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Research Article

On a conjecture of Chellali and Favaron regarding connected domination numbers

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Abstract

In a book chapter published in 2020, Chellali and Favaron listed a conjecture, which states "if G is a simple connected graph with second minimum degree δ' and connected domination number $\gamma_c(G)$ such that $\gamma_c(G) \geq n-2\delta'+1$, then G is traceable". The purpose of this article is to settle this conjecture by proving that it is true.

Keywords: connected domination number; second minimum degree; order; path.

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1. Introduction

Let $G = (V, E)$ be a connected graph with vertex set $V(G)$ and edge set $E(G)$. The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges incident with v in G. The *order* n of G is the cardinality of $V(G)$; that is, $n = |V(G)|$. If $d_1, d_2, \cdots, d_n$ are the degrees of all vertices in G with $d_1 \leq d_2 \leq \cdots \leq d_n$, then $d_1, d_2, d_3, \cdots, d_n$ is the *degree sequence* of G. The i^{th} *minimum degree*, denoted by $\delta^{(i)}$, is the i^{th} value in the degree sequence; that is, $\delta^{(i)} = d_i$. In particular, the *minimum degree* is $\delta(G) = \delta^{(1)} = d_1$ and the *second minimum degree* is $\delta' = \delta^{(2)} = d_2$. Thus, $\delta(G) \leq \delta'$. For $v \in V(G)$, if $\deg_G(v) = 1$, then v is called a *leaf vertex* or an *end vertex*. If $\deg_G(v) \geq 2$, then v is an *interior vertex*. We denote by $Int(G)$ the set of all interior vertices of G. The *leaf number* of G, denoted by $L(G)$, is the maximum number of leaf vertices contained in a spanning tree of G. The *distance* $d_G(u, v)$ between the vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. The *eccentricity* $ecc_G(v)$ of a vertex $v \in V(G)$ is the distance from v to a vertex furthest from v in G.

Consider $S \subseteq V(G)$. The set S is said to be a leaf set if there exists a subtree of G whose leaves are all the elements of S. The set S is an *independent set* of G if $xy \notin E(G)$ for every pair of vertices $x, y \in S$. If every vertex in $V(G)\backslash S$ has a neighbour in S, then S is called a *dominating set* of G. An *induced subgraph* $G[S]$ on S is a graph with the vertex set S such that $xy \in E(G[S])$ if and only if $xy \in E(G)$ for every pair of vertices $x, y \in S$. The order of the smallest connected subgraph $G[S]$ induced by a dominating set S is the *connected domination number*, denoted by $\gamma_c(G)$. If C is a cycle in G, with $V(G) \setminus V(C)$ being an independent set, then C is a *dominating cycle*. The *circumference* of a graph G, denoted by $c(G)$, is the length of a longest cycle in G. The order of a longest path in G is denoted by $p(G)$. The graph G is *Hamiltonian* if $c(G) = n$ and *traceable* if $p(G) = n$. A spanning path of G is also called a *Hamilton path*. The difference $diff(G) = p(G) - c(G)$ is called the *relative length* of G. If G has a spanning $u - v$ path for every pair of vertices $u, v \in V(G)$, then G is known as a *Hamiltonian connected graph*.

The leaf number and connected domination number, linked as $L(G) = n - \gamma_c(G)$, were introduced in a personal communication (by L. Lovász and M. E. Saks) and in [50], respectively. The determination of these parameters is known to be NP-hard [16]. The study on bounds for these two parameters is well documented [3, 19–21, 54]. These parameters have numerous applications, which are fruitful in the design and analysis of networks; for instance, see [3, 4, 19–21, 47, 54]. Other types of well-studied domination parameters include domination number, $\mathcal{F}$-domination number, outer-connected domination number, hop domination number, clone hop domination number, and restrained step triple connected domination number; for instance, see [18, 27, 36, 37, 42, 48]. The study of v-numbers of graded ideals (see for example [6, 23]) is motivated by problems in algebraic coding theory. Recently, the concept of connected domination number has been linked to the v-numbers of binomial edge ideals [23]. It appears more convenient to work with the leaf number instead of the connected domination number; using the relationship $L(G) = n - \gamma_c(G)$, the obtained results can easily be transformed into the ones involving the connected domination number.

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The computer program Graffiti.pc, introduced by DeLaViña [9], generated the following conjectures, namely Graffiti.pc 190a and Graffiti.pc 190, respectively (see also Conjecture 5 in [8]):

Conjecture 1.1. *If G is a connected graph of order n, minimum degree δ and leaf number $L(G)$ such that $L(G) \leq \delta + 1$, then G is traceable.*

Conjecture 1.2. *If G is a connected graph of order n, minimum degree δ and leaf number $L(G)$ such that $L(G) \leq 2\delta - 1$, then G contains a Hamilton path.*

Conjectures 1.1 and 1.2 have been settled completely, see [32, 39, 40].

Theorem 1.1. [39, 40] *If G is a connected graph of order n, minimum degree δ and leaf number $L(G)$, with $L(G) \leq \delta + 1$, then G has a spanning path.*

Theorem 1.2. [32] *A connected graph G of order n, minimum degree δ and leaf number $L(G)$ such that $L(G) \leq 2\delta - 1$, is traceable.*

Corresponding to Conjectures 1.1 and 1.2, Chellali and Favaron [4] posed the following two conjectures.

Conjecture 1.3. *Let G be a connected graph of order n, second minimum degree δ' and leaf number $L(G)$, such that $L(G) \leq \delta' + 1$. Then G has a Hamilton path.*

Conjecture 1.4. *If G is a connected graph of order n, second minimum degree δ' and leaf number $L(G)$ with $L(G) \leq 2\delta' - 1$, then G contains a spanning path.*

In [25], Conjecture 1.3 was settled. The purpose of the present paper is to prove that Conjecture 1.4 is also valid. By considering the families of graphs reported in [25, 26, 32, 35, 40] together with a new class of graphs constructed in this paper, it is noticed that there exist infinite families of (i) non-Hamiltonian graphs with leaf number $2\delta' - 1$, (ii) non-traceable graphs with leaf number at least $2\delta'$. Note that Conjecture 1.1 coincides with Conjecture 1.3 and Conjecture 1.2 coincides with Conjecture 1.4 when $\delta = \delta'$. Hence, by Theorem 1.2, Conjecture 1.4 is true when $\delta' = \delta$. However, the technique employed in this paper deduces Theorem 1.2 as a corollary to the main result of this paper. In [25], Conjecture 1.4 was shown to be correct for $\delta < \delta' \leq 2$. Thus, in this paper, Conjecture 1.4 is proved when $\delta' \geq 3$.

Lemma 1.1. [38] *If G satisfies the conditions of Conjecture 1.2, then G is 2-connected.*

In proving Theorem 1.2 and the results reported in [32, 34], Lemma 1.1 together with cycle- and paths-related properties [10, 41, 43, 47] played a key role. Paper [25] highlighted the possibility that the ideas similar to the ones used in [32, 34] would probably settle Conjecture 1.4; however, the present paper identifies a possibility that if G satisfies the hypotheses of Conjecture 1.4 and $\delta' > \delta$, then G is not 2-connected (contrary to the fact that G is 2-connected when $\delta = \delta'$). In fact, some graphs presented in this paper that make the main result best in a certain sense are not 2-connected and they satisfy the conditions of Conjecture 1.4. This resulted in the need for several existing results to prove the results of this paper; especially, the ones related to the concept of Hamiltonian connectedness (see for instance [24, 45, 51]). Indeed, if G satisfies the hypotheses of Dirac's Theorem 1.3 [10] or Corollary 1.1 [2, 29], then $ecc_G(v) \leq 2$ for every $v \in V(G)$ (otherwise, $n \geq 2\delta + 2$, which is not possible). Some simple observations made in the present paper (for example, "if there exist $x, y \in V(G)$ such that $\min\{\deg_G(x), \deg_G(y)\} \geq \delta'$ and $d_G(x, y) = 3$, then there is a subtree of G with $2\delta' - 2$ leaves") simplify some proofs in [32, 34] for the case $\delta = \delta'$. Also, for graphs satisfying the conditions of Conjecture 1.2, the upper bound on the order $n \leq \max\{2\delta + 6, 3\delta - 1\}$ (see [32, 34]) is slightly reduced to $n \leq \max\{2\delta + 5, 3\delta - 1\}$ for $\delta' = \delta$, which makes it possible to apply Theorem 1.10 and some techniques of the present paper to provide a short proof of Theorem 1.2 when $\delta \geq 5$.

In the rest of this introductory section, the existing results and terminology that are crucial in the establishment of the results of the present paper are presented. Dirac [10, 11], who pioneered sufficient conditions for the existence of spanning paths and cycles, proved the following two results:

Theorem 1.3. [10] *Let G be a connected graph with order $n \geq 3$ and minimum degree $\delta \geq 2$ such that $n \leq 2\delta$. Then G is Hamiltonian. Moreover, if G is 2-connected then $c(G) \geq \min\{n, 2\delta\}$.*

Theorem 1.4. [11] *Let G be an s-connected graph and C be a longest cycle in G. If x is a vertex in G such that $x \notin V(C)$, then there exist s paths starting at x and terminating in C, which are pairwise disjoint apart from x, and share with C just their terminal vertices; say, $x_1, x_2, \cdots, x_s$.*

A corollary derived from [28] is that if G has order $n \geq 3$ and G has a cut vertex, then $n \geq 2\delta + 1$ and the bound is sharp for each δ. This in conjunction with Theorem 1.3 yields the next result.

Lemma 1.2. *If G is a connected and non-Hamiltonian graph with minimum degree δ and order $n \geq 3$, then $n \geq 2\delta + 1$.*

Ore [45, 46], Erdős and Gallai [13] extended Theorem 1.3 to minimum degree conditions as well to the concept of Hamiltonicity and Hamiltonian connectedness. One of their results is the following:

Theorem 1.5. [13,46] *Let G be a connected graph with order $n \geq 3$, minimum degree $\delta \geq 2$ and $\deg(u)+\deg(v) \geq n+1$ for all pair of non-adjacent vertices $u, v \in V(G)$. Then G is Hamiltonian connected. That is, If G is not Hamiltonian connected, then $n \geq 2\delta$.*

Let G_1 and G_2 be graphs. Then G_1 and G_2 are *vertex-disjoint* if $V(G_1) \cap V(G_2) = \emptyset$. Further, G_1 and G_2 are *edge-disjoint* if they have no edge in common. The graphs G_1 and G_2 are *disjoint* if they are both vertex-disjoint and edge-disjoint. By $G_1 \cup G_2$, we mean the *union* of two edge-disjoint graphs G_1 and G_2, that is, $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$ with $E(G_1) \cap E(G_2) = \emptyset$. The *joint* $G_1 \vee G_2$, is the graph formed by taking disjoint graphs G_1 and G_2, and joining each vertex of G_1 to every vertex of G_2. For a graph H, $tK_\delta \vee H$ is the graph formed by taking H and t disjoint copies of the complete graph K_δ and joining every vertex of H to every vertex of the t copies. Shih, Su and Kao [51], defined the families $\mathcal{G}_1$ and $\mathcal{G}_2$ as follows:

Definition 1.1. [51] *Let H_i be any simple graph with i vertices. Let s, t and n be positive integers with $n \geq 3$. Then $\mathcal{G}_1 = \{H_2 \vee (K_s \cup K_t) \mid \ s+t \geq 2 \ \text{ and } \ s+t = n-2\}$ and $\mathcal{G}_2 = \{H_s \vee sK_1 \mid \ 2s = n\}$.*

The next result is an improvement of Theorem 1.5.

Theorem 1.6. [51] *Let G be a simple connected graph of order n such that $\deg_G(u) + \deg_G(v) \geq n$ for every pair of non-adjacent vertices u and v in G. Then G is Hamiltonian connected or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

A simple extension of Theorem 1.3 to the classification of non-Hamiltonian graphs satisfying $\delta \geq \lfloor \frac{n}{2} \rfloor$ was given in [2]; in [29], not only the same result was deduced as a corollary but also its traceability analogue was reported.

Corollary 1.1. [2,29] *If G is a connected graph of order n and minimum degree δ such that $n \leq 2\delta+1$, then G is Hamiltonian or $G \in \{K_1, K_2, 2K_\delta \vee K_1, K_{\delta,\delta+1} + H\}$.*

Corollary 1.2. [29] *Let G be a connected graph with order n and minimum degree δ such that $n \leq 2\delta + 2$. Then G is traceable or $G = K_{\delta,\delta+2} + H$.*

Cycle- and path-related properties are paramount in the establishment of sufficient conditions for Hamiltonicity, Hamiltonian connectedness and traceability in graphs. Such studies focus on different graph parameters that include large neighbourhood unions for non-adjacent vertices [1, 49], minimum degree and length of a longest path or cycle outside a given longest cycle [43, 44], connectivity and independence number [5, 41], Harary and Wiener indices [22], large degree sums for non-adjacent vertices [45], relative length and minimum degree [12, 47], degree, order and independence number [53]; see also [14, 17, 47] for additional details on this topic. The mentioned area of research has several applications in different fields, including electronic circuit design, optimal path computation, mapping genomes, operations research and computer graphics; for example, see [15, 52]. In addition to the already mentioned results, the properties presented in [43, 44, 47] play also a vital role in the establishment of the results of the present paper. Denote by σ_k the minimum degree sum of an independent set of k vertices, provided that the independence number is at least k; otherwise, $\sigma_k = +\infty$.

Theorem 1.7. [43] *Let G be a connected graph with minimum degree δ. If C_k is a longest cycle in G and p' is the length of a longest path in $G - C_k$, then $|V(C_k)| \geq (p'+2)(\delta - p')$.*

Theorem 1.8. [44] *Let G be a connected graph with minimum degree δ. If C_k is a longest cycle in G and c' is the length of a longest cycle in $G - C_k$, then $|V(C_k)| \geq (c'+1)(\delta - c'+1)$.*

Theorem 1.9. [47] *Let G be a 2-connected graph with connectivity κ and minimum degree δ. If $diff(G) \geq 2$ then either $c(G) \geq \sigma_3 - 3 \geq 3\delta - 3$ or $\kappa = 2$ and $p(G) \geq \sigma_3 - 1 \geq 3\delta - 1$.*

Theorem 1.10. [12] *If G is a connected graph such that $\deg(u)+\deg(v)+\deg(w) \geq n$ for every triple $\{u, v, w\} \subset V(G)$ of independent vertices, then either G is traceable or $diff(G) \leq 1$.*

Theorem 1.11. [31] *Every connected and δ-regular graph G with $L(G) \leq 2\delta - 1$ is Hamiltonian.*

For a subgraph H of G, $V(G-H) \subset V(G)$ is the set of those vertices of G that are not in H. The graph obtained from G by deleting an edge e or a vertex x is denoted by $G - e = G - \{e\}$ or $G - \{x\} = G \setminus \{x\}$, respectively. If T' is a subtree of G, then the set $S(T') \subset V(G)$ is defined by $S(T') = \{x : x \in V(G - T') \text{ and } \deg_G(x) \geq \delta'\}$.

Fact 1.1. [25] *If G is a connected graph with $\delta < \delta'$, then G has only one vertex, say u, such that $\deg_G(u) = \delta$. In addition, $\deg_G(x) \geq \delta'$ for every $x \in V(G \setminus \{u\})$. Thus, $\deg_{G\setminus\{u\}}(x) \geq \delta' - 1$ for every $x \in V(G \setminus \{u\})$.*

Lemma 1.3. [25] *Let G be a connected graph with $L(G) \leq 2\delta' - 1$ and $\delta' \geq 3$. If T' is a subtree of G such that $L(T') = 2\delta' - 1$, then $|S(T')| \leq 2$. Also, if $u \notin V(G - T')$ where $\deg_G(u) = \delta$ and $|V(G - T')| = 2$, then both the vertices of $V(G - T')$ are adjacent and they do not share a neighbour in T'.*

In addition to the already defined notation and terminology, we need the following: The *open-neighbourhood* $N_G(v)$ of a vertex v in G is defined by $N_G(v) = \{u \in V(G) : d_G(u,v) = 1\}$ and the *closed-neighbourhood* $N_G[v]$ of v is given by $N_G[v] = \{v\} \cup N_G(v)$. For a positive integer k, Danklemann and Entringer [7] defined a k-packing of G as a subset $A \subseteq V(G)$ with $d_G(a,b) > k \; \forall\, a,b \in A$. For a subgraph H of G, we write $H \leq G$. The set of neighbours of $v \in V(G)$ in H is denoted by $N_H(v)$. Let $C_k \leq G$ be a cycle with $V(C_k) = \{v_1, v_2, \cdots, v_k\}$ and $E(C_k) = \{v_i v_{i+1} : i \in \{1, 2, \cdots, k-1\}\} \cup \{v_k v_1\}$, where the subscripts are increasing on C_k in a clockwise orientation from v_1. If $v \in V(G)$ is a vertex not on C_k and $v_{t_i} \in V(C_k)$ is its neighbour, then $v_{t_i - 1}$ and $v_{t_i + 1}$ are the predecessor and successor of v_{t_i}, respectively, in a clockwise orientation. Denote by $N^+(v)$ the set of successors for elements in $N_{C_k}(v)$; that is, $N^+(v) = \{v_{t_i+1} : v_{t_i} \in N_{C_k}(v)\}$. Also, $N^-(v)$ is given by $N^-(v) = \{v_{t_i-1} : v_{t_i} \in N_{C_k}(v)$ provided $v_{t_i-1} \notin N^+(v)\}$. For distinct vertices v_s and v_t on C_k, $v_s \overrightarrow{C_k} v_t$ is a path from v_s to v_t along C_k in a clockwise orientation from v_s. Likewise, $v_s \overleftarrow{C_k} v_t$ is a $v_s - v_t$ path along C_k in an anti-clockwise orientation from v_s. Whenever there is no danger of confusion, the argument G will be dropped from the notation involving it.

