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Research Article

Stability analysis of a partially damped suspension bridge by friction

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Abstract

We consider a suspension bridge system of a given length where Timoshenko's theory models the deck. We use semigroup theory. We obtain the existence and uniqueness of the solution by the Lumer-Phillips theorem. For asymptotic behavior, we give under particular constraints a necessary and sufficient condition to guarantee exponential stability. However, regardless of the relationship between the system's coefficients, in a specific case, we lack exponential decay; in this case, we demonstrate the polynomial decay and the optimality of this rate.

Keywords: suspension bridge; Timoshenko-Enrenfest beam; partial differential equations; Lummer-Phillips theorem.

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1. Introduction

In 1921, Stephen Timoshenko and Paul Ehrenfest worked on the following pioneer beam model (see [2, 12]):

$$\rho_1 u_{tt} - k(u_x + \psi)_x = 0,$$
$$\rho_2 \psi_{tt} - b\psi_{xx} + k(u_x + \psi) = 0.$$

Compared to the Euler-Bernoulli beam model, this model has the advantage of considering both shear deformation and rotational inertia in vibrating beams. Also, by introducing a friction-type dissipative mechanism into both equations, the solution of this model has exponential decay. However, to have exponential decay, when this dissipative mechanism is introduced in just one of the equations, it is necessary and sufficient to consider $\frac{\rho_1}{k} = \frac{\rho_2}{b}$, see [11]. Many researchers have investigated the Timoshenko beam model with the most diverse stabilization mechanisms. According to [2], a simple search on Google Scholar produced about 78,000 hits on the term "Timoshenko beam".

Dealing with bridges is an essential engineering problem. Bridges were thought to have been constructed for the first time about 4000 BC, and throughout history, it has been important to comprehend how these structures respond to artificial and natural forces such as wind and earthquakes. For example, the Tacoma Narrows Bridge collapsed in November 1940 after three months of operation. Also, in 2000, the Millennium Bridge, a steel suspension bridge for pedestrians that connects Bankside to London (costing more than 20 million pounds and inaugurated by Queen Elizabeth II), suffered an unexpected lateral vibration due to a resonant structural response, causing its closure within a few days after opening, and repairs took over a year to complete.

In 1945, Timoshenko published a work about suspension bridges; namely, "The Theory of Suspension Bridges" (see [13, 14]). In 1984, Hayashikawa and Watanabe used Hamilton's principle and Timoshenko's beam theory to study the Inoshima Suspension Bridge (that connects Honshu and Shikoku in Japan), see [4]. A thermal-Timoshenko-beam system with suspenders and Kelvin-Voigt damping type, where Cattaneo's law produced the heat, was considered in [6]. A suspension bridge with internal damping was considered in [10], where it was found that the solution not only decays exponentially but is also analytical. Recently, Raposo [9] considered a suspension bridge, where laminated beams model the deck.

We study a model of a suspension bridge (see Figure 1.1), which is a mechanical structure that carries vertical loads through the main cables modeled by an elastic string $u = u(x,t)$ that is coupled to the deck employing suspension cables, where x denotes the distance along the center line of the deck in its equilibrium configuration and t denotes the time variable.

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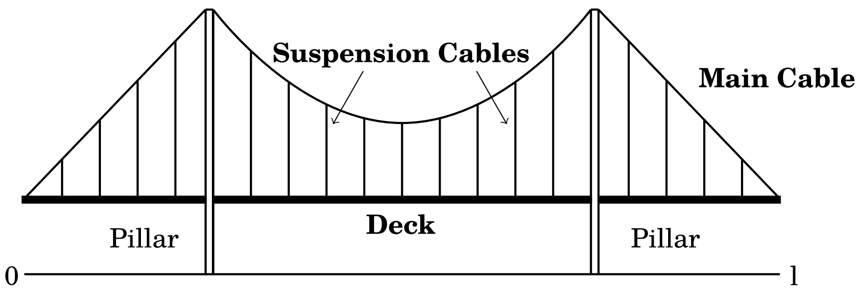

Figure 1.1: Suspension Bridge. This figure is taken from the reference [9].

Since the deck has negligible transversal section dimensions compared to the length (span of the bridge), it is modeled in Timoshenko's theory [12]. Denote by $\varphi = \varphi(x,t)$ the displacement of the cross-section on the point $x \in (0,l)$. Let $\psi = \psi(x,t)$ be the rotation angle of the cross-section. Then, we have the following coupled system:

$$u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 u_t = 0, \text{ in } (0,l)\times\mathbb{R}_+, \tag{1}$$

$$\rho_1\varphi_{tt} - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2\varphi_t = 0, \text{ in } (0,l)\times\mathbb{R}_+, \tag{2}$$

$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma_3\psi_t = 0, \text{ in } (0,l)\times\mathbb{R}_+. \tag{3}$$

The suspender cables are assumed to be linear elastic strings with standard stiffness $\lambda > 0$. The constant $\alpha > 0$ is the elastic modulus of the string (holding the main cable to the deck). The positive coefficients ρ_1 and ρ_2 are the mass density and the moment of mass inertia of the beam, respectively. Moreover, b represents the cross section's rigidity coefficient and κ denotes the elasticity's shear modulus. Finally, γ_1, γ_2, and γ_3 are non-negative parameters related to friction damping. System (1)–(3) is subject to the initial data given as

$$u(x,0) = u_0(x), \ \ u_t(x,0) = u_1(x), \ x \in (0,l),$$
$$\varphi(x,0) = \varphi_0(x), \ \ \varphi_t(x,0) = \varphi_1(x), \ x \in (0,l),$$
$$\psi(x,0) = \psi_0(x), \ \ \psi_t(x,0) = \psi_1(x), \ x \in (0,l),$$

and the following boundary conditions of Dirichlet-Dirichlet-Neumann:

$$u(0,t) = u(l,t) = \varphi(0,t) = \varphi(l,t) = \psi_x(0,t) = \psi_x(l,t) = 0, \ \ t \geq 0. \tag{4}$$

2. Energy of the system

In this section, we deduce the energy associated with the considered model and show that the energy decreases in time, which proves the system's dissipative character. We start multiplying (1) by u_t and integrating on $(0,l)$:

$$\int_0^l u_t u_{tt}dx - \alpha\int_0^l u_t u_{xx}dx - \lambda\int_0^l u_t(\varphi - u)dx = -\gamma_1\int_0^l |u_t|^2dx.$$

Integrating by parts and using boundary conditions, we have

$$\frac{d}{dt}\frac{1}{2}\int_0^l |u_t|^2dx + \alpha\int_0^l u_{tx}u_x dx - \lambda\int_0^l u_t(\varphi - u)dx = -\gamma_1\int_0^l |u_t|^2dx,$$

that is,

$$\frac{d}{dt}\frac{1}{2}\int_0^l |u_t|^2dx + \frac{d}{dt}\frac{\alpha}{2}\int_0^l |u_x|^2dx - \lambda\int_0^l u_t(\varphi - u)dx = -\gamma_1\int_0^l |u_t|^2dx. \tag{5}$$

Multiplying (2) by φ_t and integrating on $(0,l)$, we get

$$\rho_1\int_0^l \varphi_t\varphi_{tt}dx - \kappa\int_0^l \varphi_t(\varphi_x + \psi)_x dx + \lambda\int_0^l \varphi_t(\varphi - u)dx = -\gamma_2\int_0^l |\varphi_t|^2dx.$$

Integrating by parts and using boundary conditions, we have

$$\frac{d}{dt}\frac{\rho_1}{2}\int_0^l |\varphi_t|^2dx + \kappa\int_0^l \varphi_{tx}(\varphi_x + \psi)dx + \lambda\int_0^l \varphi_t(\varphi - u)dx = -\gamma_2\int_0^l |\varphi_t|^2dx. \tag{6}$$

Multiplying (3) by ψ_t and integrating on $(0,l)$, we obtain

$$\rho_2\int_0^l \psi_t\psi_{tt}dx - b\int_0^l \psi_t\psi_{xx}dx + \kappa\int_0^l \psi_t(\varphi_x+\psi)dx = -\gamma_3\int_0^l |\psi_t|^2dx,$$

Integrating by parts and using boundary conditions, we get

$$\frac{d}{dt}\frac{\rho_2}{2}\int_0^l |\psi_t|^2dx + b\int_0^l \psi_{tx}\psi_x dx + \kappa\int_0^l \psi_t(\varphi_x+\psi)dx = -\gamma_3\int_0^l |\psi_t|^2dx,$$

that is,

$$\frac{d}{dt}\frac{\rho_2}{2}\int_0^l |\psi_t|^2dx + \frac{d}{dt}\frac{b}{2}\int_0^l |\psi_x|^2dx + \kappa\int_0^l \psi_t(\varphi_x+\psi)dx = -\gamma_3\int_0^l |\psi_t|^2dx. \tag{7}$$

Adding (5), (6), and (7), and noting that

$$\begin{aligned}-\lambda\int_0^l u_t(\varphi-u)dx + \lambda\int_0^l \varphi_t(\varphi-u)dx &= \lambda\int_0^l (\varphi-u)_t(\varphi-u)_t dx\\ &= \frac{d}{dt}\frac{\lambda}{2}\int_0^l |\varphi-u|^2dx,\end{aligned}$$

and

$$\begin{aligned}-\kappa\int_0^l \varphi_t(\varphi_x+\psi)_x dx + \kappa\int_0^l \psi_t(\varphi_x+\psi)dx &= \kappa\int_0^l (\varphi_x+\psi)(\varphi_x+\psi)_t dx\\ &= \frac{d}{dt}\frac{\kappa}{2}\int_0^l |\varphi_x+\psi|^2dx,\end{aligned}$$

we deduce the following equation:

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\int_0^l \left[|u_t|^2+\alpha|u_x|^2+b|\psi_x|^2+\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\lambda|\varphi-u|^2+\kappa|\varphi_x+\psi|^2\right]dx\\ &= -\gamma_1\int_0^L |u_t|^2dx - \gamma_2\int_0^L |\varphi_t|^2dx - \gamma_3\int_0^L |\psi_t|^2dx.\end{aligned}$$

Denoting the energy by $E(t)$, we define

$$E(t) = \frac{1}{2}\int_0^l [|u_t|^2+\alpha|u_x|^2+b|\psi_x|^2+\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+\lambda|\varphi-u|^2+\kappa|\varphi_x+\psi|^2]dx,$$

and we have that

$$\frac{d}{dt}E(t) = -\gamma_1\int_0^l |u_t|^2dx - \gamma_2\int_0^l |\varphi_t|^2dx - \gamma_3\int_0^l |\psi_t|^2dx. \tag{8}$$

3. Semigroup setting: existence of solution

This section studies the existence and uniqueness of weak and strong solutions of the system (1)–(4). First, we present preliminaries including notations and technical lemmas.

