Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **6** (2023) 82–92
DOI: 10.47443/ejm.2023.055

Research Article

Fibonacci sums and divisibility properties

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(Received: 4 December 2023. Received in revised form: 26 December 2023. Accepted: 28 December 2023. Published online: 30 December 2023.)

Abstract

Based on a variant of Sury's polynomial identity established in [*Amer. Math. Monthly* **121** (2014) 236], new expressions for various finite Fibonacci (Lucas) sums are derived. The results are extended to Fibonacci and Chebyshev polynomials, and also to Horadam sequences. In addition to deriving sum relations, the main identities are shown to be very useful in establishing and discovering divisibility properties of Fibonacci and Lucas numbers.

Keywords: Fibonacci (Lucas) number; polynomial identity; Chebyshev polynomial; Horadam number; divisibility.

2020 Mathematics Subject Classification: 11B37, 11B39.

1. Introduction

As usual, we use the notation F_n for the nth Fibonacci number and L_n for the nth Lucas number, respectively. Both number sequences are defined, for $n \in \mathbb{Z}$, through the same recurrence relation $x_n = x_{n-1} + x_{n-2}, n \geq 2$, with initial values $F_0 = 0, F_1 = 1$, and $L_0 = 2, L_1 = 1$, respectively. For negative subscripts, we have $F_{-n} = (-1)^{n-1}F_n$ and $L_{-n} = (-1)^n L_n$. They possess the explicit formulas (Binet forms) given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

where $\alpha = (1+\sqrt{5})/2$ is the golden section and $\beta = -1/\alpha$. For more information about these famous sequences we refer, among others, to the books by Koshy [10] and Vajda [18]. In addition, one can consult the On-Line Encyclopedia of Integer Sequences [15], where these sequences are listed under the IDs A000045 and A000032, respectively.

In 2014, Sury [17] presented a polynomial identity in two variables u and v of the following form:

$$(2u)^{n+1} - (2v)^{n+1} = (u-v)\sum_{j=0}^{n}((2u)^j + (2v)^j)(u+v)^{n-j}. \tag{1}$$

As it contains the relation

$$2^{n+1}F_{n+1} = \sum_{j=0}^{n} 2^j L_j, \tag{2}$$

as a special instance, Sury called (1) a polynomial parent to (2). It is obvious that identity (2) can be derived directly using the geometric series. A slightly more general result is

$$2^{n+1}F_{n+r+1} - F_r = \sum_{j=0}^{n} 2^j L_{j+r},$$

and also

$$\frac{1}{5}\left(2^{n+1}L_{n+r+1} - L_r\right) = \sum_{j=0}^{n} 2^j F_{j+r},$$

or even

$$2^{n+1}G_{n+r+1} - G_r = \sum_{j=0}^{n} 2^j \left(G_{j+r+1} + G_{j+r-1}\right),$$

where r is an integer and G_n is a Gibonacci sequence, i.e., a sequence given by $G_0 = a$, $G_1 = b$, and $G_n = G_{n-1} + G_{n-2}$ for $n \geq 2$. In 2018, Philippou and Dafnis [13] generalized identity (2) to the Fibonacci and Lucas numbers of order k.

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In this paper, a variant of Sury's polynomial identity is applied to derive new expressions for various finite Fibonacci (Lucas) sums. Some divisibility properties for Fibonacci (Lucas) numbers from these sums are inferred. Some divisibility properties were also studied by Hoggatt and Bergum [7] and in the recent articles by Sury [16], Pongsriiam [14] and Onphaeng and Pongsriiam [12], among others. Extensions of the obtained results to Fibonacci (Lucas) polynomials, Chebyshev polynomials, and finally to Horadam sequences, are also provided.

2. Primary results

The polynomial identity given in the next lemma is a variant of Sury's identity and is of crucial importance in this paper.

Lemma 2.1. *If x and y are any complex variables and n is any integer, then*

$$f(x,y)=\sum_{j=0}^{n}(xy)^j\left(x^{n-2j}+y^{n-2j}\right)=\sum_{j=0}^{n}\left(\frac{x+y}{2}\right)^j\left(x^{n-j}+y^{n-j}\right) \tag{3}$$

and

$$f(x,y)=\frac{2(x^{n+1}-y^{n+1})}{x-y}\qquad (x\neq y). \tag{4}$$

In addition to deriving sum relations, identities (3) and (4) are going to be very useful in establishing and discovering divisibility properties of Fibonacci and Lucas numbers.

