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Research Article

Proper total domination in graphs

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Abstract

A vertex u in a graph G totally dominates a vertex v if v is adjacent to u. A subset S of the vertex set of a graph G is a total dominating set for G if every vertex of G is totally dominated by at least one vertex of S. The minimum cardinality of a total dominating set for G is the total domination number $\gamma_t(G)$ of G. If S is a total dominating set of a graph G, then $\sigma_S(v)$ denotes the number of vertices in S that totally dominate v. A total dominating set S in a graph G is called a proper total dominating set if $\sigma_S(u) \neq \sigma_S(v)$ for every two adjacent vertices u and v of G. Not all graphs possess a proper total dominating set. Those paths and cycles possessing a proper total dominating set are determined. It is shown that every $n \times m$ grid $P_n \square P_m$ (the Cartesian product of paths P_n and P_m of order n and m respectively) with $n \geq m \geq 2$ has a proper total dominating set. Also, for every r-regular bipartite graph H where $r \geq 2$, the graph $H \square P_2$ has a proper total dominating set. The minimum cardinality of a proper total dominating set in G is the proper total domination number $\gamma_{pt}(G)$. All pairs a, b, of positive integers are determined for which there is a graph G with a proper total dominating set such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$.

Keywords: total domination; proper total domination; proper total domination number.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

In recent decades, domination in graphs has grown in popularity in graph theory. While this area evidently began with the work of Berge [1] in 1958 and Ore [6] in 1962, it did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [3]. Since then, a large number of variations and applications of domination have surfaced. A vertex u in a graph G is said to *dominate* a vertex v if either $u = v$ or v is adjacent to u in G. That is, u dominates itself and all vertices in its neighborhood $N(u)$. A subset S of the vertex set of G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum cardinality of a dominating set of G is the *domination number* $\gamma(G)$ of G.

In their 2023 book, Haynes, Hedetniemi, and Henning [4] presented the major results that have been obtained on what they refer to as the core concepts of graph domination. One of these core concepts is standard domination where a vertex dominates itself and each neighbor. Another core concept is total domination, introduced by Cockayne, Dawes, and Hedetniemi [2] in 1977. In total domination, a vertex u in a graph G totally dominates a vertex v if v is adjacent to u. A subset S of the vertex set of G is a *total dominating set* if every vertex of G is totally dominated by at least one vertex of S. In particular, every vertex of S must be adjacent to at least one vertex of S. Therefore, a graph G has a total dominating set if and only if G contains no isolated vertices. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$ of G. The 2013 book by Henning and Yeo [5] deals exclusively with total domination in graphs.

For a total dominating set S of a graph G and a vertex v of G, the number of vertices in S that totally dominate v is denoted by $\sigma_S(v)$. Thus, $1 \leq \sigma_S(v) \leq \deg v$ for each vertex v of G, where $\deg v$ is the degree of v. It is impossible for a graph G to possess a total dominating set S such that every two vertices of G are totally dominated by different numbers of vertices of G.

Observation 1.1. *No nontrivial connected graph G possesses a total dominating set S such that every two vertices of G are totally dominated by different numbers of vertices of S.*

We assume that all graphs under consideration are nontrivial connected graphs.

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2. Proper total domination

While it is impossible for a graph G to possess a total dominating set S such that $\sigma_S(u) \neq \sigma_S(v)$ for every pair u, v of distinct vertices of G, it is possible that $\sigma_S(u) \neq \sigma_S(v)$ for every pair u, v of *adjacent* vertices of G. A total dominating set in a graph G with this property is called a *proper total dominating set* in G. Not all graphs possess a proper total dominating set. Those paths and cycles possessing a proper total dominating set are now determined.

Proposition 2.1. *For an integer $n \geq 2$, the path P_n of order n has a proper total dominating set if and only if $n \equiv 3 \pmod 4$.*

Proof. Let $P_n = (v_1, v_2, \ldots, v_n)$ where $n \geq 2$. It is straightforward to show that P_n has a proper total dominating set only when $n = 3$ for $2 \leq n \leq 5$. Thus, we may assume that $n \geq 6$. First, suppose that P_n has a proper total dominating set S. Then $\sigma_S(v_i) \in \{1, 2\}$ for $1 \leq i \leq n$. In particular, $\sigma_S(v_1) = \sigma_S(v_n) = 1$. Hence, $\sigma_S(v_i) = 2$ if i is even and $\sigma_S(v_i) = 1$ if i is odd. Since $\sigma_S(v_1) = \sigma_S(v_n) = 1$ and $\sigma_S(v_2) = \sigma_S(v_{i-1}) = 2$, it follows that n is odd and $\{v_1, v_2, v_3, v_{n-2}, v_{n-1}, v_n\} \subseteq S$. A block B of P_n with respect to S is a maximal set of consecutive vertices of P_n such that either all vertices of B belong to S or no vertices of B belong to S. Thus, the subgraph of P_n induced by a block is a path. The vertex set of P_n can be expressed as a sequence $(B_1, B_1', B_2, B_2' \ldots, B_k, B_k', B_{k+1})$ of $2k+1$ blocks for some positive integer k such that $x \in B_i$ for $1 \leq i \leq k+1$ if $x \in S$ and $y \in B_i'$ for $1 \leq i \leq k$ if $y \notin S$. First, we verify the following claim.

> **Claim:** $|B_i| = 3$ *for* $1 \leq i \leq k+1$ *and* $|B_i'| = 1$ *for* $1 \leq i \leq k$.

Suppose that $|B_i| \neq 3$ for some integer i with $1 \leq i \leq k+1$ or $|B_i'| \neq 1$ for some integer i with $1 \leq i \leq k$. First, suppose that $B_i = (v_{t+1}, v_{t+2}, \ldots, v_{t+p})$ for some integers $t \geq 0$ and $p \neq 3$. By the argument above, $i \neq 1$ and $i \neq k+1$. If $p = 1$, then $\sigma_S(v_{t+1}) = 0$. If $p = 2$, then $\sigma_S(v_{t+1}) = \sigma_S(v_{t+2}) = 1$. If $p \geq 4$, then $\sigma_S(v_{t+2}) = \sigma_S(v_{t+3}) = 2$. Thus, $1 \leq p \leq 2$ and $p \geq 4$ are impossible. Next, suppose that $B_i' = (v_{t+1}, v_{t+2}, \ldots, v_{t+q})$ for some integers $t \geq 0$ and $q \geq 2$. If $q = 2$, then $\sigma_S(v_{t+1}) = \sigma_S(v_{t+2}) = 1$. If $q \geq 3$, then $\sigma_S(v_{t+2}) = 0$. In either case, a contradiction is produced. Consequently, as claimed, $|B_i| = 3$ for $1 \leq i \leq k+1$ and $|B_i'| = 1$ for $1 \leq i \leq k$. Therefore, $n = \sum_{i=1}^{k}(|B_i| + |B_i'|) + |B_{k+1}| = 4k+3$ and so $n \equiv 3 \pmod 4$.