Notations and assumptions

We use throughout this paper the standard Lebesgue spaces $L^r(0,l)$, $r\geq 1$, with the norm denoted by $\|\cdot\|_r$. We denote by $\langle\cdot,\cdot\rangle$ the inner product in $L^2(0,l)$; that is, $\langle f\,,\,g\rangle = \int_0^l f\bar{g}\,dx$, where $\bar{g}$ is the conjugate of g. Given the boundary condition in (4), we consider the following Sobolev spaces

$$H^2_*(0,l) = \left\{ f\in H^2(0,l) : \int_0^l f\,dx = 0\right\}.$$

For the Sobolev spaces $H^1_0(0,L)$ and $H^1_*(0,L)$, the Poincaré's inequality holds: $\lambda_1\|f\|_2^2 \leq \|f_x\|_2^2$, $\forall f\in H^1_0(0,l)$ or $\forall f\in H^1_*(0,l)$, where $\lambda_1>0$ is the Poincaré's constant (the smallest eigenvalue of $-\partial_{xx}$).

To show the existence and uniqueness of the solution for the system (1)–(4), we use semi-group theory. We prove that the operator $\mathcal{A}$ defined in (12) generates a contraction semigroup in the Hilbert space $\mathcal{H}$ given by

$$\mathcal{H} := \left[H_0^1(0,l) \times L^2(0,l)\right]^2 \times H_*^1(0,l) \times L^2(0,l). \tag{9}$$

We define in $\mathcal{H}$ the following inner product:

$$\langle U, \tilde{U}\rangle_{\mathcal{H}} = \langle v\,,\, \tilde{v}\rangle + \alpha\langle u_x\,,\, \tilde{u}_x\rangle + \rho_1\langle w\,,\, \tilde{w}\rangle + \rho_2\langle z\,,\, \tilde{z}\rangle + b\langle \psi_x\,,\, \tilde{\psi}_x\rangle + \lambda\langle(\varphi - u)\,,\, (\tilde{\varphi} - \tilde{u})\rangle + k\langle(\varphi_x + \psi)\,,\, (\tilde{\varphi_x} + \tilde{\psi})\rangle,$$

where $U = (u, v, \varphi, w, \psi, z)'$, $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{w}, \tilde{\psi}, \tilde{z})'$, and $\mathcal{H}$ is a Hilbert space with $\|U\|_{\mathcal{H}}^2 = \langle U, U\rangle_{\mathcal{H}}$.

Clearly, there exists a positive constant κ_0 such that

$$\|\varphi_x\|_2^2 \le \kappa_0 \left(\|\psi_x\|_2^2 + \|\varphi_x + \psi\|_2^2\right). \tag{10}$$

By Poincaré's inequality and (10), we obtain the following useful inequality:

$$\|\varphi\|_2^2 \le \gamma \left(\|\psi_x\|_2^2 + \|\varphi_x + \psi\|_2^2\right), \tag{11}$$

where $\gamma = \kappa_0 \lambda_1^{-1}$ with λ_1 being the Poincaré's constant.

Existence and uniqueness of solutions

Let us write the system (1)–(4) as a first-order Cauchy evolution problem. We introduce the following vector function:

$$U = (u, v, \varphi, w, \psi, z)',$$

where, $u_t = v, \varphi_t = w$, and $\psi_t = z$. Then,

$$U_t = \begin{pmatrix} u_t \\ v_t \\ \varphi_t \\ w_t \\ \psi_t \\ z_t \end{pmatrix} = \begin{pmatrix} v \\ \alpha u_{xx} + \lambda(\varphi - u) - \gamma_1 v \\ w \\ \frac{\kappa}{\rho_1}(\varphi_x + \psi)_x - \frac{\lambda}{\rho_1}(\varphi - u) - \frac{\gamma_2}{\rho_1} w \\ z \\ \frac{b}{\rho_2}\psi_{xx} - \frac{\kappa}{\rho_2}(\varphi_x + \psi) - \frac{\gamma_3}{\rho_2} z \end{pmatrix} = \mathcal{A}U,$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 \\ \alpha\partial_{xx} - \lambda I & -\gamma_1 I & \lambda I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \frac{\lambda}{\rho_1} I & 0 & \frac{\kappa}{\rho_1}\partial_{xx} - \frac{\lambda}{\rho_1} I & -\frac{\gamma_2}{\rho_1} & \frac{\kappa}{\rho_1}\partial_x & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & -\frac{\kappa}{\rho_2}\partial_x & 0 & \frac{b}{\rho_2}\partial_{xx} - \frac{\kappa}{\rho_2}\partial_x & -\frac{\gamma_3}{\rho_2} I \end{pmatrix}. \tag{12}$$

System (1)–(4) is reduced to the following abstract initial problem for a first-order evolution equation:

$$\begin{cases} U_t - \mathcal{A}U = 0, \\ U(0) = U_0, \end{cases} \tag{13}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is an unbounded linear operator on the energy space $\mathcal{H}$ with domain

$$D(\mathcal{A}) = [H_0^1(0,l) \cap H^2(0,l) \times H_0^1(0,l)]^2 \times H_*^2(0,l) \times H_*^1(0,l).$$

It is clear that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

We will prove that $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{\mathcal{A}t},\ t \ge 0$, on $\mathcal{H}$. For this, we first prove two lemmas.

Lemma 3.1. *The operator $\mathcal{A}$ is dissipative.*

Proof. For any $U = (u, v, \varphi, w, \psi, z)' \in D(\mathcal{A})$, by a straightforward calculation, we have

$$\mathfrak{Re}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\gamma_1\|v\|_2^2 - \gamma_2\|w\|_2^2 - \gamma_3\|z\|_2^2 \le 0.$$

Hence, $\mathcal{A}$ is dissipative. □

Lemma 3.2. *If $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$, then $0 \in \rho(\mathcal{A})$.*

Proof. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6)' \in \mathcal{H}$, the resolvent equation $-\mathcal{A}U = F$ in $\mathcal{H}$, in terms of the component coordinates of U and F, leads to

$$-v = f_1 \text{ in } H_0^1(0,L), \tag{14}$$

$$-\alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 v = f_2 \text{ in } L^2(0,L), \tag{15}$$

$$-w = f_3 \text{ in } H_0^1(0,L), \tag{16}$$

$$-\kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w = \rho_1 f_4 \text{ in } L^2(0,L), \tag{17}$$

$$-z = f_5 \text{ in } H_0^1(0,L), \tag{18}$$

$$-b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma_3 z = \rho_2 f_6 \text{ in } L^2(0,L). \tag{19}$$

By substituting (14), (16), (18) in (15), (17), (19), respectively, we obtain

$$-\alpha u_{xx} - \lambda(\varphi - u) = \gamma_1 f_1 + f_2 := g_1 \in L^2(0,L), \tag{20}$$

$$-\kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) = \gamma_2 f_3 + \rho_1 f_4 := g_2 \in L^2(0,L), \tag{21}$$

$$-b\psi_{xx} + \kappa(\varphi_x + \psi) = \gamma_3 f_5 + \rho_2 f_6 := g_3 \in L^2(0,L). \tag{22}$$

Multiplying (20), (21), and (22) by the conjugates of the functions $\tilde{u} \in H_0^1(0,l)$, $\tilde{\varphi} \in H_0^1(0,l)$, and $\tilde{\psi} \in H_*^1(0,l)$, respectively, and integrating by parts, we obtain

$$\alpha\langle u_x, \tilde{u}_x\rangle - \lambda\langle \varphi - u, \tilde{u}\rangle = \langle g_1, \tilde{u}\rangle, \tag{23}$$

$$\kappa\langle \varphi_x + \psi, \tilde{\varphi}_x\rangle + \lambda\langle \varphi - u, \tilde{\varphi}\rangle = \langle g_2, \tilde{\varphi}\rangle, \tag{24}$$

$$b\langle \psi_x, \tilde{\psi}_x\rangle + \kappa\langle \varphi_x + \psi, \tilde{\psi}\rangle = \langle g_3, \tilde{\psi}\rangle. \tag{25}$$

