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Research Article

Berezin radius type inequalities for functional Hilbert space operators

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Abstract

In this paper, new inequalities related to the Berezin norm and Berezin radius of functional Hilbert space operators are established. The main inequalities are derived by utilizing a mapping that was recently introduced by Stojiljković and Dragomir.

Keywords: Berezin norm; Berezin radius; Cauchy-Schwarz inequality; inequalities.

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1. Introduction

A functional Hilbert space (shortly FHS) is a Hilbert space $\mathcal{H} = \mathcal{H}(O)$ of complex-valued functions on some set O such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau \in O$, are continuous on $\mathcal{H}$ and for every $\tau \in O$ there exists a function $f_\tau \in \mathcal{H}$ such that $f_\tau(\tau) \neq 0$ (or, equivalently, there is no $\tau_0 \in O$ such that $f(\tau_0) = 0$ for all $f \in \mathcal{H}$). The Riesz representation theorem guarantees the existence of a unique element k_τ for each $\tau \in O$, such that $f(\tau) = \langle f, k_\tau \rangle$ holds for every f in $\mathcal{H}$. The set $\{k_\tau : \tau \in O\}$ is referred to as the reproducing kernel of $\mathcal{H}$. The function

$$K_\tau := \frac{k_\tau}{||k_\tau||}, \quad \text{with } \tau \in O,$$

is called the normalized reproducing kernel of $\mathcal{H}$. A detailed presentation of the theory of reproducing kernels and FHSs is given in [1]. Reproducing kernels play an important role in many branches of mathematics; for instance, see Jorgensen's book [16].

Let $\mathbb{B}(\mathcal{H})$ denote the collection of all bounded linear operators defined on a complex Hilbert space $\mathcal{H}$ with the identity operator $1_{\mathcal{H}}$ in $\mathbb{B}(\mathcal{H})$. The absolute value of a positive operator P is denoted by $|P| = (P^*P)^{\frac{1}{2}}$. For a bounded linear operator P on $\mathcal{H}$ (i.e., for $P \in \mathbb{B}(\mathcal{H})$), its Berezin symbol $\widetilde{P}$ is defined (see [7, 17]) on O by

$$\widetilde{P}(\tau) := \langle PK_\tau(z), K_\tau(z) \rangle, \ \tau \in O.$$

In other words, the Berezin symbol $\widetilde{P}$ is a function on O defined by restriction of the quadratic form $\langle Px, x \rangle$ with $x \in \mathcal{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathcal{H}$. It is clear from the Cauchy-Schwarz inequality that $\widetilde{P}$ is the bounded function on O whose values lie in the numerical range of the operator $\widetilde{P}$. So, the Berezin radius $\text{ber}(P)$ and the Berezin set $\text{Ber}(P)$ of operator $\widetilde{P}$ are defined respectively by

$$\text{ber}(P) := \sup\left\{ \left| \tilde{P}(\tau) \right| : \tau \in O \right\} \text{ and } \text{Ber}(P) := \{\widetilde{P}(\tau) : \tau \in O\}.$$

It is obvious that $\text{ber}(P) \leq w(P) \leq ||P||$ and $\text{ber}(P) \subset W(P)$, where $w(P)$ denotes the numerical radius and $W(P)$ is the numerical range of operator P. Moreover, the Berezin radius of an operator P_1 satisfies the following properties:

I. $\text{ber}(P_1) = \text{ber}(P_1^*)$,

II. $\text{ber}(\zeta P_1) = |\zeta| \ \text{ber}(P_1)$ for every $\zeta \in \mathbb{C}$,

III. $\text{ber}(P_1 + P_2) \leq \text{ber}(P_1) + \text{ber}(P_2)$ for all $P_1, P_2 \in \mathbb{B}(\mathcal{H})$.

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In general, the Berezin radius does not define a norm. However, if $\mathcal{H}$ is a functional Hilbert space of analytic functions (for instance on the unit disc $\mathbb{D}$), then $\mathrm{ber}(.)$ defines a norm on $\mathbb{B}(\mathcal{H}(\mathbb{D}))$. Extensive analysis has already been conducted on the Berezin symbol, especially with regard to its relationship with Toeplitz and Hankel operators on the Hardy and Bergman spaces. For example, the Berezin symbol $\widetilde{T}_\varphi$ on the Toeplitz operator T_φ $(\varphi \in L^\infty(\delta\mathbb{D}))$ on $\mathcal{H}^2(\mathcal{D})$ coincides with the harmonic extension $\widetilde{\varphi}$ of the function φ into the unit disc $\mathbb{D}$; in particular, if $\varphi \in H^\infty(\mathbb{D})$, i.e., if the symbol function φ is a bounded analytic function on $\mathbb{D}$, then $\widetilde{T}_\varphi = \varphi$. Also, it is well known that the Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ is compact if and only if its Berezin symbol $\widetilde{T}_\varphi$ vanishes on the boundary $\partial\mathbb{D}$, i.e., if $\lim_{\tau\to\nu}\widetilde{T}_\varphi(\tau) = 0$ for every $\nu \in \partial\mathbb{D}$ (see [3]). Berezin symbol has extensive applications in a variety of analytical problems and is essential for uniquely characterizing operators. Readers who are interested in learning more about the Berezin symbol are encouraged to consult [4, 6, 8, 10–12, 18, 19, 23, 24] and the comprehensive references provided therein.

