. If the integral in (14) is the Lebesgue integral, the term is just 'regular', but one can also employ Denjoy-Perron-Henstock-Kurzweil integrable functions [19] or even the more general distributionally integrable functions [12]. Since the tradition gives the plain term regular to those distributions obtained from Lebesgue integrable functions, we will call those distributions given by (14) with f distributionally integrable $\mathfrak{dist}$-regular, while the term *DPHK*-regular could be used if f is Denjoy-Perron-Henstock-Kurzweil integrable. Independently of the integral method used, the function f needs to be locally integrable in the open interval $(0,\infty)$ which means that f is integrable in every closed interval $[a,b]$ for $0<a<b$. If f is locally Denjoy-Perron-Henstock-Kurzweil integrable in the open interval $(0,\infty)$ and the limit of the integrals exists at 0^+ then it is Denjoy-Perron-Henstock-Kurzweil integrable in every closed interval $[0,b]$ for $0<b$ and similarly if it is locally distributionally integrable in the open interval $(0,\infty)$ and the distributional limit of the integrals exists at 0^+ then the function is distributionally integrable in every closed interval $[0,b]$ for $0<b$; corresponding results hold at infinity. However, there are *improper* Lebesgue integrals at 0^+ or at ∞.

If f is $\mathfrak{dist}$-regular then the corresponding measurable function f is related to f through the distributional point values, since the values $\mathsf{f}(x)$ $(\mathfrak{dist})$ exist almost everywhere in $(0,\infty)$ and actually [12]

$$f(x)=\mathsf{f}(x)\quad \text{(a.e.)}\ . \tag{15}$$

Distributional point values were introduced by Łojasiewicz [18] as follows. The point value at the point x_0 is

$$\mathsf{f}(x_0)=\lim_{\varepsilon\to 0}\mathsf{f}(x_0+\varepsilon x)\,, \tag{16}$$

if the limit exists in $\mathcal{D}'(\mathbb{R})$, that is, if $\lim_{\varepsilon\to0}\langle\mathsf{f}(x_0+\varepsilon x),\phi(x)\rangle=\mathsf{f}(x_0)\int_{-\infty}^{\infty}\phi(x)\,\mathrm{d}x$, for each $\phi\in\mathcal{D}(\mathbb{R})$. The existence of the distributional point value $\mathsf{f}(x_0)=\gamma$ is equivalent to the existence of $n\in\mathbb{N}$, and a primitive of order n of f, that is $\mathsf{F}^{(n)}=\mathsf{f}$, which corresponds, near x_0, to a continuous function F that satisfies $\lim_{x\to x_0}n!F(x)\,(x-x_0)^{-n}=\gamma$ .One can also define point values by using the operator

$$\partial_{x_0}(\mathsf{f})=((x-x_0)\,\mathsf{f}(x))'\ , \tag{17}$$

since $\mathsf{f}_1(x_0)=\gamma$ if and only if $\mathsf{f}(x_0)=\gamma$, where $\mathsf{f}=\partial_{x_0}(\mathsf{f}_1)$. Therefore [13] f has a distributional value equal to γ at $x=x_0$ if and only if there exists $n\in\mathbb{N}$ and a function f_n, continuous at $x=x_0$, with $f_n(x_0)=\gamma$, such that $\mathsf{f}=\partial_{x_0}^n(\mathsf{f}_n)$, where $f_n\leftrightarrow\mathsf{f}_n$. We also say that the distributional lateral value $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists if $\mathsf{f}(0^+)=\lim_{\varepsilon\to0^+}\mathsf{f}(\varepsilon x)$ in $\mathcal{D}'(0,\infty)$, that is,

$$\lim_{\varepsilon\to0^+}\langle\mathsf{f}(\varepsilon x),\phi(x)\rangle=\mathsf{f}(0^+)\int_0^\infty\phi(x)\,\mathrm{d}x\quad,\quad\phi\in\mathcal{D}(0,\infty)\,. \tag{18}$$

Lateral limits can also be characterized by primitives and by the operator $\partial_{x_0}=\partial$ in the same fashion. When $\mathsf{f}\leftrightarrow f$ and $\mathsf{f}(x_0)$ $(\mathfrak{dist})$ exists we will also say that the distributional point value of f exists, and denote it also as $f(x_0)$ $(\mathfrak{dist})$.

The well-known Cesàro summability behavior of functions is extended to distributions [8,11] by using the order symbols $O(x^\alpha)$ and $o(x^\alpha)$ in the Cesàro sense. If $\mathsf{f}\in\mathcal{D}'(\mathbb{R})$ and $\alpha\in\mathbb{R}\setminus\{-1,-2,-3,...\}$, we say that $\mathsf{f}(x)=O(x^\alpha)$ as $x\to\infty$ in the Cesàro sense and write

$$\mathsf{f}(x)=O(x^\alpha)\ \ (\mathrm{C})\,,\ \text{as}\ x\to\infty\,, \tag{19}$$

if there exists $N\in\mathbb{N}$ such that every primitive F of order N, i.e., $\mathsf{F}^{(N)}=\mathsf{f}$, corresponds for large arguments to a locally integrable function, $\mathsf{F}\leftrightarrow F$, that satisfies the ordinary order relation $F(x)=p(x)+O\left(x^{\alpha+N}\right)$, as $x\to\infty$, for a suitable polynomial p of degree at most $N-1$. If we want to specify the value N, we write (C,N) instead of just (C). A similar

[†]In the study of integral equations in spaces of distributions [10] one needs to consider such spaces of mixed type, where one condition is satisfied at one endpoint and a different one at the other. In that reference the notation $\mathcal{D}'_{12}(0,\infty)$ is used for those elements of $\mathcal{D}'(0,\infty)$ with support bounded on the right while $\mathcal{D}'_{21}(0,\infty)$ is employed for those with support bounded on the left.

definition applies to the little o symbol. Using these ideas, one can define the limit of a distribution at ∞ in the Cesàro sense. We say that $\mathsf{f} \in \mathcal{D}'(\mathbb{R})$ has a limit L at infinity in the Cesàro sense and write $\lim_{x\to\infty} \mathsf{f}(x) = L\,(\mathrm{C})$, if $\mathsf{f}(x) = L + o(1)\,(\mathrm{C})$, as $x \to \infty$.