Denoting $\mathcal{V} = H_0^1(0,l) \times H_0^1(0,l) \times H_*^1(0,l)$ and adding (23), (24), (25), we build a variational problem

$$\mathbb{B}((u,\varphi,\psi),(\tilde{u},\tilde{\varphi},\tilde{\psi})) = \mathbb{L}(\tilde{u},\tilde{\varphi},\tilde{\psi}), \tag{26}$$

where $\mathbb{B} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is given by

$$\mathbb{B}((u,\varphi,\psi),(\tilde{u},\tilde{\varphi},\tilde{\psi})) = \alpha\langle u_x, \tilde{u}_x\rangle + \lambda\langle \varphi - u, \tilde{\varphi} - \tilde{u}\rangle + \kappa\langle \varphi_x + \psi, \tilde{\varphi_x} + \tilde{\psi}\rangle + b\langle \psi_x, \tilde{\psi}_x\rangle$$

and $\mathbb{L} : \mathcal{V} \to \mathbb{C}$ is a continuous and linear operator given as $\mathbb{L}(\tilde{u},\tilde{\varphi},\tilde{\psi}) = \langle g_1, \tilde{u}\rangle + \langle g_2, \tilde{\varphi}\rangle + \langle g_3, \tilde{\psi}\rangle$. We define in $\mathcal{V}$ the norm $||(u,\varphi,\psi)||_{\mathcal{V}}^2 = \mathbb{B}((u,\varphi,\psi),(u,\varphi,\psi))$. It is easy to see that with this norm, $\mathbb{B}$ is a continuous coercive sesquilinear form on $\mathcal{V}$. Therefore, by Lax-Milgram theorem, there exists a unique $(u,\varphi,\psi) \in \mathcal{V}$ solution of (26), for all $(\tilde{u},\tilde{\varphi},\tilde{\psi}) \in \mathcal{V}$. By the standard theory in the elliptic equations, (20), (21), and (22) leads to $u,\varphi,\psi \in H^2(0,l)$, and then $u,\varphi \in H_0^1(0,l) \cap H^2(0,l)$ and $\psi \in H_*^2(0,l)$. From (14), (16), and (18), we have $v,w \in H_0^1(0,l)$ and $z \in H_*^1(0,l)$. Consequently, we have $U \in D(\mathcal{A})$, and hence the unique solution of $-\mathcal{A}U = F$ follows. Note that $\|U\|_{\mathcal{H}} \leq K\|F\|_{\mathcal{H}}$, where K is a positive constant independent of U; that is, $\|\mathcal{A}^{-1}F\|_{\mathcal{H}} \leq K\|F\|_{\mathcal{H}}$. Thus, we conclude that $0 \in \rho(\mathcal{A})$. □

Proposition 3.1. *The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $S(t) = e^{\mathcal{A}t}$, $t \geq 0$, on the Hilbert space $\mathcal{H}$.*

Proof. By Lemma 3.1, $\mathcal{A}$ is dissipative and by Lemma 3.2, $0 \in \rho(\mathcal{A})$. Since $\mathcal{A}$ is densely defined, the conclusion of the lemma is a consequence of Lummer-Philips's Theorem. □

By Proposition 3.1, we have the next result.

Theorem 3.1. *If $U_0 \in \mathcal{H}$, then there exists a unique weak solution U of problem* (15) *satisfying*

$$U \in C^0([0,+\infty);\mathcal{H}). \tag{27}$$

Moreover, if $U_0 \in D(\mathcal{A})$, then

$$U \in C^0([0,+\infty);D(\mathcal{A})) \cap C^1([0,+\infty);\mathcal{H}). \tag{28}$$

Proof. By semigroup theory (see Page 100 in [7]), $U(t) = e^{t\mathcal{A}}U_0$ is the unique solution of (15) satisfying (27) and (28). □

4. Asymptotic behavior

In this section, we show that when $\gamma_1, \gamma_3 \neq 0$ and $\gamma_2 = 0$, or $\gamma_1, \gamma_2 \neq 0$ and $\gamma_3 = 0$, the relationship between the wave propagation speeds given by

$$\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}, \tag{29}$$

is a necessary and sufficient condition to guarantee exponential stability. However, when $\gamma_1 = 0$ and $\gamma_2, \gamma_3 \neq 0$, we lack exponential decay regardless of the relationship between the system's coefficients. To carry out this asymptotic analysis, we use the next two theorems.

Theorem 4.1 (Gearhart-Herbst-Prüss-Huang)**.** *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \tag{30}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{31}$$

For more details about Theorem 30, see [3, 5, 8].

Theorem 4.2 (see Theorem 2.4 in [1])**.** *Let S(t) be a C_0-semigroup in the Hilbert space $\mathcal{H}$ associated with the operator $\mathcal{A}$ such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Then,*

$$\frac{1}{|\beta|^{\epsilon}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \quad \beta \in \mathbb{R}, \textit{when } |\beta| \to \infty \iff \|S(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/\epsilon}} \tag{32}$$

where C does not depend on β.

Conditions $\gamma_1 \neq 0$, $\gamma_2 = 0$, and $\gamma_3 \neq 0$

The main results of this subsection are in the form of the next two theorems.

Theorem 4.3. *Let $\gamma_1 \neq 0$, $\gamma_2 = 0$, and $\gamma_3 \neq 0$. Then, the associated semigroup $S(t) = e^{\mathcal{A}t}$ is exponentially stable if and only if the relation (29) is true.*

Theorem 4.4. *Let $\gamma_1 \neq 0$, $\gamma_2 = 0$, and $\gamma_3 \neq 0$. If (29) is not true, then the C_0-semigroup $S(t)$ associated with the system is polynomially stable and satisfies*

$$\|S(t)U_0\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}}\|U_0\|_{\mathcal{H}} \quad \forall\, U_0 \in \mathcal{D}(\mathcal{A}), \tag{33}$$

where C is a positive constant. Also, this stability rate is optimal.

The proofs of Theorems 4.3 and 4.4 are based on some lemmas (to be established). In all these lemmas, we use the resolvent equation given by

$$(i\beta I - \mathcal{A})U = F, \tag{34}$$

where $U = (u, v, \varphi, w, \psi, z)' \in \mathcal{D}(\mathcal{A})$, $F = (f_1, f_2, f_3, f_4, f_5, f_6)' \in \mathcal{H}$, and $\beta \in \mathbb{R}$. Hence, because of (34), we obtain the following system:

$$i\beta u - v = f_1, \tag{35}$$
$$i\beta v - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 v = f_2, \tag{36}$$
$$i\beta\varphi - w = f_3, \tag{37}$$
$$i\rho_1\beta w - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) = \rho_1 f_4, \tag{38}$$
$$i\beta\psi - z = f_5, \tag{39}$$
$$i\rho_2\beta z - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma_3 z = \rho_2 f_6. \tag{40}$$

Lemma 4.1. *The set formed by the resolvents of the operator $\mathcal{A}$ contains the set $i\mathbb{R}$; that is, $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. From Lemma 3.2, it follows that $0 \in \rho(\mathcal{A})$. Now, take $\beta \in \mathbb{R}\backslash\{0\}$. Suppose that there is some $i\beta$ that is an eigenvalue of the operator $\mathcal{A}$; that is,

$$\mathcal{A}U = i\beta U \;\text{ with }\; U \neq 0. \tag{41}$$

Multiplying (41) by U, we get

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = i\beta \|U\|^2_{\mathcal{H}}.$$

Taking the real part of the last equation and using the identity obtained in Lemma 3.1, we deduce that $\|v\|_2^2 = \|z\|_2^2 = 0$, and hence $v = z = 0$. Take $F = (0,0,0,0,0,0)$. Then, because of (35) and (39), we have $u = \psi = 0$, and because of (36), we have $\varphi = 0$ and because of (37), we have $w = 0$. This is a contradiction and hence there are no imaginary eigenvalues. □

Throughout the proof of the following lemmas, we assume that $\beta \neq 0$. Thus, we routinely use the fact that there is a constant c such that $0 < \dfrac{1}{|\beta|} < c < 1$ (which will be explained later).

Lemma 4.2. *If $U = (u, v, \varphi, w, \psi, z)'$ is the solution of the system (35)–(40), then there exists a positive constant C_1 such that*

$$\|v\|_2^2 + \rho_2 \|z\|_2^2 \leq C_1 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{42}$$

Proof. Note that

$$i\beta \|U\|^2_{\mathcal{H}} - \langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \langle F, U\rangle_{\mathcal{H}},$$

from the identity obtained in Lemma 3.1, we get

$$\gamma_1 \|v\|_2^2 + \gamma_3 \|z\|_2^2 \leq \Re \langle F, U\rangle_{\mathcal{H}}.$$

Now, using Hölder's inequality, we have the required conclusion. □

Lemma 4.3. *If $U = (u, v, \varphi, w, \psi, z)'$ is the solution of system (35)–(40), then there exists a positive constant C_2 such that*

$$\alpha \|u_x\|_2^2 - \frac{C_2}{\beta^2} \|w\|_2^2 \leq C_2 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}. \tag{43}$$

Proof. Multiplying (36) by $\overline{u}$ and integrating over $(0, l)$, we obtain

$$i\beta \langle v, u\rangle + \alpha \|u_x\|_2^2 + \lambda \|u\|_2^2 - \lambda \langle \varphi, u\rangle + \gamma_1 \langle v, u\rangle = \langle f_2, u\rangle. \tag{44}$$

From (35) and (37), we have $-i\beta u = -v - f_1$ and $\varphi = -\frac{i}{\beta}(w + f_3)$. Hence, it follows that

$$-\|v\|_2^2 + \alpha \|u_x\|_2^2 + \lambda \|u\|_2^2 + \frac{i\lambda}{\beta} \langle w, u\rangle + \gamma_1 \langle v, u\rangle = \langle f_2, u\rangle + \langle v, f_1\rangle - \frac{i\lambda}{\beta} \langle f_3, u\rangle. \tag{45}$$

The required conclusion is obtained by using Hölder's inequality, Young's inequality, Poincaré's inequality, and (42). □

Lemma 4.4. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (35)–(40). Then, there exists a positive constant C_3 such that*

$$\kappa \|\varphi_x + \psi\|_2^2 - \frac{C_3}{|\beta|} \left(\|\psi_x\|_2^2 + \|z\|_2^2 + \|w\|_2^2\right) \leq C_3 \left(|\beta| \left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right| |\langle \psi_x, w\rangle| + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}\right). \tag{46}$$