The so-called Berezin norm of an operator $P \in \mathbb{B}(\mathcal{H})$ is defined as follows:

$$\|P\|_{\mathrm{ber}} := \sup_{\tau\in O} \|PK_\tau\|.$$

It is obvious that $\|P\|_{\mathrm{ber}}$ determines a new operator norm in $\mathbb{B}(\mathcal{H}(O))$. It is also trivial that $\mathrm{ber}(P) \le \|P\|_{\mathrm{ber}} \le \|P\|$. Since the family $\{k_\rho : \rho \in F\}$ is complete in $\mathcal{H}$, it is elementary to verify that $||P||_{\mathrm{ber}} = 0$ if and only if $P = 0$. Thus, it is easy to verify that $||P||_{\mathrm{ber}}$ shares the properties I–III with $\mathrm{ber}(P)$, and hence $||.||_{\mathrm{ber}}$ is the norm in $\mathbb{B}(\mathcal{H})$. Since $\mathrm{ber}(P) \le w(P)$ and $||P||_{\mathrm{ber}} \le ||P||$, the inequality $\mathrm{ber}(P) \le ||P||_{\mathrm{ber}}$ is in general better than the inequality $\mathrm{ber}(P) \le ||P||$. A significant inequality for $\mathrm{ber}(P)$ is the power inequality stating that

$$\mathrm{ber}(P^n) \le \mathrm{ber}^n(P) \tag{1}$$

for $n = 1, 2, \ldots$; more generally, if P is not nilpotent, then

$$C_1 \mathrm{ber}^n(P) \le \mathrm{ber}(P^n) \le C_2 \mathrm{ber}^n(P),$$

for some positive constants C_1 and C_2. Also, It is well known that $\frac{\|P\|}{2} \le w(P) \le \|P\|$ and

$$\mathrm{ber}(P) \le w(P) \le \|P\|, \tag{2}$$

for any $P \in \mathbb{B}(\mathcal{H})$. In 2022, Huban et al. [14, 15] proved the following inequalities:

$$\mathrm{ber}(P) \le \frac{1}{2}\||P| + |P^*|\|_{\mathrm{ber}} \le \frac{1}{2}\left(\|P\|_{\mathrm{ber}} + \left\|P^2\right\|^{\frac{1}{2}}_{\mathrm{ber}}\right) \tag{3}$$

and

$$\mathrm{ber}^{2r}(P) \le \frac{1}{2}\left\||P|^{2r} + |P^*|^{2r}\right\|_{\mathrm{ber}}, \text{ where } r \ge 1.$$

In [14, Theorem 3.1], Huban et al. significantly improved the upper bound in (2) by demonstrating that if $P \in \mathbb{B}(\mathcal{H})$, then

$$\mathrm{ber}(P) \le \frac{1}{2}\||P| + |P^*|\|_{\mathrm{ber}}. \tag{4}$$

Another improvement for the inequality (2) was provided by Huban et al. [13, Corollary 3.3.] as

$$\mathrm{ber}^2(P) \le \frac{1}{2}\left\||P|^2 + |P^*|^2\right\|_{\mathrm{ber}}, \tag{5}$$

which was further improved in [5] by Başaran and Gürdal as

$$\mathrm{ber}^2(P) \le \frac{1}{6}\left\||P|^2 + |P^*|^2\right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(P)\||P| + |P^*|\|_{\mathrm{ber}}. \tag{6}$$

It was shown in [13] and [15], respectively, that if $P \in \mathbb{B}(\mathcal{H}(O))$, then

$$\frac{1}{4}\left\||P|^2 + |P^*|^2\right\|_{\mathrm{ber}} \le \mathrm{ber}^2(P) \le \frac{1}{2}\left\||P|^2 + |P^*|^2\right\|_{\mathrm{ber}} \tag{7}$$

and

$$\mathrm{ber}^{2r}(P) \le \frac{1}{2}\left\||P|^{2r} + |P^*|^{2r}\right\|_{\mathrm{ber}} \tag{8}$$

where $r \ge 1$.

Furthermore, Huban et al. [14, Theorems 3.2 and 3.3] established refinements of (3) and (7), respectively, that can be presented as

$$\mathrm{ber}^n(P) \le \frac{1}{2}\left\| |P|^{2n\xi} + |P^*|^{2n(1-\xi)} \right\|_{\mathrm{ber}} \tag{9}$$

and

$$\mathrm{ber}^{2n}(P) \le \left\| \xi\, |P|^{2n} + (1-\xi)\, |P^*|^{2n} \right\|_{\mathrm{ber}}, \tag{10}$$

where $P \in \mathbb{B}(\mathcal{H})$, $0 \le \xi \le 1$, and $n \ge 1$.

Recently, Başaran and Gürdal [5] obtained some inequalities, showing the following for $P_1, P_2 \in \mathbb{B}(\mathcal{H})$:

$$\begin{aligned}\mathrm{ber}^2(P_1) &\le \frac{1}{12}\, \| |P_1| + |P_1^*| \|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(P_1)\, \| |P_1| + |P_1^*| \|_{\mathrm{ber}} \\ &\le \frac{1}{6}\, \| |P_1| + |P_1^*| \|_{\mathrm{ber}}^2 + \frac{1}{3}\mathrm{ber}(P_1)\, \| |P_1| + |P_1^*| \|_{\mathrm{ber}}\end{aligned} \tag{11}$$

and

$$\mathrm{ber}^2(P_2^* P_1) \le \frac{1}{6}\, \left\| |P_1|^4 + |P_1^*|^4 \right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(P_2^* P_1)\, \left\| |P_1|^2 + |P_1^*|^2 \right\|_{\mathrm{ber}}. \tag{12}$$

Başaran and Gürdal [5, Theorem 3.7 and Corollary 3.8] also obtained

$$\begin{aligned}\mathrm{ber}^n(P) &\le \frac{1}{2}\beta \left\| |P|^{2n\alpha} + |P^*|^{2n(1-\alpha)} \right\|_{\mathrm{ber}} + \frac{1}{\sqrt{2}}(1-\beta)\, \mathrm{ber}^{\frac{n}{2}}(P) \left\| |P|^{2n\alpha} + |P^*|^{2n(1-\alpha)} \right\|_{\mathrm{ber}}^{1/2} \\ &\le \frac{1}{2}\left\| |P|^{2\alpha n} + |P^*|^{2(1-\alpha)n} \right\|_{\mathrm{ber}},\end{aligned} \tag{13}$$

for all $n \ge 1$ and $0 \le \alpha, \beta \le 1$.