We will also need the following known result [12] when we make the change of variables $t = 1/x$:

Lemma 2.1. *Let* $\mathsf{f} \in \mathcal{D}'(0,\infty)$. *The distributional limit of* f *from the right at* $x = 0$ *exists and equals* γ *if and only if the Cesàro limit of* $\mathsf{f}(1/t)$ *as* $t \to \infty$ *exists and equals* γ. *A function* f *is distributionally integrable over* $[0,1]$ *if and only if* $f\left(t^{-1}\right)t^{-2}$ *is distributionally Cesàro integrable over* $[1,\infty)$ *and*

$$\int_0^1 f(x)\,\mathrm{d}x = \int_1^\infty f\left(t^{-1}\right)t^{-2}\,\mathrm{d}t \quad (\mathrm{C})\ . \tag{20}$$

3. Distributions at the origin

Let us consider the equivalence of distributional limits at the origin and the existence of a distributional form of Frullani's integral. We start with the case when $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists.

Proposition 3.1. *Suppose* $\mathsf{f} \in \mathcal{D}'(0,\infty)$. *If* $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ *exists then for all* $a, b > 0$ *any primitive of* $(\mathsf{f}(ax) - \mathsf{f}(bx))/x$ *has a distributional limit as* $x \to 0^+$. *If* $\mathsf{f} \in \mathcal{D}'(0,\infty)$ *has support bounded on the right and* $\mathsf{F}_{a,b}$ *is the primitive in* $\mathcal{D}'(0,\infty)$ *that has support bounded on the right then*

$$\mathsf{F}_{a,b}\left(0^+\right) = \mathsf{f}\left(0^+\right)\ln\left(\frac{a}{b}\right)\ .$$

Proof. For a general $\mathsf{f} \in \mathcal{D}'(0,\infty)$ we decompose it as $\mathsf{f}_1 + \mathsf{f}_2$ where $\mathsf{f}_1 \in \mathcal{D}'(0,\infty)$ has support bounded on the right and where f_2 vanishes in a neighborhood of the origin. Since any primitive of $(\mathsf{f}_2(ax) - \mathsf{f}_2(bx))/x$ is constant in a neighborhood of the origin, its limit at 0^+ exists; hence, it is enough to prove the result for f_1. Therefore, let us suppose $\mathsf{f} = \mathsf{f}_1 \in \mathcal{D}'(0,\infty)$ has support bounded on the right. Let $\phi \in \mathcal{D}(0,\infty)$ and put $\Phi(y) = \langle \mathsf{f}(yx), \phi(x)\rangle$. If $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists then Φ admits a continuous extension to $[0,\infty)$, with $\Phi(0) = \mathsf{f}(0^+)\int_0^\infty \phi(x)\,\mathrm{d}x$. On the other hand,

$$\langle \mathsf{F}_{a,b}(\varepsilon x), \phi(x)\rangle = -\int_a^b \frac{\Phi(\varepsilon x)}{x}\,\mathrm{d}x\ , \tag{21}$$

since the derivative of the left side is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\langle \mathsf{F}_{a,b}(\varepsilon x), \phi(x)\rangle\right) &= \left\langle \frac{\mathsf{f}(a\varepsilon x) - \mathsf{f}(b\varepsilon x)}{\varepsilon}, \phi(x)\right\rangle \\ &= \frac{\Phi(a\varepsilon) - \Phi(b\varepsilon)}{\varepsilon}\ ,\end{aligned}$$

while that of the right side is also

$$-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\int_a^b \frac{\Phi(\varepsilon x)}{x}\,\mathrm{d}x = -\int_a^b \Phi'(\varepsilon x)\,\mathrm{d}x = \frac{\Phi(a\varepsilon) - \Phi(b\varepsilon)}{\varepsilon}\ ,$$

and both functions in (21) vanish as $\varepsilon \to \infty$. Hence,

$$\lim_{\varepsilon\to 0^+}\langle \mathsf{F}_{a,b}(\varepsilon x), \phi(x)\rangle = -\Phi(0)\ln\left(\frac{b}{a}\right) = \mathsf{f}\left(0^+\right)\ln\left(\frac{a}{b}\right)\int_0^\infty \phi(x)\,\mathrm{d}x\ ,$$

so that $\mathsf{F}_{a,b}(0^+)$ exists and equals $\mathsf{f}(0^+)\ln(a/b)$. □

Let us now consider the converse. We need to start with a preliminary result. Suppose $\mathsf{f} \in \mathcal{D}'(0,\infty)$ has support bounded on the right. Then $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists if and only if for all $\phi \in \mathcal{D}(0,\infty)$ the function $\Phi(y) = \langle \mathsf{f}(yx), \phi(x)\rangle$ has a limit as $y \to 0^+$ in the *ordinary* sense, $\lim_{y\to 0^+}\Phi(y) = \mathsf{f}(0^+)\int_0^\infty \phi(x)\,\mathrm{d}x$. What happens if Φ has a limit in the average sense?