Proof. Multiplying (40) by $\overline{\varphi}_x + \overline{\psi}$ and integrating over $(0, l)$, we obtain

$$-i\beta\rho_2 \langle z_x, \varphi\rangle + i\beta\rho_2 \langle z, \psi\rangle + b \left\langle \psi_x, (\varphi_x + \psi)_x\right\rangle + \kappa \|\varphi_x + \psi\|_2^2 + \gamma_3 \langle z, \varphi_x + \psi\rangle = \rho_2 \langle f_6, \varphi_x + \psi\rangle. \tag{47}$$

On the other hand, from (38), it follows that

$$-i\beta\rho_1 \overline{w} - \kappa(\overline{\varphi}_x + \overline{\phi})_x + \lambda(\overline{\varphi} - \overline{u}) = \rho_1 \overline{f_4}. \tag{48}$$

Multiplying (48) by $\dfrac{b}{\kappa}\psi_x$ and integrating over $(0, l)$, we obtain

$$-i\beta \frac{b\rho_1}{\kappa} \langle \psi_x, w\rangle - b \left\langle \psi_x, (\varphi_x + \psi)_x\right\rangle + \frac{b\lambda}{\kappa} \langle \psi_x, \varphi\rangle + \frac{b\lambda}{\kappa} \langle \psi, u_x\rangle = \frac{b\rho_1}{\kappa} \langle \psi_x, f_4\rangle. \tag{49}$$

Adding (47) and (49), we get

$$-i\beta\rho_2\langle z_x,\varphi\rangle - i\beta\frac{b\rho_1}{\kappa}\langle\psi_x,w\rangle + i\beta\rho_2\langle z,\psi\rangle + \kappa\|\varphi_x+\psi\|_2^2 + \gamma_3\langle z,\varphi_x+\psi\rangle$$
$$+\frac{b\lambda}{\kappa}\langle\psi_x,\varphi\rangle + \frac{b\lambda}{\kappa}\langle\psi,u_x\rangle = \rho_2\langle f_6,\varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle.$$

From (37), it follows that $i\beta\varphi = w + f_3$ and $\varphi = -\frac{i}{\beta}(w+f_3)$. Hence,

$$\rho_2\langle z_x,w\rangle - i\beta\frac{b\rho_1}{\kappa}\langle\psi_x,w\rangle + i\beta\rho_2\langle z,\psi\rangle + \kappa\|\varphi_x+\psi\|_2^2 + \gamma_3\langle z,\varphi_x+\psi\rangle + i\frac{b\lambda}{\kappa\beta}\langle\psi_x,w\rangle + \frac{b\lambda}{\kappa}\langle\psi,u_x\rangle$$
$$= \rho_2\langle f_6,\varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle - \rho_2\langle z_x,f_3\rangle - i\frac{b\lambda}{\kappa\beta}\langle\psi_x,f_3\rangle.$$

From (39), it follows that $z = i\beta\psi - f_5$, $i\beta\psi = z + f_5$, and $\psi = -\frac{i}{\beta}(z+f_5)$. Thus,

$$i\beta\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)\langle\psi_x,w\rangle - \rho_2\|z\|_2^2 + \kappa\|\varphi_x+\psi\|_2^2 + \gamma_3\langle z,\varphi_x+\psi\rangle + i\frac{b\lambda}{\kappa\beta}\langle\psi_x,w\rangle - i\frac{b\lambda}{\kappa\beta}\langle z,u_x\rangle$$
$$= \rho_2\langle f_6,\varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle - \rho_2\langle z_x,f_3\rangle + \rho_2\langle f_{5x},w\rangle + \rho_2\langle z,f_5\rangle - i\frac{b\lambda}{\kappa\beta}\langle\psi_x,f_3\rangle + i\frac{b\lambda}{\kappa\beta}\langle f_5,u_x\rangle.$$

Consequently, we have

$$\kappa\|\varphi_x+\psi\|_2^2 = -i\beta\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)\langle\psi_x,w\rangle + \rho_2\|z\|_2^2 - \gamma_3\langle z,\varphi_x+\psi\rangle + i\frac{b\lambda}{\kappa\beta}\langle\psi_x,w\rangle - i\frac{b\lambda}{\kappa\beta}\langle z,u_x\rangle +$$
$$+\rho_2\langle f_6,\varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle - \rho_2\langle z_x,f_3\rangle + \rho_2\langle f_{5x},w\rangle + \rho_2\langle z,f_5\rangle - i\frac{b\lambda}{\kappa\beta}\langle\psi_x,f_3\rangle + i\frac{b\lambda}{\kappa\beta}\langle f_5,u_x\rangle.$$

Now, using Hölder's inequality, Young's inequality, (42), and (43), we arrive at the required conclusion. □

Lemma 4.5. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system* (35)–(40)*. Then, there exists a positive constant C_4 such that*

$$b\|\psi_x\|_2^2 - \frac{C_4}{|\beta|}\left(\|\psi_x\|_2^2 + \|z\|_2^2 + |w\|_2^2\right) \le C_4\left(|\beta|\left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right||\langle\psi_x,w\rangle| + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right). \tag{50}$$

Proof. Multiplying (40) by $\overline{\psi}$ and integrating over $(0, l)$, we obtain

$$i\beta\rho_2\langle z,\psi\rangle + b\|\psi_x\|_2^2 + \kappa\langle\varphi_x+\psi,\psi\rangle + \gamma_3\langle z,\psi\rangle = \rho_2\langle f_6,\psi\rangle.$$

By (39), we get $i\beta\psi = z + f_5$. Substituting appropriately, we get the following equation:

$$-\rho_2\|z\|_2^2 + b\|\psi_x\|_2^2 + \kappa\langle\varphi_x+\psi,\psi\rangle + \gamma_3\langle z,\psi\rangle = \rho_2\langle f_6,\psi\rangle + \rho_2\langle z,f_5\rangle.$$

Now, the required conclusion is obtained by using Hölder's inequality, Young's inequality, Poincaré's inequality, (42), and (46). □

Lemma 4.6. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system* (35)–(40)*. Then, there exists a positive constant C_5 such that*

$$\rho_1\|w\|_2^2 - \frac{C_5}{|\beta|}\left(\|\psi_x\|_2^2 + \|z\|_2^2 + \|w\|_2^2\right) \le C_5\left(|\beta|\left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right||\langle\psi_x,w\rangle| + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right). \tag{51}$$

Proof. Multiplying (38) by $\overline{\varphi}$ and integrating over $(0, l)$, we obtain

$$i\beta\rho_1\langle w,\varphi\rangle + \kappa\langle\varphi_x+\psi,\varphi_x\rangle + \lambda\|\varphi\|_2^2 - \lambda\langle u,\varphi\rangle = \rho_1\langle f_4,\varphi\rangle.$$

From (37), it follows that $i\beta\varphi = w + f_3$. Substituting appropriately, we get the following equality

$$\rho_1\|w\|_2^2 = \kappa\langle\varphi_x+\psi,\varphi_x\rangle + \lambda\|\varphi\|_2^2 - \lambda\langle u,\varphi\rangle - \rho_1\langle f_4,\varphi\rangle - \rho_1\langle w,f_3\rangle.$$

Using Hölder's inequality, Young's inequality, Poincaré's inequality, (10), (11), (43), and (46), we arrive at the required result. □

Now, we are in a position to prove Theorem 4.3.

Proof of Theorem 4.3. Assume that (29) is true. Because of Lemmas 4.4, 4.5, and 4.6, we have

$$\kappa\|\varphi_x+\psi\|_2^2-\frac{C_3}{|\beta|}\left(\|\psi_x\|_2^2+\|z\|_2^2+\|w\|_2^2\right)\le C_3\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{52}$$

$$b\|\psi_x\|_2^2-\frac{C_4}{|\beta|}\left(\|\psi_x\|_2^2+\|z\|_2^2+\|w\|_2^2\right)\le C_4\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}, \tag{53}$$

and

$$\rho_1\|w\|_2^2-\frac{C_5}{|\beta|}\left(\|\psi_x\|_2^2+\|z\|_2^2+\|w\|_2^2\right)\le C_5\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{54}$$

Using Hölder's inequality, Young's inequality, Poincaré's inequality, and (11), we obtain a constant C' such that

$$\|U\|_{\mathcal{H}}^2\le C'\left(\|v\|_2^2+\rho_1\|w\|_2^2+\rho_2\|z\|_2^2+\alpha\|u_x\|_2^2+\kappa\,\|\varphi_x+\psi\|_2^2+b\|\psi_x\|_2^2\right).$$

Also, by Lemmas 4.2, 4.3, 4.4, 4.5, and 4.6, we obtain a positive constant C such that

$$\|U\|_{\mathcal{H}}^2-\frac{C}{|\beta|}\left(\|\psi_x\|_2^2+\|z\|_2^2+\|w\|_2^2+\right)\le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Taking $c=\dfrac{1}{2C}$, we obtain

$$\|U\|_{\mathcal{H}}\le 2C\|F\|_{\mathcal{H}}. \tag{55}$$

Therefore, $\overline{\lim}_{|\beta|\to\infty}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\le 2C<\infty$. Furthermore, from Lemma 4.1, it follows that $i\mathbb{R}\subset\rho(\mathcal{A})$. Therefore, Gearhart-Herbst-Prüss-Huang theorem guarantees that the semigroup S(t) is exponentially stable.