For the converse, suppose that $n \equiv 3 \pmod 4$. Let $S = \{v_i : i \not\equiv 0 \pmod 4\}$. This is illustrated in Figure 2.1 for $n = 3, 7$, where the solid vertices are those that belong to a proper total dominating set. Since $\sigma_S(v_i) = 1$ if i is odd and $\sigma_S(v_i) = 2$ if i is even, it follows that S is a proper total dominating set of P_n. □

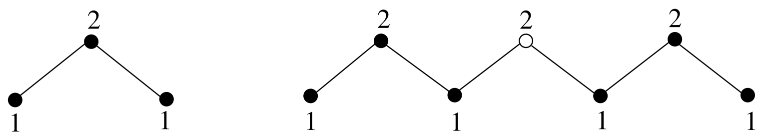


Figure 2.1: Proper total dominating sets in P_3 and P_7.

Proposition 2.2. *For an integer $n \geq 3$, the cycle C_n of order n has a proper total dominating set if and only if $n \equiv 0 \pmod 4$.*

Proof. Clearly, C_3 does not have a proper total dominating set. So, we may assume that $n \geq 4$. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ where $n \geq 4$. First, suppose that C_n has a proper total dominating set S. Since $\sigma_S(v_i) \in \{1, 2\}$ for $1 \leq i \leq n$, it follows that n must be even. Hence, we may assume that n is even. A block B of C_n with respect to S is a maximal set of consecutive vertices of C_n such that either all vertices of B belong to S or no vertices of B belong to S. Thus, the subgraph of C_n induced by a single block is a path. The vertex set of C_n can be expressed therefore as a sequence $(B_1, B_1', B_2, B_2' \ldots, B_k, B_k')$ of $2k$ blocks for some positive integer k such that $x \in B_i$ if $x \in S$ and $y \in B_i'$ if $y \notin S$ where $1 \leq j \leq k$. First, we verify the following claim.

> **Claim:** *For each integer i with $1 \leq i \leq k$, $|B_i| = 3$ and $|B_i'| = 1$.*

Suppose that this claim is false. Then either $|B_1| \neq 3$ or $|B_1'| \neq 1$. Let $B_1 = (v_1, v_2, v_3, \ldots, v_p)$ and $B_1' = (v_{p+1}, v_{p+2}, \ldots, v_{p+q})$ where either $p \neq 3$ or $q \geq 2$. If $p \geq 4$, then $\sigma_S(v_2) = \sigma_S(v_3) = 2$. If $p = 2$, then $\sigma_S(v_1) = \sigma_S(v_2) = 1$. If $p = 1$, then $\sigma_S(v_1) = 0$. Thus, $1 \leq p \leq 2$ and $p \geq 4$ are impossible. Therefore, $|B_i| = 3$ for $1 \leq i \leq k$. If $q \geq 3$, then $\sigma_S(v_{p+2}) = 0$; while if $q = 2$, then $\sigma_S(v_{p+1}) = \sigma_S(v_{p+2}) = 1$. In either case, a contradiction is produced. Consequently, $|B_i'| = 1$ for $1 \leq i \leq k$. Therefore, $n = \sum_{i=1}^{k}(|B_i| + |B_i'|) = 4k$ and so $n \equiv 0 \pmod 4$.

For the converse, suppose that $n \equiv 0 \pmod 4$. Let $S = \{v_i : i \not\equiv 0 \pmod 4\}$. This is illustrated in Figure 2.2 for $n = 4, 8$. Since $\sigma_S(v_i) = 1$ if i is odd and $\sigma_S(v_i) = 2$ if i is even, it follows that S is a proper total dominating function of C_n. □

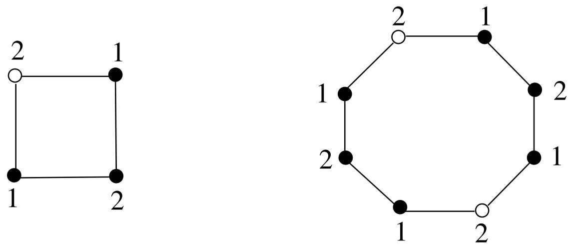


Figure 2.2: Proper total dominating sets of C_4 and C_8.

We now consider a class of graphs, every member of which has a proper total dominating set. For integers n and m with $n \geq m \geq 1$, the Cartesian product $G_{n,m} = P_n \,\square\, P_m$ of the path P_n of order n and the path P_m of order m is referred to as a *grid graph*. In particular, $G_{n,1} = P_n \,\square\, P_1 \cong P_n$. The graph $G_{n,2} = P_n \,\square\, K_2$ is often referred to as a *ladder graph*. Next, we show that for all integers n and m with $n \geq m \geq 2$, the grid $G_{n,m}$ possesses a proper total dominating set. We begin with ladders.

Proposition 2.3. *For each integer $n \geq 2$, the ladder $G_{n,2}$ possesses a proper total dominating set.*

Proof. Let $G = G_{n,2}$ be constructed from the two n-paths $P = (u_1, u_2, \ldots, u_n)$ and $P' = (v_1, v_2, \ldots, v_n)$ by adding the edges $u_i v_i$ for $1 \leq i \leq n$. We consider two cases, according to whether n is odd or n is even.

Case 1. *$n \geq 3$ is odd*. Let $S = V(P) \cup \{v_i : i \text{ is even and } 2 \leq i \leq n-1\}$. Then

$$\sigma_S(x) = \begin{cases} 1 & \text{if } x = u_1, u_n \text{ or } x = v_i, \text{ where } i \text{ is even and } 2 \leq i \leq n-1`, \\ 2 & \text{if } x = v_1, v_n \text{ or } x = u_i, \text{ where } i \text{ is odd and } 3 \leq i \leq n-2, \\ 3 & \text{if } x = u_i \text{ where } i \text{ is even and } 2 \leq i \leq n-1 \text{ or} \\ & \text{if } x = v_i \text{ where } i \text{ is odd and } 3 \leq i \leq n-2. \end{cases}$$

Since $\sigma_S(x) \neq \sigma_S(y)$ for every two adjacent vertices x and y in G, it follows that S is a proper total dominating set of G. This is illustrated in Figure 2.3 for $n = 3, 5, 7$.

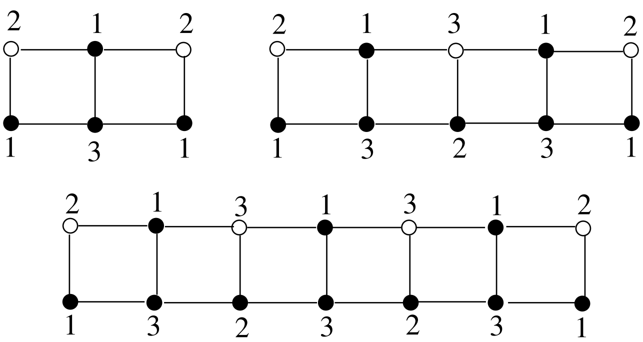


Figure 2.3: Proper total dominating sets in $G_{n,2}$ for $n = 3, 5, 7$.

Case 2. *$n \geq 2$ is even*. First, $G_{2,2} = C_4$ has a proper total dominating set by Proposition 2.2 and $G_{4,2}$ has a proper total dominating set as shown in Figure 2.4.

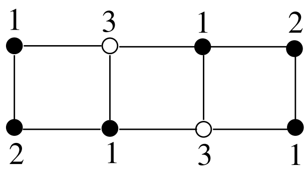


Figure 2.4: A proper total dominating set in $G_{4,2}$.