Theorem 2.1. *If r and n are any integers, then*

$$\sum_{j=0}^{n}(-1)^{rj}L_{r(n-2j)}=\sum_{j=0}^{n}\left(\frac{L_r}{2}\right)^j L_{r(n-j)}=\frac{2F_{r(n+1)}}{F_r}. \tag{5}$$

Proof. Set $x=\alpha^r$ and $y=\beta^r$ in (3), and use (4) and the Binet formulas. □

Theorem 2.1 offers a new simple proof of a well-known fact concerning the divisibility of Fibonacci numbers.

Corollary 2.1. *If m and r are integers, then F_r divides F_{mr}.*

Theorem 2.2. *If r and n are any integers, then*

$$\sum_{j=0}^{2n}(-1)^{j(r+1)}L_{2r(n-j)}=\sum_{j=0}^{n}\left(\frac{F_r}{2}\right)^{2j}5^jL_{2r(n-j)}+\sum_{j=1}^{n}\left(\frac{F_r}{2}\right)^{2j-1}5^jF_{r(2n-2j+1)}=\frac{2L_{r(2n+1)}}{L_r}, \tag{6}$$

and

$$\sum_{j=0}^{2n-1}(-1)^{j(r+1)}F_{r(2n-2j-1)}=\sum_{j=0}^{n-1}\left(\frac{F_r}{2}\right)^{2j}5^jF_{r(2n-2j-1)}+\sum_{j=1}^{n}\left(\frac{F_r}{2}\right)^{2j-1}5^{j-1}L_{r(2n-2j)}=\frac{2F_{2rn}}{L_r}. \tag{7}$$

Proof. Write (3) as

$$\begin{aligned}\sum_{j=0}^{2n}(xy)^j\left(x^{2n-2j}+y^{2n-2j}\right)&=\sum_{j=0}^{n}\left(\frac{x+y}{2}\right)^{2j}\left(x^{2n-2j}+y^{2n-2j}\right)\\&\quad+\sum_{j=1}^{n}\left(\frac{x+y}{2}\right)^{2j-1}\left(x^{2n-2j+1}+y^{2n-2j+1}\right);\end{aligned} \tag{8}$$

set $x=\alpha^r$ and $y=-\beta^r$ and combine according to the Binet formulas; thereby proving (6). To prove (7), write (3) as

$$\begin{aligned}\sum_{j=0}^{2n-1}(xy)^j\left(x^{2n-2j-1}+y^{2n-2j-1}\right)&=\sum_{j=0}^{n-1}\left(\frac{x+y}{2}\right)^{2j}\left(x^{2n-2j-1}+y^{2n-2j-1}\right)\\&\quad+\sum_{j=1}^{n}\left(\frac{x+y}{2}\right)^{2j-1}\left(x^{2n-2j}+y^{2n-2j}\right);\end{aligned} \tag{9}$$

set $x=\alpha^r$ and $y=-\beta^r$. Finish the proof in both cases using (4). □

Corollary 2.2. *If m is an odd integer, then L_r divides L_{mr}. Also, if m is an even integer, then L_r divides F_{mr}.*

Theorem 2.3. *If r is a positive integer such that $r \neq 1$, then*

$$\begin{aligned}\sum_{j=0}^{n} F_r^j F_{r-1}^{n-j} L_j &= \sum_{j=0}^{n} \frac{1}{2^{j+1}} \left(F_r^{n-j} L_{n+j(r-1)} + F_{r-1}^{n-j} L_{rj} \right) \\ &= \frac{F_r^{n+2} L_n + F_{r-1} F_r^{n+1} L_{n+1} + F_r F_{r-1}^{n+1} - 2F_{r-1}^{n+2}}{F_r^2 + F_r F_{r-1} - F_{r-1}^2}\end{aligned} \tag{10}$$

and

$$\begin{aligned}\sum_{j=0}^{n} F_r^j F_{r-1}^{n-j} F_j &= \sum_{j=0}^{n} \frac{1}{2^{j+1}} \left(F_r^{n-j} F_{n+j(r-1)} + F_{r-1}^{n-j} F_{rj} \right) \\ &= \frac{F_r^{n+2} F_n + F_{r-1} F_r^{n+1} F_{n+1} - F_r F_{r-1}^{n+1}}{F_r^2 + F_r F_{r-1} - F_{r-1}^2}.\end{aligned} \tag{11}$$