Now, assume that (29) is not true. Our goal now is to show that (31) does not happen. To this end, let us make explicitly a sequence $\beta_n\in\mathbb{R}$ and a sequence of bounded functions $F_n=(f_{1n},f_{2n},f_{3n},f_{4n},f_{5n},f_{6n})'$ such that

$$\limsup_{|\beta|\to\infty}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}=\infty.$$

By (34), we have

$$i\beta u-v=f_{1n}, \tag{56}$$
$$i\beta v-\alpha u_{xx}-\lambda(\varphi-u)+\gamma_1 v=f_{2n}, \tag{57}$$
$$i\beta\varphi-w=f_{3n}, \tag{58}$$
$$i\rho_1\beta w-\kappa(\varphi_x+\psi)_x+\lambda(\varphi-u)=\rho_1 f_{4n}, \tag{59}$$
$$i\beta\psi-z=f_{5n}, \tag{60}$$
$$i\rho_2\beta z-b\psi_{xx}+\kappa(\varphi_x+\psi)+\gamma_3 z=\rho_2 f_{6n}. \tag{61}$$

Taking $f_{1n}=f_{3n}=f_{5n}=0$, we have

$$-\beta^2 u-\alpha u_{xx}-\lambda(\varphi-u)+i\gamma_1\beta u=f_{2n}, \tag{62}$$
$$-\rho_1\beta^2\varphi-\kappa(\varphi_x+\psi)_x+\lambda(\varphi-u)=\rho_1 f_{4n}, \tag{63}$$
$$-\rho_2\beta^2\psi-b\psi_{xx}+\kappa(\varphi_x+\psi)+i\gamma_3\beta\psi=\rho_2 f_{6n}. \tag{64}$$

Because of the boundary conditions (4), we can suppose that $u(x)=A\sin(\theta_n x)$, $\varphi(x)=B\sin(\theta_n x)$, and $\psi(x)=C\cos(\theta_n x)$, where $\theta_n=\frac{n\pi}{l}$, with $n\in N$. Hence, we have

$$[P_1A-\lambda B]\sin(\theta_n x)=f_{2n}, \tag{65}$$
$$[-\lambda A+P_2B+\kappa\theta_n C]\sin(\theta_n x)=\rho_1 f_{4n}, \tag{66}$$
$$[\kappa\theta_n B+P_3C]\cos(\theta_n x)=\rho_2 f_{6n}, \tag{67}$$

where

$$P_1=-\beta^2+\alpha\theta_n^2+\lambda+i\gamma_1\beta,\quad P_2=\lambda-\rho_1\beta^2+\kappa\theta_n^2,\quad\text{and}\quad P_3=-\rho_2\beta^2+b\theta_n^2+\kappa+i\gamma_3\beta.$$

Now, taking $f_{2n}=0$, $f_{4n}=\rho_1^{-1}\sin(\theta_n x)$, and $f_{6n}=\rho_2^{-1}\cos(\theta_n x)$, we have the following system:

$$P_1A-\lambda B=0, \tag{68}$$

$$-\lambda A+P_2B+\kappa\theta_n C=1, \tag{69}$$

$$\kappa\theta_n B+P_3C=0. \tag{70}$$

Solving the this system, we obtain

$$B=\frac{P_1P_3}{P_3\left(P_1P_2-\lambda^2\right)-\kappa^2\theta_n^2P_1}. \tag{71}$$

Choosing $\beta=\sqrt{\frac{\kappa}{\rho_1}}\theta_n$, we get

$$\begin{aligned}P_1P_3=&\rho_2\left(\alpha-\frac{\kappa}{\rho_1}\right)\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)\theta_n^4+i\left[\left(\alpha-\frac{\kappa}{\rho_1}\right)\gamma_3+\gamma_1\rho_2\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)\right]\sqrt{\frac{\kappa}{\rho_1}}\theta_n^3\\&+\left[\kappa\left(\alpha-\frac{\kappa}{\rho_1}\right)+\lambda\rho_2\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)-\gamma_1\gamma_3\frac{\kappa}{\rho_1}\right]\theta_n^2+\left[i\left(\lambda\gamma_3+\gamma_1\kappa\right)\sqrt{\frac{\kappa}{\rho_1}}\right]\theta_n+\lambda\kappa,\end{aligned}$$

and

$$\begin{aligned}P_3\left(P_1P_2-\lambda^2\right)-\kappa^2\theta_n^2P_1=&\lambda\rho_2\left[\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)-\kappa^2\right]\left(\alpha-\frac{\kappa}{\rho_1}\right)\theta_n^4+i\lambda\left[\left(\alpha-\frac{\kappa}{\rho_1}\right)\gamma_3+\gamma_1\rho_2\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)-\gamma_1\kappa^2\right]\sqrt{\frac{\kappa}{\rho_1}}\theta_n^3\\&+\lambda\left[\kappa\left(\alpha-\frac{\kappa}{\rho_1}\right)-\gamma_1\gamma_3\frac{\kappa}{\rho_1}-\kappa^2\right]\theta_n^2+i\lambda\gamma_1\kappa\sqrt{\frac{\kappa}{\rho_1}}\theta_n.\end{aligned}$$

Thus, for $\dfrac{\kappa}{\rho_1}\neq\dfrac{b}{\rho_2}$, we get

$$|B|\to\frac{\rho_2\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)}{\lambda\rho_2\left(\frac{b}{\rho_2}-\frac{\kappa}{\rho_1}\right)-\kappa^2}\neq 0,\quad\text{when}\quad\beta\to\infty. \tag{72}$$

Therefore, when $\beta\to\infty$, we have

$$\|U\|_{\mathcal{H}}\geq\rho_1\|w\|_2^2=\frac{\rho_1 l}{2}\beta^2|B|^2\to+\infty. \tag{73}$$

This completes the proof of Theorem 4.3. □

Proof of Theorem 4.4. Assume that (29) is not true; that is,

$$\frac{\kappa}{\rho_1}\neq\frac{b}{\rho_2}.$$

Because of Lemmas 4.2, 4.3, 4.4, 4.5, and 4.6, we obtain a constant C', which is independent of β, such that

$$\|U\|_{\mathcal{H}}^2\leq C'\left(|\beta|\,|\langle\psi_x,w\rangle|+\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right). \tag{74}$$

Using (37) and (39), we get

$$\langle\psi_x,w\rangle=\langle z,\varphi_x\rangle+\langle f_6,\varphi_x\rangle-\langle\psi_x,f_3\rangle.$$

Using Hölder's inequality, we obtain

$$\|U\|_{\mathcal{H}}^2\leq C\left(|\beta||\langle z,\varphi_x\rangle|+2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right).$$

Using Young's inequality and Lemma 4.2, we obtain a positive constant C such that

$$\|U\|_{\mathcal{H}}^2\leq C'|\beta|^2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{75}$$

Thus,

$$\frac{1}{|\beta|^2}\|(i\beta I-\mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})}\leq C,\text{when }|\beta|\to\infty. \tag{76}$$

Using Theorem 4.2, we obtain

$$\|S(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}}, \tag{77}$$

implying that

$$\|S(t)\mathcal{A}^{-1}F\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}}\|F\|_{\mathcal{H}}, \tag{78}$$

for all $F \in \mathcal{H}$. As zero is in the resolvent of the operator $\mathcal{A}$, it follows that $\mathcal{A}$ is onto over $\mathcal{H}$. Now, taking $U_0 \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U_0 = F$, we get

$$\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}}\|U_0\|_{\mathcal{H}}, \quad \forall\, t \geq 0. \tag{79}$$

Therefore, the semigroup $S(t)$ is polynomially stable. To show that the polynomial stability rate in question is optimal, we prove it by contradiction argument. Suppose that the rate $t^{-1/2}$ can be improved; that is, there exists $0 < \epsilon < 2$ such that

$$\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{t^{1/(2-\epsilon)}}\|U_0\|_{\mathcal{H}}, \quad \forall\, t \geq 0. \tag{80}$$

Using Theorem 4.2, we obtain

$$\frac{1}{|\beta|^{2-\epsilon}}\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \text{when } |\beta| \to \infty. \tag{81}$$

However, by (73), we obtain a sequence of real numbers (β) and a sequence of limited functions F in $\mathcal{H}$ such that

$$\frac{1}{|\beta|^{2-\epsilon}}\|U\|_{\mathcal{H}} \geq \frac{\rho_1 l}{2}|\beta|^{\epsilon}B^2 \to \infty, \text{when } |\beta| \to \infty. \tag{82}$$

Hence,

$$\frac{1}{|\beta|^{2-\epsilon}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty, \text{when } |\beta| \to \infty. \tag{83}$$

So, we have a contradiction. Therefore, the rate cannot be improved. This completes the proof of Theorem 4.4. □

Conditions $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, and $\gamma_3 = 0$

The main results of this subsection are in the form of the following two theorems:

Theorem 4.5. *Assume that $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, and $\gamma_3 = 0$. The associated semigroup $S(t) = e^{\mathcal{A}t}$ is exponentially stable if and only if the relation (29) is true.*

Theorem 4.6. *Assume that $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, and $\gamma_3 = 0$. If (29) is not true, then the C_0-semigroup $S(t)$ associated with the system is polynomially stable and satisfies*

$$\|S(t)U_0\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}}\|U_0\|_{\mathcal{D}(\mathcal{A})} \;\; \forall\, U_0 \in \mathcal{D}(\mathcal{A}), \tag{84}$$

where C is a positive constant. Furthermore, this stability rate is optimal.