Thus, we may assume that $n \geq 6$ and so $n - 3 \geq 3$. Let $H = G_{n-3,2}$ be the subgraph of $G_{n,3}$ constructed from the subpaths $Q = (u_1, u_2, \ldots, u_{n-3})$ and $Q' = (v_1, v_2, \ldots, v_{n-3})$ by adding the edges $u_i v_i$ for $1 \leq i \leq n-3$. Next, let S' be the proper total dominating set of H as defined in Case 1 and $S'' = \{u_{n-2}, u_{n-1}, u_n, v_{n-1}\}$. Now, let $S = S' \cup S''$. If $x \in \{u_i, v_i\}$ where $1 \leq i \leq n-4$, then $\sigma_S(x)$ is the same as described in Case 1. Furthermore, $(\sigma_S(u_{n-3}), \sigma_S(u_{n-2}), \sigma_S(u_{n-1}), \sigma_S(u_n)) = (3, 1, 3, 1)$ and $(\sigma_S(v_{n-3}), \sigma_S(v_{n-2}), \sigma_S(v_{n-1}), \sigma_S(v_n)) = (1, 3, 1, 2)$. Since $\sigma_S(u_{n-4}) = 1$ and $\sigma_S(v_{n-4}) = 3$, it follows that S is proper total dominating set. This is illustrated in Figure 2.5 for $n = 6, 8$. $\square$

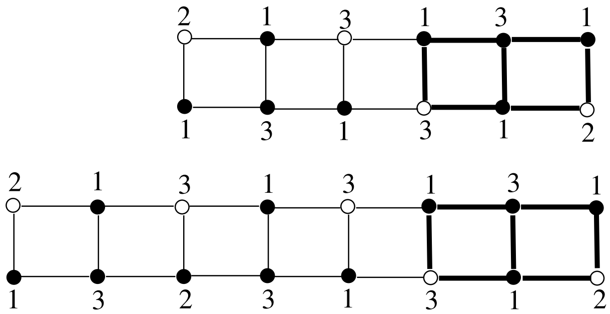


Figure 2.5: Proper total dominating sets in $G_{n,2}$ for $n = 6, 8$.

Let the grid $G_{n,m}$ be constructed from m copies $Q_1, Q_2, \ldots, Q_m$ of n-paths where $Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ for $1 \le i \le m$ such that $u_{i,j}u_{i+1,j} \in E(G)$ for $1 \le i \le m-1$ and $1 \le j \le n$. For a path $Q = (x_1, x_2, \ldots, x_n)$ of order n in a graph G and a set S of vertices in G, let

$$\sigma_S(Q) = (\sigma_S(x_1), \sigma_S(x_2), \ldots, \sigma_S(x_n)).$$

Theorem 2.1. *For every two integers $n, m \ge 2$, the grid graph $G_{n,m}$ possesses a proper total dominating set.*

Proof. By Proposition 2.3, we may assume that $n, m \ge 3$. Let $G = G_{n,m}$ be constructed from m copies $Q_1, Q_2, \ldots, Q_m$ of n-paths where $Q_i = (u_{i,1}, u_{i,2}, \ldots, u_{i,n})$ for $1 \le i \le m$ such that $u_{i,j}u_{i+1,j} \in E(G)$ for $1 \le i \le m-1$ and $1 \le j \le n$. We consider two cases, according to whether at least one of n and m is odd or both n and m are even.

Case 1. *At least one of n and m is odd, say n is odd.* Then m is either odd or even. For $1 \le i \le m$, the set S_i is defined by

$$S_i = \begin{cases} \{u_{i,j} : j \text{ is even and } 2 \le j \le n-1\} & \text{if } i \text{ is odd,} \\ V(Q_i) & \text{if } i \text{ is even.} \end{cases}$$

Let $S = S_1 \cup S_2 \cup \cdots \cup S_m$. We show that S is a proper total dominating set.

- ⋆ If $n = 3$, then

$$\begin{aligned} \sigma_S(Q_1) &= (2,1,2) \\ \sigma_S(Q_i) &= \begin{cases} (1,4,1) & \text{if } i \text{ is even and } 2 \le i \le m-1 \\ (3,2,3) & \text{if } i \text{ is odd and } 3 \le i \le m-1 \end{cases} \\ \sigma_S(Q_m) &= \begin{cases} (2,1,2) & \text{if } m \text{ is odd} \\ (1,3,1) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

- ⋆ If $n \ge 5$, then

$$\begin{aligned} \sigma_S(Q_1) &= (2,1,\ \underline{3,1},\ \underline{3,1}, \ldots,\ \underline{3,1},\ 2) \\ \sigma_S(Q_i) &= \begin{cases} (1,4,\ \underline{2,4}, \underline{2,4}, \ldots, \underline{2,4},\ 1) & \text{if } i \text{ is even and } 2 \le i \le m-1 \\ (3,2,\ \underline{4,2}, \underline{4,2}, \ldots, \underline{4,2},\ 3) & \text{if } i \text{ is odd and } 3 \le i \le m-1 \end{cases} \\ \sigma_S(Q_m) &= \begin{cases} (2,1,\ \underline{3,1},\ \underline{3,1}, \ldots,\ \underline{3,1},\ 2) & \text{if } m \text{ is odd} \\ (1,3,\ \underline{2,3},\ \underline{2,3}, \ldots,\ \underline{2,3},\ 1) & \text{if } m \text{ is even.} \end{cases} \end{aligned}$$

This is illustrated in Figure 2.6. Since $\sigma_S(x) \ne \sigma_S(y)$ for every two adjacent vertices x and y in G, it follows that S is a proper total dominating set of G.

Case 2. *Both n and m are even.* Then $n, m \ge 4$. Let $G_{n-1,m}$ be the subgraph of $G_{n,m}$ be constructed from $Q_i - u_{i,n}$ for $1 \le i \le m$. Let S_1 be the proper total dominating set of $G_{n-1,m}$ as defined in Case 1 and $S_2 = \{u_{i,n} : 1 \le i \le m-1\}$. Let $S = S_1 \cup S_2$. We show that S is a proper total dominating set. Observe that

$$\begin{aligned} \sigma_S(Q_1) &= (2,1,\ \underline{3,1},\ \underline{3,1}, \ldots,\ \underline{3,1}) \\ \sigma_S(Q_i) &= \begin{cases} (1,\ \underline{4,2}, \underline{4,2}, \ldots, \underline{4,2},\ 3) & \text{if } i \text{ is even and } 2 \le i \le m-2 \\ (3,\ \underline{2,4}, \underline{2,4}, \ldots, \underline{2,4},\ 2) & \text{if } m \ge 6,\ i \text{ is odd, and } 3 \le i \le m-3 \end{cases} \end{aligned}$$