Lemma 3.1. *Let* $\mathsf{f} \in \mathcal{D}'(0,\infty)$ *with support bounded on the right. Suppose that for all* $\phi \in \mathcal{D}(0,\infty)$ *the function* $\Phi(y) = \langle \mathsf{f}(yx), \phi(x)\rangle$ *satisfies that*

$$\int_{0^+}^1 \Phi(y)\,\mathrm{d}y \ \textit{exists and}\ \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\int_{0^+}^{\varepsilon}\Phi(y)\,\mathrm{d}y\ \textit{exists.} \tag{22}$$

Then $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ *exists and the limit in (22) equals* $\mathsf{f}(0^+)\int_0^\infty \phi(x)\,\mathrm{d}x$.

Proof. Let us define the distribution $\mathsf{F} \in \mathcal{D}'(0,\infty)$ as

$$\langle \mathsf{F}(x), \phi(x) \rangle = \int_0^1 \langle \mathsf{f}(yx), \phi(x) \rangle \, \mathrm{d}y = \int_{0^+}^1 \Phi(y) \, \mathrm{d}y \,, \tag{23}$$

for $\phi \in \mathcal{D}(0,\infty)$. Notice that the distributional limit $\mathsf{F}(0^+)$ $(\mathfrak{dist})$ exists since

$$\lim_{\varepsilon \to 0^+} \langle \mathsf{F}(\varepsilon x), \phi(x) \rangle = \lim_{\varepsilon \to 0^+} \int_0^1 \langle \mathsf{f}(\varepsilon yx), \phi(x) \rangle \, \mathrm{d}y = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^\varepsilon \langle \mathsf{f}(yx), \phi(x) \rangle \, \mathrm{d}y \,,$$

the limit in (22), which exists for all ϕ. If we show that $\partial \mathsf{F} = \mathsf{f}$ it would follow that $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists and equals $\mathsf{F}(0^+)$ $(\mathfrak{dist})$. But we have that

$$\langle \partial \mathsf{F}(x), \phi(x) \rangle = -\langle \mathsf{F}(x), x\phi'(x) \rangle = -\int_0^1 \langle \mathsf{f}(yx), x\phi'(x) \rangle \, \mathrm{d}y = \int_0^1 \left\langle \frac{\partial}{\partial x}(x\mathsf{f}(yx)), \phi(x) \right\rangle \mathrm{d}y = \int_0^1 \frac{\partial}{\partial y} \langle y\mathsf{f}(yx), \phi(x) \rangle \, \mathrm{d}y = \langle \mathsf{f}(x), \phi(x) \rangle \,.$$

□

We can now prove the following result:

Proposition 3.2. *Suppose* $\mathsf{f} \in \mathcal{D}'(0,\infty)$. *Suppose that for all* $a, b > 0$ *any primitive of* $(\mathsf{f}(ax) - \mathsf{f}(bx))/x$ *has a distributional limit as* $x \to 0^+$. *Then* $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ *exists.*

Proof. Denote as before the primitive as $\mathsf{F}_{a,b}$. If $\phi \in \mathcal{D}(0,\infty)$ and $\Phi(y) = \langle \mathsf{f}(yx), \phi(x) \rangle$ then formula (21) holds. A simple manipulation thus gives

$$\langle \mathsf{F}_{a,b}(\varepsilon x), \phi(x) \rangle = \int_\varepsilon^\infty \frac{\Phi(ax) - \Phi(bx)}{x} \, \mathrm{d}x \,. \tag{24}$$

Since the distributional limit $\mathsf{F}_{a,b}(0^+)$ $(\mathfrak{dist})$ exists it follows that the limit of (24) when $\varepsilon \to 0^+$ exists. Iyengar equivalence implies that $\int_{0^+}^1 \Phi(t) \, \mathrm{d}t$ exists and the limit (22) also exists. Lemma 3.1 yields that $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ exists, as required. □

Corresponding results hold at infinity.

Proposition 3.3. *Let* $\mathsf{f} \in \mathcal{D}'(0,\infty)$. *The Cesàro limit* $\mathsf{f}(\infty)$ (C) *exists if and only if for all* $a, b > 0$ *any primitive of* $(\mathsf{f}(ax) - \mathsf{f}(bx))/x$ *has a Cesàro limit as* $x \to \infty$. *If* $\operatorname{supp} \mathsf{f} \subset [c,\infty)$ *for some* $c > 0$ *and* $\mathsf{G}_{a,b}$ *is the primitive with support in* $[c,\infty)$ *then*

$$\mathsf{G}_{a,b}(\infty) = \mathsf{f}(\infty) \ln\left(\frac{a}{b}\right) \,. \tag{25}$$

Proof. We just need to apply Propositions 3.1 and 3.2 and Lemma 2.1 to the distribution $\mathsf{h}(x) = \mathsf{f}(1/x)$. □

Combining the results at the origin and at infinity, we obtain the ensuing summary.