Before proving Theorems 4.5 and 4.6, we first establish some lemmas. Consider the resolvent equation

$$(i\beta I - \mathcal{A})U = F, \tag{85}$$

where $U = (u, v, \varphi, w, \psi, z)' \in \mathcal{D}(\mathcal{A})$, $F = (f_1, f_2, f_3, f_4, f_5, f_6)' \in \mathcal{H}$, and $\beta \in \mathbb{R}$. Because of (34), we obtain the following system:

$$i\beta u - v = f_1, \tag{86}$$
$$i\beta v - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 v = f_2, \tag{87}$$
$$i\beta\varphi - w = f_3, \tag{88}$$
$$i\rho_1\beta w - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w = \rho_1 f_4, \tag{89}$$
$$i\beta\psi - z = f_5, \tag{90}$$
$$i\rho_2\beta z - b\psi_{xx} + \kappa(\varphi_x + \psi) = \rho_2 f_6. \tag{91}$$

Lemma 4.7. *The set formed by the resolvents of the operator $\mathcal{A}$ contains the set $i\mathbb{R}$; that is, $i\mathbb{R} \subset \rho(\mathcal{A})$.*

Proof. From Lemma 3.2, it follows that $0 \in \rho(\mathcal{A})$. Now, take $\beta \in \mathbb{R}\backslash\{0\}$. Suppose that there is some $i\beta$ that is an eigenvalue of operator A; that is,

$$AU = i\beta U \ \text{ with } \ U \neq 0. \tag{92}$$

On the other hand, multiplying (92) by U, we get

$$\langle AU, U\rangle_{\mathcal{H}} = i\beta\|U\|^2_{\mathcal{H}}.$$

Taking the real part of the last equation and using the identity obtained in Lemma 3.1, we have $\|v\|_2^2 = \|w\|_2^2 = 0$, and hence $v = w = 0$. Next, take $F = (0,0,0,0,0,0)$. Because of (86) and (88), we have $u = \varphi = 0$. Because of (89), we have $\psi = 0$, and because of (90), we obtain $z = 0$. This is a contradiction and hence there are no imaginary eigenvalues. □

Throughout the proof of the subsequent lemmas, we assume that $\beta \neq 0$. Consequently, we routinely use the fact that there is a constant c such that $0 < \dfrac{1}{|\beta|} < c < 1$ (which will be explained later).

Lemma 4.8. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (86)–(91). Then, there exists a positive constant C such that*

$$\|v\|_2^2 + \rho_1\|w\|_2^2 \leq C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{93}$$

Proof. Note that

$$i\beta\|U\|^2_{\mathcal{H}} - \langle AU, U\rangle_{\mathcal{H}} = \langle F, U\rangle_{\mathcal{H}}.$$

By Lemma 3.1, we get

$$\gamma_1\|v\|_2^2 + \gamma_2\|w\|_2^2 = \Re\langle F, U\rangle_{\mathcal{H}}.$$

Now, by Hölder's inequality, the required conclusion is obtained. □

Lemma 4.9. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (35)–(40). Then, there exists a positive constant C_1 such that*

$$\alpha\|u_x\|_2^2 \leq C_1\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{94}$$

Proof. Multiplying (36) by $\overline{u}$ and integrating over $(0, l)$, we obtain

$$i\beta\langle v, u\rangle + \alpha\|u_x\|_2^2 + \lambda\|u\|_2^2 - \lambda\langle\varphi, u\rangle + \gamma_1\langle v, u\rangle = \langle f_2, u\rangle. \tag{95}$$

From (35) and (37), we have $-i\beta u = -v - f_1$ and $\varphi = -\frac{i}{\beta}(w + f_3)$. Hence,

$$-\|v\|_2^2 + \alpha\|u_x\|_2^2 + \lambda\|u\|_2^2 + \frac{i\lambda}{\beta}\langle w, u\rangle + \gamma_1\langle v, u\rangle = \langle f_2, u\rangle + \langle v, f_1\rangle - \frac{i\lambda}{\beta}\langle f_3, u\rangle. \tag{96}$$

Now, by using Hölder's inequality, Young's inequality, Poincaré's inequality, (93), and $\dfrac{1}{|\beta|} < 1$, we arrive at the desired conclusion. □

Lemma 4.10. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (35)–(40). Then, there exists a positive constant C_2 such that*

$$b\|\psi_x\|_2^2 - C_2\|z\|_2^2 \leq C_2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{97}$$

Proof. Multiplying (40) by $\overline{\psi}$ and integrating over $(0, l)$, we obtain

$$i\beta\rho_2\langle z, \psi\rangle + b\|\psi_x\|_2^2 + \kappa\langle\varphi_x + \psi, \psi\rangle = \rho_2\langle f_6, \psi\rangle. \tag{98}$$

Multiplying (38) by $-\int_0^x \overline{\psi}(s)\,ds$ and integrating over $(0, l)$, we obtain

$$\begin{aligned} -i\beta\rho_1\left\langle w, \int_0^x \psi(s)\,ds\right\rangle - \kappa\langle\varphi_x + \psi, \psi\rangle + \lambda\left\langle \varphi, \int_0^x \psi(s)\,ds\right\rangle - \lambda\left\langle u, \int_0^x \psi(s)\,ds\right\rangle - \gamma_2\left\langle w, \int_0^x \psi(s)\,ds\right\rangle \\ = -\rho_1\left\langle f_4, \int_0^x \psi(s)\,ds\right\rangle. \end{aligned} \tag{99}$$

Adding (98) and (99), we get

$$i\beta\rho_2\left\langle z,\psi\right\rangle - i\beta\rho_1\left\langle w,\int_0^x\psi(s)\,ds\right\rangle + b\|\psi_x\|_2^2 + \lambda\left\langle \varphi,\int_0^x\psi(s)\,ds\right\rangle - \lambda\left\langle u,\int_0^x\psi(s)\,dx\right\rangle - \gamma_2\left\langle w,\int_0^x\psi(s)\,dx\right\rangle$$
$$= \rho_2\left\langle f_6,\psi\right\rangle - \rho_1\left\langle f_4,\int_0^x\psi(s)\,dx\right\rangle.$$

From (37) and (39), we get $i\beta\varphi = w + f_3$ and $i\beta\psi = z + f_5$, respectively. So, substituting appropriately, we get the following equation:

$$-\rho_2\|z\|_2^2 + \rho_1\left\langle w,\int_0^x z(s)\,ds\right\rangle + b\|\psi_x\|_2^2 - i\frac{\lambda}{\beta}\left\langle w,\int_0^x\psi(s)\,ds\right\rangle - \lambda\left\langle u,\int_0^x\psi(s)\,ds\right\rangle - \gamma_2\left\langle w,\int_0^x\psi(s)\,ds\right\rangle$$
$$= \rho_2\left\langle f_6,\psi\right\rangle - \rho_1\left\langle f_4,\int_0^x\psi(s)\,ds\right\rangle + \rho_2\left\langle z,f_5\right\rangle - \rho_1\left\langle w,\int_0^x f_5(s)\,dx\right\rangle + i\frac{\lambda}{\beta}\left\langle f_3,\int_0^x\psi(s)\,ds\right\rangle.$$

Now, using Hölder's inequality, Young's inequality, Poincaré's inequality, (93), and $\frac{1}{|\beta|} < 1$, we arrive at the required conclusion. □

Lemma 4.11. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (35)–(40). Then, there exists a positive constant C_3 such that*

$$\kappa\|\varphi_x + \psi\|_2^2 - \frac{C_3}{|\beta|}\|z\|_2^2 \le C_3\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{100}$$

Proof. Multiplying (38) by $\overline{\varphi}$ and integrating over $(0,l)$, we obtain

$$i\beta\rho_1\langle w,\varphi\rangle + \kappa\left\langle \varphi_x + \psi,\varphi_x\right\rangle + \lambda\|\varphi\|_2^2 - \lambda\langle u,\varphi\rangle + \gamma_2\langle w,\varphi\rangle = \rho_1\langle f_4,\varphi\rangle.$$

Note that

$$\left\langle \varphi_x + \psi,\varphi_x\right\rangle = \|\varphi_x + \psi\|_2^2 - \left\langle \varphi_x + \psi,\psi\right\rangle, \tag{101}$$

hence, $i\beta\rho_1\langle w,\varphi\rangle + \kappa\|\varphi_x+\psi\|_2^2 - \kappa\left\langle\varphi_x+\psi,\psi\right\rangle + \lambda\|\varphi\|_2^2 - \lambda\langle u,\varphi\rangle + \gamma_2\langle w,\varphi\rangle = \rho_1\langle f_4,\varphi\rangle$. From (88) and (90), we get $i\beta\varphi = w + f_3$ and $\psi = -\frac{i}{\beta}(z + f_5)$. So, substituting appropriately, we obtain the following equation:

$$-\rho_1\|w\|_2^2 + \kappa\|\varphi_x + \psi\|_2^2 - i\frac{\kappa}{\beta}\left\langle \varphi_x + \psi, z\right\rangle + \lambda\|\varphi\|_2^2 - i\frac{\lambda}{\beta}\langle u,w\rangle + i\frac{\gamma_2}{\beta}\|w\|_2^2$$
$$= \rho_1\langle f_4,\varphi\rangle + \rho_1\langle w,f_3\rangle + i\frac{\lambda}{\beta}\langle u,f_3\rangle - i\frac{\gamma_2}{\beta}\langle w,f_3\rangle + i\frac{\kappa}{\beta}\left\langle \varphi_x + \psi, f_5\right\rangle.$$

Now, using Hölder's inequality, Young's inequality, Poincaré's inequality, (93), and (97), we arrive at the desired result. □

Lemma 4.12. *Let $U = (u, v, \varphi, w, \psi, z)'$ be the solution of system (86)–(91). Then, there exists a positive constant C_4 such that*

$$\rho_2\|z\|_2^2 - C_4\left(\frac{1}{|\beta|}|\langle\psi_x,w\rangle| + |\langle\psi,u_x\rangle| + |\left\langle\psi_x,w\right\rangle|\right) \le C_4\left(|\beta|\left|\frac{\rho_1}{\kappa} - \frac{\rho_2}{b}\right||\langle\psi_x,w\rangle| + \|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}\right). \tag{102}$$