2. Main results

The main aim of this section is to settle Conjecture 1.4 completely. We start by presenting some observations and lemmas that are crucial in the proofs of the main results. As mentioned before, Conjecture 1.4 is true for $\delta' \leq 2$, see [25]. Here, we consider the case $\delta' \geq 3$. Theorem 1.2 implies that Conjecture 1.4 is true for $\delta' = \delta$ and hence we could have considered only the case $\delta' > \delta$; however, the proof of the considered conjecture is unified in this paper for $\delta' \geq \delta$. Indeed, Theorem 1.2 is deduced as a corollary of the main results of this paper.

Throughout this section, a vertex $u \in V(G)$ is fixed such that $\deg_G(u) = \delta$. Then $\deg_G(x) \geq \delta'$ for every $x \in V(G \setminus \{u\})$. Also, if $\delta' > \delta$ then u is the only vertex of degree δ in G (see Fact 1.1). The following observation is derived from the proofs of Corollaries 1.1 and 1.2 of [2, 29].

Observation 2.1. *Let G be a connected graph with minimum degree $\delta = \delta(G) \geq 2$ and order n such that $3 \leq n \leq 2\delta + 2$. If G is not 2-connected, then G has a cut vertex u_{cc} such that $G - \{u_{cc}\}$ has 2 components G' and G'' with the property that $G[\{u_{cc}\} \cup V(G')]$ forms a complete graph $K_{\delta+1}$ such that $\deg_G(y) = \delta$ for every $y \in V(K_{\delta+1}) \setminus \{u_{cc}\}$ and $G[\{u_{cc}\} \cup V(G'')]$ contains a spanning path that has u_{cc} as its end vertex. That is, G has a spanning subgraph which is a lollipop whose head is $K_{\delta+1}$ and its tail is formed by the vertex set $\{u_{cc}\} \cup V(G'')$.*

Although by Theorem 1.6, every graph $G \in \mathcal{G}_1 \cup \mathcal{G}_2$ is non-Hamiltonian connected, we note the following crucial observation concerning this result.

Observation 2.2. *If G satisfies the conditions of Theorem 1.6 and $G \in \mathcal{G}_1$, then for any distinct pair of vertices $x, y \in V(G)$ with $x \notin V(H_2)$, there exists an $x - y$ spanning path of G. Also, if $G \in \mathcal{G}_2$ and x, y, are distinct vertices in different partite sets of G, then there is an $x - y$ spanning path of G.*

Lemma 2.1. Assume that G satisfies the hypotheses of Conjecture 1.4. If $\delta' \geq 4$ and if there is a tree $T' \leq G$ with $S(T') \subseteq V(G - T')$ such that $L(T') = 2\delta' - 2$ and $S(T') = \{w : w \in V(G - T')$ and $\deg_G(w) \geq \delta'\}$, then $|S(T')| \leq 4$. That is, $|V(G - T')| \leq 4$ if $V(G - T')$ has no vertex of degree δ.

Proof. If there is $x \in Int(T')$ and $w \in V(G - T')$ such that $xw \in E(G)$, then $T' \cup \{xw\}$ is a tree with $2\delta' - 1$ leaves and the result follows from Lemma 1.3. Next, assume that no element of $Int(T)$ has a neighbour in $V(G - T')$. Since $\delta' \geq 4$, no vertex of $G \setminus \{u\}$ has at least 3 neighbours in $V(G - T')$; otherwise, we obtain a tree with at least $2\delta'$ leaves, a contradiction. If there is a leaf x of T' that has 2 neighbours, say w_1 and w_2 in $V(G - T')$, then $T \cup \{xw_1, xw_2\}$ has $2\delta' - 1$ leaves and the desired result follows from Lemma 1.3. Now, assume that each leaf of T' has at most one neighbour in $V(G - T')$. Then T' receives at most $2\delta' - 2$ edges from $V(G - T')$. Since each element of $S(T')$ has at least $\delta' - 2$ neighbours in T', we have $|S(T')| \leq 3$; otherwise, $4(\delta' - 2) > 2\delta' - 2$ for every $\delta' > 3$, which is a contradiction to the fact that T' receives at most $2\delta' - 2$ edges from $V(G - T')$. □

Observation 2.3. *Assume that G satisfies the conditions of Conjecture 1.4 and $\delta' \geq 4$. If $x, y \in V(G) \setminus \{u\}$ such that either (i) $xy \in E(G)$ and $N_G(x) \cap N_G(y) = \emptyset$ or (ii) $|N_G(x) \cap N_G(y)| = 1$ and $N_G(x) \cap N_G(y) = \{u\}$, then $n \leq 2\delta' + 5$.*

Proof. Suppose that (i) holds. By Fact 1.1 $\min\{\deg_G(x), \deg_G(y)\} \geq \delta'$. Take the edge xy, attach $\delta' - 1$ neighbours of x to x and $\delta' - 1$ neighbours of y to y and form a tree with order $2\delta'$ and leaf number $2\delta' - 2$. By Lemma 2.1, $n \leq 2\delta' + 5$. By taking the path x, u, y, and applying similar arguments with the help of Lemma 2.1, we establish the result when (ii) holds. □

Observation 2.4. *Assume that G satisfies the conditions of Conjecture 1.4 such that $\delta' \geq 4$. If $T' \leq G$ is a tree such that $L(T') = 2\delta' - 1$, $u \in V(T')$ and $|V(G - T')| = 2$, then $n \leq 2\delta' + 5$.*

Proof. Let $w_1, w_2 \in V(G-T')$. Since $u \in V(T')$, $\min\{\deg_G(w_1), \deg_G(w_2)\} \geq \delta'$ and $|V(G-T')| \leq 2$ (see Fact 1.1 and Lemma 1.3). Also, by Lemma 1.3, $w_1w_2 \in E(G)$. Furthermore, w_1 and w_2 do not share any neighbour in T'; otherwise, we obtain a tree with $2\delta'$ leaves, which is not permitted. This in conjunction with $|V(G-T')| \leq 2$ implies that $N_G(w_1) \cap N_G(w_2) = \emptyset$. Hence, w_1 and w_2 satisfy condition (i) of Observation 2.3. Therefore, $n \leq 2\delta' + 5$, as required. □

Lemma 2.2. *If G is a connected graph with order n and second minimum degree $\delta' \geq 4$ such that $L(G) \leq 2\delta' - 1$, then $n \leq \max\{2\delta' + 5, 3\delta' - 1\}$. In addition, if $ux \in E(G)$ such that $ecc_G(x) \geq 3$, then $n \leq 2\delta' + 5$.*

Proof. Consider $ux \in E(G)$ such that $ecc_G(x) \leq 2$. Since $\deg_G(x) \geq \delta'$ (by Fact 1.1), take x and attach to it δ' of its neighbours that include u to form a subgraph of G, say $K'_{1,\delta'}$.

If there exists $w \in V(G - K'_{1,\delta'})$ such that w has no neighbour in $K'_{1,\delta'}$, then (because of the fact that $w \neq u$) let $K''_{1,\delta'}$ be a subgraph of G outside $K'_{1,\delta'}$ formed by taking w and attaching δ' of its neighbours. Since $ecc_G(x) \leq 2$, one of the neighbours, say w', of w is a neighbour of x. The subgraph $K''_{1,\delta'}$ is chosen in such a way that $w' \in V(K''_{1,\delta'})$. Join $K'_{1,\delta'}$ and $K''_{1,\delta'}$ by inserting an edge xw' to form a tree of order $2\delta' + 2$ and leaf number $2\delta' - 1$. Hence, by Lemma 1.3, $n \leq 2\delta' + 4$.

Next, consider the case when every vertex outside $K'_{1,\delta'}$ has a neighbour in $K'_{1,\delta'}$. If every vertex in $V(G - K'_{1,\delta'})$ has at least 2 neighbours in $K'_{1,\delta'}$, then $|V(G - K'_{1,\delta'})| \leq 2\delta' - 2$; otherwise, $\{u\} \cup V(G - K'_{1,\delta'})$ forms a leaf set with at least $2\delta'$ leaves of the tree formed by attaching every vertex of $V(G - K'_{1,\delta'})$ to one of its neighbours (which is not u) in $K'_{1,\delta'}$, a contradiction. Thus, $n \leq 3\delta' - 1$ in the considered subcase.

Now, assume that there is a vertex $w \in V(G - K'_{1,\delta'})$ such that w has only one neighbour, say x', in $K'_{1,\delta'}$. To $K'_{1,\delta'}$, add the edge $x'w$ and attach to w, $\delta' - 1$ of its neighbours from $N_G(w) \setminus \{x'\}$. Thus, we have a tree $T'' \leq G$ with order $2\delta' + 1$, leaf number $2\delta' - 2$ when $x' \neq x$, and leaf number $2\delta' - 1$ if $x' = x$. Since $u \in V(T'')$, it follows from Lemma 1.3 or Lemma 2.1 that $n \leq 2\delta' + 5$. Hence, the lemma holds whenever $ecc_G(x) \leq 2$.

Consider $ux \in E(G)$ with $ecc_G(x) \geq 3$ for some $x \in N_G(u)$. Take $y \in V(G)$ such that $d_G(x, y) = 3$. By Fact 1.1, $\min\{\deg_G(x), \deg_G(y)\} \geq \delta'$. Let $A = \{x_1, x_2, \cdots, x_{\delta'}\} \subseteq N_G(x)$ and $B = \{y_1, y_2, \cdots, y_{\delta'}\} \subseteq N_G(y)$. Choose A such that $u \in A$. Assume that $P_{xy} = x, x_1, y_1, y$ is a shortest $x - y$ path in G. Then $A \setminus \{x_1\} \cup B \setminus \{y_1\}$ is a leaf set of a binary star, say $R \leq G$, whose interior vertices are the ones that lie on P_{xy}. Thus, $L(R) = 2\delta' - 2$ and by Lemma 2.1, $|V(G - R)| \leq 4$ since $u \in V(R)$. If $|V(G - R)| \leq 3$, then $n \leq 2\delta' + 5$ as desired. Assume that $|V(G - R)| = 4$. Following the proof of Lemma 2.1, it is possible only if there is no interior vertex of R that has a neighbour in $V(G - R)$ and there is a leaf, say x_2, of R that has 2 neighbours, say w' and w'', in $V(G - R)$. Now, $T = R \cup \{x_2w', x_2w''\}$ is a tree with $2\delta' - 1$ leaves, $u \in V(T)$ and $|V(G-T)| = 2$. By Observation 2.4, we have $n \leq 2\delta' + 5$; otherwise, T is not a subtree of G. This completes the proof of the lemma. □

Lemma 2.3. *Assume that the hypotheses of Conjecture 1.4 hold in G. If $\delta' \geq 4$ and if $x, y \in V(G)$ such that*

$$\min\{\deg_G(x), \deg_G(y)\} \geq \delta'$$

and $d_G(x, y) = 3$, then $n \leq 2\delta' + 5$.

Proof. For $\delta' = 4$ and $\delta' = 5$, the result follows from Lemma 2.2. Assume that $\delta' \geq 6$. Let $A = \{x_1, x_2, \cdots, x_{\delta'}\} \subseteq N_G(x)$ and $B = \{y_1, y_2, \cdots, y_{\delta'}\} \subseteq N_G(y)$. Assume that $P_{xy} = x, x_1, y_1, y$ is a shortest $x - y$ path in G. As in Lemma 2.2, let R be a binary star with the leaf set $A \setminus \{x_1\} \cup B \setminus \{y_1\}$ and $Int(R) = V(P_{xy})$. Then $L(R) = 2\delta' - 2$ and R has order $2\delta' + 2$. If $u \in V(R)$, then by Lemma 2.2 we are done because either $ux \in E(G)$ and $ecc_G(x) \geq 3$ or $uy \in E(G)$ and $ecc_G(y) \geq 3$. Hence, consider the case when $u \notin V(R)$. If there exist $x' \in Int(R)$ and $w \in V(G - R)$ such that $wx' \in E(G)$, then $T' = R \cup \{wx'\}$ has order $2\delta' + 3$ and $L(T') = 2\delta' - 1$. By Lemma 1.3, $|S(T')| \leq 2$. If $|S(T')| \leq 1$, then $n \leq 2\delta' + 5$ as desired. Assume that $|S(T')| = 2$. Let $w', w'' \in S(T')$. Then $V(G - T') = \{u, w', w''\}$. Note that w' and w'' do not share a neighbour in T'; otherwise, we obtain a tree with $2\delta'$ leaves, a contradiction. By the same argument, neither w' nor w'' has 2 neighbours in $V(G - T')$. If $w'w'' \in E(G)$, then these arguments imply that w' and w'' satisfy Observation 2.3 and $n \leq 2\delta' + 5$. Similarly, if $uw', uw'' \in E(G)$, then $N_G(w') \cap N_G(w'') = \{u\}$ and we are done by Observation 2.3.

Now, consider the case when $V(G - T') = \{u, w', w''\}$ with $w'w'' \notin E(G)$ such that either w' or w'' is not adjacent to u. Note that exactly one of the vertices w' and w'' is adjacent to u; otherwise, w' and w'' would share a neighbour in T', which is not permitted. Consider $uw' \in E(G)$. Then $uw'' \notin E(G)$. Since $V(G) = V(T') \cup V(G - T')$ with w' and w'' being neither adjacent nor share a neighbour in $V(T') \cup V(G-T')$, we have $w'w'' \notin E(G)$ and $N_G(w') \cap N_G(w'') = \emptyset$. Thus, $d_G(w', w'') \geq 3$ and hence $uw' \in E(G)$ with $ecc_G(w') \geq 3$. Therefore, by Lemma 2.2, $n \leq 2\delta' + 5$ as needed. Hence, it suffices to assume that no interior vertex of R has a neighbour in $V(G - R)$.

Observing that no vertex in $Int(R)$ has a neighbour out, assume first that each leaf of R has at most one neighbour out. Then R receives at most $2\delta' - 2$ edges from $V(G-R)$. Recall, as in the proof of Lemma 2.1, that each vertex of $S(R)$ has at least $\delta' - 2$ neigbhours in R. Thus, $|S(R)| \le 2$; otherwise, $3(\delta' - 2) > 2\delta' - 2$ for every $\delta' > 4$, which is a contradiction to the fact that R receives at most $2\delta' - 2$ edges from $V(G-R)$. Thus, $|V(G-R)| \le 3$ and we are done because $|V(R)| = 2\delta' + 2$.

To complete the proof, assume that no interior vertex of R has a neighbour in $V(G-R)$ and there is a leaf, say x_2, that has neighbours, say w_1 and w_2, in $V(G-R)$. Recall that no vertex of R has at least 3 neighbours in $V(G-R)$ because $L(G) \le 2\delta' - 1$. Now, $T'' = R \cup \{x_2w_1, x_2w_2\}$ is a tree with order $2\delta' + 4$ and $L(T'') = 2\delta' - 1$. If $|V(G-T'')| \le 1$, then we are done. We claim that $|V(G-T'')| \le 1$; otherwise, T'' is not a subtree of G. Assume the contrary, then $|V(G-T'')| = 2$ or $|V(G-T'')| = 3$ by an application of Lemma 1.3 and since u may possibly not be in T''. If $S(T'') = 2$, let $w_3, w_4 \in S(T'')$. Then $V(G-T'') = \{u, w_3, w_4\}$. As before, since w_3 and w_4 do not share a neighbour in T'' and none of them can have 2 neighbours in $V(G-T'')$, if either $w_3w_4 \in E(G)$ or $uw_3, uw_4 \in E(G)$, then by Observation 2.3, we must have $n \le 2\delta' + 5$; otherwise, T'' is not a subtree of G. Again as before, if $uw_4 \notin E(G)$, then $uw_3 \in E(G)$ with $d_G(w_3, w_4) \ge 3$ and we are done by Lemma 2.2, since $ecc_G(w_3) \ge 3$. Therefore, it is enough to assume that $|S(T'')| = 1$.