The upper bounds on the Berezin radius that are important to us in the present study also include the one due to Huban et al. in [13, Theorem 3.11]. If $P_1, P_2 \in \mathbb{B}(\mathcal{H})$ and $n \ge 1$, then

$$\mathrm{ber}^n(P_2^* P_1) \le \frac{1}{2}\left\| |P_1|^{2n} + |P_2|^{2n} \right\|_{\mathrm{ber}}. \tag{14}$$

The primary aim of the present study is to make several improvements to the Berezin radius inequalities mentioned above for functional Hilbert space operators.

2. Some preliminary inequalities

To prove our Berezin radius inequalities, we need some existing results.

The Schwarz inequality states that for all vectors x and y in an inner product space,

$$|\langle x, y\rangle|^2 \le \|x\|^2 \|y\|^2. \tag{15}$$

On the other hand, the classical Schwarz inequality for positive operators reads that if $P \in \mathbb{B}(\mathcal{H})$ is a positive operators, then

$$|\langle Px, y\rangle|^2 \le \langle Px, x\rangle \langle Py, y\rangle \tag{16}$$

for any $x, y \in \mathcal{H}$.

A companion of Schwarz inequality (16) known as Kato's inequality, or the so-called mixed Cauchy-Schwarz inequality, was first proposed by Kato [20] in 1952. It states:

$$|\langle Px, y\rangle|^2 \le \langle |P|^{2\alpha} x, x\rangle \langle |P^*|^{2(1-\alpha)} y, y\rangle, \ \alpha \in [0,1] \tag{17}$$

for any operator $P \in \mathbb{B}(H)$ and any $x, y \in \mathcal{H}$. In order to generalize this result, Furuta [9] established the following inequality:

$$|\langle P|P|^{\alpha+\beta-1} x, y\rangle|^2 \le \langle |P|^{2\alpha} x, x\rangle \langle |P^*|^{2\beta} y, y\rangle, \tag{18}$$

for any $x, y \in \mathcal{H}$ and $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge 1$.

The following lemma follows from the spectral theorem for positive operators and Jensen's inequality (see [21]):

Lemma 2.1. *Let $P \in \mathbb{B}(\mathcal{H})$ and $P \ge 0$. Let $x \in \mathcal{H}$ be any unit vector. Then*

$$\langle Px, x\rangle^n \le (\ge)\, \langle P^n x, x\rangle, \quad n \ge 1 \ (n \in [0,1]). \tag{19}$$

The next result is concerned with non-negative convex functions and can be found in [2].

Lemma 2.2. *Let f be a non-negative convex function on $[0,+\infty)$ and $P_1, P_2 \in \mathbb{B}(\mathcal{H})$ be positive operators. Then*

$$\left\| f\left(\frac{P_1+P_2}{2}\right)\right\| \leq \left\|\frac{f(P_1)+f(P_2)}{2}\right\|. \tag{20}$$

Lemma 2.3 (see [22])**.** *Let $u, v \in \mathcal{H}$ and let $n \geq 1$. Let J be a set such that $(0,1) \subset J \subset \mathbb{R}$. Let g be a mapping such that $g : J \to \mathbb{R}^+$, provided that $g(\Theta) + g(1-\Theta) = 1$. Then*

$$|\langle u,v\rangle|^{2n} \leq g(\Theta)\,\|u\|^{2n}\,\|v\|^{2n} + g(1-\Theta)|\langle u,v\rangle|^n\,\|u\|^n\,\|v\|^n \leq \|u\|^{2n}\,\|v\|^{2n}. \tag{21}$$

3. Main results

Throughout this section, we assume that $g : J \to \mathbb{R}^+$ is a mapping such that $g(\Theta)+g(1-\Theta) = 1$ and $(0,1) \subset J \subset \mathbb{R}$, unless otherwise stated.

Lemma 3.1. *If $P_1, P_2, P_3, P_4 \in \mathbb{B}(\mathcal{H})$, then*

$$|\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^2 \leq \langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle. \tag{22}$$

Proof. If we take $u = P_2P_1K_\tau$ and $v = P_3^*P_4^*K_p$, then we have $\langle u,v\rangle = \langle P_4P_3P_2P_1K_\tau, K_p\rangle$, $\|u\|^2 = \langle P_1^*\,|P_2|^2\,P_1K_\tau, K_\tau\rangle$ and $\|v\|^2 = \langle P_4\,|P_3^*|^2\,P_4^*K_p, K_p\rangle$. Now, by utilizing (15), we deduce the desired result. □

The inequality given in the following theorem is a refinement of (22).