Proof. Multiplying (91) by $\overline{\varphi}_x + \overline{\psi}$ and integrating over $(0,l)$, we obtain

$$-i\beta\rho_2\langle z_x,\varphi\rangle + i\beta\rho_2\langle z,\psi\rangle + b\left\langle\psi_x,(\varphi_x+\psi)_x\right\rangle + \kappa\|\varphi_x+\psi\|_2^2 = \rho_2\left\langle f_6,\varphi_x+\psi\right\rangle. \tag{103}$$

From (89), it follows that

$$-i\beta\rho_1\overline{w} - \kappa(\overline{\varphi}_x + \overline{\phi})_x + \lambda(\overline{\varphi} - \overline{u}) + \gamma_2\overline{w} = \rho_1\overline{f_4}. \tag{104}$$

Multiplying (104) by $\frac{b}{\kappa}\psi_x$ and integrating over $(0,L)$, we obtain

$$-i\beta\frac{b\rho_1}{\kappa}\langle\psi_x,w\rangle - b\left\langle\psi_x,(\varphi_x+\psi)_x\right\rangle + \frac{b\lambda}{\kappa}\langle\psi_x,\varphi\rangle + \frac{b\lambda}{\kappa}\langle\psi,u_x\rangle + \frac{b\gamma_2}{\kappa}\left\langle\psi_x,w\right\rangle = \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle. \tag{105}$$

Adding (103) and (105), we get

$$-i\beta\rho_2\langle z_x,\varphi\rangle - i\beta\frac{b\rho_1}{\kappa}\langle\psi_x,w\rangle + i\beta\rho_2\langle z,\psi\rangle + \kappa\|\varphi_x+\psi\|_2^2 + \frac{b\lambda}{\kappa}\langle\psi_x,\varphi\rangle + \frac{b\lambda}{\kappa}\langle\psi,u_x\rangle + \frac{b\gamma_2}{\kappa}\left\langle\psi_x,w\right\rangle = \rho_2\left\langle f_6,\varphi_x+\psi\right\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x,f_4\rangle.$$

From (88), it follows that $i\beta\varphi = w + f_3$. Hence, we have

$$\rho_2\langle z_x, w\rangle - i\beta\frac{b\rho_1}{\kappa}\langle\psi_x, w\rangle + i\beta\rho_2\langle z,\psi\rangle + \kappa\|\varphi_x+\psi\|_2^2 + i\frac{b\lambda}{\kappa\beta}\langle\psi_x, w\rangle + \frac{b\lambda}{\kappa}\langle\psi, u_x\rangle + \frac{b\gamma_2}{\kappa}\langle\psi_x, w\rangle$$
$$= \rho_2\langle f_6, \varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x, f_4\rangle - \rho_2\langle z_x, f_3\rangle - i\frac{b\lambda}{\kappa\beta}\langle\psi_x, f_3\rangle.$$

From (90), we obtain $z = i\beta\psi - f_5$ and $i\beta\psi = z + f_5$. Hence, we have

$$i\beta\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)\langle\psi_x, w\rangle - \rho_2\|z\|_2^2 + \kappa\|\varphi_x+\psi\|_2^2 + i\frac{b\lambda}{\kappa\beta}\langle\psi_x, w\rangle + \frac{b\lambda}{\kappa}\langle\psi, u_x\rangle + \frac{b\gamma_2}{\kappa}\langle\psi_x, w\rangle$$
$$= \rho_2\langle f_6, \varphi_x+\psi\rangle + \frac{b\rho_1}{\kappa}\langle\psi_x, f_4\rangle - i\frac{b\lambda}{\kappa\beta}\langle\psi_x, f_3\rangle - \rho_2\langle z_x, f_3\rangle + \rho_2\langle f_{5x}, w\rangle + \rho_2\langle z, f_5\rangle.$$

Consequently, we have

$$\rho_2\|z\|_2^2 = \kappa\|\varphi_x+\psi\|_2^2 + i\beta\left(\rho_2 - \frac{b\rho_1}{\kappa}\right)\langle\psi_x, w\rangle + i\frac{b\lambda}{\kappa\beta}\langle\psi_x, w\rangle + \frac{b\lambda}{\kappa}\langle\psi, u_x\rangle + \frac{b\gamma_2}{\kappa}\langle\psi_x, w\rangle$$
$$- \rho_2\langle f_6, \varphi_x+\psi\rangle - \frac{b\rho_1}{\kappa}\langle\psi_x, f_4\rangle + i\frac{b\lambda}{\kappa\beta}\langle\psi_x, f_3\rangle + \rho_2\langle z_x, f_3\rangle - \rho_2\langle f_{5x}, w\rangle - \rho_2\langle z, f_5\rangle.$$

Now, using Hölder's inequality, Young's inequality, and (100), we arrive at the desired conclusion. □

Now, we are ready to prove Theorem 4.3.

Proof of Theorem 4.3. Assuming that (29) is true. Because of Lemma 4.5, we obtain

$$\rho_2\|z\|_2^2 - C_4\left(\frac{1}{|\beta|}|\langle\psi_x, w\rangle| + |\langle\psi, u_x\rangle| + |\langle\psi_x, w\rangle|\right) \le C_4\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}$$

and

$$b\|\psi_x\|_2^2 - C_2C_4\left(\frac{1}{|\beta|}|\langle\psi_x, w\rangle| + |\langle\psi, u_x\rangle| + |\langle\psi_x, w\rangle|\right) \le C_2\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Using Hölder's inequality, Young's inequality, Poincaré's inequality, and (11), we obtain a positive constant C' such that

$$\|U\|_{\mathcal{H}}^2 \le C'\left(\|v\|_2^2 + \rho_1\|w\|_2^2 + \rho_2\|z\|_2^2 + \alpha\|u_x\|_2^2 + \kappa\|\varphi_x+\psi\|_2^2 + b\|\psi_x\|_2^2\right).$$

By Lemmas 4.8, 4.9, 4.10, 4.11, and 4.12, we obtain a positive constant C_6 such that

$$\|U\|_{\mathcal{H}}^2 - C_6\left(\frac{1}{|\beta|}|\langle\psi_x, w\rangle| + |\langle\psi, u_x\rangle| + |\langle\psi_x, w\rangle|\right) - \frac{C_6}{|\beta|}\|z\|_2^2 \le C_6\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Hence, there exists a positive constant C satisfying

$$\|U\|_{\mathcal{H}}^2 - \frac{C}{|\beta|}(\rho_2\|z\|_2^2 + b\|\psi_x\|_2^2 + \rho_1\|w\|_2^2) \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}},$$

and hence

$$\|U\|_{\mathcal{H}}^2 - \frac{C}{|\beta|}\|U\|_{\mathcal{H}}^2 \le C\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}.$$

Taking $c = \dfrac{1}{2C}$, we obtain

$$\|U\|_{\mathcal{H}} \le 2C\|F\|_{\mathcal{H}}.$$

Therefore,

$$\overline{\lim_{|\beta|\to\infty}}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \le 2C < \infty.$$

Furthermore, from Lemma 4.1, it follows that $i\mathbb{R} \subset \rho(\mathcal{A})$. Consequently, Gearhart-Herbst-Prüss-Huang theorem guarantees that the semigroup $S(t)$ is exponentially stable.

Next, we assume that (29) is not true. Our goal now is to show that (31) does not happen. For this, let us make explicitly a sequence $\beta_n \in \mathbb{R}$ and a sequence of bounded functions $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n})' \in \mathcal{H}$ such that

$$\limsup_{|\beta|\to\infty}\|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Using (85), we obtain

$$\begin{aligned} i\beta u - v &= f_{1n}, \\ i\beta v - \alpha u_{xx} - \lambda(\varphi - u) + \gamma_1 v &= f_{2n}, \\ i\beta\varphi - w &= f_{3n}, \\ i\rho_1\beta w - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w &= \rho_1 f_{4n}, \\ i\beta\psi - z &= f_{5n}, \\ i\rho_2\beta z - b\psi_{xx} + \kappa(\varphi_x + \psi) &= \rho_2 f_{6n}. \end{aligned}$$

Taking $f_{1n} = f_{3n} = f_{5n} = 0$, we arrive at the following equations:

$$\begin{aligned} -\beta^2 u - \alpha u_{xx} - \lambda(\varphi - u) + i\gamma_1\beta u &= f_{2n}, \\ -\rho_1\beta^2\varphi - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + i\gamma_2\beta\varphi &= \rho_1 f_{4n}, \\ -\rho_2\beta^2\psi - b\psi_{xx} + \kappa(\varphi_x + \psi) &= \rho_2 f_{6n}. \end{aligned}$$

Because of the boundary conditions (4), we suppose that $u(x) = A\sin(\theta_n x)$, $\varphi(x) = B\sin(\theta_n x)$, and $\psi(x) = C\cos(\theta_n x)$, where $\theta_n = \dfrac{n\pi}{l}$ with $n \in N$. Hence, we have

$$\begin{aligned} [P_1 A - \lambda B]\sin(\theta_n x) &= f_{2n}, \\ [-\lambda A + P_2 B + \kappa\theta_n C]\sin(\theta_n x) &= \rho_1 f_{4n}, \\ [\kappa\theta_n B + P_3 C]\cos(\theta_n x) &= \rho_2 f_{6n}, \end{aligned}$$

where

$$\begin{aligned} P_1 &= -\beta^2 + \alpha\theta_n^2 + \lambda + i\gamma_1\beta, \\ P_2 &= \lambda - \rho_1\beta^2 + \kappa\theta_n^2 + i\gamma_2\beta, \\ P_3 &= -\rho_2\beta^2 + b\theta_n^2 + \kappa. \end{aligned}$$