Consider $|S(T'')| = 1$ and let $w' \in S(T'')$. Then $V(G-T'') = \{u, w'\}$. If $uw' \in E(G)$, then w' cannot have neighbours in both the components G' and G'' of $T'' - \{x_1y_1\}$; otherwise, if x' and y' are neighbours of w' in the aforementioned 2 components, then add to T'' edges $x'w', y'w', uw'$ and delete the edge x_1y_1 to get a tree with $2\delta'$ leaves, which is a contradiction. Hence either $d_G(x, w') \ge 3$ or $d_G(w', y) \ge 3$ and by Lemma 2.2 we must have $n \le 2\delta' + 5$, since $ecc_G(w') \ge 3$. Assume that $uw' \notin E(G)$, then u has a neighbour, say x', in the component, say G', of $T'' - \{x_1y_1\}$. Again x' has no neighbour in the other component, say G''; otherwise, by adding suitable edges and deleting the edge x_1y_1, we get a contradiction. Thus, either $d_G(x, x') \ge 3$ or $d_G(y, x') \ge 3$ and we are done by Lemma 2.2.

Therefore, in all possible cases, we have $n \le 2\delta' + 5$ as required. □

Lemma 2.4. *For $\delta' \ge 3$, if G satisfies the conditions of Conjecture 1.4 and $G \setminus \{u\}$ is not connected, then G is traceable, provided that $n \le 10$ for $\delta' = 3$.*

Proof. Note that u is a cut vertex of G. If $G \setminus \{u\}$ is a forest, then $\delta' \le 2$ and hence the result holds by [25]. Next, assume that $G \setminus \{u\}$ is neither a tree nor a forest. It is claimed that $G \setminus \{u\}$ has 2 components. Assume on the contrary that $G \setminus \{u\}$ has at least 3 components. Let G_1, G_2, and G_3 be three of its components. Then u has a neighbour in every component because G is connected. Also, for every i, it holds that $\deg_{G_i}(x) \ge \delta' - 1$ for every $x \in V(G_i)$ because $\deg_G(x) \ge \delta'$. Let x, y and z be neighbours of u in G_1, G_2 and G_3, respectively. Attach each of them (via an edge) to u and attach to each of them at least $\delta' - 1$ neighbours of it, from its respective component. This yields a tree with at least $3(\delta' - 1) \ge 2\delta'$ leaves, which is not allowed. Thus, the claim holds.

Let G_1 and G_2 be the components of $G \setminus \{u\}$. Note that $\min\{\delta(G_1), \delta(G_2)\} \ge \delta' - 1$. Thus, $|V(G_i)| \ge \delta' \ge 3$ for $i = 1, 2$. Also, $N_G(x) \subseteq \{u\} \cup V(G_i)$ for a fixed i and for every $x \in V(G_i)$. Hence, if $|V(G_i)| = \delta'$, then $\deg_G(u) \ge \delta' + 1$ because every vertex in G_i has degree at least δ' in G and u has a neighbour in both components. This contradicts the fact that $\delta \le \delta'$. Consequently, we consider the case when $|V(G_i)| \ge \delta' + 1$. First, assume that $\delta' \ge 5$. Then $|V(G_i)| \le 2(\delta' - 1)$; otherwise, $n = |V(G_1)| + |V(G_2)| + |\{u\}| \ge 2(\delta' - 1) + 1 + (\delta' + 1) + 1 > \max\{2\delta' + 5, 3\delta' - 1\}$, which is a contradiction to Lemma 2.2. Thus, by Theorem 1.3, both the components are Hamiltonian and hence G is traceable for $\delta' \ge 5$. By similar arguments, we have $|V(G_i)| \le 2(\delta' - 1) + 1$ when $\delta' = 4$. We may assume that $|V(G_1)| = 2(\delta' - 1) + 1$; otherwise, both the components are Hamiltonian and we are done as before. Then $|V(G_2)| = \delta' + 1$ because $n \le 2\delta' + 5$ (see Lemma 2.2). By Lemma 1.2, G_2 is 2-connected and by Theorem 1.5 it is Hamiltonian connected. Assume that G_1 is not Hamiltonian; otherwise, we are done. Since $\delta(G_1) = \delta' - 1$ (note that it cannot exceed this; otherwise, we arrive at a contradiction because of the order of G, see Lemma 1.2), it follows from Corollary 1.1 or Observation 2.1 that $G_1 = K_{\delta'-1,\delta'} + H$ or G_1 has a subgraph isomorphic to a lollipop whose head is $K_{\delta'}$. By Observation 2.1 and our choice of u, if G_1 has a subgraph isomorphic to a lollipop whose head is $K_{\delta'}$, then there is a vertex in $K_{\delta'}$ that is not a cut vertex of the lollipop and must be adjacent to u; otherwise, such a vertex would have degree at most $\delta' - 1$ in G, which is not allowed. This implies that G is traceable because both $K_{\delta'}$ and G_2 are Hamiltonian connected. If $G_1 = K_{\delta'-1,\delta'} + H$, then u has a neighbour in the larger partite set of $K_{\delta'-1,\delta'}$; otherwise, every such vertex would have degree less than δ' in G, which is a contradiction to Fact 1.1. Thus, G is traceable. Similar arguments hold for $\delta' = 3$. □

Lemma 2.5. *If G satisfies the hypotheses of Conjecture 1.4 and if $G \setminus \{u\}$ is not 2-connected, then G is traceable for $\delta' \ge 4$. The result also holds for $\delta' = 3$ when $n \le 10$.*

Proof. We may assume that $G \setminus \{u\}$ is connected (by Lemma 2.4). Since $G \setminus \{u\}$ is not 2-connected and $\delta(G \setminus \{u\}) \ge \delta' - 1 \ge 2$ (that is, $|V(G \setminus \{u\})| \ge 3$), let u_c be a cut vertex of $G \setminus \{u\}$. We claim that $G \setminus \{u, u_c\}$ has 2 components. Contrarily, assume that $G \setminus \{u, u_c\}$ has at least 3 components. Let G_1, G_2 and G_3 be three of its components. Then $\delta(G_i) \ge \delta' - 2$ and $|V(G_i)| \ge \delta' - 1$

for $i = 1, 2, 3$. Examine first the occurrence that there is a vertex $x_i \in V(G_i)$ such that $ux_i \notin E(G)$ for $i = 1, 2, 3$. Then by the choice of u and u_c, $\deg_{G_i}(x_i) \geq \delta' - 1$ because $\deg_G(x_i) \geq \delta'$; that is, $|V(G_i)| \geq \delta'$. Also, it holds that $|V(G_i)| \leq \delta'$; otherwise, $n \geq \sum_{i=1}^{3} |V(G_i)| + |\{u, u_c\}| \geq 3\delta' + 3$, which is a contradiction to Lemma 2.2. By symmetry, $|V(G_i)| = \delta'$. Thus, $x_i u_c \in E(G)$ because $N_G(x_i) \subseteq \{u_c\} \cup V(G_i)$ with $\deg_G(x_i) \geq \delta'$ by the choice of u and u_c. Now, take a star with edges $u_c x_i$ and to every x_i attach $\delta' - 1$ of its neighbours from G_i to get a tree with $3\delta' - 3 \geq 2\delta'$ leaves, which is not permitted.

Consider the case when every vertex in G_i is a neighbour of u for some i. We may assume that every vertex in G_1 is a neighbour of u. Then $|V(G_1)| \leq \delta'$; otherwise, $\delta = \deg_G(u) > \delta'$, which is a contradiction. This together with $|V(G_1)| \geq \delta' - 1$ implies that $|V(G_1)| = \delta' - 1$ or $|V(G_1)| = \delta'$. If $|V(G_1)| = \delta' - 1$, then by the choice of u and u_c every vertex in G_1 is adjacent to u_c and u has no neighbour in one of the components, say G_3, because $\delta \leq \delta'$. Let x and y be neighbours of u_c in G_2 and G_3, respectively. Form a tree T' by attaching all vertices of G_1 to u_c, add the edges $u_c x, u_c y$, and apart from u_c, attach to x, $\delta' - 1$ of its neighbours and attach $\delta' - 1$ neighbours of y to y. Then $L(T') = 3\delta' - 3 \geq 2\delta'$, which contradicts $L(G) \leq 2\delta' - 1$. To complete the proof, let $|V(G_1)| = \delta'$. Then u has no neighbour in $V(G_2) \cup V(G_3)$. Let w, x and y be neighbours of u_c in G_1, G_2 and G_3, respectively. Take the edges $u_c w, u_c x, u_c y$ and to each element of $\{w, x, y\}$ attach $\delta' - 1$ of its neighbours apart from u_c. This again builds a contradiction. Therefore, $G \setminus \{u, u_c\}$ has 2 components.

Let G_1 and G_2 be the components of $G \setminus \{u, u_c\}$. Then the following property holds:

Property 2.1. $n = |V(G)| = |V(G_1)| + |V(G_2)| + |\{u, u_c\}|$, $N_G(x) \subseteq \{u, u_c\} \cup V(G_i)$ *and* $\delta(G_i) \geq \delta' - 2$ *(that is,* $|V(G_i)| \geq \delta' - 1$*) for* $i = 1, 2$ *and for every* $x \in V(G_i)$*.*

Claim 2.1. *The vertex* u *has a neighbour in* G_1 *or* G_2*.*

Proof of Claim 2.1. Assume to the contrary that $uu_c \in E(G)$ and $\deg_G(u) = 1$. Let x and y be neighbours of u_c in G_1 and G_2, respectively. Apart from u_c, let $\{x_1, x_2, \cdots, x_{\delta'-1}\} \subseteq N_G(x)$ and $\{y_1, y_2, \cdots, y_{\delta'-1}\} \subseteq N_G(y)$. Set

$$T' = \{uu_c, u_c x, u_c y, xx_i, yy_i : \ i \in \{1, 2, \cdots, \delta' - 1\}\}.$$

Then $L(T') = 2\delta' - 1$ and $|V(G - T')| \leq 2$ (see Lemma 1.3). Assume that $|V(G - T')| = 2$ and take $w', w'' \in V(G - T')$. Then, by Lemma 1.3, $w'w'' \in E(G)$. Thus, both the vertices w' and w'' are either in G_1 or in G_2. We may assume that $w', w'' \in V(G_1)$. By Property 2.1 and because of the fact that no interior vertex of T' can have a neighbour in $V(G - T')$, possible neighbours for w' and w'' in T' are only found in the set $\{x_1, x_2, \cdots, x_{\delta'-1}\}$. Therefore, w' and w'' share a neighbour in T' because each of them has at least $\delta' - 1$ neighbours in T', which contradicts $L(G) \leq 2\delta' - 1$. In fact, $V(G - T') = \emptyset$; otherwise, if w is the only vertex in $V(G - T')$, then by similar arguments, w has at most $\delta' - 1$ neighbours in T', which is a contradiction to $\deg_G(w) \geq \delta'$. Here, $V(G - T') = \emptyset$ means $|V(G_i)| = \delta'$ for $i = 1, 2$. Since $\deg_G(z) \geq \delta'$ for every $z \in V(G_i)$ and $\deg_G(u) = 1$, it follows from Property 2.1 that every vertex in G_i must be adjacent to u_c. Thus, $\deg_G(u_c) > 2\delta' - 1$, which is a contradiction to $L(G) \leq 2\delta' - 1$. Therefore, u must have a neighbour in G_1 or G_2. This completes the proof of Claim 2.1. □

Claim 2.2. *If* $\min\{|V(G_1)|, |V(G_2)|\} = \delta' - 1$*, then* G *is traceable.*

Proof of Claim 2.2. We may assume that $|V(G_1)| = \delta' - 1$. Since $\deg_G(x) \geq \delta'$ for every $x \in V(G_1)$, by employing Property 2.1 we have $ux, xu_c \in E(G)$. Thus, u has at most one neighbour in G_2; otherwise, $\delta = \deg_G(u) \geq |V(G_1)| + 2$, which contradicts $\delta \leq \delta'$.

Case 1. Assume that u and u_c share a neighbour in G_2. Let $y \in V(G_2)$ be a common neighbour of u and u_c. Let T'' be a tree formed by attaching every vertex of G_1 to u_c, adding the edges $u_c y, yu$, and attaching $\delta' - 2$ neighbours, say $y_1, y_2, \cdots, y_{\delta'-2}$, of y to y. Then $L(T'') = 2\delta' - 2$. By Lemma 2.1, $|V(G - T'')| \leq 4$ because $n \leq 10$ for $\delta' = 3$. Note that, apart from y, u has no neighbour in G_2. Hence, for every $z \in V(G_2) \setminus \{y\}$, we have $N_G(z) \subseteq \{u_c\} \cup V(G_2)$. In what follows, we show that $|V(G - T'')| = 1$, so that $|V(G_2)| = \delta'$ and G is traceable. To do this, we note that $V(G - T'') \neq \emptyset$; otherwise, since the vertices in $\{u\} \cup V(G_1)$ are not possible neighbours of y_i for $i \in \{1, 2, \cdots, \delta' - 2\}$, we have $\deg_G(y_i) \leq \delta' - 1$, which is not permitted (see Fact 1.1). Also, we note that if $|V(G - T'')| = 1$ and $w \in V(G - T'')$, then by previous arguments and because of the fact that $V(G - T'') \subseteq V(G_2)$, $N_G(w) = \{u_c, y, y_1, y_2, \cdots, y_{\delta'-2}\}$. Similarly, for $i \in \{1, 2, \cdots, \delta' - 2\}$, we have $N_G(y_i) = (N_G[y] \setminus \{u, y_i\}) \cup \{u_c, w\}$. That is, the graph induced by $\{u_c\} \cup V(G_2)$ forms the complete subgraph $K_{\delta'+1}$ of G that is Hamiltonian connected. This together with the fact that $uy \in E(G)$ and that the graph induced by $\{u_c\} \cup V(G_1)$ is Hamiltonian, implies that G is traceable.

Assume that there is $x' \in Int(T'')$ such that $x'w \in E(G)$ for some $w \in V(G - T'')$. Take $T''' = T'' \cup \{x'w\}$. Then by Lemma 1.3, $|V(G - T''')| \leq 2$. Suppose that $|V(G - T''')| = 2$ and set $V(G - T''') = \{w', w''\}$, then by Lemma 1.3, $w'w'' \in E(G)$. Since $V(G - T''') \subseteq V(G_2)$, either w' and w'' share a neighbour in T''' or one of them is adjacent to an interior vertex of T''', thereby producing a tree with at least $2\delta'$ leaves, which is not allowed. Assume that $|V(G-T''')|=1$ and let w' be the only vertex not in T'''. Since w' cannot be adjacent to an interior vertex of T''' and $V(G-T''')\subseteq V(G_2)$, $w, y_1, y_2, \cdots, y_{\delta'-2}$ are the only possible neighbours of w' in T'''. It means that $\deg_G(w')\leq \delta'-1$, which is a contradiction. Thus, in this instance, $V(G-T''')=\emptyset$ implies that $|V(G-T'')|\leq 1$ and G is traceable as shown before.

Suppose that no interior vertex of T'' has a neighbour in $V(G-T'')$. Then, for a vertex in $V(G-T'')$, $y_1, y_2, \cdots, y_{\delta'-2}$ are the only possible neighbours of it because $V(G-T'')\subseteq V(G_2)$ (see the first paragraph of the proof of the considered case, that is Case 1). Evidently, no leaf of T'' has at least 3 neighbours in $V(G-T'')$. Assume that there exists a leaf, say y_1, that has 2 neighbours, say w' and w'', in $V(G-T'')$. Again by Lemma 1.3, $|V(G-T''')|\leq 2$, where $T'''=T''\cup\{y_1w', y_1w''\}$. By the same arguments as in the previous paragraph, we have $V(G-T''')=\emptyset$. Since no interior vertex of T'' has a neighbour outside T'', the elements of $\{w''\}\cup\{y_1, y_2, \cdots, y_{\delta'-2}\}$ are the only possible neighbours of w', which is a contradiction to $\deg_G(w')\geq\delta'$. Therefore, no leaf of T'' can have 2 neighbours outside T''. Now, consider each leaf of T'' having at most one neighbour in $V(G-T'')$. Since $y_1, y_2, \cdots, y_{\delta'-2}$ are the only possible neighbours in T'' of a vertex in $V(G-T'')$, T'' receives at most $\delta'-2$ edges from $V(G-T'')$. This in conjunction with the fact that each vertex in $V(G-T'')$ has at least $\delta'-2$ neighbours in T'' means that $|V(G-T'')|\leq 1$; otherwise, $2(\delta'-2)>\delta'-2$ for every $\delta'>2$, which is a contradiction to the fact that T'' receives at most $\delta'-2$ edges from outside. Thus, in all instances, either the case is impossible or G is traceable as mentioned before.