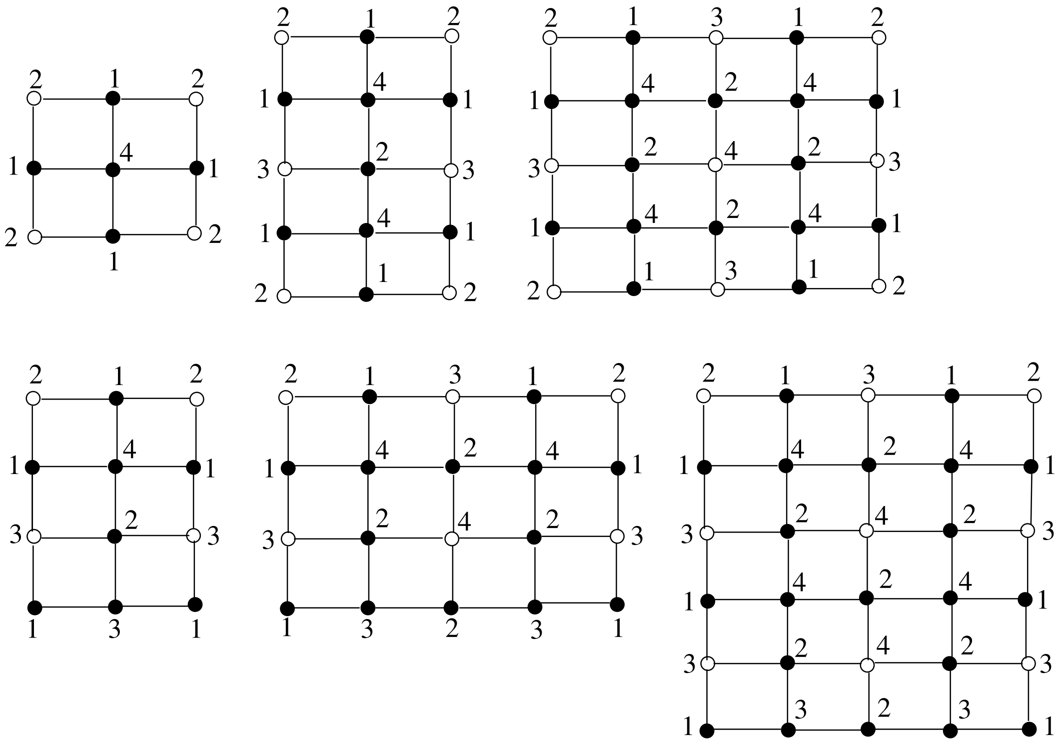


Figure 2.6: Proper total dominating sets in Case 1 of the proof of Theorem 2.1.

$$\begin{aligned} \sigma_S(Q_{m-1}) &= (3,\ \underline{2,4}, \underline{2,4}, \ldots, \underline{2,4},\ 1) \\ \sigma_S(Q_m) &= (1,3,\ \underline{2,3},\ \underline{2,3}, \ldots,\ \underline{2,3},\ 1,2). \end{aligned}$$

This is illustrated in Figure 2.7. Since $\sigma_S(x) \neq \sigma_S(y)$ for every two adjacent vertices x and y in G, it follows that S is a proper total dominating set of G. □

The grids $G_{n,m} = P_n \,\square\, P_m$ are a class of graphs defined as the Cartesian product of two well-known graphs. We saw that $G_{n,m}$ has a proper total dominating set for every two integers $n, m \geq 2$. Another much studied class of graphs defined as the Cartesian product of two well-known graphs are prisms $C_n \,\square\, K_2$, We investigate this class next. First, the following result will be useful for this purpose.

Proposition 2.4. *If H is an r-regular bipartite graph for some integer $r \geq 2$, then $H \,\square\, K_2$ contains a proper total dominating set.*

Proof. Let H and H' be two vertex disjoint copies of the graph H in the construction of $G = H \,\square\, K_2$ where a vertex v' of H' corresponds to the vertex v in H. Thus, $vv' \in E(G)$. Let U and W be the partite sets of H and let U' and W' be the partite sets of H' corresponding to U and W. Let $S = V(H) \cup U'$. Then

$$\sigma_S(x) = \begin{cases} r+1 & \text{if } x \in U \text{ or } x \in W' \\ r & \text{if} \in W \\ 1 & \text{if } x \in U'. \end{cases}$$

Since $\sigma_S(x) \neq \sigma_S(y)$ for every two adjacent vertices x and y in G, it follows that S is a proper total dominating set of G. □

The *n-cube* Q_n is K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as the Cartesian product $Q_{n-1} \,\square\, K_2$ of Q_{n-1} and K_2. Since the n-cube Q_{n-1} is an $(n-1)$-regular bipartite graph for $n \geq 2$, it follows that the n-cube Q_n possesses a proper total dominating set for each integer $n \geq 2$. Furthermore, since each even cycle is a 2-regular bipartite graph, the following is a consequence of Proposition 2.4.

Corollary 2.1. *For each even integer $n \geq 4$, the prism $C_n \,\square\, K_2$ possesses a proper total dominating set.*

While the prism $C_n \,\square\, K_2$ possesses a proper total dominating set for each even integer $n \geq 4$, such is not the case when $n \geq 3$ is odd. For example, let $G = C_3 \,\square\, K_2$ (see Figure 2.8). Suppose that G possesses a proper total dominating set S. Since the numbers $\sigma_S(u)$, $\sigma_S(v)$, $\sigma_S(w)$ are distinct and G is 3-regular, we may assume that $\sigma_S(u) = 3$, $\sigma_S(v) = 2$, and $\sigma_S(w) = 1$. Thus, $\{v, w, x\} \subseteq S$. Since $\sigma_S(w) = 1$, it follows that $\{u, z\} \cap S = \emptyset$. Thus, $\sigma_S(y) = 2$, which contradicts the fact that $\sigma_S(v) = 2$ since $vy \in E(G)$. Therefore, $C_3 \,\square\, K_2$ does not possess a proper total dominating set.