Proof. Set $(x, y) = (\alpha F_r, F_{r-1})$ and $(x, y) = (\beta F_r, F_{r-1})$, in turn, in (3) and (4), respectively. Use the relations

$$\alpha F_r + F_{r-1} = \alpha^r \qquad \text{and} \qquad \beta F_r + F_{r-1} = \beta^r$$

and combine using the Binet formulas. □

Corollary 2.3. *If r is a non-zero integer and n is any positive integer, then*

$$F_r^2 + F_r F_{r-1} - F_{r-1}^2 \mid F_r^{n+2} L_n + F_{r-1} F_r^{n+1} L_{n+1} + F_r F_{r-1}^{n+1} - 2F_{r-1}^{n+2},$$
$$F_r^2 + F_r F_{r-1} - F_{r-1}^2 \mid F_r^{n+2} F_n + F_{r-1} F_r^{n+1} F_{n+1} - F_r F_{r-1}^{n+1}.$$

In particular,

$$5 \mid (2^n L_{n+1} - 1),$$
$$11 \mid (3^n (F_{n+2} + L_{n+1}) - 2^{n+1}),$$
$$11 \mid (3^{n+1} (L_{n+2} + 5F_{n+1}) - 2^{n+1}).$$

Theorem 2.4. *If r is a non-zero integer and n is a non-negative integer, then*

$$\begin{aligned}2\sum_{j=0}^{n} (-1)^{r(n-j)} L_{2rj} &= \sum_{j=0}^{n} \left(\frac{L_r}{2}\right)^j \left(L_{r(2n-j)} + (-1)^{r(n-j)} L_{rj} \right) \\ &= 2\frac{(-1)^{r+1} L_{2r(n+1)} - (-1)^{r(n+1)} L_{2r} + L_{2rn} + 2(-1)^{rn}}{(-1)^{r+1} 5F_r^2}\end{aligned} \tag{12}$$

and

$$\begin{aligned}2\sum_{j=0}^{n} (-1)^{r(n-j)} F_{2rj} &= \sum_{j=0}^{n} \left(\frac{L_r}{2}\right)^j \left(F_{r(2n-j)} + (-1)^{r(n-j)} F_{rj} \right) \\ &= 2\frac{(-1)^{r+1} F_{2r(n+1)} + (-1)^{r(n+1)} F_{2r} + F_{2rn}}{(-1)^{r+1} 5F_r^2}.\end{aligned} \tag{13}$$

Proof. Set $(x, y) = (\alpha^{2r}, (-1)^r)$ and $(x, y) = (\beta^{2r}, (-1)^r)$, in turn, in (3) and (4), respectively. Use the relations

$$\alpha^{2r} + (-1)^r = \alpha^r L_r \qquad \text{and} \qquad \beta^{2r} + (-1)^r = \beta^r L_r$$

and combine using the Binet formulas. □

Corollary 2.4. *If r is a non-zero integer and n is any positive integer, then*

$$5F_r^2 \mid (-1)^{r+1}F_{2r(n+1)} + (-1)^{r(n+1)}F_{2r} + F_{2rn}.$$

Remark 2.1. *We also have that*

$$5F_r^2 \mid (-1)^{r+1}L_{2r(n+1)} - (-1)^{r(n+1)}L_{2r} + L_{2rn} + 2(-1)^{rn}.$$

But this is obvious as

$$(-1)^{r+1}L_{2r(n+1)} - (-1)^{r(n+1)}L_{2r} + L_{2rn} + 2(-1)^{rn} = (-1)^{r+1}5F_rF_{2rn+r} - (-1)^{r(n+1)}5F_r^2$$

and because $F_r|F_{r(2n+1)}$.