Theorem 3.1. *If $P_1, P_2, P_3, P_4 \in \mathbb{B}(\mathcal{H})$ and $n \geq 1$, then*

$$\begin{aligned} |\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^{2n} &\leq g(\Theta)\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle^n\,\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle^n \\ &\quad + g(1-\Theta)|\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^n\,\sqrt[\frac{n}{2}]{\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle}\,\sqrt[\frac{n}{2}]{\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle} \\ &\leq \langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle^n\,\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle^n. \end{aligned} \tag{23}$$

Proof. Let $\tau, p \in O$ be arbitrary. Elementary calculations show that

$$\|P_2P_1K_\tau\|^2 = \langle P_2P_1K_\tau, P_2P_1K_\tau\rangle = \langle P_1^*P_2^*P_2P_1K_\tau, K_\tau\rangle = \langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle$$

and $\|P_3^*P_4^*K_p\|^2 = \langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle$. If we take $u = P_2P_1K_\tau$ and $v = P_3^*P_4^*K_p$ in the inequality (21), we obtain

$$\begin{aligned} |\langle P_2P_1K_\tau, P_3^*P_4^*K_p\rangle|^{2n} &\leq g(\Theta)\,\|P_2P_1K_\tau\|^{2n}\,\|P_3^*P_4^*K_p\|^{2n} + g(1-\Theta)|\langle P_2P_1K_\tau, P_3^*P_4^*K_p\rangle|^n\,\|P_2P_1K_\tau\|^n\,\|P_3^*P_4^*K_p\|^n \\ &\leq \|P_2P_1K_\tau\|^{2n}\,\|P_3^*P_4^*K_p\|^{2n} \end{aligned}$$

and hence

$$\begin{aligned} |\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^{2n} &\leq g(\Theta)\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle^n\,\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle^n \\ &\quad + g(1-\Theta)\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^n\,\sqrt[\frac{n}{2}]{\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle}\,\sqrt[\frac{n}{2}]{\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle} \\ &\leq \langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle^n\,\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle^n, \end{aligned}$$

as desired. □

We get the inequality (22) by putting $n = 1$ in (23). Also, the following corollaries are consequences of (23).

Corollary 3.1. *If $P \in \mathbb{B}(\mathcal{H})$, then for $\alpha, \beta \geq 0$ with $\alpha+\beta \geq 1$, we have*

$$\begin{aligned} |\langle P|P|^{\alpha+\beta-1}K_\tau, K_p\rangle|^{2n} &\leq g(\Theta)\langle |P|^{2\alpha}K_\tau, K_\tau\rangle^n\,\langle |P^*|^{2\beta}K_p, K_p\rangle^n \\ &\quad + g(1-\Theta)|\langle P|P|^{\alpha+\beta-1}K_\tau, K_p\rangle^n|\,\sqrt[\frac{n}{2}]{\langle |P|^{2\alpha}K_\tau, K_\tau\rangle}\,\sqrt[\frac{n}{2}]{\langle |P^*|^{2\beta}K_p, K_p\rangle} \\ &\leq \langle |P|^{2\alpha}K_\tau, K_\tau\rangle^n\,\langle |P^*|^{2\beta}K_\tau, K_\tau\rangle^n. \end{aligned} \tag{24}$$

Proof. Let $P = U|P|$ be the polar decomposition of the operator P, where U is partial isometry and the kernel

$$N(U) = N(|P|).$$

If we take $P_4 = U, P_3 = |P|^\beta, P_2 = I$, and $P_1 = |P|^\alpha$ in (23), then after simplification we obtain the desired result. □

The proof of the next corollary is similar to that of Corollary 3.1.

Corollary 3.2. *If $P \in \mathbb{B}(\mathcal{H})$ and $\alpha, \beta \geq 1$, then*

$$\begin{aligned}|\langle P|P|^{\beta-1}P|P|^{\alpha-1}K_\tau, K_p\rangle|^{2n} &\leq g(\Theta)\langle |P|^{2\alpha}K_\tau, K_\tau\rangle^n \langle |P^*|^{2\beta}K_p, K_p\rangle^n \\ &\quad + g(1-\Theta)|\langle P|P|^{\beta-1}P|P|^{\alpha-1}K_\tau, K_p\rangle^n \sqrt[\frac{n}{2}]{\langle |P|^{2\alpha}K_\tau, K_\tau\rangle} \sqrt[\frac{n}{2}]{\langle |P^*|^{2\beta}K_p, K_p\rangle} \\ &\leq \langle |P|^{2\alpha}K_\tau, K_\tau\rangle^n \langle |P^*|^{2\beta}K_p, K_p\rangle^n. \end{aligned} \tag{25}$$

By taking $\alpha = \beta = 1$ and $n = 1$ in (25), we obtain

$$\begin{aligned}|\langle P^2K_\tau, K_p\rangle|^2 &\leq g(\Theta)\langle |P|^2K_\tau, K_\tau\rangle\langle |P^*|^2K_p, K_p\rangle + g(1-\Theta)|\langle P^2K_\tau, K_p\rangle\sqrt{\langle |P|^2K_\tau, K_\tau\rangle}\sqrt{\langle |P^*|^2K_p, K_p\rangle} \\ &\leq \langle |P|^2K_\tau, K_\tau\rangle\langle |P^*|^2K_p, K_p\rangle. \end{aligned} \tag{26}$$

Corollary 3.3. *If $P \in \mathbb{B}(\mathcal{H})$ and $\gamma, \delta \geq 0$, then*

$$\begin{aligned}|\langle |P|^\gamma P^2|P|^\gamma K_\tau, K_p\rangle|^{2n} &\leq g(\Theta)\langle |P|^{2+2\delta}K_\tau, K_\tau\rangle^n \langle |P^*|P|^\gamma|^2K_p, K_p\rangle^n \\ &\quad + g(1-\Theta)\,|\langle |P|^\gamma P^2|P|^\gamma K_\tau, K_p\rangle|^n \sqrt[\frac{n}{2}]{\langle |P|^{2+2\delta}K_\tau, K_\tau\rangle} \sqrt[\frac{n}{2}]{\langle |P^*|P|^\gamma|^2K_p, K_p\rangle} \\ &\leq \langle |P|^{2+2\delta}K_\tau, K_\tau\rangle^n \langle |P^*|P|^\gamma|^2K_p, K_p\rangle^n. \end{aligned} \tag{27}$$

Next, we derive an inequality involving the Berezin norm.