Proposition 3.4. *Let* $\mathsf{f} \in \mathcal{D}'(0,\infty)$. *The distributional limit* $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ *and the Cesàro limit* $\mathsf{f}(\infty)$ (C) *both exist if and only if for all* $a, b > 0$ *any primitive* $\mathsf{H}_{a,b}$ *of* $(\mathsf{f}(ax) - \mathsf{f}(bx))/x$ *has a distributional limit at* 0^+ *and a Cesàro limit as* $x \to \infty$. *In such a case,*

$$\mathsf{H}_{a,b}(\infty) - \mathsf{H}_{a,b}(0^+) = \left(\mathsf{f}(\infty) - \mathsf{f}(0^+)\right) \ln\left(\frac{a}{b}\right) \,. \tag{26}$$

4. Integrals

Let f be distributionally integrable in each compact of $(0,\infty)$. Then the corresponding distribution $\mathsf{f} \leftrightarrow f$, given by the formula

$$\langle \mathsf{f}(x), \phi(x) \rangle = \int_0^\infty f(x)\phi(x) \, \mathrm{d}x \,, \tag{27}$$

belongs to $\mathcal{D}'(0,\infty)$. The primitives of $(\mathsf{f}(ax) - \mathsf{f}(bx))/x$ are also regular and correspond to the functions

$$x \rightsquigarrow A + \int_1^x \frac{f(at) - f(bt)}{t} \, \mathrm{d}t \,, \tag{28}$$

for a certain constant A. If $\mathsf{f} \in \mathcal{D}'(0,\infty)$ has support bounded on the right then the primitive $\mathsf{F}_{a,b} \in \mathcal{D}'(0,\infty)$ with support bounded on the right corresponds to the Łojasiewics function $F_{a,b}$, $\mathsf{F}_{a,b} \leftrightarrow F_{a,b}$, given by

$$F_{a,b}(x) = -\int_x^\infty \frac{f(at) - f(bt)}{t}\,\mathrm{d}t\,. \tag{29}$$

The distributional limit $\mathsf{F}_{a,b}(0^+)$ $(\mathfrak{dist})$ exists when $(f(ax) - f(bx))/x$ is distributionally integrable in $[0,\infty)$ and

$$\mathsf{F}_{a,b}\left(0^+\right) = -\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x\,. \tag{30}$$

Propositions 3.1 and 3.2 thus immediately give the following equivalence.

Proposition 4.1. *Let f be distributionally integrable in each compact of $(0,\infty)$. The distributional limit $f(0^+)$ $(\mathfrak{dist})$ exists if and only if the distributional integral*

$$\int_0^1 \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x\,, \tag{31}$$

exists for all $a, b > 0$. If additionally f has support bounded on the right then

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = -f\left(0^+\right)\ln\left(\frac{a}{b}\right)\,. \tag{32}$$

Observe that the existence of $f(0^+)$ $(\mathfrak{dist})$ implies that f is distributionally integrable in $[0,c]$ for any $c > 0$. The converse is not true, of course, and thus there are functions that are distributionally integrable in $[0,c]$ for any $c > 0$ for which the integrals (31) do not exist for some a and b.

Furthermore, Proposition 3.3 yields the ensuing result.

Proposition 4.2. *Let f be distributionally integrable in each compact of $(0,\infty)$. The Cesàro limit $f(\infty)$ (C) exists if and only if for all $a, b > 0$ the integral*

$$\int_1^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x \quad (C)\,, \tag{33}$$

is Cesàro summable at infinity. If additionally $f(x) = 0$ for $x < c$ for some $c > 0$ then

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = f(\infty)\ln\left(\frac{a}{b}\right) \quad (C)\,. \tag{34}$$

If we combine the conclusions of Propositions 4.1 and 4.2, we obtain that $(f(ax) - f(bx))/x$ is Cesàro distributionally integrable over $[0,\infty)$ for all $a, b > 0$ if and only if $f(0^+)$ $(\mathfrak{dist})$ and $f(\infty)$ (C) both exist, and then Frullani's integral equals $(f(\infty) - f(0^+))\ln(a/b)$.

5. Finite parts

There is a connection between Frullani's integrals and the Hadamard finite part of the generically divergent integral $\int_0^\infty f(x)\ \mathrm{d}x/x$ (see [20]).

Suppose f has support bounded on the right. When the Hadamard finite part of the divergent integral $\int_0^\infty f(x)\ \mathrm{d}x/x$,

$$\text{F.p.}\int_0^\infty \frac{f(x)}{x}\,\mathrm{d}x = \int_0^\infty \frac{f(x) - H(1-x)f(0)}{x}\,\mathrm{d}x\,, \tag{35}$$

is a convergent integral, then the change of variables formula for finite part integrals [11],

$$\text{F.p.}\int_0^\infty \frac{f(\lambda x)}{\lambda x}\,\mathrm{d}x = \text{F.p.}\int_0^\infty \frac{f(x)}{x}\,\mathrm{d}x - f(0)\ln\lambda\,, \tag{36}$$

immediately yields Frullani's formulas, by writing a convergent integral as a difference of divergent integrals,

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = \text{F.p.}\int_0^\infty \frac{f(ax)}{ax}\,\mathrm{d}x - \text{F.p.}\int_0^\infty \frac{f(bx)}{bx}\,\mathrm{d}x\,. \tag{37}$$

However, this nice derivation cannot be applied to all cases, since there are functions for which Frullani's integrals converge but for which the Hadamard finite part (35) does not. Indeed, let $\eta \in (0,1)$ and consider the function

$$g(x) = \frac{H(\eta - x)}{\ln x}\,. \tag{38}$$

It is continuous in $[0,\infty)$ with $g(0)=0$. Consequently, Frullani's integrals

$$\int_0^\infty \left(\frac{H(\eta-ax)}{\ln ax}-\frac{H(\eta-bx)}{\ln bx}\right)\frac{\mathrm{d}x}{x}\,, \tag{39}$$

are convergent at $x=0$ for any $a,b>0$, and in fact are equal to 0.

On the other hand, the improper integral

$$\int_0^\infty \frac{g(x)}{x}\,\mathrm{d}x\,, \tag{40}$$

diverges, to $-\infty$. The Hadamard finite part

$$\mathrm{F.p.}\int_0^\infty \frac{g(x)}{x}\,\mathrm{d}x=\int_0^\infty \frac{g(x)-H(1-x)\,g(0)}{x}\,\mathrm{d}x\,, \tag{41}$$

does not exist, since it just reduces to (40).