Theorem 2.5. *If n and r are any integers, then*

$$\begin{aligned}\sum_{j=0}^{n} L_{2r}^j 2^{n-j+1} &= \begin{cases} \sum_{j=0}^{n}\left(\frac{L_r^2}{2}\right)^j \left(L_{2r}^{n-j}+2^{n-j}\right), & \text{if } r \text{ is even;}\\ \sum_{j=0}^{n}\left(\frac{5F_r^2}{2}\right)^j \left(L_{2r}^{n-j}+2^{n-j}\right), & \text{if } r \text{ is odd;}\end{cases}\\ &= \begin{cases} \frac{2(L_{2r}^{n+1}-2^{n+1})}{5F_r^2}, & \text{if } r \text{ is even;}\\ \frac{2(L_{2r}^{n+1}-2^{n+1})}{L_r^2}, & \text{if } r \text{ is odd.}\end{cases}\end{aligned} \tag{14}$$

Proof. Set $x = L_{2r}$ and $y = 2$ in (3) and (4), respectively, and use

$$L_{2r} + 2 = \begin{cases} 5F_r^2, & r \text{ odd;}\\ L_r^2, & r \text{ even.}\end{cases}$$

□

Corollary 2.5. *If r is a non-zero integer and n is any positive integer, then*

$$5F_r^2 \mid (L_{2r}^{n+1} - 2^{n+1}), \text{ if } r \text{ is even,}$$

$$L_r^2 \mid (L_{2r}^{n+1} - 2^{n+1}), \text{ if } r \text{ is odd.}$$

Theorem 2.6. *If r and n are any integers, then*

$$\begin{aligned}2\sum_{j=0}^{n}(-1)^{rj}4^jF_r^{2n-2j}5^{n-j} &= \sum_{j=0}^{n}\left(\frac{L_r^2}{2}\right)^j\left(5^{n-j}F_r^{2(n-j)} + (-1)^{r(n-j)}4^{n-j}\right)\\ &= 2\frac{(5F_r^2)^{n+1} - (-1)^{r(n+1)}4^{n+1}}{5F_r^2 - (-1)^r4}.\end{aligned} \tag{15}$$

Proof. Set $x = 5F_r^2$ and $y = (-1)^r4$ in (3) and (4), respectively, and use the identity $5F_r^2 + (-1)^r4 = L_r^2$. □

Theorem 2.7. *If r, n and t are any integers, then*

$$\begin{aligned}\sum_{j=0}^{2n} L_r^jL_{r-1}^{2n-j}L_{j+t} &= \sum_{j=0}^{n}\frac{5^j}{2^{2j+1}}\left(L_r^{2n-2j}L_{2n-2j+2jr+t} + L_{r-1}^{2n-2j}L_{2jr+t}\right)\\ &\quad + \sum_{j=1}^{n}\frac{5^j}{2^{2j}}\left(L_r^{2n-2j+1}F_{2n-2j+(2j-1)r+t} + L_{r-1}^{2n-2j+1}F_{(2j-1)r+t}\right)\\ &= \frac{L_r^{2n+1}\left(L_rL_{2n+t} + L_{r-1}L_{2n+t+1}\right) - L_{r-1}^{2n+1}\left(L_rL_{t-1} + L_{r-1}L_t\right)}{L_{r-2}L_{r+1} + L_rL_{r-1}}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\sum_{j=0}^{2n} L_r^j L_{r-1}^{2n-j} F_{j+t} &= \sum_{j=0}^{n} \frac{5^j}{2^{2j+1}} \left(L_r^{2n-2j} F_{2n-2j+2jr+t} + F_{r-1}^{2n-2j} F_{2jr+t}\right)\\ &\quad + \sum_{j=1}^{n} \frac{5^j}{2^{2j}} \left(L_r^{2n-2j+1} L_{2n-2j+(2j-1)r+t} + L_{r-1}^{2n-2j+1} L_{(2j-1)r+t}\right)\\ &= \frac{L_r^{2n+1}\left(L_r F_{2n+t} + L_{r-1} F_{2n+t+1}\right) - L_{r-1}^{2n+1}\left(L_r F_{t-1} + L_{r-1} F_t\right)}{L_{r-2}L_{r+1} + L_r L_{r-1}}.\end{aligned} \tag{17}$$

Proof. Set $x = \alpha L_r$ and $y = L_{r-1}$ in (8), noting that

$$\alpha L_r + L_{r-1} = \alpha^r \sqrt{5}. \tag{18}$$

Multiply through the resulting equation by α^t. Use $2\alpha^s = L_s + F_s\sqrt{5}$ to reduce the resulting equation. Finally, compare the coefficients of $\sqrt{5}$. □