Lemma 3.2. *If $P_1, P_2, P_3, P_4 \in \mathbb{B}(\mathcal{H})$, then*

$$\operatorname{ber}(P_4P_3P_2P_1)^2 \leq \left\|P_1^*|P_2|^2P_1\right\|_{\operatorname{ber}} \left\|P_4|P_3^*|^2P_4^*\right\|_{\operatorname{ber}}. \tag{28}$$

Proof. By taking the supremum over $\tau, p \in O$ in (22), we have

$$\begin{aligned}\|P_4P_3P_2P_1\|_{\operatorname{ber}}^2 &= \sup_{\tau,p\in O} |\langle P_4P_3P_2P_1K_\tau, K_p\rangle|^2 \\ &\leq \sup_{\tau,p\in O} \langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle \\ &= \sup_{\tau\in O}\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle \sup_{p\in O}\langle P_4|P_3^*|^2P_4^*K_p, K_p\rangle \\ &= \left\|P_1^*|P_2|^2P_1\right\|_{\operatorname{ber}} \left\|P_4|P_3^*|^2P_4^*\right\|_{\operatorname{ber}}, \end{aligned}$$

as desired. □

Lemma 3.3. *If $P_1, P_2, P_3, P_4 \in \mathbb{B}(\mathcal{H})$, then*

$$\operatorname{ber}^n(P_4P_3P_2P_1) \leq \frac{1}{2}\left\|\left(P_1^*|P_2|^2P_1\right)^n + \left(P_4|P_3^*|^2P_4^*\right)^n\right\|_{\operatorname{ber}}$$

for any $n \geq 1$.

Proof. Let $\tau, p \in O$ be arbitrary. If we take $\tau = p$ in the inequality (22), then we have

$$\begin{aligned}|\langle P_4P_3P_2P_1K_\tau, K_\tau\rangle| &\leq \left(\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle\langle P_4|P_3^*|^2P_4^*K_\tau, K_\tau\rangle\right)^{1/2} \\ &\leq \frac{\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle + \langle P_4|P_3^*|P_4^*K_\tau, K_\tau\rangle}{2} \\ &\leq \left(\frac{\langle P_1^*|P_2|^2P_1K_\tau, K_\tau\rangle^n + \langle P_4|P_3^*|^2P_4^*K_\tau, K_\tau\rangle^n}{2}\right)^{1/n}. \end{aligned} \tag{29}$$

Now, using the inequality (19), we obtain

$$\frac{\langle P_1^*|P_2|^2P_1K_\tau,K_\tau\rangle^n+\langle P_4|P_3^*|^2P_4^*K_\tau,K_\tau\rangle^n}{2}\le\frac{\langle\left(P_1^*|P_2|^2P_1\right)^nK_\tau,K_\tau\rangle+\langle\left(P_4|P_3^*|^2P_4^*\right)^nK_\tau,K_\tau\rangle}{2}$$
$$=\left\langle\frac{\left(P_1^*|P_2|^2P_1\right)^n+\left(P_4|P_3^*|^2P_4^*\right)^n}{2}K_\tau,K_\tau\right\rangle. \tag{30}$$

From the inequalities (29) and (30), we have

$$|\langle P_4P_3P_2P_1K_\tau,K_\tau\rangle|^n\le\left\langle\frac{\left(P_1^*|P_2|^2P_1\right)^n+\left(P_4|P_3^*|^2P_4^*\right)^n}{2}K_\tau,K_\tau\right\rangle$$

and

$$\sup_{\tau\in O}|\langle P_4P_3P_2P_1K_\tau,K_\tau\rangle|^n\le\sup_{\tau\in O}\left\langle\frac{\left(P_1^*|P_2|^2P_1\right)^n+\left(P_4|P_3^*|^2P_4^*\right)^n}{2}K_\tau,K_\tau\right\rangle.$$

Therefore,

$$\mathrm{ber}^n\left(P_4P_3P_2P_1\right)\le\frac{1}{2}\left\|\left(P_1^*|P_2|^2P_1\right)^n+\left(P_4|P_3^*|^2P_4^*\right)^n\right\|_{\mathrm{ber}},$$

as desired. □

In the following corollary, we present a number of particular cases of Lemmas 3.2 and 3.3:

Corollary 3.4. **(i).** *If $P\in\mathbb{B}(\mathcal{H})$, $n\ge1$ and $\alpha,\beta\ge0$ with $\alpha+\beta\ge1$, then*

$$\mathrm{ber}^n\left(P\left|P\right|^{\alpha+\beta-1}\right)\le\frac{1}{2}\left\||P|^{2\alpha n}+|P^*|^{2\beta n}\right\|_{\mathrm{ber}},$$

$$\mathrm{ber}^n\left(P\left|P\right|^{2\alpha-1}\right)\le\frac{1}{2}\left\||P|^{2\alpha n}+|P^*|^{2\alpha n}\right\|_{\mathrm{ber}},\ \alpha\ge\frac{1}{2},$$

and

$$\mathrm{ber}^n\left(P\left|P\right|\right)\le\frac{1}{2}\left\||P|^{2n}+|P^*|^{2n}\right\|_{\mathrm{ber}}.$$

(ii). *If $P\in\mathbb{B}(\mathcal{H})$, $n\ge1$ and $\alpha,\beta\ge1$, then*

$$\mathrm{ber}^n\left(P\left|P\right|^{\beta-1}P\left|P\right|^{\alpha-1}\right)\le\frac{1}{2}\left\||P|^{2\alpha n}+|P^*|^{2\beta n}\right\|_{\mathrm{ber}},$$

$$\mathrm{ber}^n\left(\left(P\left|P\right|^{\alpha-1}\right)^2\right)\le\frac{1}{2}\left\||P|^{2\alpha n}+|P^*|^{2\alpha n}\right\|_{\mathrm{ber}},\ \alpha\ge1,$$

and

$$\mathrm{ber}^n\left(P^2\right)\le\frac{1}{2}\left\||P|^{2n}+|P^*|^{2n}\right\|_{\mathrm{ber}}.$$