Interestingly, we have the ensuing result, observed by Ostrowski [22] for improper Lebesgue integrals.

Proposition 5.1. *Suppose f is distributionally integrable in $(0,\infty)$. There is at most one constant c such that*

$$\int_0^1 \frac{f(x)-c}{x}\,\mathrm{d}x\,, \tag{42}$$

exists. In such a case, $f(0^+)$ $(\mathfrak{dist})$ exists and equals c. There are functions f such that $f(0^+)=c$ $(\mathfrak{dist})$ exists but with (42) divergent.

Proof. Let $g(x)=(f(x)-c)\,H(1-x)$. If $g(x)/x$ is distributionally integrable at 0, then it is valid to make changes of variables, to obtain

$$\int_0^\infty \frac{g(x)}{x}\,\mathrm{d}x=\int_0^\infty \frac{g(ax)}{x}\,\mathrm{d}x\,, \tag{43}$$

for any $a>0$. Therefore the Frullani integrals $\int_0^\infty (g(ax)-g(bx))\,\mathrm{d}x/x$ exist and vanish for all $a,b>0$. We thus conclude that $g(0^+)=0$ $(\mathfrak{dist})$ and so $f(0^+)=c$ $(\mathfrak{dist})$. □

6. Sequences

We can obtain information on the Cesàro behavior of a sequence $\{c_n\}_{n=1}^\infty$ by studying the Cesàro behavior of the associated train of deltas

$$\mathsf{f}(x)=\sum_{n=1}^\infty c_n\delta(x-n)\ . \tag{44}$$

In fact $\lim_{n\to\infty} c_n=L$ (C) if and only if $\lim_{x\to\infty}\mathsf{f}(x)=L$ (C), see [11]. Also

$$\sum_{n=1}^\infty c_n=S\ \ (\mathrm{C})$$

if and only if $\langle \mathsf{f}(x),1\rangle=S$ (C). Let us observe that if $a\geq b>0$ then

$$\left\langle \frac{\mathsf{f}(ax)-\mathsf{f}(bx)}{x},1\right\rangle=\lim_{x\to\infty}\sum_{bx\leq n<ax}\frac{c_n}{n}\,, \tag{45}$$

as ordinary evaluation and limit or as both in the Cesàro sense. The equivalence result of Proposition 3.3 immediately gives us the ensuing result.

Proposition 6.1. *If $\{c_n\}_{n=1}^\infty$ is a sequence of complex numbers then*

$$\lim_{n\to\infty} c_n=L\quad (C)\ , \tag{46}$$

if and only if

$$\lim_{x\to\infty}\sum_{bx\leq n<ax}\frac{c_n}{n}=L\ln\left(\frac{a}{b}\right)\quad (C)\ , \tag{47}$$

whenever $a\geq b>0$.

It is interesting to observe that if the *ordinary* limit $\lim_{n\to\infty} c_n = L$ exists, then it is true that

$$\lim_{x\to\infty} \sum_{bx\leq n<ax} c_n/n = L\ln(a/b)$$

for $a \geq b > 0$ since $\sum_{n\leq ax} 1/n = \ln ax + \gamma + O(1)$ as $x\to\infty$. However, if $c_n = (-1)^n$ then we have

$$\lim_{x\to\infty} \sum_{bx\leq n<ax} (-1)^n/n = 0$$

but $\lim_{n\to\infty} c_n$ does not exist. The equivalence of Proposition 6.1 does not hold in the ordinary sense.

We also obtain the next corollary.

Corollary 6.1. *If the series $\sum_{n=1}^{\infty} a_n$ is Cesàro summable then $\lim_{n\to\infty} na_n = 0\ (C)$. The converse does not hold.*

Proof. If $c_n = na_n$ then the Cesàro summability of $\sum_{n=1}^{\infty} c_n/n$ implies that the limit in (47) vanishes and thus (46) yields the result. That the converse is not true can be seen by taking $a_n = (n\ln n)^{-1}$. □

Notice that it is well known that the convergence of the series $\sum_{n=1}^{\infty} a_n$ does not give that $\lim_{n\to\infty} na_n = 0$ but it does yield the average limit $\lim_{n\to\infty} N^{-1}\sum_{n=1}^{N} na_n = 0$.

Consider now a sequence $\{a_n\}_{n=-\infty}^{\infty}$ with indices in $\mathbb{Z}$. For convergence or for a sense of summability we can consider several different meanings for the series $S = \sum_{n=-\infty}^{\infty} a_n$. The series is convergent (summable) if both $S_- = \sum_{n=-\infty}^{-1} a_n$ and $S_+ = \sum_{n=0}^{\infty} a_n$ are convergent (summable) and then $S = S_- + S_+$. The series is principal value convergent (summable) if

$$\text{p.v.} \sum_{n=-\infty}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n = S\,. \tag{48}$$

Summability is too strong but principal value summability is too weak. A convenient intermediate notion is that of e.v. summability,

$$\text{e.v.} \sum_{n=-\infty}^{\infty} a_n = \lim_{x\to\infty} \sum_{-N\leq n\leq \alpha N} a_n = S\,. \tag{49}$$

In fact [7], if $\mathsf{g} \in \mathcal{D}'(\mathbb{R})$ is a periodic distribution of period 2π with Fourier series $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$, then the distributional point value $\mathsf{g}(\theta_0)$ $(\mathfrak{dist})$ exists if and only if

$$\text{e.v.} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \mathsf{g}(\theta_0) \quad (\mathrm{C})\ . \tag{50}$$