Corollary 2.6. *If* r, n *and* t *are any integers, then*

$$L_{r-2}L_{r+1} + L_r L_{r-1} \mid L_r^{2n+1}\left(L_r L_{2n+t} + L_{r-1} L_{2n+t+1}\right) - L_{r-1}^{2n+1}\left(L_r L_{t-1} + L_{r-1} L_t\right), \tag{19}$$

$$L_{r-2}L_{r+1} + L_r L_{r-1} \mid L_r^{2n+1}\left(L_r F_{2n+t} + L_{r-1} F_{2n+t+1}\right) - L_{r-1}^{2n+1}\left(L_r F_{t-1} + L_{r-1} F_t\right). \tag{20}$$

In particular,

$$11 \mid 3^{2n+1}\left(L_{2n} + 5F_{2n+1}\right) + 1,$$

$$11 \mid 9^n\left(F_{2n} + L_{2n+1}\right) - 1.$$

Theorem 2.8. *If* r, k, s *and* n *are any integers with* $k \neq -r$ *and* $k \neq -s$*, then*

$$\begin{aligned}2\sum_{j=0}^{n} (-1)^{(k+s)j} L_{r-s}^j L_{2k+r+s}^{n-j} &= \sum_{j=0}^{n} \left(\frac{L_{k+r}L_{k+s}}{2}\right)^j \left(L_{2k+r+s}^{n-j} + (-1)^{(k+s)(n-j)} L_{r-s}^{n-j}\right)\\ &= 2\frac{L_{2k+r+s}^{n+1} - (-1)^{(k+s)(n+1)} L_{r-s}^{n+1}}{5F_{k+r}F_{k+s}}.\end{aligned} \tag{21}$$

Proof. Set $x = L_{2k+r+s}$ and $y = (-1)^{k+s} L_{r-s}$ in (3) and (4), respectively, and use the identities [18]

$$L_{2k+r+s} + (-1)^{k+s} L_{r-s} = L_{k+r}L_{k+s},$$

$$L_{2k+r+s} - (-1)^{k+s} L_{r-s} = 5F_{k+r}F_{k+s}.$$

□

Corollary 2.7. *If* r, k, s *are integers and* n *is any non-negative integer, then*

$$5F_{k+r}F_{k+s} \mid (L_{2k+r+s}^{n+1} - (-1)^{(k+s)(n+1)} L_{r-s}^{n+1}). \tag{22}$$

In particular,

$$5F_{k+r}^2 \mid (L_{2(k+r)}^{n+1} - (-1)^{(k+r)(n+1)} 2^{n+1}). \tag{23}$$

3. Extension to Fibonacci polynomials

Fibonacci (Lucas) polynomials are polynomials that can be defined by the Fibonacci-like recursion and generalizing Fibonacci (Lucas) numbers. For any integer $n \geq 0$, the Fibonacci polynomials $\{F_n(x)\}_{n\geq 0}$ are defined by the second-order recurrence relation

$$F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x),$$

while the Lucas polynomials $\{L_n(x)\}_{n\geq 0}$ follow the rule

$$L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x).$$

Their Binet forms are given by

$$F_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \qquad L_n(x) = \alpha^n(x) + \beta^n(x),$$

where

$$\alpha(x) = \frac{x + \sqrt{x^2+4}}{2} \quad \text{and} \quad \beta(x) = \frac{x - \sqrt{x^2+4}}{2}.$$

One checks easily that $F_{-n}(x) = (-1)^{n-1}F_n(x)$ and $L_{-n}(x) = (-1)^n L_n(x)$.

Theorem 3.1. *For any non-negative integer n we have*

$$\sum_{j=0}^{n} (-1)^j L_{n-2j}(x) = \sum_{j=0}^{n} \left(\frac{x}{2}\right)^j L_{n-j}(x) = 2F_{n+1}(x). \tag{24}$$

Proof. Apply Lemma 2.1 inserting $x = \alpha(x)$ and $y = \beta(x)$. □

Corollary 3.1. *For any non-negative integer n,*

$$\sum_{j=0}^{n} (-1)^j Q_{n-2j} = \sum_{j=0}^{n} Q_{n-j} = 2P_{n+1}, \tag{25}$$

where $P_n = F_n(2)$ and $Q_n = L_n(2)$ are the Pell and Pell-Lucas numbers, respectively.