Case 2. We now consider the scenario where $|V(G_1)|=\delta'-1$ such that u and u_c have no common neighbour in G_2. Clearly, $\delta(G_2)\geq\delta'-1$. Since by the construction, the graph induced by $V(G_1)$ is the complete graph $K_{\delta'-1}$, G_1 is Hamiltonian connected. This together with the fact that every vertex in G_1 is adjacent to the vertices u and u_c implies that the graph induced by $\{u, u_c\}\cup V(G_1)$ has a u_c-u spanning path. Hence, to prove that G is traceable, it is enough to show that the subgraph induced by $\{u_c\}\cup V(G_2)$ or $\{u\}\cup V(G_2)$ contains a spanning path that has either u or u_c as an end vertex. For $\delta'\geq 6$, $|V(G_2)|\leq 2\delta'-2\leq 2\delta(G_2)$; otherwise, by Property 2.1 it holds that $n>3\delta'-1$, which contradicts Lemma 2.2. Thus, by Theorem 1.3, G_2 is Hamiltonian and we are done. For $3\leq\delta'\leq 5$, we set $\delta(G_2)=\delta'-1$; otherwise, $|V(G_2)|\leq 2\delta'$ and G_2 is Hamiltonian as before.

Contemplating at $\delta'=5$, we see that $|V(G_2)|\leq 2(\delta'-1)+1$ because $n\leq 2\delta'+5$ (see Property 2.1 and Lemma 2.2). We may assume that G_2 is not Hamiltonian; otherwise, we are done as before. By Corollary 1.1 and Observation 2.1, $G_2=K_{\delta'-1,\delta'}+H$ or G_2 has a spanning subgraph isomorphic to a lollipop whose head is $K_{\delta'}$. If G_2 has a spanning subgraph that is a lollipop whose head is K'_{δ}, then u or u_c is adjacent to a vertex $y\in V(K_{\delta'})$, where y is not a cut vertex of the lollipop (since $\deg_{G_2}(y)=\delta'-1$ in contrast to $\deg_G(y)\geq\delta'$). Thus, in this scenario, the graph induced by $\{u_c\}\cup V(G_2)$ or $\{u\}\cup V(G_2)$ contains a spanning path that has either u or u_c as an end vertex; that is, G is traceable. Similarly, if $G_2=K_{\delta'-1,\delta'}+H$, then u or u_c has a neighbour in the larger partite set of the subgraph $K_{\delta'-1,\delta'}$ of G_2 and hence the desired conclusion follows.

Now, we consider the possibility $\delta'\in\{3,4\}$. Again by Lemma 2.2 and Property 2.1, $|V(G_2)|\leq 2(\delta'-1)+2$. Hence, by using Observation 2.1, Corollaries 1.1 and 1.2, if G_2 is Hamiltonian or G_2 is not 2-connected, then we proceed as in the previous paragraph. Assume that G_2 is non-Hamiltonian and 2-connected. It suffices to consider $|V(G_2)|=2(\delta'-1)+2$; otherwise, we are done by the same argument as for $\delta'=5$. Let $C(G_2)$ be a longest cycle in G_2. By Theorem 1.3, $|C(G_2)|\geq 2(\delta'-1)$. If $|V(G_2-C(G_2))|=1$ and v is the only vertex in G_2 not on $C(G_2)$, then by Property 2.1, either $uv\in E(G)$ or $u_cv\in E(G)$ because v cannot have at least δ' neighbours on $C(G_2)$, that is, $|V(G_2)|\leq 2\delta'$. This together with the fact that v has a neighbour on $C(G_2)$, implies that G is traceable. Therefore, consider $|V(G_2-C(G_2))|=2$ and let $u_1, u_2\in V(G_2-C(G_2))$. If u_1 and u_2 are adjacent such that at least one of them is a neighbour of u or u_c, then we are done by noticing that either u_1 or u_2 has a neighbour on $C(G_2)$. Assume that $u_1u_2\in E(G)$ such that neither of these two vertices is adjacent to u or u_c. By Property 2.1, at least $\delta'-1$ neighbours of u_i are on $C(G_2)$. By the choice of $C(G_2)$, neither u_1 nor u_2 is adjacent to a vertex in $N^+(u_1)$. This in conjunction with the fact that $N^+(u_1)$ is an independent set and that every $x\in N^+(u_1)$ has degree at least δ' in G (x has at most $\delta'-1$ neighbours in G_2 by the aforementioned arguments), implies that $xu\in E(G)$ or $xu_c\in E(G)$. Thus, again the graph induced by $\{u_c\}\cup V(G_2)$ or $\{u\}\cup V(G_2)$ contains a spanning path that has either u or u_c as an end vertex and G is traceable.

Next, we consider the possibility when $u_1u_2\notin E(G)$. Since u has at most one neighbour in G_2, either $uu_1\notin E(G)$ or $uu_2\notin E(G)$. We may assume that $uu_1\notin E(G)$. Then $u_1u_c\in E(G)$ because u_1 has at most $\delta'-1$ neighbours on $C(G_2)$ (see also Property 2.1). Note that the set $\{u_1, u_2\}\cup N^+(u_1)$ is an independent set; otherwise, G is traceable. In addition, every vertex in the mentioned set has at most $\delta'-1$ neighbours in G_2 because of the choice of $C(G_2)$. Thus, at least δ' elements of this set are adjacent to u_c, see Property 2.1. Now, take u_c, attach to it all vertices of $V(G_1)$ and at least δ' elements of $\{u_1, u_2\}\cup N^+(u_1)$ (including u_1), then attach to u_1 the $\delta'-1$ of its neighbours from $C(G_2)$; thus, this newly formed tree has at least $|V(G_1)|+(|\{u_1,u_2\}\cup N^+(u_1)|-2)+|N_{C(G_2)}(u_1)|=3\delta'-3$ leaves, which is a contradiction to $L(G)\leq 2\delta'-1$.

Therefore, the considered subcase is impossible. □

Claim 2.3. *If* $\min\{|V(G_1)|, |V(G_2)|\} = \delta'$*, then* G *is traceable.*

Proof of Claim 2.3. We may assume that $|V(G_1)| = \delta'$.

Case 1. suppose that u has no neighbour in G_1. By Property 2.1 and because of the fact that $\deg_G(x) \geq \delta'$ for every $x \in V(G_1)$, every vertex in G_1 is adjacent to u_c. In fact, the subgraph of G induced by the set $\{u_c\} \cup V(G_1)$ forms a complete graph $K_{\delta'+1}$. Let $y \in V(G_2)$ be a neighbour of u_c and $y_1, y_2, \cdots, y_{\delta'-1}$ be neighbours of y apart from u_c. Define $T' \leq G$ as follows

$$T' = \{u_c x, u_c y, y y_i \mid \text{ for } i \in \{1, 2, \cdots, \delta' - 1\}, \text{ for every } x \in V(G_1)\}.$$

Then $L(T') = 2\delta' - 1$. By Lemma 1.3, $|S(T')| \leq 2$ and hence $|V(G - T')| \leq 3$. After doing an analogous analysis to that of Case 1 in the proof of Claim 2.2, we have

$$n \leq \begin{cases} 2\delta' + 3 \text{ if } u \notin V(T'), \\ 2\delta' + 1 \text{ if } u \in V(T'). \end{cases}$$

Thus, if $u \in V(T')$, then the graph induced by $\{u, u_c\} \cup V(G_2)$ has a subgraph isomorphic to $K_{\delta'+1} - e$ and hence G is traceable. Consider the subcase when $u \notin V(T')$. If u is the only vertex not in T', then we are done by following similar arguments as before. Hence, let $V(G - T') = \{u, w\}$. By Property 2.1 and because of the fact that no interior vertex of T' has a neighbour in $V(G - T')$, we have $N_G(w) = \{u, y_1, y_2, \cdots, y_{\delta'-1}\}$. Since u and w cannot share a neighbour in T', it follows by the choice of u that $\deg_G(u) = 1$. Also, $u_c y_i \notin E(G)$ for every $i \in \{1, 2, \cdots, \delta' - 1\}$; otherwise, if $u_c y_j \in E(G)$ for some fixed j, the tree

$$(T' - \{y y_i, y u_c \mid \ i \in \{1, 2, \cdots, \delta' - 1\} \text{ and } i \neq j) \cup \{u_c y_j, wz \mid \ z \in N_G(w)\}$$

has $2\delta'$ leaves, which is not permitted. Therefore, G_2 has a subgraph isomorphic to $K_{\delta'+1} - e$ where $e = wy$. This together with the fact that $uw \in E(G)$ and that the graph induced by $\{u_c\} \cup V(G_1)$ forms a $K_{\delta'+1}$ graph, implies that G is traceable.

Case 2. Suppose that u has a neighbour in G_1. Since $\delta(G_1) \geq \delta' - 2$, it follows from Theorem 1.3 and Corollary 1.1 that G_1 is Hamiltonian or $G_1 = P_3$. If $G_1 = P_3$ then the end vertices of P_3 must be adjacent to both u and u_c. If G_1 is Hamiltonian, then there exist distinct vertices x' and x'' on the Hamilton cycle $C(G_1)$ of G_1 such that $x'u, x''u_c \in E(G)$ because both u and u_c have neighbours in G_1 with each vertex in $V(G_1)$ being adjacent to u or u_c. In all instances, the graph induced by $\{u, u_c\} \cup V(G_1)$ has a spanning $u_c - u$ path. As before it suffices to show that the graph induced by $\{u_c\} \cup V(G_2)$ or $\{u\} \cup V(G_2)$ has a spanning path that has u or u_c as an end vertex. Consider first the subcase when $\delta(G_2) \geq \delta' - 1$. It follows by employing Property 2.1 and Lemma 2.2 that $|V(G_2)| \leq 2(\delta' - 1) + 1$. Therefore, by using Corollary 1.1 together with Observation 2.1 as in Claim 2.2, we conclude that G is traceable.

Now, consider the subcase when $\delta(G_2) = \delta' - 2$. Note that u and u_c share a neighbour, say $y \in V(G_2)$. Take the path u, y, u_c, attach every vertex of G_1 to either u or u_c without creating cycles and attach to y, $\delta' - 2$ of its neighbours, apart from those already mentioned. Let $T'' \leq G$ be a tree formed by these operations. Then $L(T'') \geq 2\delta' - 2$ and $|V(G - T'')| \leq 4$ (by Lemma 1.3, Lemma 2.1 and the fact that $n \leq 10$ for $\delta' = 3$). Using similar arguments to that of Case 1 of Claim 2.2's proof, we have $|V(G - T'')| \leq 1$ and $n \leq 2\delta' + 2$. That is, $|V(G_2)| \leq \delta'$. Again by Theorem 1.3 and Corollary 1.1, $G_2 = P_3$ or G_2 is Hamiltonian and we are done by the same arguments as in the previous paragraph. □

Claim 2.4. *If* $\min\{|V(G_1)|, |V(G_2)|\} \geq \delta' + 1$*, then* G *is traceable.*

Proof of Claim 2.4. Note that $|V(G_i)| \leq 2(\delta' - 2) + 2 \leq 2\delta(G_i) + 2$ for $i = 1, 2$; otherwise, from Property 2.1, it follows that $n \geq 3\delta' + 2 \geq \max\{2\delta' + 5, 3\delta' - 1\}$, which is not allowed (see also Lemma 2.2). Thus, by Corollaries 1.1 and 1.2, G_i is Hamiltonian, or G_i is traceable or $G_i = K_{\delta(G_i), \delta(G_i)+2} + H$.

We show that $G_i \neq K_{\delta(G_i), \delta(G_i)+2} + H$. Suppose to the contrary that $G_i = K_{\delta(G_i), \delta(G_i)+2} + H$. Denote by $V_{\delta(G_i)}$ and $V_{\delta(G_i)+2}$ the smaller and the larger partite sets of the subgraph $K_{\delta(G_i), \delta(G_i)+2}$ of G_i, respectively. Note that $\deg G_i(z) = \delta(G_i)$ for every $z \in V_{\delta(G_i)+2}$. Consider first the subcase when $\delta(G_i) = \delta' - 2$. By Property 2.1, the choice of u and u_c, and the fact that $\deg_G(z) \geq \delta'$ for every $z \in V_{\delta'}$, we have $uz, u_c z \in E(G)$. Hence, $\delta = \deg_G(u) \geq |V_{\delta'}| = \delta'$. This together with $\delta \leq \delta'$ implies that $\deg_G(u) = \delta'$. Thus, u cannot have a neighbour in the other component. This implies that u_c is also a cut vertex of G, which is a contradiction to Lemma 1.1; that is, G must be 2-connected whenever $\delta = \delta'$ and $L(G) \leq 2\delta - 1$. Hence, the considered subcase is impossible. Next, consider the subcase when $\delta(G_i) = \delta' - 1$. Since $\deg G_i(z) = \delta' - 1$ for every $z \in V_{\delta'+1}$, u_c must have a neighbour, say z', in the larger partite set $V_{\delta'+1}$; otherwise, by Property 2.1, u is adjacent to every vertex in $V_{\delta'+1}$, which is a contradiction to $\delta \leq \delta'$. In G_i, take z', attach to it every vertex of $V_{\delta'-1}$, and to one of its neighbours, say $z'' \in V_{\delta'-1}$, attach δ' of its neighbours from $V_{\delta'+1} \setminus \{z'\}$. If $T' \leq G_i$ is a tree formed by these operations, then $L(T') = 2\delta' - 2$. We may assume that $i = 2$. Let $x \in V(G_1)$ be a neighbour of u_c. To T', add edges $u_c z, u_c x$, and attach to x, $\delta' - 1$ of its neighbours, apart from u_c. Then the resulting tree has at least $2\delta'$ leaves, which is a contradiction again. For $\delta(G_i) \geq \delta'$, by similar operations, we get a tree with at least $2\delta'$ leaves, again a contradiction. Since $\delta(G_i) \geq \delta' - 2$, we conclude that $G_i \neq K_{\delta(G_i),\delta(G_i)+2} + H$. □

Claim 2.5. *If* $\min\{|V(G_1)|, |V(G_2)|\} \geq \delta' + 1$*, then* u *and* u_c *do not share a neighbour. That is,* $\delta(G_i) \geq \delta' - 1$ *for* $i = 1, 2$.

Proof of Claim 2.5. First, we consider the case when all components are Hamiltonian. We assume the contrary and let $y_0 \in V(G_2)$ be a common neighbour of u and u_c. For integers s and t such that $\min\{s, t\} \geq \delta'$, let $C' = x_0, x_1, x_2, \cdots, x_s, x_0$ and $C'' = y_0, y_1, y_2, \cdots, y_t, y_0$ be spanning cycles for G_1 and G_2, in that order. Let $x_0 \in V(G_1)$ be a neighbour of u_c. Using C' and C'' form a connected spanning subgraph G' of G by adding edges $u_c x_0, u_c y_0, u y_0$. From G' form a tree T'' by deleting consecutive vertices $x_1, x_2, \cdots, x_{\delta'}$ of C' and $y_1, y_2, \cdots, y_{\delta'}$ of C''. Then by the choice of u, u_c, G_1 and G_2, each of the deleted vertices has at most $\delta' - 1$ neighbours in $V(G - T'')$. Thus, each of the deleted vertices has a neighbour in T''. Join each of the deleted vertices to one of its neighbours in T'' to get a tree with at least $2\delta'$ leaves, which opposes $L(G) \leq 2\delta' - 1$. If any of the components is traceable but not Hamiltonian, then we consider its spanning path and use similar arguments as before; that is, we delete δ' vertices from the respective component in a consecutive manner from at least one of its end vertices in such a way that the remaining component is tree of G. Therefore, u and u_c cannot share a neighbour. □

Claim 2.5 in conjunction with Property 2.1 and Lemma 2.2 implies that $|V(G_i)| \leq 2(\delta' - 1) \leq 2\delta(G_i)$ for $i = 1, 2$. Hence, by Theorem 1.6, G_i is Hamiltonian connected, or $G_i \in \mathcal{G}_1$ or $G_i \in \mathcal{G}_2$. Note also that G_i is Hamiltonian (see Theorem 1.3). By Claim 2.1, we may consider that u has a neighbour in G_1. If G_1 is Hamiltonian connected, then G is traceable because a neighbour of u is not a neighbour of u_c, by Claim 2.5. Now, consider the case when $G_1 \in \mathcal{G}_2$. Then, for s defined in Theorem 1.6, we have $\delta' - 1 \leq s < \delta'$; otherwise, we have a contradiction to either $\delta(G_1) \geq \delta' - 1$ or $|V(G_1)| \leq 2\delta' - 2$. Thus, $s = \delta' - 1$. Since G_1 is a bipartite graph, let V' and V'' be its partite sets. By the definition of G_1, $\deg_{G_1}(x) = \delta' - 1$ for every $x \in V(G_1)$. Hence, every vertex of G_1 must be adjacent to either u or u_c (see Property 2.1). This together with the fact that u has a neighbour in G_1 implies that there exist distinct vertices $x' \in V'$ and $x'' \in V''$ such that $ux' \in E(G)$ and $u_c x'' \in E(G)$. Thus, by Observation 2.2, the subgraph induced by $\{u, u_c\} \cup V(G_1)$ has a $u_c - u$ spanning path. Therefore, G is traceable.