Proposition 6.2. *The series $\sum_{n=-\infty}^{\infty} a_n$ is* e.v. *Cesàro summable if and only if it is* p.v. *Cesàro summable and*

$$\lim_{n\to\infty} na_n = 0\ (C)\,.$$

The periodic distribution $\mathsf{g}(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ has a distributional point value $\mathsf{g}(\theta_0)$ $(\mathfrak{dist})$ at the point θ_0 if and only if

$$\text{p.v.} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = \mathsf{g}(\theta_0) \quad (C) \quad \textit{and} \quad \lim_{n\to\infty} na_n e^{in\theta_0} = 0 \quad (C)\ . \tag{51}$$

Proof. The first part follows at once from Proposition 6.1, while the second part is a particular case of the first. □

In fact, the distributional jump behavior of periodic distributions was characterized by the Cesàro behavior of the e.v. finite sums [27, 28]. A distribution f has distributional jump behavior at x_0 if the two lateral distributional limits $\mathsf{f}(x_0^{\pm}) = \gamma^{\pm}$ $(\mathfrak{dist})$ exist and f has no delta functions at x_0; this can also be expressed by saying that f has the following asymptotic behavior,

$$\langle \mathsf{f}(x_0+\varepsilon x), \phi(x)\rangle = \gamma_- \int_{-\infty}^{0} \phi(x)\ \mathrm{d}x + \gamma_+ \int_{0}^{\infty} \phi(x)\ \mathrm{d}x + o(1)\ , \tag{52}$$

as $\varepsilon \to 0^+$. Using Proposition 6.1 we obtain the ensuing result.

Proposition 6.3. *The periodic distribution $\mathsf{g}(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ has a distributional jump behavior at θ_0 if and only if*

$$\text{p.v.} \sum_{n=-\infty}^{\infty} a_n e^{in\theta_0} = d_1 = \frac{\mathsf{g}(\theta_0^-) + \mathsf{g}(\theta_0^+)}{2} \qquad (C)\ , \tag{53}$$

and

$$\lim_{n\to\infty} a_n e^{in\theta_0} = \frac{d_2}{2\pi i} = \frac{\mathsf{g}(\theta_0^+) - \mathsf{g}(\theta_0^-)}{2\pi i} \qquad (C)\ . \tag{54}$$

Proof. Indeed [27,28], g has distributional jump behavior at θ_0 if and only if

$$\lim_{x\to\infty}\sum_{-x\le n<ax} a_n e^{in\theta_0} = d_1 + \frac{d_2}{2\pi i}\ln a \qquad (\mathrm{C}) \ , \tag{55}$$

where $d_1 = \left(\mathrm{g}\left(\theta_0^-\right) + \mathrm{g}\left(\theta_0^+\right)\right)/2$ and $d_2 = \mathrm{g}\left(\theta_0^+\right) - \mathrm{g}\left(\theta_0^-\right)$. □

7. Other results

It was observed by several authors that it is enough to assume the existence of the Frullani integrals for some set of pairs (a,b) in order to obtain the existence for all $a,b>0$. In fact, the set $S=\{a/b$:integral exists$\}$ is a subgroup of the multiplicative group $(0,\infty)$, and it is not hard to see that such subgroups have zero measure or coincide with $(0,\infty)$. Hence if the set of quotients a/b for which the integral exist, at 0^+ or at infinity, has positive measure then the integrals exist for all $a,b>0$.

The function $f(x)=x^i H(x-1)$ provides an example where $S=\{e^{2k\pi}: k\in\mathbb{Z}\}$ is non empty but not the whole $(0,\infty)$. Employing a Wiener-type Tauberian theorem, Agnew [2] proves that if the function f is bounded in a neighborhood of the origin and vanishes for $x\ge c$, and S contains two elements λ_1 and λ_2 such that $\ln\lambda_1/\ln\lambda_2$ is irrational, then $S=(0,\infty)$, and a corresponding result at infinity; he also gives an example of an unbounded function for which $S\ne(0,\infty)$.

For the distributional limits, we can employ the notion of distributional boundedness of [13] to obtain the following.

Proposition 7.1. *Suppose* $\mathsf{f}\in\mathcal{D}'(0,\infty)$. *Suppose that for two pairs* (a_1,b_1) *and* (a_2,b_2) *such that* $\ln(a_1/b_1)/\ln(a_2/b_2)$ *is irrational there are primitives of* $(\mathsf{f}(a_jx)-\mathsf{f}(b_jx))/x$ *that have distributional limits as* $x\to0^+$. *If* f *is bounded at* 0^+ *then* $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ *exists.*

Proof. Indeed, let $\phi\in\mathcal{D}(0,\infty)$. The fact that f is bounded at 0^+ means that the function $\Phi(y)=\langle\mathsf{f}(yx),\phi(x)\rangle$, $y>0$, is bounded as $y\to0^+$. A primitive of $(\mathsf{f}(ax)-\mathsf{f}(bx))/x$ has a limit for a pair (a,b) if and only if the improper integrals

$$\int_{0^+}^{1}\frac{\Phi(ax)-\Phi(bx)}{x}\,\mathrm{d}x\ , \tag{56}$$

exist for all ϕ. They exist for the two pairs (a_1,b_1) and (a_2,b_2), so Agnew result shows that they exist for all pairs with $a,b>0$. Therefore all primitives of $(\mathsf{f}(ax)-\mathsf{f}(bx))/x$ for any $a,b>0$ have distributional limits at 0^+, and the existence of $\mathsf{f}(0^+)$ $(\mathfrak{dist})$ follows from Proposition 3.2. □

We immediately obtain the next result for the distributional integration.