(iii). *If $P\in\mathbb{B}(\mathcal{H})$, $n\ge1$ and $\beta\ge0$, then*

$$\mathrm{ber}^n\left(P\left|P^*\right|^{\beta}P\right)\le\frac{1}{2}\left\||P|^{2n}+\left(P|P^*|^{2\beta}P^*\right)^n\right\|_{\mathrm{ber}},$$

$$\mathrm{ber}^n\left(P\left|P\right|^{\beta}P\right)\le\frac{1}{2}\left\||P|^{2n}+\left(P|P^*|^{2\beta}P^*\right)^n\right\|_{\mathrm{ber}},$$

and

$$\mathrm{ber}^n\left(P\left|P^*\right|P\right)\le\frac{1}{2}\left\||P|^{2n}+\left(P^2\left(P^*\right)^2\right)^n\right\|_{\mathrm{ber}},$$

$$\mathrm{ber}^n\left(P\left|P\right|P\right)\le\frac{1}{2}\left\||P|^{2n}+|P^*|^{4n}\right\|_{\mathrm{ber}}.$$

Theorem 3.2. *If $P_1,P_2,P_3,P_4\in\mathbb{B}(\mathcal{H})$ and $n\ge1$, then*

$$\mathrm{ber}\left(P_4P_3P_2P_1\right)^{2n}\le g(\Theta)\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^n\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^n+g(1-\Theta)\mathrm{ber}\left(P_4P_3P_2P_1\right)^n\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^{n/2}\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^{n/2}$$
$$\le\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^n\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^n. \tag{31}$$

Proof. By taking the supremum in the inequality (23), we have

$$\begin{aligned}\sup_{\tau,p\in O}|\langle P_4P_3P_2P_1K_\tau,K_p\rangle|^{2n}&\le\sup_{\tau,p\in O}\Big(g(\Theta)\langle P_1^*|P_2|^2P_1K_\tau,K_\tau\rangle^n\ \langle P_4|P_3^*|^2P_4^*K_p,K_p\rangle^n\\&\quad+g(1-\Theta)|\langle P_4P_3P_2P_1K_\tau,K_p\rangle|^n\ \sqrt[\frac{n}{2}]{\langle P_1^*|P_2|^2P_1K_\tau,K_\tau\rangle}\ \sqrt[\frac{n}{2}]{\langle P_4|P_3^*|^2P_4^*K_p,K_p\rangle}\Big)\\&\le\sup_{\tau,p\in O}\langle P_1^*|P_2|^2P_1K_\tau,K_\tau\rangle^n\ \langle P_4|P_3^*|^2P_4^*K_p,K_p\rangle^n,\end{aligned}\tag{32}$$

whose simplification yields the left inequality in (31). To obtain the right inequality in (31), we start from the right-hand side of (32) and apply (28), and then use the properties of the mapping g as well as of the supremum. Hence, we obtain

$$\begin{aligned}\mathrm{ber}\,(P_4P_3P_2P_1)^{2n}&\le g(\Theta)\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^n\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^n+g(1-\Theta)\mathrm{ber}\,(P_4P_3P_2P_1)\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^{n/2}\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^{n/2}\\&\le\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}^n\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}^n,\end{aligned}$$

as desired. □

We obtain the following result by taking $n=1$ in Theorem 3.2:

Corollary 3.5. *If $P_1,P_2,P_3,P_4\in\mathbb{B}(\mathcal{H})$, then*

$$\mathrm{ber}\,(P_4P_3P_2P_1)^2\le\left\|P_1^*|P_2|^2P_1\right\|_{\mathrm{ber}}\left\|P_4|P_3^*|^2P_4^*\right\|_{\mathrm{ber}}.$$

Theorem 3.3. *If $P_1\in\mathbb{B}(\mathcal{H})$, $n\ge2$ and $\alpha\in[0,1]$, then*

$$\begin{aligned}\mathrm{ber}^n(P_1)&\le\frac{g(\Theta)}{2}\left\||P_1|^{2\alpha n}+|P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}}+\frac{g(1-\Theta)}{2}\mathrm{ber}^{\frac{n}{2}}(P_1)\left\||P_1|^{\alpha n}+|P_1^*|^{n(1-\alpha)}\right\|_{\mathrm{ber}}\\&\le\frac12\left\||P_1|^{2\alpha n}+|P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}}.\end{aligned}\tag{33}$$

Proof. Assume that $K_\tau\in\mathcal{H}$ is a normalized reproducing kernel. By using the inequality (17), the inequality (19) and the AG-inequality, we have