Let us examine the case when G_1 belongs to $\mathcal{G}_1$. Then $\delta(G_i) = \min\{s+1, t+1\} \geq \delta' - 1$. Hence, $\delta' - 2 = \min\{s, t\}$ because $\min\{s, t\} < \delta' - 1$; otherwise, by Theorem 1.6, we have $|V(G_1)| = s + t + 2 > 2\delta' - 2$, a contradiction. We may consider that $\min\{s, t\} = s = \delta' - 2$. Then, either u or u_c has a neighbour in K_s because every vertex $x \in V(K_s)$ satisfies $\deg_G(x) \geq \delta'$, whereas $\deg_{G_1}(x) = \delta' - 1$. Since, by Claim 2.5, u and u_c have no common neighbour, it follows from Observation 2.2 that the subgraph induced by $\{u, u_c\} \cup V(G_1)$ has a $u_c - u$ spanning path. Therefore, G is traceable because G_2 is Hamiltonian.

The proof of Lemma 2.5 is now complete. □

The following claim is crucial when $G \setminus \{u\}$ is 2-connected.

Claim 2.6. *Let* G *be a graph that satisfies the conditions of Conjecture 1.4 and* $A \subset V(G)$ *be an independent set such that there exists* $v \in A$ *with* $\deg_G(v) \geq \delta'$*. Define* B *by* $B = V(G) - (N_G[v] \cup A)$*. If* $ecc_G(v) \leq 2$*, then* $|A| \leq \delta' + 1$*. Also, if* $d_G(v, x) \leq 2$ *for every* $x \in V(G) \setminus \{u\}$*, then* $|A| \leq \delta' + 2$ *when* $u \notin A$ *and* $|A| \leq \delta' + 1$ *when* $u \in A$ *or* u *has only one neighbour in* A*. In addition, if* $w \neq u$ *such that* $d_G(v, x) \leq 2$ *for every* $x \in V(G) \setminus \{w\}$ *and* $ww' \in E(G)$ *for some* $w' \in B$*, then* $|A| \leq \delta' + 1$.

Proof. For non-negative integers d, l and p, set $\deg_G(v) = d = p + \delta'$ and $l = |B|$. Assume to the contrary that $ecc_G(v) \leq 2$ but $|A| \geq \delta' + 2$. Since $ecc_G(v) \leq 2$, we have $l \leq \delta' - 3$; otherwise, $A \cup B$ is a leaf set with at least $2\delta'$ leaves of the tree formed by attaching every element of $A \cup B$ to one of its neighbours in the star graph formed by $N_G[v]$. Choose $p + 1 + l$ neighbours of v in such a way that every element in B, and possibly u, has a neighbour among them, and attach them to v. Let T' be the tree formed by this operation. Then, $|N_{G-T'}(v)| = d - (p + 1 + l) = \delta' - l - 1$. Since A is an independent set, every element of A has at most $|N_{G-T'}(v)| + |B| = \delta' - 1$ neighbours outside T'. Thus, every element of A has at least one neighbour in T' because either $\deg_G(x) \geq \delta'$, for $x \in A$, or if u is in A then already one of its neighbours has been chosen among the neighbours of v in T'. Consequently, every vertex in $V(G - T')$ has a neighbour in T'. Thus, $V(G - T')$ forms a leaf set of a tree in G. Now, $|V(G - T')| = |N_{G-T'}(v)| + |A \setminus \{v\}| + |B| \geq (\delta' - l - 1) + (\delta' + 1) + l = 2\delta'$, which is a contradiction to $L(G) \leq 2\delta' - 1$.

To prove the second part, we note that $d_G(u, v) \leq 3$. We may assume that $d_G(u, v) = 3$; otherwise, we are done as before. Let x'' be a neighbour of u. If u has a neighbour in A, then we choose x'' such that $x'' \in A$. Since $d_G(v, x) \leq 2$ for every $x \in V(G) \setminus \{u\}$, there exists $x' \in N_G(v)$ such that $x'x'' \in E(G)$. Since $d_G(u, v) = 3$, choose $P_{uv} = u, x'', x', v$ as a shortest $u - v$ path in G. Then, P_{uv} contains at most 2 elements of A because A is an independent set. Also, if $u \in A$, then $x'' \notin A$ and hence $x'' \in B$. Similarly, if $u \notin A$, then $u \in B$. Thus, P_{uv} contains at least one element of B. Let $s \leq l-1$ be the number of elements of B not on P_{uv}. Choose $p+s$ neighbours of v such that every element of B not on P_{uv} has a neighbour among them, and attach them to v to form a tree T''. Then, $|N_{G-T''}(v)| = d-(p+1+s) = \delta'-s-1$. Since A is an independent set, every element of A has at most $|N_{G-T''}(v)|+s = \delta'-1$ neighbours outside T'. As before, every element of $V(G-T'')$ has a neighbour in T''. Hence, if $|A| \geq \delta'+3$, then $|V(G-T'')| \geq 2\delta'$ and as before we have a contradiction. If u has at most one neighbour in A, then by the choice of x'', either $x'' \in A$ or $u \in A$. Thus, every vertex of A not in T'' cannot be adjacent to u. Thus, every element of A not in T'' has a neighbour in $V(T'') \setminus \{u\}$ and by the choice of T'', it follows that every vertex in $V(G-T'')$ has a neighbour in $V(T'') \setminus \{u\}$. Thus, $\{u\} \cup V(G-T'')$ forms a leaf set of a tree in G. Therefore, if

$$|A| \geq \delta'+2,$$

then

$$|\{u\} \cup V(G-T')| = |\{u\}| + |N_{G-T''}(v)| + (|A|-2) + s \geq 1 + (\delta'-s-1) + (\delta') + s = 2\delta',$$

which is a contradiction.

Now, to settle the last part of the claim, we may consider that $d_G(v,w) = 3$; otherwise, the previous cases prove the result. Since $d_G(v,x) \leq 2$ for every $x \in V(G) \setminus \{w\}$, let $v' \in N(v)$ be a neighbour of w'. Let $P_{vw} = v, v', w', w$ be a shortest $v-w$ path in G. If $uv \notin E(G)$ or u has no neighbour on P_{vw}, then to the path, attach to v, $p+1+s$ of its neighbours, which are chosen in such a way that the elements of B not on the path, together with u, have a neighbour among them. If $uv \in E(G)$ or u has a neighbour on the path, we attach $p+s$ neighbours of v to v. In both cases we proceed as before to show that there is tree with at least $2\delta'$ leaves in G whenever $|A| \geq \delta'+2$, which is a contradiction.

Hence, in all cases, if we assume the contrary, then G does not satisfy the hypotheses of Conjecture 1.4. Therefore, Claim 2.6 holds. □

Lemma 2.6. *If G satisfies the conditions of Conjecture 1.4 and $G \setminus \{u\}$ is 2-connected with $diff(G \setminus \{u\}) \leq 1$, then G is traceable for $\delta' \geq 4$. The result also holds for $\delta' = 3$ when $|V(G)| \leq 10$.*

Proof. If $diff(G\setminus\{u\}) = 0$, then $G\setminus\{u\}$ is Hamiltonian and G is traceable. Consider the case when $diff(G\setminus\{u\}) = 1$. Let $C_k = v_0, v_1, v_2, \cdots, v_k, v_0$ be a longest cycle in $G\setminus\{u\}$. Assume that $G\setminus\{u\}$ is not Hamiltonian. Let $v \in V(G\setminus\{u\} - C_k)$ be arbitrary. Set $\deg_G(v) = d = p+\delta'$ for $p \in \mathbb{Z}^+ \cup \{0\}$. Since $diff(G\setminus\{u\}) = 1$, let $N_{C_k}(v) = \{v_{t_1}, \cdots, v_{t_s},\ \delta'-1 \leq s \leq d\}$. Then $N^+(v) = \{v_{t_1+1}, v_{t_2+1}, \cdots, v_{t_s+1}\}$. We observe that $V(G\setminus\{u\} - C_k) \cup N^+(v)$ is an independent set because $diff(G\setminus\{u\}) = 1$ and C_k is a longest cycle in $G\setminus\{u\}$. Let $B = V(G) - [N_G[v] \cup N^+(v)]$ and set $l = |B|$.

Consider first the case when every vertex in $V(G\setminus\{u\} - C_k)$ is adjacent to u. We show that $|V(G\setminus\{u\} - C_k)| \leq 2$, which implies that G is traceable. Assume to the contrary that $|V(G\setminus\{u\} - C_k)| \geq 3$. Let $u_1, u_2 \in V(G\setminus\{u\} - C_k)$, apart from v. Then $ecc_G(v) \geq 3$; otherwise, $V(G\setminus\{u\} - C_k) \cup N^+(v)$ is an independent set of at least $\delta'+2$ elements, which contradicts Claim 2.6. Let $w \in V(G)$ such that $d_G(v,w) = 3$. Evidently, $w \in B \cap V(C_k)$. Set $w = v_s$ for $t_1 < s < t_2$. By the choice of w, $|V(C_k)| \geq 2\delta'$, since $s \geq \delta'-1$ and $v_{t_2-1} \notin N(v) \cup N^+(v)$. By Lemma 2.3, $n \leq 2\delta'+5$. So, $|N^-(v)| \leq 2$. Neither $|N^-(v)| = 2$ nor $|V(G\setminus\{u\} - C_k)| \geq 4$; otherwise, $d_G(v,x) \leq 2$ for every $x \in V(G)\setminus\{w\}$ and $V(G\setminus\{u\} - C_k) \cup N^+(v)$ is an independent set of at least $\delta'+2$ elements, which is not allowed. Hence, consider that $|N^-(v)| = 1$. Take the edge uv, join to u every vertex of $V(G\setminus\{u\} - C_k)$ and add the path $v, v_{t_2}\overleftarrow{C_k}v_{t_2-2}$. Let T' be the tree formed following these procedures. Then each vertex in $N^+(v)$ is adjacent to at most $\delta'-2$ vertices in $V(G-T')$ and to at most one leaf of T', since $V(G\setminus\{u\} - C_k) \cup N^+(v)$ is an independent set. Thus, every vertex in $N^+(v)$ not in T' is adjacent to some interior vertex of T'. This together with the fact that v is an interior vertex of T' and that $V(G-T') \subset N_G(v) \cup N^+(v)$ implies that every vertex of $V(G-T')$ is adjacent to a vertex in $Int(T')$. Hence, the set $\{u_1, u_2, v_{t_2+2}\} \cup V(G-T')$ forms a leaf set of at least $2\delta'$ elements, which is a contradiction. Thus, G must be traceable in the considered case.

Next, consider the case when there exists a vertex of $V(G\setminus\{u\} - C_k)$ that is not a neighbour of u. We consider that $uv \notin E(G)$. Then, by previous arguments, all neighbours of v are on C_k. Note that $ecc_G(v) \leq 2$. Then, $p = 0$; otherwise, $\{v\} \cup N^+(v)\}$ is an independent set with at least $\delta'+2$ vertices, which contradicts Claim 2.6. By the same argument, v is the only vertex in $G\setminus\{u\}$ which is not on C_k; otherwise, $V(G\setminus\{u\} - C_k) \cup N^+(v)$ is an independent set with at least $\delta'+2$ elements. Thus, u has a neighbour in $N^+(v)$; if not, the set $\{u,v\} \cup N^+(v)$ is an independent set with $\delta'+2$ vertices, which is not permitted. Hence, in this case too, G is traceable.

Now, consider that $uv \notin E(G)$ and $ecc_G(v) \geq 3$. Assume first that $d_G(v,w) \leq 2$ for every $w \in B \setminus \{u\}$. Then, $d_G(v,x) \leq 2$ for every $x \in V(G) \setminus \{u\}$. So, $d_G(u,v) = 3$. Evidently, $|V(G \setminus \{u\} - C_k)| \leq 2$ because of Claim 2.6 and the fact that $V(G \setminus \{u\} - C_k) \cup N^+(v)$ is an independent set. Consider $|V(G \setminus \{u\} - C_k)| = 2$. Then, u has at least 2 neighbours in $N^+(v)$; otherwise, $V(G \setminus \{u\} - C_k) \cup N^+(v)$ is an independent set with at least $\delta' + 2$ elements, which violates Claim 2.6. Therefore, G is traceable, since there exists a cycle in G whose vertex set is $\{u,v\} \cup V(C_k)$. If $|V(G \setminus \{u\} - C_k)| = 1$, then u has a neighbour in $N^+(v)$; or else, $\{u,v,\} \cup N^+(v)$ is an independent set of $\delta' + 2$ elements, which contradicts Claim 2.6. Thus, again, G is traceable.

We now consider the case when there is a vertex $w \in B \setminus \{u\}$ such that $d_G(v,w) \geq 3$. Since $d_G(v,x) \leq 2$ for every $x \in V(G) \setminus B$, we can choose w such that $d_G(v,w) = 3$ or else u has a distance 3 from v on a shortest $v - w$ path (that is, u has a neighbour of eccentricity at least 3 in G). In both instances, $n \leq 2\delta' + 5$ (see Lemma 2.2 or Lemma 2.3). Thus, $l \leq 4$. By the choice of C_k and because of the assumption $diff(G \setminus \{u\}) = 1$, no vertex in $V(G \setminus \{u\} - C_k)$ has a neighbour in $N^-(v) \cup N^+(v)$. This together with the facts that $V(G \setminus \{u\} - C_k)$ is an independent set and $l \leq 4$, implies that $d_G(u_i, v) \leq 2$ for every $u_i \in V(G \setminus \{u\} - C_k)$. That is, either $|N^-(v)| \geq 1$ or $|V(G \setminus \{u\} - C_k)| \geq 3$ and hence u_i has at most 2 neighbours in B. Thus, as before, $w \in V(C_k)$ and we consider it between v_{t_1} and v_{t_2}, as stated previously. Hence, $|V(C_k)| \geq 2\delta' + 2$. Since $u \notin V(C_k)$, one of the neighbours of w on C_k is in $N^-(v) \cup N^+(v)$. Therefore, $d_G(v,w) = 3$. Also, since $n \leq 2\delta' + 5$, we have $|V(G \setminus \{u\} - C_k)| \leq 2$.

Assume that $|V(G \setminus \{u\} - C_k)| = 2$. If u has at least 2 neighbours in $N^+(v)$, then G is traceable. Let $u_1 \in V(G \setminus \{u\} - C_k)$, apart from v. We claim that $uu_1 \in E(G)$. Consider the opposite case when $uu_1 \notin E(G)$. If u has a neighbour, say v_{t_i+1}, in $N^+(v)$, then by considering the tree $vv_{t_i}, v_{t_i}v_{t_i+1}, v_{t_i+1}u, vv_{t_2}, v_{t_2}v_{t_2-1}, v_{t_2-1}v_{t_2-2}$, we proceed using similar arguments as before to build a tree with at least $2\delta'$ leaves by noting that apart from v_{t_1+1}, no vertex in $N^+(v)$ is a neighbour of u and $V(G \setminus \{u\} - C_k) \cup N^+(v)$ is an independent set. This yields a contradiction. So, consider the case when u has no neighbour in $N^+(v)$. Then, $\{u, u_1, v\} \cup N^+(v)$ is an independent set. Hence, by Claim 2.6, u has no neighbour in $N_G(v)$; otherwise, $d_G(v,x) \leq 2$ for every $x \in V(G) \setminus \{w\}$. Thus, v_{t_2-1} and $w = v_{t_2-2}$ are the only possible neighbours of u in G. Take the path $v, v_{t_2}\overleftarrow{C_k}v_{t_2-2}$ and add the edge $x'u$ for some $x' \in \{w, v_{t_2-1}\}$. For a fixed i, let $v_{t_i} \in N_G(v)$ be a neighbour of u_1 and add the edges $vv_{t_i}, v_{t_i}u_1$. If T'' denotes the newly formed tree, then as before, in T'' every vertex of $V(G - T'')$ has a neighbour in $V(T'') \setminus \{u, u_1\}$. This implies that there is a tree with at least $2\delta'$ leaves, which is a contradiction. Thus, $uu_1 \in E(G)$.