Proposition 7.2. *Let* f *be a distributionally integrable function in* $[0,\infty)$. *Suppose* f *is distributionally bounded at* 0^+. *If the distributional integrals*

$$\int_{0}^{1}\frac{f(ax)-f(bx)}{x}\,\mathrm{d}x\ , \tag{57}$$

exists for two pairs (a_1,b_1) *and* (a_2,b_2) *such that* $\ln(a_1/b_1)/\ln(a_2/b_2)$ *is irrational, then* $f(0^+)$ $(\mathfrak{dist})$ *exists.*

8. Examples

Here we give several examples of distributional Frullani's integrals.

Example 8.1. Recently the integral

$$\int_0^\infty\frac{|\sin\sqrt{ax}|-|\sin\sqrt{bx}|}{x}\,\mathrm{d}x\ , \tag{58}$$

was considered by several researchers in an online forum [26]; they gave various methods of solution. We would like to point out that this integral is already covered by the Iyengar-Ostrowski-Agnew results, since a substitution yields the Frullani integral

$$2\int_0^\infty\frac{|\sin\sqrt{a}u|-|\sin\sqrt{b}u|}{u}\,\mathrm{d}x\ . \tag{59}$$

Of course $m_*=\lim_{x\to0^+}|\sin x|=0$ while $f(x)=|\sin x|$ is a periodic function with period π, so that

$$M_*=\frac{1}{\pi}\int_0^\pi|\sin x|\,\mathrm{d}x=\frac{2}{\pi}\ . \tag{60}$$

Thus, Frullani's integral formula gives the result of (58) as $2((2/\pi)-0)\ln\left(\sqrt{a}/\sqrt{b}\right)=(2/\pi)\ln(a/b)$.

Example 8.2. We now consider a very general integral that generalizes Example 8.1, namely the *finite part* integral

$$\mathrm{F.p.}\int_0^\infty \frac{|\sin(ax+c)|^\beta - |\sin(bx+c)|^\beta}{x}\,\mathrm{d}x\,, \tag{61}$$

where $a, b > 0$ and $0 \le c \le \pi/2$. If $\mathsf{H}_{a,b}$ is a primitive of the pseudofunction distribution

$$\mathcal{P}f\left(\frac{|\sin(ax+c)|^\beta - |\sin(bx+c)|^\beta}{x}\right)\,, \tag{62}$$

then the finite part integral is actually equal to the expression $\mathsf{H}_{a,b}(\infty) - \mathsf{H}_{a,b}(0^+)$, the value at 0^+ in the distributional sense and that at infinity in the Cesàro sense. The integral is an ordinary integral when $\Re e\,\beta > -1$ but it becomes a distributional evaluation otherwise, if we take the finite part at all the zeros of $\sin(ax+c)$ and of $\sin(bx+c)$. We can apply Frullani's integral formula with $f(x) = |\sin(x+c)|^\beta$. First, if $c > 0$ and any β or if $c = 0$ and $\Re e\,\beta > 0$

$$m_* = \lim_{x\to 0^+} |\sin(x+c)|^\beta = \sin^\beta c\,.$$

The integral (61) will be divergent if $c = 0$ and $\Re e\,\beta \le 0$ and $\beta \ne 0$. Next, observe that $f(x)$ is a periodic distribution with a period π so that [11, Chapter 6] the Cesàro limit $f(\infty)$ (C) exists and equals the value

$$M_*(\beta) = \frac{1}{\pi}\mathrm{F.p.}\int_0^\pi |\sin(x+c)|^\beta\,\mathrm{d}x\,. \tag{63}$$

The formula

$$M_*(\beta) = \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}+1\right)}\,, \tag{64}$$

holds when $\Re e\,\beta > -1$, that is, when the integral is an ordinary integral [17]. Formula (64) remains true by analytic continuation when the integral (63) is the finite part of a divergent integral at all points where the right side of (64) is analytic, namely, whenever $\beta \ne -1, -3, -5, \ldots$. Thus, Proposition 3.4 gives that (61) equals

$$\left(\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\beta}{2}+1\right)} - \sin^\beta c\right)\ln\left(\frac{a}{b}\right)\,,$$

if $c > 0$ and $\beta \ne -1, -3, -5, \ldots$ or if $c = 0$ and $\Re e\,\beta > 0$. Notice the special case

$$\mathrm{F.p.}\int_0^\infty \frac{\sin^{-2k}(ax+c) - \sin^{-2k}(bx+c)}{x}\,\mathrm{d}x = -\sin^{-2k} c\,\ln\left(\frac{a}{b}\right)\,,$$

for $c > 0$ and $k = 1, 2, 3, \ldots$. Interestingly, the distributional evaluation is possible if $c > 0$ even for $\beta = -1, -3, -5, \ldots$ since the finite part still exists (but it is not obtained by analytic continuation). Formula (64) cannot be applied, of course. We need to take the finite part of the analytic function $M_*(\beta)$ at each of the poles, that is, at $-1, -3, -5, \ldots$. In fact, $\Gamma(z)$ has poles at precisely $0, -1, -2, \ldots$ with residue $(-1)^k/k!$ and finite part

$$\mathrm{F.p.}_{z=-k}\Gamma(z) = \mathrm{F.p.}\lim_{z\to -k}\Gamma(z) = \lim_{z\to -k}\left(\Gamma(z) - \frac{(-1)^k}{k!\,(z+k)}\right) = \frac{(-1)^k\,\psi(k+1)}{k!}\,,$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function [11]; notice that $\psi(k+1) = -\gamma + \sum_{n=1}^{k}(1/n)$ [17]. The result of (63) when $\beta = -2k-1$ is

$$\mathrm{F.p.}_{\beta=-2k-1}M_*(\beta) = \frac{1}{\sqrt{\pi}}\frac{(-1)^k\,\psi(k+1)}{k!\Gamma\left(\frac{-2k+1}{2}\right)}\,,$$

and (61) equals

$$\left(\frac{1}{\sqrt{\pi}}\frac{(-1)^k\,\psi(k+1)}{k!\Gamma\left(\frac{-2k+1}{2}\right)} - \sin^{-2k-1} c\right)\ln\left(\frac{a}{b}\right)\,.$$