Proof. Insert $x = 2$ and use $F_n(2) = P_n$ and $L_n(2) = Q_n$, respectively. □

Remark 3.1. *We mention that a different proof of Theorem 2.1 can be provided by inserting $x = L_r$, r odd, and $x = iL_r$, r even, $i = \sqrt{-1}$, in Theorem 3.1 and making use of*

$$L_n(L_r) = L_{rn}, \qquad F_n(L_r) = \frac{F_{rn}}{F_r}, \qquad r \text{ odd},$$

and

$$L_n(iL_r) = i^n L_{rn}, \qquad F_n(iL_r) = i^{n-1}\frac{F_{rn}}{F_r}, \qquad r \text{ even}.$$

Theorem 3.2. *For any non-negative integer n and any $x \neq 0$,*

$$\sum_{j=0}^{n} L_{2(n-2j)}(x) = \sum_{j=0}^{n} \left(\frac{x^2+2}{2}\right)^j L_{2(n-j)}(x) = \frac{2}{x}F_{2(n+1)}(x). \tag{26}$$

Proof. Apply Theorem 3.1 with $x = i(x^2+1), i = \sqrt{-1}$, and use

$$F_n(i(x^2+1)) = i^{n-1}\frac{F_{2n}(x)}{x} \qquad \text{and} \qquad L_n(i(x^2+1)) = i^n L_{2n}(x).$$

□

Theorem 3.3. *For any non-negative integer n, any positive integer r, and any $x \neq 0$,*

$$2\sum_{j=0}^{n} F_{r-1}^j(x) F_{r+1}^{n-j}(x) = \sum_{j=0}^{n} \left(\frac{L_r(x)}{2}\right)^j \left(F_{r+1}^{n-j}(x) + F_{r-1}^{n-j}(x)\right) = 2\frac{F_{r+1}^{n+1}(x) - F_{r-1}^{n+1}(x)}{xF_r(x)}. \tag{27}$$

Corollary 3.2. *For any $n \geq 0$ and $m \geq 1$,*

$$F_m^n L_m F_{rm} \mid F_{m(r+1)}^{n+1} - F_{m(r-1)}^{n+1}, \qquad m \text{ odd}, \tag{28}$$

and

$$F_m^n L_m F_{rm} \mid F_{m(r+1)}^{n+1} + (-1)^n F_{m(r-1)}^{n+1}, \qquad m \text{ even}. \tag{29}$$

4. Extension to Chebyshev polynomials

Recall that, for any integer $n \geq 0$, the Chebyshev polynomials $\{T_n(x)\}_{n\geq 0}$ of the first kind are defined by the second-order recurrence relation [11] given as

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \tag{30}$$

while the Chebyshev polynomials $\{U_n(x)\}_{n\geq 0}$ of the second kind are defined by

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x). \tag{31}$$

The sequences $T_n(x)$ and $U_n(x)$ have the exact (Binet) formulas

$$T_n(x) = \frac{1}{2}\left((x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right), \tag{32}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left((x+\sqrt{x^2-1})^{n+1} - (x-\sqrt{x^2-1})^{n+1}\right). \tag{33}$$

Also, we have $T_{-n}(x) = T_n(x)$ and $U_{-n}(x) = -U_{n-2}(x)$. More information about these polynomials can be found in the book by Mason and Handscomb [11] and also in the recent articles by Frontczak and Goy [6], Fan and Chu [5], and Adegoke et al. [3].

Theorem 4.1. *For any integer n we have*

$$\sum_{j=0}^{n} T_{n-2j}(x) = \sum_{j=0}^{n} x^j T_{n-j}(x) = U_n(x). \tag{34}$$

Proof. Apply Lemma 2.1 inserting $x \mapsto x + \sqrt{x^2+1}$ and $y \mapsto x - \sqrt{x^2+1}$. □

Although (34) offers a very appealing relation we have learnt that it is not new. It was proved by completely other methods in 1985 by Boscarol [4].

5. Extension to the Horadam sequence

Lemma 2.1 in the form

$$f(x,y) = \sum_{j=0}^{n} x^j y^{n-j} = \sum_{j=0}^{n} \frac{(x+y)^j}{2^{j+1}}\left(x^{n-j} + y^{n-j}\right) = \frac{x^{n+1} - y^{n+1}}{x-y} \tag{35}$$

readily allows sum relations to be derived for the Horadam sequence and divisibility properties to be established.