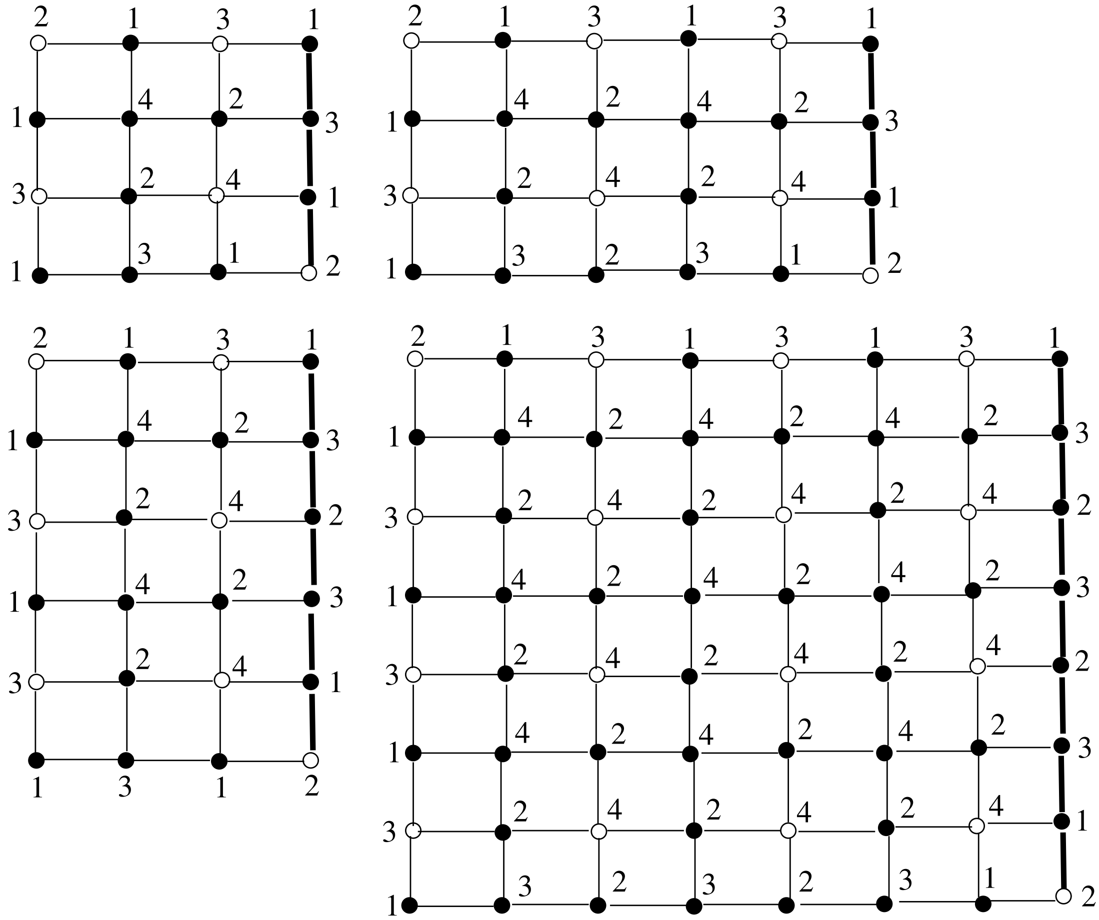


Figure 2.7: Proper total dominating sets in Case 2 of the proof of Theorem 2.1.

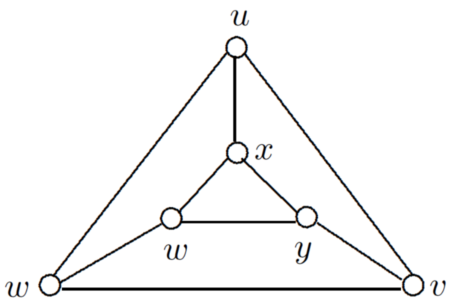


Figure 2.8: The graph $C_3 \square K_2$.

Not only does $C_3 \square K_2$ fail to possess a proper total dominating set but $C_n \square K_2$ fails to possess a proper total dominating set for every odd integer $n \geq 5$.

Theorem 2.2. *For every odd integer $n \geq 3$, the prism $C_n \square K_2$ does not possess a proper total dominating set.*

Proof. Since we know that $C_3 \square K_2$ does not possess a proper total dominating set, we may assume that $n \geq 5$. Suppose, to the contrary, that there is an odd integer $n \geq 5$ such that $G = C_n \square K_2$ possesses a proper total dominating set S. Let G be constructed from the two cycles $C = (u_1, u_2, \ldots, u_n, u_1)$ and $C' = (v_1, v_2, \ldots, v_n, v_1)$ by adding the edges $u_i v_i$ for $1 \leq i \leq n$. Since $\chi(C) = \chi(C') = 3$, there are vertices x on C and vertices y on C' such that $\sigma_S(x) = \sigma_S(y) = 3$. Suppose that there are k vertices x on C with $\sigma_S(x) = 3$ and k' vertices y on C' with $\sigma_S(y) = 3$. Thus, $k \geq 1$ and $k' \geq 1$. Hence, there are k blocks on C and k' blocks on C' such that $\sigma_S(z) \in \{1, 2\}$ for every vertex z in the block. Since n is odd, there is at least one block B consisting of an even number of vertices (an even block) on each of C and C'. We may assume that $B = (u_1, u_2, \ldots, u_{2p})$ is such a block where $2p \geq 2$. Thus, $\sigma_S(u_n) = \sigma_S(u_{2p+1}) = 3$, where $u_n = u_{2p+1}$ if $k = 1$. For every two integers i and j of opposite parity where $1 \leq i, j \leq 2p$, it follows that $\{\sigma_S(u_i), \sigma_S(u_j)\} = \{1, 2\}$. In particular, $\{\sigma_S(u_1), \sigma_S(u_{2p})\} = \{1, 2\}$. Since $\sigma_S(u_n) = \sigma_S(u_{2p+1}) = 3$, it follows that $u_1, v_n, u_{2p}, v_{2p+1} \in S$. Consequently, $\sigma_S(v_1) \geq 2$ and $\sigma_S(v_{2p}) \geq 2$.

We claim that $\{\sigma_S(v_1), \sigma_S(v_{2p})\} = \{2,3\}$; for otherwise, either $\sigma_S(v_1) = \sigma_S(v_{2p}) = 2$ or $\sigma_S(v_1) = \sigma_S(v_{2p}) = 3$. If $\sigma_S(v_1) = \sigma_S(v_{2p}) = 2$, then $\sigma_S(u_1) = \sigma_S(u_{2p}) = 1$, which contradicts the fact that $\{\sigma_S(u_1), \sigma_S(u_{2p})\} = \{1,2\}$. Suppose that $\sigma_S(v_1) = \sigma_S(v_{2p}) = 3$. Thus, $v_2, v_{2p-1} \in S$ and so $\sigma_S(u_2) = \sigma_S(u_{2p-1}) = 2$. Since 2 and $2p-1$ are of opposite parity, this is a contradiction. Therefore, $\{\sigma_S(v_1), \sigma_S(v_{2p})\} = \{2,3\}$, as claimed. Hence, we may assume that $\sigma_S(v_1) = 2$ and $\sigma_S(v_{2p}) = 3$. Thus, $\sigma_S(u_1) = 1$, $\sigma_S(u_{2p}) = 2$, and $\sigma_S(u_{2p-1}) = 1$. Since $\sigma_S(v_{2p}) = 3$, it follows that $u_{2p}, v_{2p-1} \in S$. However then, $\sigma_S(u_{2p-1}) \geq 2$, which is a contradiction. □

3. Proper total domination numbers

While every graph G without isolated vertices possesses a total dominating set, it is those sets of minimum cardinality that have drawn the most interest. This minimum cardinality is of course the total domination number $\gamma_t(G)$ of G. Consequently, we now turn our attention to the corresponding parameter for graphs possessing a proper total dominating set. The minimum cardinality of a proper total dominating set in a graph G is the *proper total domination number* $\gamma_{pt}(G)$ of G, namely

$$\gamma_{pt}(G) = \min\{|S| : S \text{ is a proper total dominating set of } G\}.$$

While every total dominating set in a graph must consist of at least two vertices, every proper total dominating set must consist of at least three vertices, for suppose that $S = \{u, v\}$ is a proper total dominating set in a graph G. Then S is also a total dominating set of G and so $uv \in E(G)$. However then, $\sigma_S(u) = \sigma_S(v) = 1$, which is impossible. Thus, we have the following observation.

Observation 3.1. *If a graph G has a proper total dominating set S, then $\gamma_{pt}(G) \geq 3$.*

As we saw in the proof of Proposition 2.1, the construction of a proper total dominating set in the path P_n of order n is unique when $n \geq 3$ and $n \equiv 3 \pmod 4$. Furthermore, the construction of a proper total dominating set in the cycle C_n of order n is also unique (up to isomorphism) when $n \geq 4$ and $n \equiv 0 \pmod 4$. Thus, we have the following.