Now, we show that u has a neighbour in $N^+(v)$ and we are done. Assume the contrary. Again, let $v_{t_i} \in N_G(v)$ be a neighbour of u_1. Take v and add the edges $vv_{t_i}, v_{t_i}u_1, uu_1, vv_{t_2}, v_{t_2}v_{t_2-1}, v_{t_2-1}v_{t_2-2}(= w)$ to form the tree T'''. If $i = 1$, then every element of $N^+(v)$ not in T''' has at most $\delta' - 1$ neighbours in $V(G - T''')$ and hence every such vertex has a neighbour in T''' among the vertices of $V(T''') \setminus \{u\}$. Thus, by the choice of T''', every vertex of $V(G - T''')$ has a neighbour in T''' among the vertices of $V(T''') \setminus \{u\}$. Hence, $\{u\} \cup V(G - T')$ is a leaf set of at least $2\delta'$ elements, which is a contradiction. By similar arguments, if $i \neq 1$, then every vertex of $V(G - T''')$ has a neighbour in T''' among the vertices of $V(T''') \setminus \{u, w\}$. Again, $V(G - T''') \cup \{u, w\}$ is a leaf set of cardinality $2\delta'$, which is a contradiction. Consequently, u has a neighbour in $N^+(v)$ and hence G is traceable.

To complete the proof of the lemma, assume that v is the only vertex in $G \setminus \{u\}$ that is not on C_k. We show that u has a neighbour in $N^-(v) \cup N^+(v)$. To do this, assume to the contrary that u has no neighbour in $N^-(v) \cup N^+(v)$. Then $\{u, v,\} \cup N^+(v)$ is an independent set of $\delta' + 2$ elements. So, if $|N^-(v)| = 2$, then u has no neighbour in $N_G(v)$; otherwise, $d_G(v,x) \leq 2$ for every $x \in V(G) \setminus \{w\}$, which is a contradiction to the fact that no independent set in G can have more than $\delta' + 1$ elements in this instance. Thus, $w = v_{t_2-2}$ is the only possible neighbour of u in G, whenever $|N^-(v)| = 2$. For a fixed positive integer i with $i \neq 2$, take the path $P_{v_{t_i-1},u} = v_{t_i-1}, v_{t_i}, v, v_{t_2}\overleftarrow{C_k}v_{t_2-2}, u$. Then, as before, every element of $V(G - P_{v_{t_i-1},u})$ is adjacent to some interior vertex of this path and a suitable operation yields a tree with $2\delta'$ leaves of the set $\{v_{t_i-1}, u\} \cup V(G - P_{v_{t_i-1},u})$, which is a contradiction.

Now, assume that $|N^-(v)| = 1$. Then, elements of B, apart from u, are consecutive on C_k. Consider first the case when u has no neighbour in $N_G(v)$. Then, by our assumption, v_{t_2-2} and v_{t_2-3} are the only possible neighbours of u in G, provided that $v_{t_2-3} \notin N^+(v)$. Take the path $P_{v,v_{t_2-3}} = v, v_{t_2}\overleftarrow{C_k}v_{t_2-3}$ and without creating a cycle, add an edge $x'u$ for $x' \in \{v_{t_2-2}, v_{t_2-3}\}$ and $x' \notin N^+(v)$. Let T^{iv} be the tree formed by the aforementioned operations. If $v_{t_2-3} \notin N^+(v)$, then every vertex of $V(G - T^{iv})$ has a neighbour in T^{iv} among vertices of $V(T^{iv}) \setminus \{u\}$; if $v_{t_2-3} \in N^+(v)$, then every vertex of $V(G - T^{iv})$ has a neighbour in T^{iv} among vertices of $V(T^{iv}) \setminus \{u, v_{t_2-3}\}$. In either case, $\{u\} \cup V(G - T^{iv})$ or $\{u, v_{t_2-3}\} \cup V(G - T^{iv})$ is a leaf set of at least $2\delta'$ elements, which is not permitted. Now, consider the case when u has a neighbour in $N_G(v)$. Let $v_{t_i} \in N_G(v)$ be a neighbour of u, where $i \geq 1$. Take v and add the edges $vv_{t_2}, v_{t_2}v_{t_2-1}, v_{t_2-1}v_{t_2-2}, v_{t_2-2}v_{t_2-3}, vv_{t_i}, v_{t_i}u$. Since $\{u\} \cup N^+(v)$ is an independent set, we proceed as before to build a tree with at least $2\delta'$ leaves in both subcases; that is, $i = 2$ and $i \neq 2$. This again contradicts $L(G) \leq 2\delta' - 1$.

Therefore, if v is the only vertex in $V(G \setminus \{u\} - C_k)$, then u must have a neighbour in $N^-(v) \cup N^+(v)$. This implies that G is traceable. This completes the proof of the lemma. $\square$

It is interesting here to note that Lemma 1.1, Lemma 2.2, and arguments similar to the ones given in the proof of Lemma 2.6, in conjunction with Theorem 1.10, would provide a short proof to Theorem 1.2 for $\delta \geq 5$. That is, $n \leq max\{2\delta+5, 3\delta-1\}$ implies that $\deg(u)+\deg(v)+\deg(w) \geq n$ for all triples of independent sets $\{u, v, w\}$, and hence G is traceable or $diff(G) \leq 1$. Thus, the proof for the case $diff(G) \geq 2$ would have been eliminated in the proof of Theorem 1.2.

Claim 2.7. *Assume that G satisfies the conditions of Conjecture 1.4. If $w \notin A$ is a vertex in G such that w has exactly one neighbour, say z, in A, $d_G(v, x) \leq 2$ for every $x \in V(G) \setminus \{w\}$ and $N_G(w) \cap N_G(z) \neq \emptyset$, then $|A| \leq \delta'$ provided that $u \notin A$; where v, A and B are defined in Claim 2.6.*

Proof. Let $y \in N_G(w) \cap N_G(z)$. Then $y \notin A$, since $z \in A$. Suppose to the contrary that $|A| \geq \delta'+1$. Let P_{vy} be a shortest $v-y$ path in G. Then P_{vy} has at most 3 vertices and contains exactly one element of A, since $d_G(v, y) \leq 2$ and $v \in A$. To P_{vy}, add the edges wy, yz, and join to v, $p+s$ of its neighbours which are chosen in such a way that each of the s elements of B not on $P_{vy} \cup \{wy, yz\}$ has a neighbour among them. Let T' be the tree formed by these operations. Then, $|N_{G-T'}(v)| = \delta' - s - 1$. Also, every element of A not in T' has at most $\delta' - 1$ neighbours in $V(G - T')$ and it has no neighbour in $\{v, w, z\}$, since A is an independent set such that w has only one neighbour z in A. Thus, by the choice of T', every vertex in $V(G - T')$ has a neighbour in T' among the vertices of the set $V(T') \setminus \{w, z\}$. Hence, $\{w, z\} \cup V(G - T')$ is a leaf set with at least $2\delta'$ leaves, which is a contradiction. $\square$

Lemma 2.7. *Let G be a connected graph G such that $L(G) \leq 2\delta' - 1$ and $G \setminus \{u\}$ is 2-connected with $diff(G \setminus \{u\}) \geq 2$. If $\delta' \geq 4$, then G has a spanning path. The result holds also for $\delta' = 3$ when $n \leq 10$.*

Proof. When C_k is not a dominating cycle, we consider $\delta' \geq 6$ and along the way, we give an outline of the proof for $3 \leq \delta' \leq 5$. Let C_k be the same as defined in Lemma 2.6. By Theorem 1.9, $|V(C_k)| \geq 3\delta' - 6$, since $\delta(G \setminus \{u\}) \geq \delta' - 1$. Hence, $|V(G \setminus \{u\} - C_k)| \leq 4$. For $\delta' \geq 8$, Theorems 1.7 and 1.8 imply that P_4 and C_4 are not subgraphs of $G[V(G \setminus \{u\} - C_k)]$; otherwise, $n \geq k + 4 + |\{u\}| > 3\delta' - 1$, which is not allowed (see Lemma 2.2). However, here we give a unified proof for $\delta' \geq 6$.

Assume first that $C_4 \leq G[V(G \setminus \{u\} - C_k)]$. Fix $C_4 = C'_4 = v, w, x, y, v$. Then each vertex on C'_4 has at most 3 neighbours on C'_4. If u has a neighbour on C'_4, then G is traceable. Suppose u has no neighbour on C'_4. Then, every vertex on C'_4 has at least $\delta' - 3$ neighbours on C_k. Let $a = |N_{C_k}(v) \cap N_{C_k}(w)|$. Whenever $v_{t_i} \in N_{C_k}(v) \cap N_{C_k}(w)$, then $v_{t_i+1}, v_{t_i+2}, v_{t_i+3}, v_{t_i+4} \notin N_G(v) \cup N_G(w)$; otherwise, we have a contradiction with the choice of C_k. Thus, $|V(C_k)| \geq 5a$. Also, whenever $v_{t_j} \in N_{C_k}(v) \cup N_{C_k}(w)$ such that $v_{t_j} \notin N_{C_k}(v) \cap N_{C_k}(w)$, then $v_{t_j+1} \notin N_G(v) \cup N_G(w)$. So, $k \geq 5a + 2(|N_{C_k}(v)| - a) + 2(|N_{C_k}(w)| - a)$. Thus,

$$|V(C_k)| \geq \max\{5a, 5a + 2(|N_{C_k}(v)| - a) + 2(|N_{C_k}(w)| - a)\}$$

$$\geq \max\{5a, a + 4\delta' - 12\}, \text{ since } \min\{|N_{C_k}(v)|, |N_{C_k}(w)|\} \geq \delta' - 3.$$

Hence, $n \geq k+4+1 \geq a+4\delta'-7 > 3\delta'-1$ for $\delta' \geq 6-a$, which is a contradiction for $\delta' \geq 6$ whenever $a \geq 1$ (see Lemma 2.2). Now, consider the case when $a = 0$. Then, for some integers i, j, r and s with $i \leq j$, $r \leq s$, there exist $v_{t_i}, v_{t_j} \in N_{C_k}(v)$ and $v_{t_r}, v_{t_s} \in N_{C_k}(w)$ such that, apart from v_{t_i} and v_{t_j}, the path $v_{t_i}\overrightarrow{C_k}v_{t_s}$ contains neither neighbours of v nor w and this path has at least 6 vertices; otherwise, we obtain a cycle in $G \setminus \{u\}$, longer than C_k. Likewise to the path $v_{t_r}\overrightarrow{C_k}v_{t_j}$. Therefore, $|C_k| \geq 6 + 2|N_{C_k}(v)| + 2|N_{C_k}(w)| \geq 4\delta' - 6$. Thus, $n \geq 4\delta' - 1$, which is a contradiction to Lemma 2.2. It follows that G must be traceable whenever C_4 is a subgraph of $V(G \setminus \{u\} - C_k)$.

For $3 \leq \delta' \leq 5$, the ideas similar to the ones that are used for the case $\delta' \geq 6$, together with Theorem 1.4, establish the result whenever C_k is not a dominating cycle in $G \setminus \{u\}$. Note that $|V(G \setminus \{u\} - C_k)| \leq 6$ in this case, since $n \leq 2\delta'+5$. Hence, one should start by considering an event where C_6 is a subgraph of $G[V(G \setminus \{u\} - C_k)]$. For instance, if $w \in V(G \setminus \{u\} - C_k)$ is a vertex such that it has no neighbour on C_k, then for distinct vertices $v_t, v_s \in V(C_k)$, let P_{wv_s} and P_{wv_t} be disjoint paths from w to C_k (see Theorem 1.4), since $G \setminus \{u\}$ is 2-connected. Take $P = P_{wv_s} \cup P_{wv_t}$ and let b be its length. Then, $b \geq 4$ and $k \geq 2b$, since w has no neighbour on C_k and C_k is a longest cycle in $G \setminus \{u\}$. Now, $n \geq k + (b - 1) + |\{u\}| \geq 12$, which is a contradiction for $\delta' = 3$. Thus, for $\delta' = 3$, it suffices to consider that every vertex in $V(G \setminus \{u\} - C_k)$ has a neighbour on C_k. For $\delta' = 4$, $b = 4$ and for $\delta' = 5$, $b \leq 5$, since $n \leq 2\delta' + 5$. Such arguments together with an analysis similar to that we apply for $\delta' \geq 6$, can be used to show that G is traceable whenever C_k is not a dominating cycle or else the case fails.

Assume for example that P_5, P_6, C_5 and C_6 are not subgraphs of $G[V(G \setminus \{u\} - C_k)]$ and suppose that C'_4 is its subgraph. Then, for $a \geq 3$, we have a contradiction, since $k \geq \max\{5a, a + 4\delta' - 12\}$ and $n \geq \max\{5a + 5, a + 4\delta' - 7\}$. The same analysis applies for $\delta' = 3$ and $\delta' = 4$ when $a = 2$. If $a = 0$, then (as above) $n \geq 4\delta' - 1$, which is not allowed; for instance, if

$\delta' = 3$ or $\delta' = 4$, let v_{t_1} be a neighbour of v on C_k, then $v_{t_1-4}, v_{t_1-3}, v_{t_1-2}, v_{t_1-1}, v_{t_1+1}, v_{t_1+2}, v_{t_1+3}, v_{t_1+4} \notin N_{C_k}(w)$. Hence, by considering a neighbour of w on C_k, we have a contradiction. For $a = 1$ and $\delta' = 3$ or 4, we assume that v_{t_1} is the common neighbour of v and w, then we apply Theorem 1.4 if w has no other neighbour on C_k; otherwise, we are done by previous arguments. For $a = 1$ and $\delta' = 5$, again set $N_{C_k}(v) \cap N_{C_k}(w) = \{v_{t_1}\}$, then by looking at the position of the second neighbour of v on C_k and second neighbour of w on C_k, we see that $k > 10$, which is a contradiction.

If $a = 2$ and $\delta' = 5$, set $N_{C_k}(v) \cap N_{C_k}(w) = \{v_{t_1}, v_{t_2}\}$, then $n = 15$; otherwise, we have a contradiction. Either x or y has a neighbour on C_k that is neither v_{t_1} nor v_{t_2} and $k > 10$, which is not allowed, or both x and y are adjacent to v_{t_1} and v_{t_2}. If both x and y are adjacent to v_{t_1} and v_{t_2}, then $k = 10$ or we have a contradiction. We analyze on possible neighbours of v_{t_1-1} (the analysis for v_{t_1+1}, v_{t_2-1} and v_{t_2+1} follows by symmetry). If $uv_{t_1-1} \in E(G)$, then G is traceable. Assume that $uv_{t_1-1} \notin E(G)$. Then v_{t_1-1} has no neighbour in the set $\{v_{t_1+1}, v_{t_1+2}, v_{t_1+3} = v_{t_2-2}, v_{t_1+4} = v_{t_2-1}\}$; otherwise, there is a cycle in $G \setminus \{u\}$ that contains all vertices of C_4' and misses at most 3 vertices of C_k, which is a contradiction to our choice of C_k. Thus, $v_{t_1-1}v_{t_1}, v_{t_1-1}v_{t_2} \in E(G)$, since $\deg_G(v_{t_1-1}) \geq \delta'$ and v_{t_1-1} has no neighbour in $V(G - C_k)$. By symmetry, all vertices in $\{v_{t_1+1}, v_{t_2-1}, v_{t_2+1}\}$ are adjacent to both v_{t_1} and v_{t_2}. That is, $\deg_G(v_{t_1}) \geq 8$. Now, consider the star subgraph $K_{1,8}'$ formed by the set $\{v_{t_1}, v_{t_1-1}, v_{t_1+1}, v_{t_2-1}, v_{t_2+1}, v, w, x, y\}$ and whose center vertex is v_{t_1}. Then, $V(G - K_{1,8}') = \{u, v_{t_2}, v_{t_1-2}, v_{t_1+2}, v_{t_2-2}, v_{t_2+2}\}$, which is a contradiction to Lemma 2.1. Therefore, G is traceable.

For the case when C_k is not dominating in $G \setminus \{u\}$, analysis similar to those in the preceding 3 paragraphs establish the result for $3 \leq \delta' \leq 5$. Thus, to shorten the length of the proof, in what follows we consider $\delta' \geq 6$ whenever C_k is not a dominating cycle in $G \setminus \{u\}$. Assume that P_4 is a subgraph of $G[V(G \setminus \{u\} - C_k)]$. Fix $P_4' = v, w, x, y$. Then $vy \notin E(G)$, or else we get a C_4 and we are done by previous arguments. Also, $uv, uy \notin E(G)$ or G is traceable. Thus, $\min\{|N_{C_k}(v)|, |N_{C_k}(y)|\} \geq \delta' - 2$. Hence, as before, if $a = |N_{C_k}(v) \cap N_{C_k}(y)|$, then $|V(C_k)| \geq \max\{5a, a + 4\delta' - 8\}$, which is a contradiction to our choice of n.