Let $\{w_n(a,b;p,q)\}_{n\geq 0}$ be the Horadam sequence [9] defined for all non-negative integers n by the recurrence

$$w_0 = a, \;\; w_1 = b; \quad w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2, \tag{36}$$

where a, b, p and q are arbitrary complex numbers, with $p \neq 0$ and $q \neq 0$. Extension of the definition of $w_n(a,b;p,q)$ to negative subscripts is provided by writing the recurrence relation as

$$w_{-n} = \frac{1}{q}(pw_{-n+1} - w_{-n+2})$$

where, for brevity, we wrote (and will write) w_n for $w_n(a,b;p,q)$.

Two important cases of w_n are the Lucas sequences of the first kind, $u_n(p,q) = w_n(0,1;p,q)$, and of the second kind, $v_n(p,q) = w_n(2,p;p,q)$. The most well-known Lucas sequences are the Fibonacci sequence $F_n = u_n(1,-1)$ and the sequence of Lucas numbers $L_n = v_n(1,-1)$.

The Binet formulas for sequences u_n, v_n and w_n in the non-degenerate case, $p^2 - 4q > 0$, are

$$u_n = \frac{\tau^n - \sigma^n}{\sqrt{p^2-4q}} = \frac{\tau^n - \sigma^n}{\Delta}, \qquad v_n = \tau^n + \sigma^n, \qquad w_n = A\tau^n + B\sigma^n, \tag{37}$$

with

$$\Delta = \sqrt{p^2-4q}, \quad A = \frac{b - a\sigma}{\Delta}, \quad \text{and} \quad B = \frac{a\tau - b}{\Delta},$$

where

$$\tau = \tau(p,q) = \frac{p+\Delta}{2} \quad \text{and} \quad \sigma = \sigma(p,q) = \frac{p-\Delta}{2}$$

are the distinct zeros of the characteristic polynomial $x^2 - px + q$ of the Horadam sequence.

In this section, we will make use of the following known results.

Lemma 5.1. *If a, b, c and d are rational numbers and λ is an irrational number, then*

$$a + b\,\lambda = c + d\,\lambda \quad\iff\quad a = c,\ \ b = d.$$

Lemma 5.2. *For any integer s,*

$$q^s + \tau^{2s} = \tau^s v_s, \qquad q^s - \tau^{2s} = -\Delta\tau^s u_s, \tag{38}$$

$$q^s + \sigma^{2s} = \sigma^s v_s, \qquad q^s - \sigma^{2s} = \Delta\sigma^s u_s. \tag{39}$$

In particular,

$$(-1)^s + \alpha^{2s} = \alpha^s L_s, \qquad (-1)^s - \alpha^{2s} = -\sqrt{5}\alpha^s F_s, \tag{40}$$

$$(-1)^s + \beta^{2s} = \beta^s L_s, \qquad (-1)^s - \beta^{2s} = \sqrt{5}\beta^s F_s. \tag{41}$$

Lemma 5.3. *Let r and s be any integers. Then*

$$v_{r+s} - \tau^r v_s = -\Delta\sigma^s u_r, \tag{42}$$

$$v_{r+s} - \sigma^r v_s = \Delta\tau^s u_r, \tag{43}$$

$$u_{r+s} - \tau^r u_s = \sigma^s u_r, \tag{44}$$

$$u_{r+s} - \sigma^r u_s = \tau^s u_r. \tag{45}$$

In particular, [8],

$$L_{r+s} - L_r\alpha^s = -\sqrt{5}\beta^r F_s, \qquad L_{r+s} - L_r\beta^s = \sqrt{5}\alpha^r F_s, \tag{46}$$

$$F_{r+s} - F_r\alpha^s = \beta^r F_s, \qquad F_{r+s} - F_r\beta^s = \alpha^r F_s. \tag{47}$$

Lemma 5.4. *For any integer n,*

$$A\tau^n - B\sigma^n = \frac{w_{n+1} - qw_{n-1}}{\Delta}, \tag{48}$$

$$A\sigma^n + B\tau^n = q^n w_{-n}. \tag{49}$$

Proof. See [2, Lemma 1] for a proof of (48). Identity (49) is a consequence of the Binet formula. □