Corollary 3.1. *Let P_n be a path of order $n \geq 2$ and let C_n be a cycle of order $n \geq 3$. Then*

$$\gamma_{pt}(P_n) = \frac{3(n+1)}{4} \quad \text{if } n \geq 3 \text{ and } n \equiv 3 \pmod 4$$

$$\gamma_{pt}(C_n) = \frac{3n}{4} \quad \text{if } n \geq 4 \text{ and } n \equiv 0 \pmod 4.$$

Since proper total domination is more restrictive than total domination, it follows that $\gamma_t(G) \leq \gamma_{pt}(G)$ for every graph G with a proper total dominating set. For the inequality $\gamma_t(G) \leq \gamma_{pt}(G)$, both strict inequality and equality are possible. For example, for $P_3 = (u, v, w)$, the set $V(P_3)$ is a minimum proper total dominating set of P_3 and $\{u, v\}$ is a minimum total dominating set. Thus, $\gamma_t(P_3) = 2$ and $\gamma_{pt}(P_3) = 3$. To illustrate equality, let G be the corona $cor(K_{2,3})$ of the graph $K_{2,3}$, where G is obtained from $K_{2,3}$ by adding a pendant edge at each vertex of $K_{2,3}$. Then the subset $S = V(K_{2,3})$ of $V(G)$ is both a minimum total dominating set and a minimum proper total dominating set of G. Therefore, $\gamma_t(G) = \gamma_{pt}(G) = 5$. This leads to the following question.

For which pairs a, b of positive integers with $a \leq b$, does there exist a graph G such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$?

The primary goal of this section is to provide an answer to this question. We saw in Observation 3.1 that if G is a graph with a proper total dominating set, then $\gamma_{pt}(G) \geq 3$. First, we show that if $3 \leq \gamma_{pt}(G) \leq 4$, then $\gamma_t(G) < \gamma_{pt}(G)$.

Proposition 3.1. *If G is a graph with $\gamma_{pt}(G) \in \{3,4\}$, then $\gamma_t(G) \leq \gamma_{pt}(G) - 1$. In particular, if $\gamma_{pt}(G) = 3$, then $\gamma_t(G) = 2$.*

Proof. Let S be a minimum proper total dominating set of G. Since $|S| \in \{3,4\}$, it follows that $G[S]$ is a locally irregular subgraph of order 3 or 4. Thus, either $G[S] = K_{1,2}$ or $G[S] = K_{1,3}$. Let w be the central vertex of $G[S]$, where $G[S] = \{w, y, z\}$ if $|S| = 3$ or $G[S] = \{w, y_1, y_2, z\}$ if $|S| = 4$. Let $T = S - \{z\}$. We show that T is a total dominating set of G. Since every vertex of S is totally dominated by a vertex of T, it remains only to show that every vertex not in S is totally dominated by a vertex of T. Suppose that there is a vertex v of G not in S that is not totally dominated by a vertex of T. Since S is a proper total dominating set of G, it follows that z is the only vertex of S that totally dominates v. Thus, $\sigma_S(v) = 1$. However, z is only totally dominated by w and so $\sigma_S(z) = 1$, which is impossible. □

By Proposition 3.1, if G is a graph with $\gamma_{pt}(G) = 3$, then $\gamma_t(G) = 2$. On the other hand, if $\gamma_{pt}(G) = 4$, then it is possible that $\gamma_t(G) = 3$ or $\gamma_t(G) = 2$, as we will see later. Furthermore, by Proposition 3.1, there is no graph G such that $\gamma_t(G) = \gamma_{pt}(G) = k$ if $k \in \{3,4\}$. We saw, however, that if $G = cor(K_{2,3})$, then $\gamma_t(G) = \gamma_{pt}(G) = 5$. This example can be extended to provide a proof of the next result.

Proposition 3.2. *For each integer $k \geq 5$, there exists a connected graph G such that $\gamma_{pt}(G) = \gamma_t(G) = k$.*

Proof. Let $H = K_{s,t}$ where $2 \leq s < t$ and $s + t = k$ and let $G = cor(H)$ be the corona of H. Let $S = V(H)$. Thus, $\{\sigma_S(u), \sigma_S(w)\}$ is a 2-element subset of $\{1, s, t\}$ for every two adjacent vertices u and w of G. Thus, S is a proper total dominating set of G and so $\gamma_{pt}(G) \leq k$. Since S is a minimum total dominating set of G, it follows that $\gamma_t(G) = k$. Therefore, $\gamma_{pt}(G) = \gamma_t(G) = k$. □

We saw that there is a graph G such that $\gamma_t(G) = 2$ and $\gamma_{pt}(G) = 3$. We show next that for each integer $k \geq 3$, there is a graph G such that $\gamma_t(G) = 2$ and $\gamma_{pt}(G) = k$. Figure 3.1 shows graphs G such that $\gamma_t(G) = 2$ and $\gamma_{pt}(G) \in \{4, 5\}$, where the vertices in a minimum proper total dominating set are indicated by solid vertices and the set $\{u, v\}$ is a minimum total dominating set in each graph.

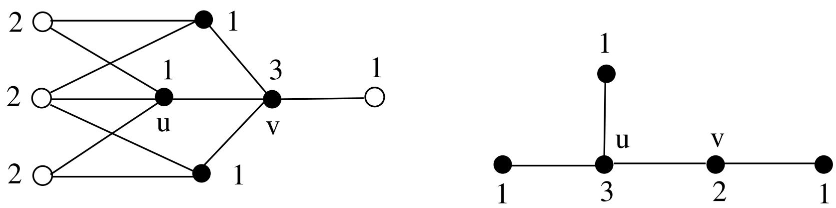


Figure 3.1: Graphs G with $\gamma_t(G) = 2$ and $\gamma_{pt}(G) \in \{4, 5\}$.

Two (nonadjacent) vertices u and v in a graph G are called *twins* (or *false twins*) if $N(u) = N(v)$. If u and v are adjacent vertices in a graph G such that $N[u] = N[v]$, then u and v are *adjacent twins* (or *true twins*). Suppose that u and v are adjacent twins in a graph G and S is a total dominating set of G. If $\{u, v\} \subseteq S$ or $\{u, v\} \cap S = \emptyset$, then $\sigma_S(u) = \sigma_S(v)$. This observation yields the following result.

Observation 3.2. *Let G be a nontrivial connected graph. If S is a proper total dominating set of G and u and v are adjacent twins of G, then exactly one of u and v belongs to S.*

Proposition 3.3. *For each integer $k \geq 3$, there is a connected graph G such that $\gamma_t(G) = 2$ and $\gamma_{pt}(G) = k$.*

Proof. Since the statement is known to be true for $k \in \{3, 4, 5\}$, we may assume that $k \geq 6$. First, let $H = K_2 \vee (k-3)K_2$ be the join of K_2 and $(k-3)K_2$, where $V(H) = \{u, v\} \cup \{u_i, v_i : 1 \leq i \leq k-3\}$ and $uv \in E(H)$ with $\deg_H u = \deg_H v = 2k - 5$ and $u_iv_i \in E(H)$ with $\deg_H u_i = \deg_H v_i = 3$ for $1 \leq i \leq k-3$. The graph G is obtained from H by adding the pendant edge uu' at u and the pendant edge vv' at v. This graph is shown in Figure 3.2 for $k = 6$. Since $\{u, v\}$ is a total dominating set of G, it follows that $\gamma_t(G) = 2$. It remains to show that $\gamma_{pt}(G) = k$.