Now, assume that C_3 is a subgraph of $[V(G \setminus \{u\} - C_k)]$. Fix $C_3 = C_3' = v, w, x, v$. If u has no neighbour on C_3', then $\min\{|N_{C_k}(v)|, |N_{C_k}(w)|\} \geq \delta' - 2$ or else we get P_4 or C_4 outside C_k and we are done. Thus, as before, if $a = |N_{C_k}(v) \cap N_{C_k}(w)|$, then $|V(C_k)| \geq \max\{4a, 4\delta' - 8\}$. So, $n \geq k + |V(C_3')| + |\{u\}| \geq 4\delta' - 4 > 3\delta' - 1$, which is a contradiction. Now, suppose that u has a neighbour, say w, on C_3'. Then, G is traceable or there is another vertex $y \in V(G \setminus \{u\} - C_k)$, since $|V(G \setminus \{u\} - C_k)| \leq 4$. Assume that such y exists. Then, y has no neighbour on C_3'; otherwise, we get a P_4 or C_4 and we are done. If $uy \in E(G)$, then G has a spanning path. Suppose that $uy \notin E(G)$. Then all neighbours of y are on C_k. We claim that either u, v or x has a neighbour in $N^+(y)$, so that G is traceable. To prove this, assume the contrary, then $ecc_G(v) \geq 3$; otherwise, $\{v, y\} \cup N^+(y)$ is an independent set, which contradicts Claim 2.6. Since $d_G(u, v) \leq 2$, there is a vertex $x' \neq u$ such that $d_G(v, x') = 3$. Hence, $n \leq 2\delta' + 5$, see Lemma 2.3. So, $V(C_k) = N(y) \cup N^+(y)$. Further, w has at most one neighbour in $N^+(y)$; otherwise, we have a contradiction to the choice of C_k. We may assume that v_{s_1+1} is the only neighbour of w in $N^+(y)$, where $v_{s_1} \in N_{C_k}(y)$. Then, every vertex in $N^+(y) \setminus \{v_{s_1+1}\}$ has at least δ' neighbours in $N(y)$. Note here that $|N(y)| = \delta'$. Let $T' = \{yv_{s_i} | \ v_{s_i} \in N(y)\} \cup \{v_{s_1}v_{s_i+1} | \ v_{s_i+1} \in N^+(y)\}$. Then, $L(T') = 2\delta' - 1$ and $V(G - T') = \{u, v, w, x\}$, which is a contradiction to Lemma 1.3. Therefore, G must be traceable in the considered case.

Arguments similar to the ones that are used for P_4' and C_3', prove that G must be traceable when P_3 is a subgraph of $G[V(G \setminus \{u\} - C_k)]$.

Now, assume that K_2 is a maximal subgraph of $G[V(G \setminus \{u\} - C_k)]$. Set $K_2 = vw$ and consider first that at least one vertex of K_2 is not adjacent to u. We may assume that $uv \notin E(G)$. Then v has at least $\delta' - 1$ neighbours on C_k. Let $a = |N_{C_k}(v) \cap N_{C_k}(w)|$. Then $|V(C_k)| \geq \max\{3a, 4\delta' - 6 - a\}$. If $a = \delta - 2$, then set $N_{C_k}(v) \cap N_{C_k}(w) = \{v_{t_1}, v_{t_2}, \cdots, v_{t_{\delta'-2}}\}$ for $t_1 < t_2 < t_3 \cdots < t_{\delta'-2} < t_{\delta'-1}$, where $v_{t_{\delta'-1}} \in N_{C_k}(v)$. Then $v_{t_{\delta'-1}} \notin \{v_{t_1-1}, v_{t_1-2}\}$. Thus,

$$|V(C_k)| \geq \begin{cases} 3a \geq 3\delta' - 3 & \text{for } a \geq \delta' - 1, \\ 3a + |\{v_{t_1-1}, v_{t_1-2}, v_{t_{\delta'-1}}\}| = 3\delta' - 3 & \text{for } a = \delta' - 2, \\ 4\delta' - 6 - a = 3\delta' - 3 & \text{for } a \leq \delta' - 3. \end{cases}$$

Hence, $n \geq k + 3 > 3\delta' - 1$, which is a contradiction to Lemma 2.2.

Assume that both v and w are adjacent to u. Then, it is enough to consider the case when $|N_{C_k}(v) \cap N_{C_k}(w)| \leq \delta' - 2$ and neither v nor w have more than $\delta' - 2$ neighbours on C_k; otherwise, we are done by the arguments similar to the ones used in the previous paragraph. Now, $k \geq 3a + 2(\delta' - 2 - a) + 2(\delta' - 2 - a) = 4\delta' - 8 - a$. Consider first $a \leq \delta' - 3$. Then, $k \geq 3\delta' - 5$ and $|V(G \setminus \{u\} - C_k)| \leq 3$, since $n \leq 3\delta' - 1$. We may take a look at $|V(G \setminus \{u\} - C_k)| = 3$, or else G has a spanning path. Let $y \in V(G \setminus \{u\} - C_k)$ be a vertex apart from v and w. As before, all the neighbours of y are on C_k; otherwise, G is traceable. Again by the arguments similar to the ones used before, either u, v or w has a neighbour in $N^+(y)$, or else $ecc_G(v) \geq 3$ and the tree $\{yv_{s_i} | \ v_{s_i} \in N(y)\} \cup \{v_{s_1}v_{s_i+1} | \ v_{s_i+1} \in N^+(y)\} \cup \{x'v, vw, uv | \text{ where } x' \in N(y)\}$ has $2\delta'$ leaves, which is prohibited. Thus, G is traceable.

Next, assume that $a = \delta' - 2$. Then, $k \geq 3\delta' - 6$ and $|V(G \setminus \{u\} - C_k)| \leq 4$. It suffices to consider $k = 3\delta' - 6$ and $|V(G \setminus \{u\} - C_k)| = 4$; otherwise, we are done by the same arguments as given in the previous paragraph. Now, $V(C_k) = N^-(v) \cup N_{C_k}(v) \cup N^+(v)$. Let x and y be vertices in $V(G \setminus \{u\} - C_k)$, apart from v and w. We claim that $xy \in E(G)$. Assume to the contrary that $xy \notin E(G)$. Then x has at most one neighbour in $N^-(v) \cup N^+(v)$; otherwise, we obtain a cycle longer than C_k in $G \setminus \{u\}$. This together with the facts that $xv, xw \notin E(G)$ and $\deg_G(x) \geq \delta'$ implies that x must be adjacent to u, to all the $\delta' - 2$ neighbours of v on C_k and to exactly one vertex in $N^-(v) \cup N^+(v)$. This again yields a contradiction, since the neighbour of x in $N^-(v) \cup N^+(v)$ and one of the neighbours of x in $N_{C_k}(v)$ are consecutive on C_k. Hence, $xy \in E(G)$ as desired. Now, x must have a neighbour in $\{u\} \cup N^-(v) \cup N^+(v)$ because $xv, xw \notin E(G)$ and $\deg_G(x) \geq \delta'$ (see Fact 1.1). Therefore, G is traceable.

To complete the proof, it is enough to consider the case when C_k is a dominating cycle in $G \setminus \{u\}$. Let v, l and B be the same as defined in the proof of Lemma 2.6. In the case when there exists a vertex $w \in V(C_k)$ such that $d_G(v, w) = 3$, we set w same as in Lemma 2.6; that is, $w = v_s$ for $t_1 < s < t_2$. If $V(G \setminus \{u\} - C_k) \cup N^+(v)$ is an independent set, then we are done by the proof of Lemma 2.6. So, we assume that there is a vertex $u_1 \in V(G \setminus \{u\} - C_k)$ such that $u_1 \neq v$ and u_1 has a neighbour in $N^+(v)$. For a fixed i, let $v_{t_i+1} \in N^+(v)$ be a neighbour of u_1. Note that every vertex in $V(G \setminus \{u\} - C_k)$ has at most one neighbour in $N^+(v)$; otherwise, we have a contradiction to our choice of C_k.

Assume that every vertex in $V(G \setminus \{u\} - C_k)$ is a neighbour of u. Recall that $|V(G \setminus \{u\} - C_k)| \leq 4$ for $\delta' \geq 6$. This also holds for $\delta' = 3$ and $\delta' = 4$; otherwise, $\delta = \deg_G(u) > \delta'$, which is not permitted. Let us show that it also holds for $\delta' = 5$. We realize first that $|V(G \setminus \{u\} - C_k)| \leq 5$, since $\deg_G(u) \leq 5$. Assume that $|V(G \setminus \{u\} - C_k)| = 5$. Consider a binary star, say R', formed by $N_G[u] \cup N_G[v]$, which has 8 leaves. Then, $R' \cup \{v_{t_1}v_{t_1-1}, v_{t_1}v_{t_1+1}, v_{t_3}v_{t_3-1}, v_{t_3}v_{t_3+1}\}$ is a tree with 10 leaves, which is a contradiction. Thus, $|V(G \setminus \{u\} - C_k)| \leq 4$ for $\delta' \geq 3$. If $|V(G \setminus \{u\} - C_k)| \leq 3$, then G is traceable; that is, since u_1 has a neighbour in $N^+(v)$, there is a cycle C' in G that contains u, u_1, v together with all vertices in $V(C_k)$ and $|V(G - C')| \leq 1$. Take a look at $|V(G \setminus \{u\} - C_k)| = 4$ and let $V(G \setminus \{u\} - C_k) = \{v, u_1, u_2, u_3\}$. Either u_2 or u_3 has a neighbour in $N^+(v) \setminus \{v_{t_i+1}\}$; otherwise, $V(G \setminus \{u\} - C_k) \cup (N^+(v) \setminus \{v_{t_i+1}\})$ is an independent set of at least $\delta' + 2$ elements with $ecc_G(v) \leq 2$ or $d_G(v, x) \leq 2$ for every $x \in V(G) \setminus \{w\}$, which is a contradiction to Claim 2.6. For a fixed j such that $i < j$, we may choose v_{t_j+1} in such a way that it is a neighbour of u_2 in $N^+(v)$. Then, the path $u_2, v_{t_j+1}\overrightarrow{C_k}v_{t_i}, v, v_{t_j}\overleftarrow{C_k}v_{t_i+1}, u_1, u, u_3$ spans G.

Let us examine the case when at least one vertex in $V(G \setminus \{u\} - C_k)$ is not a neighbour of u. We may consider that $uv \notin E(G)$. Then, all neighbours of v are on C_k, since C_k is a dominating cycle. Hence, this together with Theorem 1.9 implies that $|V(C_k)| \geq \max\{2\delta', 3\delta' - 6\}$. Hence, $|V(G \setminus \{u\} - C_k| \leq 4$ for $\delta' \geq 3$. Thus, $l \leq 4$. Again $u_1v_{t_i+1} \in E(G)$ for some fixed i. Consider first $\delta' = 3$. If $n \leq 9$, then $V(G - C_k) = \{u, u_1, v\}$ and $k = 6$ or else G is traceable. Since $u_1v_{t_i+1} \in E(G)$, we have $u_1v_{t_i}, u_1v_{t_{i+1}} \notin E(G)$; otherwise, we have a contradiction to our choice of C_k. Thus, u_1 has at most $\delta' - 2$ neighbours in $N_G(v)$. This together with the facts that $\deg_G(u_1) \geq 3$ and that u_1 has only one neighbour in $N^+(v)$ implies that $uu_1 \in E(G)$. Hence, G is traceable. Next, consider $n = 10$. If $V(G - C_k) = \{u, u_1, v\}$, then as before $uu_1 \in E(G)$ or u has at least 2 neighbours in $N^+(v)$ and G is traceable; otherwise, $\{u_1, v\} \cup N^-(v) \cup N^+(v) \setminus \{v_{t_i+1}\}$ is an independent set with $\delta' + 2$ vertices, which contradicts Claim 2.6. So, let us consider that there exists $u_2 \in V(G - C_k)$, apart from u, u_1 and v. Then $k = 6$. Now, u_2 has a neighbour in $N^+(v)$; otherwise, $\{u_2, v\} \cup N^+(v)$ is an independent set with $\delta' + 2$ elements, which is a contradiction to Claim 2.6. Thus, as before, $uu_1, uu_2 \in E(G)$ or else $\min\{\deg_G(u_1), \deg_G(u_2)\} < \delta'$, which contradicts Fact 1.1. Therefore, $v, v_{t_i}\overleftarrow{C_k}v_{t_i+1}, u_1, u, u_2$ is a spanning path of G as required.

Now, consider the case when $\delta' \geq 4$. Since $V(G \setminus \{u\} - C_k)$ is an independent set such that its every vertex has at most one neighbour in $N^+(v)$ and $l \leq 4$, $d_G(v, x) \leq 2$ for every $x \in V(G \setminus \{u\} - C_k)$; otherwise, $|V(G \setminus \{u\} - C_k)| = 2$ and $uu_1 \in E(G)$ imply that G has a Hamilton path. We show first that $N_G(u_1) \cap N_G(v_{t_i+1}) \neq \emptyset$. Assume the contrary. Recall that no vertex in $N^+(v) \setminus \{v_{t_i+1}\}$ is adjacent to v_{t_i+1} or u_1, since $N^+(v)$ is an independent set and u_1 has only one neighbour in $N^+(v)$. So, there is a tree T'' with $Int(T'') = \{u_1, v_{t_i+1}\}$, and $V(T'') \subseteq N[u_1] \cup N[v_{t_i+1}]$ such that $L(T'') = 2\delta' - 2$. Clearly, $\{v\} \cup N^+(v) \setminus \{v_{t_i+1}\} \subseteq V(G - T'')$ with $|\{v\} \cup N^+(v) \setminus \{v_{t_i+1}| = \delta'$, which is a contradiction to Lemma 2.1 for $\delta' \geq 5$. For $\delta' = 4$, we may set $i = 1$ (other cases follow by symmetry). Then $T''' = T'' \cup \{v_{t_2}v_{t_2+1}, v_{t_2}v\}$ is a tree with $2\delta' - 1$ leaves. Now, in T'', each vertex in $\{v_{t_3+1}, v_{t_4+1}\}$ has at least 3 neighbours in the leaf set of T''' and none of their neighbours belongs to $\{v, v_{t_2+1}\}$, since $\{v\} \cup N^+(v)$ is an independent set. Hence, v_{t_3+1} and v_{t_4+1} share a neighbour in T'', which yields a tree with 8 leaves, a contradiction. Thus, we must have $N_G(u_1) \cap N_G(v_{t_i+1}) \neq \emptyset$ for $\delta' \geq 4$.

We claim that $ecc_G(v) \geq 3$. If u has no neighbour in $N^+(v)$ then the claim holds; otherwise, $\{u, v\} \cup N^+(v)$ is an independent set in G, which is not allowed (see Claim 2.6). Suppose that u has a neighbour in $N^+(v)$. Then, we are done again since $N_G(u_1) \cap N_G(v_{t_i+1}) \neq \emptyset$ and $\{v\} \cup N^+(v)$ is an independent set of $\delta' + 1$, see Claim 2.7. By the same arguments, there exists $w \neq u$ such that $d_G(v, w) \geq 3$, or else we have a contradiction to Claim 2.6 or Claim 2.7. By the same arguments as in the previous paragraph, w is on C_k. Thus, $|V(C_k)| \geq 2\delta' + 2$. Also, by Lemma 2.3, $n \leq 2\delta' + 5$. Therefore, v and u_1 are the only vertices in $G \setminus \{u\}$ which are not on C_k and $|V(C_k)| = 2\delta' + 2$. Recall that $w = v_s$ for $t_1 < s < t_2$.