$$\begin{aligned}|\langle P_1K_\tau,K_\tau\rangle|^n&=g(\Theta)|\langle P_1K_\tau,K_\tau\rangle|^n+g(1-\Theta)|\langle P_1K_\tau,K_\tau\rangle|^n\\&\le g(\Theta)\langle|P_1|^{2\alpha}K_\tau,K_\tau\rangle^{\frac{n}{2}}\langle|P_1^*|^{2(1-\alpha)}K_\tau,K_\tau\rangle^{\frac{n}{2}}\\&\quad+g(1-\Theta)|\langle P_1K_\tau,K_\tau\rangle|^{\frac{n}{2}}\langle|P_1|^{2\alpha}K_\tau,K_\tau\rangle^{\frac{n}{4}}\langle|P_1^*|^{2(1-\alpha)}K_\tau,K_\tau\rangle^{\frac{n}{4}}\\&\le\frac{g(\Theta)}{2}\left(\langle|P_1|^{2\alpha}K_\tau,K_\tau\rangle^n+\langle|P_1^*|^{2(1-\alpha)}K_\tau,K_\tau\rangle^n\right)\\&\quad+\frac{g(1-\Theta)}{2}|\langle P_1K_\tau,K_\tau\rangle|^{\frac{n}{2}}\left(\langle|P_1|^{2\alpha}K_\tau,K_\tau\rangle^{\frac{n}{2}}+\langle|P_1^*|^{2(1-\alpha)}K_\tau,K_\tau\rangle^{\frac{n}{2}}\right)\\&\le\frac{g(\Theta)}{2}\left(\langle|P_1|^{2\alpha n}K_\tau,K_\tau\rangle+\langle|P_1^*|^{2n(1-\alpha)}K_\tau,K_\tau\rangle\right)\\&\quad+\frac{g(1-\Theta)}{2}|\langle P_1K_\tau,K_\tau\rangle|^{\frac{n}{2}}\left(\langle|P_1|^{\alpha n}K_\tau,K_\tau\rangle+\langle|P_1^*|^{n(1-\alpha)}K_\tau,K_\tau\rangle\right)\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau\in O}|\langle P_1K_\tau,K_\tau\rangle|^n&\le\frac{g(\Theta)}{2}\sup_{\tau\in O}\langle|P_1|^{2\alpha n}+|P_1^*|^{2n(1-\alpha)}K_\tau,K_\tau\rangle\\&\quad+\frac{g(1-\Theta)}{2}\sup_{\tau\in O}|\langle P_1K_\tau,K_\tau\rangle|^{\frac{n}{2}}\langle|P_1|^{\alpha n}+|P_1^*|^{n(1-\alpha)}K_\tau,K_\tau\rangle.\end{aligned}$$

Therefore, we have

$$\mathrm{ber}^n(P_1)\le\frac{g(\Theta)}{2}\left\||P_1|^{2\alpha n}+|P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}}+\frac{g(1-\Theta)}{2}\mathrm{ber}^{\frac{n}{2}}(P_1)\left\||P_1|^{\alpha n}+|P_1^*|^{n(1-\alpha)}\right\|_{\mathrm{ber}}.$$

By using the inequality (9) given by Huban et al. [14, Theorem 3.2] for $\xi = \alpha$, we obtain

$$\begin{aligned}\mathrm{ber}(P_1)^n &\le \frac{g(\Theta)}{2}\left\||P_1|^{2\alpha n} + |P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\mathrm{ber}^{\frac{n}{2}}(P_1)\left\||P_1|^{\alpha n} + |P_1^*|^{n(1-\alpha)}\right\|_{\mathrm{ber}}\\ &\le \frac{g(\Theta)}{2}\left\||P_1|^{2\alpha n} + |P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{4}\left\||P_1|^{\alpha n} + |P_1^*|^{n(1-\alpha)}\right\|^2_{\mathrm{ber}}\\ &\le \frac{g(\Theta)}{2}\left\||P_1|^{2\alpha n} + |P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\left\||P_1|^{2\alpha n} + |P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}}\\ &= \frac{1}{2}\left\||P_1|^{2\alpha n} + |P_1^*|^{2n(1-\alpha)}\right\|_{\mathrm{ber}}\end{aligned}$$

which proves the second inequality in (33). □

We arrive at the following result by taking $n = 2$, $g = I$, $\Theta \in [0, 1]$, and $\alpha = \frac{1}{2}$ in Theorem 3.3.

Corollary 3.6. *If $P_1 \in \mathbb{B}(\mathcal{H})$, then*

$$\begin{aligned}\mathrm{ber}(P_1)^2 &\le \frac{g(\Theta)}{2}\left\||P_1|^2 + |P_1^*|^2\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\mathrm{ber}(P_1)\left\||P_1| + |P_1^*|\right\|_{\mathrm{ber}}\\ &\le \frac{1}{2}\left\||P_1|^2 + |P_1^*|^2\right\|_{\mathrm{ber}}\end{aligned}$$

Corollary 3.6 refines the upper bound of the refinement given by Gürdal and Tapdigoglu [12]. Also, we obtain the following chain of inequalities by putting $g(\Theta) = \Theta, \Theta = \frac{1}{3}$, $n = 2$, and $\alpha = \frac{1}{2}$ in (33):

$$\mathrm{ber}(P_1)^2 \le \frac{1}{6}\left\||P_1|^2 + |P_1^*|^2\right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(P_1)\left\||P_1| + |P_1^*|\right\|_{\mathrm{ber}} \le \frac{1}{2}\left\||P_1|^2 + |P_1^*|^2\right\|_{\mathrm{ber}},$$

which is a refinement of (7) given by Başaran and Gürdal via (6).

Theorem 3.4. *If $P_1, P_2 \in \mathbb{B}(\mathcal{H})$ and $n \ge 1$, then*

$$\begin{aligned}\mathrm{ber}^{2n}(P_2^*P_1) &\le \frac{g(\Theta)}{2}\left\||P_1|^{4n} + |P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\mathrm{ber}^n(P_2^*P_1)\left\||P_1|^{2n} + |P_2|^{2n}\right\|_{\mathrm{ber}}\\ &\le \frac{1}{2}\left\||P_1|^{4n} + |P_2|^{4n}\right\|_{\mathrm{ber}}.\end{aligned} \tag{34}$$