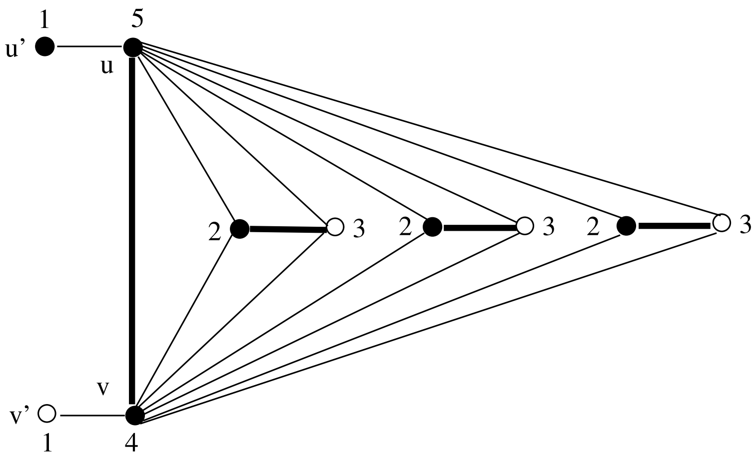


Figure 3.2: A graph G with $\gamma_t(G) = 2$ and $\gamma_{pt}(G) = 6$.

Let $S = \{v, u, u', u_1, u_2, \ldots, u_{k-3}\}$. Then

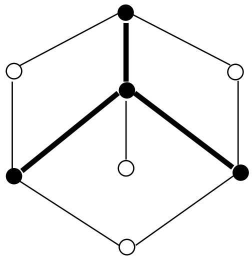
$$\sigma_S(x) = \begin{cases} 1 & \text{if } x = u' \text{ or } x = v' \\ 2 & \text{if } x = u_i \text{ where } 2 \leq i \leq k-3 \\ 3 & \text{if } x = v_i \text{ where } 2 \leq i \leq k-3 \\ k-2 & \text{if } x = v \\ k-1 & \text{if } x = u. \end{cases}$$

Since S is a proper total dominating set of G, it follows that $\gamma_{pt}(G) \le |S| = k$. Next, we show that $\gamma_{pt}(G) \ge k$. Assume, to the contrary, that $\gamma_{pt}(G) \le k-1$. Let T be a proper total dominating set of G. Since each of u and v is adjacent to an end-vertex of G, it follows that $u, v \in T$. For $1 \le i \le k-3$, the vertices u_i and v_i are adjacent twins of G and so exactly one of u_i and v_i belongs to T by Observation 3.2. We may assume that $u_i \in T$ for $1 \le i \le k-3$. Thus, $A = \{u, u', u_1, u_2, \ldots, u_{k-3}\} \subseteq T$. Since $\sigma_T(u) = \sigma_T(v) = k-2$, it follows that $A \subset T$. Hence, $|T| \ge k$ and so $\gamma_{pt}(G) \ge k$. Therefore, $\gamma_{pt}(G) = k$. □

We saw that there is a graph G such that $\gamma_{pt}(G) = k$ and $\gamma_t(G) = k-1$ for $k = 3$. We now show that there is such a graph when $k \ge 4$ as well.

Proposition 3.4. *For each integer $k \ge 4$, there exists a connected graph G such that $\gamma_t(G) = k-1$ and $\gamma_{pt}(G) = k$.*

Proof. Let $H = K_{1,k-1}$ be the star of order $k \ge 4$, where $V(H) = \{v_0, v_1, v_2, \ldots, v_{k-1}\}$ with $\deg v_0 = k-1$. The graph G is constructed from H by adding $\binom{k-1}{2} + 1$ vertices u_0 and $u_{i,j}$, where $1 \le i < j \le k-1$, the edge u_0v_0, and the edges $u_{i,j}v_i$ and $u_{i,j}v_j$ for all i, j with $1 \le i < j \le k-1$. Thus, G has order $\binom{k}{2} + 2$ and size $(k-1)^2 + 1$. This is illustrated in Figure 3.3 for $k = 4$ and $k = 5$, where the $H = K_{1,k-1}$ is drawn in bold. We show that $\gamma_{pt}(G) = k$ and $\gamma_t(G) = k-1$.



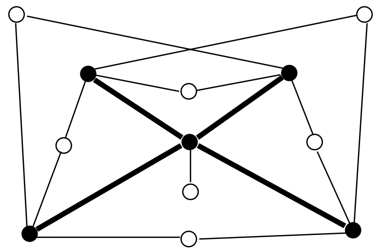


Figure 3.3: Graphs constructed from $H = K_{1,k-1}$ for $k = 4, 5$.

First, we show that $\gamma_{pt}(G) = k$. Let $S = V(H)$. Then

$$\sigma_S(v) = \begin{cases} k-1 & \text{if } v = v_0 \\ 1 & \text{if } v = v_i \text{ where } 0 \le i \le k-1 \\ 2 & \text{if } v = u_{i,j} \text{ where } 1 \le i < j \le k-1. \end{cases}$$

Since $\sigma_S(x) \ne \sigma_S(y)$ for every two adjacent vertices x and y, it follows that S is a proper total dominating set of G and so $\gamma_{pt}(G) \le |S| = k$. Assume, to the contrary, that there is a proper total dominating set T of G where $|T| \le k-1$. Necessarily, $v_0 \in T$. Suppose that $v_i \notin T$ for some integer i with $1 \le i \le k-1$, say $v_1 \notin T$. Since $\sigma_T(u_{i,j}) \ge 1$ for $1 \le i < j \le k-1$, it follows that $T = \{v_0, v_2, v_3, \ldots, v_{k-1}\}$. However then, $\sigma_T(v_1) = \sigma_T(u_{1,2}) = 1$, which is impossible. Therefore, $\gamma_{pt}(G) \ge k$ and so $\gamma_{pt}(G) = k$.

Next, we show that $\gamma_t(G) = k-1$. Since $\{v_0, v_1, v_2, \ldots, v_{k-2}\}$ is a total dominating set of G, $\gamma_t(G) \le k-1$. Let S be a total dominating set of G. Necessarily, $v_0 \in S$. Suppose that two of the vertices $v_1, v_2, \ldots, v_{k-2}$ do not belong to S, say $v_1, v_2 \notin S$. Then $u_{1,2}$ is not totally dominated by any vertex of S. Thus, S must contain at least $k-2$ of the vertices $v_1, v_2, \ldots, v_{k-1}$ and so $\gamma_t(G) \ge k-1$. Therefore, $\gamma_t(G) = k-1$. □

Not only for each integer $k \ge 3$ is there a graph G such that $\gamma_{pt}(G) = 2$ and $\gamma_t(G) = k$, for each integer $k \ge 4$ there is a graph G such that $\gamma_{pt}(G) = 3$ and $\gamma_t(G) = k$.

Proposition 3.5. *For each integer $k \ge 4$, there is a connected graph G such that $\gamma_t(G) = 3$ and $\gamma_{pt}(G) = k$.*

Proof. By Proposition 3.4, there is a connected graph G with $\gamma_t(G) = 3$ and $\gamma_{pt}(G) = 4$. The graph G of in Figure 3.4 has $\gamma_t(G) = 3$ and $\gamma_{pt}(G) = 5$. Thus, we may assume that $k \ge 6$.

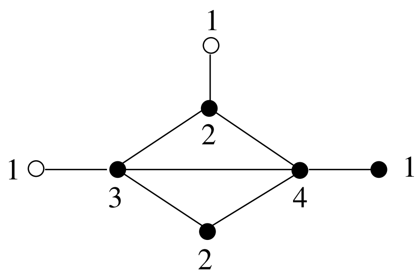


Figure 3.4: A graph G with $\gamma_t(G) = 3$ and $\gamma_{pt}(G) = 5$.