We show that $uu_1 \in E(G)$. Assume to the contrary that $uu_1 \notin E(G)$. Consider first $i = 1$; that is, $u_1 v_{t_i+1} \in E(G)$. Then, v_{t_1} and $w = v_{t_1+2}$ are not neighbours of u_1 or else we violate our choice of C_k. Hence, $N_G(u_1) \subseteq \{v_{t_1+1}, v_{t_2-1}\} \cup (N(v) \setminus \{v_{t_1}\})$. Note here that $\deg_G(v) = \delta'$. Hence, either $u_1 v_{t_2} \in E(G)$ or $u_1 v_{t_2-1} \in E(G)$, since $\deg_G(u_1) \geq \delta'$. Also, since $\deg_G(w) \geq \delta'$ and $d_G(v, w) \geq 3$, for some fixed j with $j \neq 1$, we have $wv_{t_j+1} \in E(G)$, where $v_{t_j+1} \in N^+(v)$. Now, the cycle $v, v_{t_{j+1}} \overrightarrow{C_k} v_{t_1+1}, u_1, v_{t_2-1}, w, v_{t_j+1} \overleftarrow{C_k} v_{t_2}, v$ or $v, v_{t_{j+1}} \overrightarrow{C_k} v_{t_1+1}, u_1, v_{t_2} \overleftarrow{C_k} w, v_{t_j+1} \overleftarrow{C_k} v_{t_3}, v$ is longer than C_k in $G \setminus \{u\}$, which is not permitted. Now, consider the case when $i \neq 1$. Then, the neighbours of v, which are v_{t_i} and $v_{t_{i+1}}$, are not neighbours of u_1, since $u_1 v_{t_i+1} \in E(G)$ and C_k is a longest cycle in $G \setminus \{u\}$. Also, $N^+(v) \setminus \{v_{t_1+1}\} \cup \{v_{t_2-1}\}$ is an independent set, or else we have a contradiction to the choice of C_k. Since $\deg_G(u_1) \geq \delta'$ and $uu_1 \notin E(G)$, we have $u_1 w \in E(G)$, since u_1 has at most $\delta' - 2$ neighbours in $N(v)$. Hence, the cycle $v, v_{t_{i+1}} \overrightarrow{C_k} w, u_1, v_{t_i+1} \overleftarrow{C_k} v_{t_2}, v$ is longer than C_k, which is prohibited. Hence, $uu_1 \in E(G)$ and consequently, G is traceable. □

Lemma 2.8. *If G is a connected graph with $\delta' = 3$ and $L(G) \leq 5$, then G has a spanning path.*

Proof. Clearly, $\deg_G(x) \leq 5$ for every $x \in V(G)$. Let $v \in V(G)$ be a vertex of maximum degree in G. Let $K'_{1,\deg_G(v)}$ be the star graph formed on $N[v]$ with v as its center vertex. Consider first the case when $\deg_G(v) = 5$. By Lemma 1.3 and Fact 1.1, $|V(G - K'_{1,5})| \leq 3$. Hence, $n \leq 9$ and the result follows from Lemmas 2.4-2.7. Next, suppose that $\deg_G(v) = 4$. Let $N(v) = \{v_1, v_2, v_3, v_4\}$. Evidently, every vertex of $K'_{1,4}$ has at most 2 neighbours in $V(G - K'_{1,4})$. Assume that there is a leaf, say v_1, of $K'_{1,4}$ that has 2 neighbours, say w_1 and w_2, outside the star $K'_{1,4}$. Again $|V(G - (K'_{1,4} \cup \{v_1 w_1, v_1 w_2\}))| \leq 3$. So, $n \leq 10$ and by Lemmas 2.4-2.7, G is traceable. Consider the case when every vertex in $K'_{1,4}$ has at most one neighbour out. Then $K'_{1,4}$ receives at most 4 edges from $V(G - K'_{1,4})$. Clearly, if $d_G(v, x) \leq 2$ for every $x \in V(G) \setminus \{u\}$, then $|V(G - K'_{1,4})| \leq 5$, $n \leq 10$ and hence the desired conclusion holds.

Let us examine the case when there exists $w \in V(G)$, $w \neq u$, such that $d_G(v, w) = 3$. Let R be a binary star with $V(R) = N[v] \cup N[w]$. Then, $\deg_G(w) = 3$; otherwise, $L(R) \geq 6$, which is not needed. Let $N(w) = \{w_1, w_2, w_3\}$. We may assume that the path v, v_1, w_1, w contains the interior vertices of R. Assume first that u is not in R. Then, $|V(G - R)| \leq 3$ and $n \leq 12$. As before, it is enough to consider $n \in \{11, 12\}$. Since every vertex in $K'_{1,4}$ has at most one neighbour in $V(G - K'_{1,4})$, by Corollary 1.1, either $G[N[v]]$ is Hamiltonian or $G[N[v]] = K_2 \vee K_1 \vee K_2$. If $G[N[v]] = K_2 \vee K_1 \vee K_2$, then $K_1 = \{v\}$, in the considered. Set $G[N[v]] = \{v_1 v_2\} \vee \{v\} \vee \{v_3 v_4\}$. Then, there are $v_1 - v_3$ and $v_1 - v_4$ spanning paths of $G[N[v]]$. Similarly, if $G[N[v]]$ is Hamiltonian, there is a $v_i - v_j$ spanning path of $G[N[v]]$, since v is adjacent to every vertex on the Hamilton cycle. Thus, in both cases, the graph $G[N[v]]$ has $v_1 - v_3$ and $v_1 - v_4$ spanning paths. Note that both v_1 and w_1 have no neighbours in $V(G - R)$ or we have a contradiction to $L(G) \leq 5$. Also, R receives at most 5 edges from $V(G - R)$, since no vertex in $Int(R)$ has a neighbour in $V(G - R)$ and every leaf of R has at most one neighbour outside R.

A leaf of a tree is *dead* if it has no neighbour outside the tree; otherwise, it is *alive*. Assume first that w_1 has at least 2 neighbours in $K'_{1,4}$. Then, apart from v_1, a neighbour of w_1 in $K'_{1,4}$ is dead in R. Thus, R receives at most 4 edges from $V(G - R)$. If a vertex in $V(G - R)$ has a neighbour in $V(G - R)$, then it can not have neighbours in both components of $R \setminus \{v_1 w_1\}$; otherwise, by adding 3 suitable edges and deleting the edge $v_1 w_1$, we get a tree with at least 6 leaves, which is a contradiction. By a similar argument, if $w' \neq u$ is a vertex in $V(G - R)$, then w' cannot have 2 neighbours in $V(G - R)$. Consider $n = 12$ and set $V(G - R) = \{u, w', w''\}$. Then, by these arguments, the only scenario that can occur is where $w'u, w''u \in E(G)$; otherwise, R receives more than 4 edges from $V(G - R)$, which is prohibited. Furthermore, there are 4 edges from $V(G - R)$ to R, since $\min\{\deg_G(w'), \deg_G(w'')\} \geq 3$. Thus, both w_2 and w_3 cannot have neighbours in $K'_{1,4}$; otherwise, we increase dead leaves in R. Hence, $w_2 w_3 \in E(G)$, since neither w_2 nor w_3 has 2 neighbours outside R and they cannot be both adjacent to w_1; for otherwise, $\{w, w_2, w_3, v_2, v_3, v_4\}$ is a leaf set. In this case, either w' or w'' has 2 neighbours in $K'_{1,4}$, possibly in the set $\{v_2, v_3, v_4\}$. We may assume that w' has 2 neighbours in $K'_{1,4}$. Then, by aforementioned arguments, $w'v_3 \in E(G)$ or $w'v_4 \in E(G)$. Therefore, G is traceable, since the graph $G[N[v]]$ has $v_1 - v_3$ and $v_1 - v_4$ spanning paths, and the graph induced by $\{v_1\} \cup N[w]$ has a spanning path whose set of end vertices include v_1. Similar arguments hold for $n = 11$, in the considered case. Likewise, if either w_2 or w_3 has a neighbour in $K'_{1,4}$, we are done by gaining dead leaves.

Suppose that w_1 has no neighbour in $K'_{1,4}$. Since w_1 has no neighbour outside R and $\deg_G(w_1) \geq 3$, either $w_1 w_2 \in E(G)$ or $w_1 w_3 \in E(G)$. We may assume that $w_1 w_2 \in E(G)$. Then, $w_1 w_3 \notin E(G)$; otherwise, $\{w, w_2, w_3, v_2, v_3, v_4\}$ is a leaf set, which is a contradiction. Thus, $w_2 w_3 \in E(G)$, since $\deg_G(w_3) \geq 3$ and w_3 cannot have 2 neighbours in $V(G - R)$. Hence, the graph induced by $\{v_1\} \cup N[w]$ has $v_1 - w_2$ and $v_1 - w_3$ spanning paths. Furthermore, w_2 is now dead in R; otherwise, the set of its neighbours in $V(G - R) \cup \{v_2, v_3, v_4, w, w_3\}$ is a leaf set of at least 6 elements, which is not allowed. Thus, by the arguments similar to the ones given in the previous paragraph, G is traceable.

Now, consider the case when u belongs to R. Then, by Lemma 1.3, $n \leq 11$. In this subcase, consider $n = 11$, or else we are done by Lemmas 2.4-2.7. Consider the only vertices, say w' and w'', not in R. Then $w'w'' \in E(G)$, see Lemma 1.3.

Again, neither w' nor w'' has neighbours in both components of $R - \{v_1 w_1\}$ or else we have a tree with $2\delta'$ leaves, which is not allowed. This together with the fact that no interior vertex of R has a neighbour outside R implies that either w' or w'', but not both, is adjacent to w_2 and w_3. We may assume that $w''w_2, w''w_3 \in E(G)$. Then, as before, neither w_2 nor w_3 has a neighbour in $K'_{1,4}$. Also, at least 2 neighbours of w' are in $K'_{1,4}$ among the vertices of the set $\{v_2, v_3, v_4\}$. We may consider the case when $w'v_2, w'v_3 \in E(G)$ (other subcases follow by symmetry). Now, either w_2 or w_3, say w_2, is not u. This together with the aforementioned arguments implies that $w_2w_3 \in E(G)$ or $w_2w_1 \in E(G)$. In either case, the subgraph induced by $\{v_1, w', w''\} \cup N_G[w]$ has a $v_1 - w'$ spanning path, say $P_{v_1w'}$. Similarly, either v_2 or v_3, say v_2, is not u; that is, $\deg_G(v_2) \geq 3$. Since $\deg_{G[N[v]]}(v_2) \geq 2$, $v_2v_1 \in E(G)$, or $v_2v_3 \in E(G)$ or $v_2v_4 \in E(G)$. Therefore, $v_4, v, v_2, P_{v_1w'}, v_3$; or $v_4, v, P_{v_1w'}, v_3, v_2$; or $v_4, v_2, v, P_{v_1w'}, v_3$ is a spanning path of G. Hence, G is traceable in the considered case.

To complete the proof, consider the case when $\deg_G(v) = 3$. In this case, u is adjacent to a vertex of degree 3. So, we may choose v such that $uv \in E(G)$. Using the Danklemann-Entringer technique [7], let A be a maximal 2-packing of G that emanates from v. Then, by the arguments similar to the ones that are used in [7, 33], there is a tree T' such that $L(T') \geq |A|(\delta' - 2) + 2$ and $V(T') = N[A]$, where $N[A] = \cup_{x \in A} N[x]$. Thus, $|A| \leq 3$ and every vertex not in T' has a neighbour in T'. Assume that $|A| = 3$, then $|V(G - T')| \leq 2$. Thus, $12 \leq n \leq 14$. Rename the vertices of T' as $A = \{x, y, z\}$ with $N(x) = \{x_1, x_2, x_3\}$, $N(y) = \{y_1, y_2, y_3\}$, $N(z) = \{z_1, z_2, z_3\}$ and consider that $E(T') = \{x_1y_1, y_2z_1, xx_i, yy_i, zz_i\ i \in \{1, 2, 3\}\}$. Take a look at $n = 14$. Let $w', w'' \in V(G - T')$. Then, $w'w'' \in E(G)$, see Lemma 1.3. Neither w' nor w'' has neighbours in different components of $T' - \{x_1y_1\}$ or $T' - \{y_2z_1\}$ or else we obtain a tree with 6 leaves, which is prohibited. This in conjunction with the fact that each vertex not in T' is adjacent to at least 2 leaves of T' implies that $w'y_3, w''y_3 \notin E(G)$; otherwise, we construct a tree with 6 leaves. Thus, neighbours of w' or w'', but not both, are all in $\{x_2, x_3\}$ or are all in $\{z_2, z_3\}$. We may assume that $w'x_2, w'x_3, w''z_2, w''z_3 \in E(G)$. Now, the only graphical degree sequences are $1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3$ and $3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3$. So, $\delta = 3$ or $u = y_3$ and $\deg_G(y_3) = 1$. Now, neighbours of x_2 and x_3 are in $\{w'\} \cup N_G[x]$, or else we have a contradiction to the choice of $L(G)$. Likewise, x_2 and x_3 cannot be both adjacent to x_1. Thus, $x_2x_3 \in E(G)$. Similarly, $z_2z_3 \in E(G)$. Now, $x_1x_2, x_1x_3 \notin E(G)$; otherwise, $L(G) > 5$, which is a contradiction. Likewise, apart from x and y_1, the only possible neighbour of x_1 is y_3. Hence, $x_1y_3 \in E(G)$, since $\deg_G(x_1) \geq 3$. By symmetry, $z_1y_3 \in E(G)$. Therefore, $y_1, y, y_2, z_1, y_3, x_1, x, x_2, x_3, w', w'', z_2, z_3$ is a spanning path of G or $\{x_1y_3, z_1y_3\} \cup T' - \{x_1y_1, y_2z_1\}$ is a tree with 6 leaves. For $n = 13$, the only degree sequence is $2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3$ and we are done by the similar arguments used just before. For $n = 12$, if G is a 3-regular graph then we are done by Theorem 1.11; otherwise, the only graphical degree sequence is $1, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3$ and we use the arguments similar to the ones used for the case $n = 14$.

If $|A| \leq 2$, then $|V(G - T')| \leq 4$ and $n \leq 12$. By the previous arguments, one can easily study the degree sequences for the cases when $n \in \{11, 12\}$; otherwise, we are done by Lemmas 2.4-2.7. □

The main result of this paper, which settles Conjecture 1.4, is the following:

Theorem 2.1. *Let G be a connected graph with second minimum degree δ', order n and leaf number $L(G)$ such that $L(G) \leq 2\delta' - 1$. Then G is traceable and the result is best possible in certain senses.*

Proof. For $\delta' \leq 2$, the result is deduced from [25] as mentioned before. For $\delta' \geq 3$, Lemmas 2.4-2.8 yield the proof.

Now, we show that the result is best possible in certain senses. Let $K^*_{\delta'+1} - e_1$ and $K^{**}_{\delta'+1} - e_2$ be the graphs obtained from the complete graph $K_{\delta'+1}$ by deleting edges e_1 and e_2, respectively, where $e_1 = wx$ and $e_2 = yz$ for distinct vertices w, x, y, z. Set $2K_2 \vee K_1 = \{v_1v_2\} \vee \{v\} \vee \{v_3v_4\}$. In addition, let $u_c \in V(K_{\delta'+1})$ be fixed. Furthermore, let u and v_5 be distinct vertices not in $V(K_{\delta'+1}) \cup V(K^*_{\delta'+1} - e_1) \cup V(K^{**}_{\delta'+1} - e_2) \cup V(2K_2 \vee K_1)$. Define $G'_{1,\delta'}, G''_{1,\delta'}$ and $G'''_{1,\delta'}$ by $G'_{1,\delta'} = K_{\delta'+1} \cup (K^*_{\delta'+1} - e_1) \cup \{u_cx, wu\}$, $G''_{1,\delta'} = (K^*_{\delta'+1} - e_1) \cup (K^{**}_{\delta'+1} - e_2) \cup \{yx, wz, wu\}$ and $G'''_{1,\delta'} = (K^*_4 - e_1) \cup (2K_2 \vee K_1) \cup \{wu, v_2v_5, v_3v_5, v_4v_5\}$. For the integers s and p' with $s \geq \delta > 1$ and $0 \leq p' < s$, let $K_{s,s+1} - p'e$ be the graph obtained from the complete bipartite graph $K_{s,s+1}$ by deleting those p' edges that are incident with only one vertex of the larger partite set of $K_{s,s+1}$. Also, define $\mathcal{G}_{2\delta'-1}$ by $\mathcal{G}_{2\delta'-1} = \{G'_{1,\delta'}, G''_{1,\delta'}, G'''_{1,\delta'}, K_{s,s+1} - p'e\}$, which is a family of graphs with leaf number $2\delta' - 1$. Note that the result is best possible in the sense that every graph isomorphic to a graph belonging to either $\mathcal{G}_{2\delta'-1}$ or $\mathcal{F}_4$ (see [26]) is connected, traceable and non-Hamiltonian with leaf number $2\delta' - 1$. That is, if G satisfies the hypotheses of the theorem, then G is not necessarily Hamiltonian.

Also, the result is best possible in the sense that every graph of some families reported in [25, 32, 35, 40] is connected with leaf number at least $2\delta'$ and is non-traceable. That is, if $L(G) \leq 2\delta'$, then G may or may not contain a spanning path. Moreover, if $L(G) \geq 2\delta'$, then G is not necessarily traceable. Such families of graphs for every δ and $\delta' \geq 3$ include $K_{\delta,\delta+p}$ (see [32]) for an integer $p \geq 2$ and $K_{s,s+p} - p'e$ (see [25]); here, for integers p, p' and s with $1 \leq p' < s$, $p \geq 2$ and $s = \delta + p'$, $K_{s,s+p} - p'e$ is obtained from the complete bipartite graph $K_{s,s+p}$ by deleting those p' edges that are incident with one vertex x, which is in the larger partite set. For $\delta' \leq 2$, see families of graphs reported in [25, 35, 40]. □

3. Conclusion

The validity of Conjecture 1.4 has been established. It has been demonstrated that the result corresponding to Conjecture 1.4 is best possible in certain senses. Although it has been found that a connected graph G, with $L(G) \leq 2\delta' - 1$, is not necessarily Hamiltonian, researchers may attempt to classify all non-Hamiltonian graphs that satisfy the aforementioned condition. Likewise, readers may attempt to classify all non-traceable but connected graphs with the leaf number at most $2\delta'$. Also, it seems to be natural to mention here that generalizations of Conjectures 1.1 and 1.2 to the problems that involve the i^{th} minimum degree were raised in [26] and are still open. Furthermore, there are still challenging open problems given in [4, 8, 9, 35] on the connected domination number of graphs.

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