Proof. Setting $P_2 = I$, $P_3 = I$, $P_4^* = P_2$, $K_\tau = K_p$ in Theorem 3.1 and using the AG-inequality and the inequality (19), we obtain

$$\begin{aligned}|\langle P_2^*P_1K_\tau, K_\tau\rangle|^{2n} &\le g(\Theta)\langle |P_1|^2K_\tau, K_\tau\rangle^n\ \langle |P_2|^2K_\tau, K_\tau\rangle^n\\ &\quad + g(1-\Theta)|\langle P_2^*P_1K_\tau, K_\tau\rangle|^n\ \sqrt[\frac{n}{2}]{\langle |P_1|^2K_\tau, K_\tau\rangle}\ \sqrt[\frac{n}{2}]{\langle |P_2|^2K_\tau, K_\tau\rangle}\\ &\le \frac{g(\Theta)}{2}\left(\langle |P_1|^2K_\tau, K_\tau\rangle^{2n} + \langle |P_2|^2K_\tau, K_\tau\rangle^{2n}\right)\\ &\quad + \frac{g(1-\Theta)}{2}|\langle P_2^*P_1K_\tau, K_\tau\rangle|^n\left(\langle |P_1|^2K_\tau, K_\tau\rangle^n + \langle |P_2|^2K_\tau, K_\tau\rangle^n\right)\\ &\le \frac{g(\Theta)}{2}\left(\langle |P_1|^{4n}K_\tau, K_\tau\rangle + \langle |P_2|^{4n}K_\tau, K_\tau\rangle\right)\\ &\quad + \frac{g(1-\Theta)}{2}|\langle P_2^*P_1K_\tau, K_\tau\rangle|^n\left(\langle |P_1|^{2n}K_\tau, K_\tau\rangle + \langle |P_2|^{2n}K_\tau, K_\tau\rangle\right)\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau\in O}|\langle P_2^*P_1K_\tau, K_\tau\rangle|^{2n} &\le \frac{g(\Theta)}{2}\sup_{\tau\in O}\langle |P_1|^{4n} + |P_2|^{4n}K_\tau, K_\tau\rangle\\ &\quad + \frac{g(1-\Theta)}{2}\sup_{\tau\in O}|\langle P_2^*P_1K_\tau, K_\tau\rangle|^n\ \langle (|P_1|^{2n} + |P_2|^{2n})K_\tau, K_\tau\rangle.\end{aligned}$$

Therefore, we have

$$\mathrm{ber}^{2n}(P_2^*P_1) \leq \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\mathrm{ber}^n(P_2^*P_1)\left\||P_1|^{2n}+|P_2|^{2n}\right\|_{\mathrm{ber}}.$$

Again, using the inequality (14), the inequality (20) and the convexity of $f(x)=x^2$, we have

$$\begin{aligned}
\mathrm{ber}^{2n}(P_2^*P_1) &\leq \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\mathrm{ber}^n(P_2^*P_1)\left\||P_1|^{2n}+|P_2|^{2n}\right\|_{\mathrm{ber}} \\
&\leq \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{4}\left\||P_1|^{2n}+|P_2|^{2n}\right\|^2_{\mathrm{ber}} \\
&\leq \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{4}\left\|\left(|P_1|^{2n}+|P_2|^{2n}\right)^2\right\|_{\mathrm{ber}} \\
&= \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{4}\left\|\left(\frac{2|P_1|^{2n}+2|P_2|^{2n}}{2}\right)^2\right\|_{\mathrm{ber}} \\
&\leq \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{4}\left\|\frac{\left(2|P_1|^{2n}\right)^2+\left(2|P_2|^{2n}\right)^2}{2}\right\|_{\mathrm{ber}} \\
&= \frac{g(\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{g(1-\Theta)}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} \\
&= \frac{1}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}}.
\end{aligned}$$

This completes the proof. □

If we take $g=I$ and $\Theta\in[0,1]$ in Theorem 3.4, then we obtain the following result, which was given by Başaran and Gürdal [5, Theorem 3.1]:

Corollary 3.7. *If $P_1,P_2\in\mathbb{B}(\mathcal{H})$ and $n\geq 1$, then*

$$\begin{aligned}
\mathrm{ber}^{2n}(P_2^*P_1) &\leq \frac{\Theta}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}} + \frac{1-\Theta}{2}\mathrm{ber}^n(P_2^*P_1)\left\||P_1|^{2n}+|P_2|^{2n}\right\|_{\mathrm{ber}} \\
&\leq \frac{1}{2}\left\||P_1|^{4n}+|P_2|^{4n}\right\|_{\mathrm{ber}}
\end{aligned}$$

The substitutions $g(\Theta)=\Theta$, $\Theta=\frac{l}{1+l}$, $l\geq 0$, and $n=1$ in Theorem 3.4 yield the next result, which was given by Gürdal and Tapdigoglu [12, Theorem 3.1].

Corollary 3.8. *If $P_1,P_2\in\mathbb{B}(\mathcal{H})$, then*

$$\begin{aligned}
\mathrm{ber}^2(P_2^*P_1) &\leq \frac{1}{2l+2}\mathrm{ber}(P_2^*P_1)\left\||P_1|^2+|P_2|^2\right\|_{\mathrm{ber}} + \frac{l}{2l+2}\left\||P_1|^4+|P_2|^4\right\|_{\mathrm{ber}} \\
&\leq \frac{1}{2}\left\||P_1|^4+|P_2|^4\right\|_{\mathrm{ber}}.
\end{aligned} \tag{35}$$

By taking $l=\frac{1}{2}$ in (35), we obtain the next result, which was given by Başaran and Gürdal [5].

Corollary 3.9. *If $P_1,P_2\in\mathbb{B}(\mathcal{H})$, then*

$$\mathrm{ber}^2(P_2^*P_1) \leq \frac{1}{6}\left\||P_1|^4+|P_1^*|^4\right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(P_2^*P_1)\left\||P_1|^2+|P_1^*|^2\right\|_{\mathrm{ber}}.$$

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