Let H be the graph constructed in the proof of Proposition 3.3, that is, $H = K_2 \vee (k-3)K_2$ is the join of K_2 and $(k-3)K_2$, where $V(H) = \{u, v\} \cup \{u_i, v_i : 1 \le i \le k-3\}$ and $uv \in E(H)$ with $\deg_H u = \deg_H v = 2k-5$ and $u_iv_i \in E(H)$ with $\deg_H u_i = \deg_H v_i = 3$ for $1 \le i \le k-3$. Let F be the graph obtained from H by adding the pendant edge uu' at u and the pendant edge vv' at v. As in the proof of Proposition 3.3, $\gamma_t(F) = 2$ and $\gamma_{pt}(F) = k$. We now construct a graph G from F by subdividing the edge uv in F exactly once, denoting the resulting new vertex of degree 2 by w, and then adding the pendant edge ww' at w. Since $\{u, v, w\}$ is the unique minimum total dominating set of G, it follows that $\gamma_t(G) = 3$.

It remains to show that $\gamma_{pt}(G) = k$. Necessarily, every proper total dominating set must contain u, v, and w. For each integer i with $1 \le i \le k-3$, the vertices u_i and v_i are adjacent twins of G and so exactly one of u_i and v_i belongs to every proper total dominating set by Observation 3.2. Thus, $\gamma_{pt}(G) \ge k$. For the set $S = \{u, v, w\} \cup \{u_i : 1 \le i \le k-3\}$, it follows that

$$\sigma_S(x) = \begin{cases} 1 & \text{if } x \in \{u', v', w'\} \\ 2 & \text{if } x = w \text{ or } x \in \{u_i : 1 \le i \le k-3\} \\ 3 & \text{if } x \in \{v_i : 1 \le i \le k-3\} \\ k-2 & \text{if } x \in \{u, v\}. \end{cases}$$

Since $\sigma_S(x) \ne \sigma_S(y)$ for every two adjacent vertices x and y, it follows that S is a proper total dominating set of G. Thus, $\gamma_{pt}(G) \le |S| = k$ and so $\gamma_{pt}(G) = k$. □

We are now prepared to determine all pairs a, b of integers with $2 \le a \le b$ that are realizable as the total domination number and the proper total domination number, respectively, of some graph.

Theorem 3.1. *For each pair a, b of integers with $2 \le a \le b$, there exists a connected graph G such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$ if and only if* **(1)** *$a \in \{2, 3, 4\}$ and $b \ge a+1$ or* **(2)** *$5 \le a \le b$.*

Proof. First, suppose that G is a graph such that $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$. If $a \in \{2, 3, 4\}$, then $b \ge a+1$ by Observation 3.1 and Proposition 3.1 and so (1) holds. Since $a \le b$, it follows that (2) hold if $a \ge 5$. Thus, it remains to verify the converse. By Propositions 3.3 and 3.5, we may assume that $a \ge 4$. By Propositions 3.4 and 3.2, we may further assume that $a \ge 4$ and $b \ge a+2$. We construct a connected graph G for which $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$. Let F be the corona of the complete bipartite graph $K_{2,a-2}$ with partite sets $U = \{u, v\}$ and $W = \{w_1, w_2, \ldots, w_{a-2}\}$ and let $H = (b-2)K_2$ with $V(H) = \{u_i, v_i : 1 \le i \le b-a\}$ where $u_iv_i \in E(H)$. Let G be the graph obtained from F and H joining u and v to every vertex of H. This is illustrated in Figure 3.5 for $(a, b) = (4, 6)$ and $(a, b) = (5, 7)$. Since $V(K_{2,a-2}) = U \cup W$ is the unique minimum total dominating set of G, it follows that $\gamma_t(G) = a$.

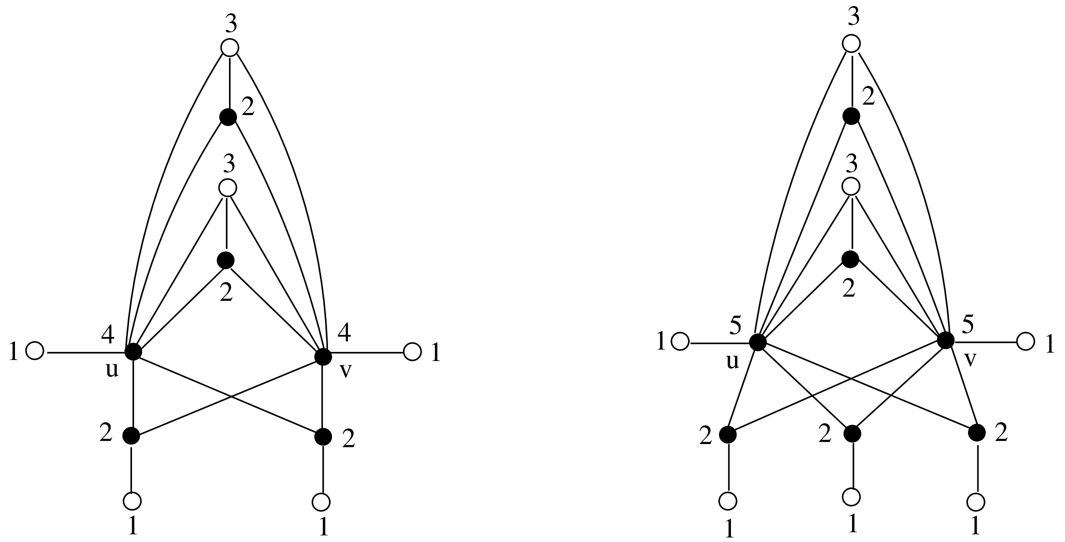


Figure 3.5: Graphs G with $\gamma_t(G) = a$ and $\gamma_{pt}(G) = b$, where $(a, b) = (4, 6)$ or $(a, b) = (5, 7)$.

It remains to show that $\gamma_{pt}(G) = b$. Let S be a proper total dominating set of G. Since each vertex in $U \cup W$ is adjacent to an end-vertex of G, it follows that $U \cup W \subseteq S$. For $1 \le i \le b-a$, the vertices u_i and v_i are adjacent twins of G and so exactly one of u_i and v_i belongs to S by Observation 3.2. Thus,

$$|S| \ge |U \cup W| + (b-a) = b.$$

Consequently, $\gamma_{pt}(G) \ge b$.

Now, let $S = U \cup W \cup \{u_i :\ 1 \le i \le b-a\}$. Thus, $|S| = b$. Then

$$\sigma_S(x) = \begin{cases} 1 & \text{if } \deg x = 1 \\ 2 & \text{if } x = w_i \text{ for } 1 \le i \le a-2 \text{ or } x = u_i \text{ for } 1 \le i \le b-a \\ 3 & \text{if } x = v_i \text{ for } 1 \le i \le b-a \\ b-2 & \text{if } x = u \text{ or } x = v. \end{cases}$$

Since $\sigma_S(x) \ne \sigma_S(y)$ for every two adjacent vertices x and y, it follows that S is a proper total dominating set of G. Thus, $\gamma_{pt}(G) \le b$ and so $\gamma_{pt}(G) = b$. □

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