Now, taking $f_{2n} = f_{4n} = 0$ and $f_{6n} = \rho_2^{-1}\cos(\theta_n x)$, we have the following system:

$$\begin{aligned} P_1 A - \lambda B &= 0, \\ -\lambda A + P_2 B + \kappa\theta_n C &= 0, \\ \kappa\theta_n B + P_3 C &= 1. \end{aligned}$$

Solving this system, we obtain

$$C = \frac{P_1P_2 - \lambda^2}{P_3(P_1P_2 - \lambda^2) - \kappa\theta_n^2 P_1}.$$

Choosing $\beta = \sqrt{\frac{b}{\rho_2}}\theta_n$, we obtain

$$\begin{aligned} P_1P_2 - \lambda^2 = {} & \rho_1\left(\alpha - \frac{b}{\rho_2}\right)\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right)\theta_n^4 + i\left[\left(\alpha - \frac{b}{\rho_2}\right)\gamma_2 + \gamma_1\rho_1\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right)\right]\sqrt{\frac{b}{\rho_2}}\theta_n^3 \\ & + \left[\lambda\left(\alpha - \frac{b}{\rho_2}\right) - \gamma_1\gamma_2\frac{b}{\rho_2}\right]\theta_n^2 + i\lambda\left[\gamma_1 + \gamma_2\right]\sqrt{\frac{b}{\rho_2}}\theta_n, \end{aligned}$$

and

$$\begin{aligned} P_3(P_1P_2 - \lambda^2) - \kappa^2\theta_n^2 P_1 = {} & \kappa\left[\rho_1\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right) - \kappa\right]\left(\alpha - \frac{b}{\rho_2}\right)\theta_n^4 + i\kappa\left[\left(\alpha - \frac{b}{\rho_2}\right)\gamma_2 + \gamma_1\rho_1\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right) - \kappa\gamma_1\right]\sqrt{\frac{b}{\rho_2}}\theta_n^3 \\ & + \kappa\left[\lambda\left(\alpha - \frac{b}{\rho_2}\right) - \gamma_1\gamma_2\frac{b}{\rho_2} - \lambda\kappa\right]\theta_n^2 + i\lambda\kappa\left[\gamma_1 + \gamma_2\right]\sqrt{\frac{b}{\rho_2}}\theta_n. \end{aligned}$$

Thus, for $\dfrac{\kappa}{\rho_1} \neq \dfrac{b}{\rho_2}$, we get

$$|C| \to \frac{\rho_1\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right)}{\kappa\rho_1\left(\frac{b}{\rho_2} - \frac{\kappa}{\rho_1}\right) - \kappa^2} \neq 0, \quad \text{when} \quad \beta \to \infty.$$

Consequently, when $\beta \to \infty$, we have

$$\|U\|_{\mathcal{H}} \geq \rho_2 \|z\|_2^2 = \frac{\rho_2 l}{2} \beta^2 |C|^2 \to +\infty.$$

Therefore, Gearhart-Herbst-Prüss-Huang theorem guarantees the lack of exponential stability of the semigroup $S(t)$. This completes the proof of Theorem 4.3 □

Proof of Theorem 4.6. Assume that (29) is not true; that is,

$$\frac{\kappa}{\rho_1} \neq \frac{b}{\rho_2}.$$

Because of Lemmas 4.8, 4.9, 4.10, 4.11, and 4.12, we obtain a constant C', which is independent of β, such that

$$\|U\|_{\mathcal{H}}^2 \leq C' \left(|\beta| \, |\langle \psi_x, w \rangle| + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \right).$$

Using Young's inequality and (97), we obtain a positive constant C such that

$$\|U\|_{\mathcal{H}}^2 \leq C |\beta|^2 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}}.$$

Hence,

$$\frac{1}{|\beta|^2} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \text{ when } |\beta| \to \infty.$$

Using Theorem 4.2, we obtain

$$\|S(t)\mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{t^{1/2}}.$$

Thus, for every $F \in \mathcal{H}$, we have

$$\|S(t)\mathcal{A}^{-1} F\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|F\|_{\mathcal{H}}.$$

As 0 belongs to the resolvent of the operator $\mathcal{A}$, it follows that $\mathcal{A}$ is onto over $\mathcal{H}$. Taking $U_0 \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U_0 = F$, we obtain

$$\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|U_0\|_{\mathcal{H}}, \quad \forall\, t \geq 0.$$

Therefore, the semigroup $S(t)$ is polynomially stable. To prove that the polynomial stability rate in question is optimal, we use the approach of contradiction. Suppose that the rate $t^{-1/2}$ can be improved; that is, there exists $0 < \epsilon < 2$ such that

$$\|S(t)U_0\|_{\mathcal{H}} \leq \frac{C}{t^{1/(2-\epsilon)}} \|U_0\|_{\mathcal{H}}, \quad \forall\, t \geq 0.$$

Using Theorem 4.2, we have

$$\frac{1}{|\beta|^{2-\epsilon}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C, \text{when } |\beta| \to \infty.$$

However, by (73), we obtain a sequence of real numbers (β) and a sequence of bounded functions F in $\mathcal{H}$ such that

$$\frac{1}{|\beta|^{2-\epsilon}} \|U\|_{\mathcal{H}} \geq \frac{\rho_2 l}{2} |\beta|^{\epsilon} B^2 \to \infty, \text{when } |\beta| \to \infty.$$

Hence,

$$\frac{1}{|\beta|^{2-\epsilon}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \to \infty, \text{when } |\beta| \to \infty.$$

So, we arrive at a contradiction. Therefore, the rate cannot be improved. This completes the proof of Theorem 4.6. □

Conditions $\gamma_1 = 0$, $\gamma_2 \neq 0$, and $\gamma_3 \neq 0$

The main result of this subsection is in the form of the following theorem:

Theorem 4.7. *Assume that $\gamma_1 = 0$, $\gamma_2 \neq 0$, and $\gamma_3 \neq 0$. The associated semigroup $S(t) = e^{\mathcal{A}t}$ is not exponentially stable.*

Proof. We take explicitly a sequence $\beta_n \in \mathbb{R}$ and a sequence of bounded functions $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n})'$ such that

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = \infty.$$

Similar to system (86)–(91), we have

$$\begin{aligned} i\beta u - v &= f_{1n}, \\ i\beta v - \alpha u_{xx} - \lambda(\varphi - u) &= f_{2n}, \\ i\beta\varphi - w &= f_{3n}, \\ i\rho_1\beta w - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w &= \rho_1 f_{4n}, \\ i\beta\psi - z &= f_{5n}, \\ i\rho_2\beta z - b\psi_{xx} + \kappa(\varphi_x + \psi) + \gamma_3 z &= \rho_2 f_{6n}. \end{aligned}$$

Taking $f_{1n} = f_{3n} = f_{5n} = 0$, we arrive at

$$\begin{aligned} -\beta^2 u - \alpha u_{xx} - \lambda(\varphi - u) &= f_{2n}, \\ -\rho_1\beta^2\varphi - \kappa(\varphi_x + \psi)_x + \lambda(\varphi - u) + i\gamma_2\beta\varphi &= \rho_1 f_{4n}, \\ -\rho_2\beta^2\psi - b\psi_{xx} + \kappa(\varphi_x + \psi) + i\gamma_3\beta\psi &= \rho_2 f_{6n}. \end{aligned}$$

Because of the boundary conditions (4), we suppose that $u(x) = A\sin(\theta_n x)$, $\varphi(x) = B\sin(\theta_n x)$, and $\psi(x) = C\cos(\theta_n x)$, where $\theta_n = \dfrac{n\pi}{l}$ with $n \in N$. Then, we have

$$\begin{aligned} [P_1 A - \lambda B]\sin(\theta_n x) &= f_{2n}, \\ [-\lambda A + P_2 B + \kappa\theta_n C]\sin(\theta_n x) &= \rho_1 f_{4n}, \\ [\kappa\theta_n B + P_3 C]\cos(\theta_n x) &= \rho_2 f_{6n}, \end{aligned}$$

where

$$\begin{aligned} P_1 &= -\beta^2 + \alpha\theta_n^2 + \lambda, \\ P_2 &= \lambda - \rho_1\beta^2 + \kappa\theta_n^2 + i\gamma_2\beta, \\ P_3 &= -\rho_2\beta^2 + b\theta_n^2 + \kappa + i\gamma_3\beta. \end{aligned}$$

Taking $f_{2n} = \sin(\theta_n x)$ and $f_{4n} = f_{6n} = 0$, we obtain the following system:

$$\begin{aligned} P_1 A - \lambda B &= 1, \\ -\lambda A + P_2 B + \kappa\theta_n C &= 0, \\ \kappa\theta_n B + P_3 C &= 0. \end{aligned} \tag{106}$$

Choosing $\beta = \sqrt{\alpha\theta_n^2 + \lambda}$, we have $P_1 = 0$. By (106), we obtain $B = -\frac{1}{\lambda}$. Consequently, when $\beta \to \infty$, we have

$$\|U\|_{\mathcal{H}} \geq \rho_1\|w\|_2^2 = \frac{\rho_1 l}{2}\beta^2|B|^2 \to +\infty.$$

Therefore, Gearhart-Herbst-Prüss-Huang theorem guarantees the lack of exponential stability of the semigroup $S(t)$. □

5. Conclusion

In this work, we have used the Timoshenko-Ehrenfest theory to study a suspension bridge with partial internal frictional damping. We also have used a suitable Hilbert space to build a semigroup to prove that its energy is dissipative and have applied the Lummer-Phillips theorem to obtain the system's solution. Furthermore, if the condition $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ is valid, we have found that the mentioned semigroup has exponential decay; otherwise, the semigroup has polynomial decay. The suspension bridge with other types of damping can be studied in further work.

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