Example 8.3. The function

$$f(x) = \frac{1+cx^\alpha}{x^\alpha}\cos\frac{1}{x^\beta}\,,$$

satisfies $f(0^+) = 0$ $(\mathfrak{dist})$ for any $\alpha, \beta > 0$. Also $f(\infty) = c$. We thus obtain

$$\int_0^\infty \frac{b^\alpha(1+ca^\alpha x^\alpha)\cos\dfrac{1}{a^\beta x^\beta} - a^\alpha(1+cb^\alpha x^\alpha)\cos\dfrac{1}{b^\beta x^\beta}}{x^{\alpha+1}}\,\mathrm{d}x = ca^\alpha b^\alpha\ln\left(\frac{a}{b}\right)\,.$$

This is a distributional integral; in general it is not a convergent improper Lebesgue integral.

Example 8.4. Let the series $\sum_{n=1}^{\infty} c_n$ be Cesàro summable to S. Putting $f(x) = \sum_{n\leq x} c_n$ we obtain

$$\int_0^{\infty} \frac{f(ax) - f(bx)}{x} \, \mathrm{d}x = \int_0^{\infty} \frac{1}{x} \sum_{bx<n\leq ax} c_n \, \mathrm{d}x = S \ln\left(\frac{a}{b}\right) \quad (\mathrm{C}) \ .$$

Also, $x^{-1} \sum_{b/x<n\leq a/x} c_n$ is distributionally integrable at 0 and

$$\int_0^{\infty} \frac{1}{x} \sum_{b/x<n\leq a/x} c_n \, \mathrm{d}x = S \ln\left(\frac{a}{b}\right) \ .$$

In particular, we obtain the following integral

$$\int_0^{\infty} \frac{1}{x} \sum_{bx<n\leq ax} (-1)^n n^{\alpha} \, \mathrm{d}x = \left(2^{\alpha+1} - 1\right) \zeta(-\alpha) \ln\left(\frac{a}{b}\right) \quad (\mathrm{C}) \ ,$$

for any $\alpha \in \mathbb{C}$ (for $\alpha = -1$ we take the limit of the right side) since [11]

$$\sum_{n=1}^{\infty} (-1)^n n^{\alpha} = \left(2^{\alpha+1} - 1\right) \zeta(-\alpha) \quad (\mathrm{C}) \ .$$

Example 8.5. Using the ideas of Example 8.4 we obtain

$$\int_0^{\infty} \frac{1}{x} \sum_{bx<n\leq ax} n^{\alpha} \, \mathrm{d}x = \zeta(-\alpha) \ln\left(\frac{a}{b}\right) \ , \ \Re e \ \alpha < -1 \ .$$

The series $\sum_{n=1}^{\infty} n^{\alpha}$ is not Cesàro summable if $\Re e \ \alpha \geq -1$, but we have [11, Example 132]

$$\lim_{x\to\infty} \left(\sum_{n\leq x} n^{\alpha} - \frac{x^{\alpha+1}}{\alpha+1} \right) = \zeta(-\alpha) \quad (\mathrm{C}) \ , \ \Re e \ \alpha > -1 \ .$$

Thus

$$\int_0^{\infty} \left\{ \frac{1}{x} \sum_{bx<n\leq ax} n^{\alpha} - \frac{a^{\alpha+1} - b^{\alpha+1}}{\alpha+1} x^{\alpha} \right\} \mathrm{d}x = \zeta(-\alpha) \ln\left(\frac{a}{b}\right) \quad (\mathrm{C}) \ , \ \Re e \ \alpha > -1 \ .$$

When $\alpha = k \in \mathbb{N}$ then

$$\sum_{n\leq x} n^k = \frac{1}{k+1} \left(B_{k+1}\left(⟦ x ⟧ + 1\right) - B_{k+1}(1) \right) \ ,$$

where $B_{k+1}(x)$ is the Bernoulli polynomial of order $k+1$ (see [11]). Hence,

$$\int_0^{\infty} \left\{ \frac{1}{x} \left(B_{k+1}\left(⟦ ax ⟧ + 1\right) - B_{k+1}\left(⟦ bx ⟧ + 1\right) \right) - \left(a^{k+1} - b^{k+1}\right) x^k \right\} \mathrm{d}x = B_{k+1} \ln\left(\frac{a}{b}\right) \quad (\mathrm{C}) \ ,$$

since $(k+1)\zeta(-k)$ equals the Bernoulli number B_{k+1}. Observe that the function

$$g_k(x) = \frac{1}{k+1} \left(B_{k+1}\left(⟦ 1/x ⟧ + 1\right) - B_{k+1}(1) - \frac{1}{x^{k+1}} \right) \ ,$$

has a distributional limit at 0^+, equal to $B_{k+1}/(k+1)$. This limit is the mean value limit

$$\lim_{x\to 0^+} x^{-1} \int_0^x g_0(x) \ \mathrm{d}x$$

when $k = 0$, but not for $k \geq 1$, since the integral $\int_0^x g_k(x) \ \mathrm{d}x$ diverges.

Naturally, many more examples can be given.

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