Lemma 5.5. *The following identities hold for integers n, m and r:*

$$\tau^r u_{m-s} = \tau^m u_{r-s} - q^{m-s}\tau^s u_{r-m}\,, \tag{50}$$

$$\sigma^r u_{m-s} = \sigma^m u_{r-s} - q^{m-s}\sigma^s u_{r-m}\,, \tag{51}$$

$$\tau^r u_{m-s}\Delta = \tau^m v_{r-s} - q^{m-s}\tau^s v_{r-m} \tag{52}$$

amd

$$\sigma^r u_{m-s}\Delta = -\sigma^m v_{r-s} + q^{m-s}\sigma^s v_{r-m}\,. \tag{53}$$

Proof. These are immediate consequences of the Binet formulas. □

Lemma 5.6. *If m and n are integers, then [1]*

$$u_{n+m} - q^m u_{n-m} = u_m v_n, \tag{54}$$

$$v_{n+m} - q^m v_{n-m} = \Delta^2 u_m u_n, \tag{55}$$

$$u_{n+m} + q^m u_{n-m} = v_m u_n, \tag{56}$$

and

$$v_{n+m} + q^m v_{n-m} = v_m v_n. \tag{57}$$

In the next result, we give a generalization of Theorem 2.1.

Theorem 5.1. *If r, n and t are any integers with $r \neq 0$, then*

$$\begin{aligned}\sum_{j=0}^{n} q^{rj} w_{r(n-2j)+t} &= w_t \sum_{j=0}^{n} \frac{v_r^j v_{r(n-j)}}{2^{j+1}}\\ &= \frac{w_{t+1+r(n+1)} - q^{r(n+1)} w_{t+1-r(n+1)}}{u_r \Delta^2} - \frac{q\left(w_{t-1+r(n+1)} - q^{r(n+1)} w_{t-1-r(n+1)}\right)}{u_r \Delta^2}\end{aligned} \tag{58}$$

Proof. Set $(x, y) = (\tau^r, \sigma^r)$ and $(x, y) = (\sigma^r, \tau^r)$, in turn in (35) and use the Binet formulas and Lemma 5.4. Note also the use of [9, Equation (3.16)]:

$$w_{r+s} + q^s w_{r-s} = v_s w_r.$$

□

Corollary 5.1. *If r and n are any integers, then*

$$\sum_{j=0}^{n} q^{rj} u_{r(n-2j)} = 0.$$

Corollary 5.2. *If n is any integer, then*

$$\sum_{j=0}^{n} q^j v_{n-2j} = \sum_{j=0}^{n} \left(\frac{p}{2}\right)^j v_{n-j} = 2u_{n+1}.$$

Corollary 5.3. *If r, n and t are any integers with $r \neq 0$, then*

$$u_r \Delta^2 \mid w_{t+1+r(n+1)} - q^{r(n+1)} w_{t+1-r(n+1)} - q\left(w_{t-1+r(n+1)} - q^{r(n+1)} w_{t-1-r(n+1)}\right),$$

provided both quantities are integers.

In particular, on account of (54) and (55), we have

$$u_r \Delta^2 \mid u_{r(n+1)},$$

$$u_r \mid u_{r(n+1)}.$$

Remark 5.1. *Doing the transformation $j \to n - j$ followed by $t \to t + rn$, we get an equivalent form of* (58) *given by*

$$\begin{aligned}2\sum_{j=0}^{n} q^{r(n-j)} w_{2rj+t} &= \sum_{j=0}^{n} \left(\frac{v_r}{2}\right)^j \left(w_{r(2n-j)+t} + q^{r(n-j)} w_{rj+t}\right)\\ &= 2\frac{w_{r(2n+1)+t+1} - q w_{r(2n+1)+t-1} - q^{r(n+1)}\left(w_{t-r+1} - q w_{t-r-1}\right)}{u_r \Delta^2}.\end{aligned}$$

In the next theorem, we present a generalization of Theorem 2.3.

Theorem 5.2. *Let m, n, r, s and t be any integers. Then*

$$\begin{aligned}&\sum_{j=0}^{n} (-1)^j q^{(m-s)j} u_{r-s}^{n-j} u_{r-m}^j w_{(s-m)j+mn+t}\\ &= \sum_{j=0}^{n} \frac{u_{m-s}^j}{2^{j+1}} \left(u_{r-s}^{n-j} w_{(r-m)j+mn+t} + (-1)^{n-j} q^{(m-s)(n-j)} u_{r-m}^{n-j} w_{s(n-j)+t+rj}\right)\\ &= \frac{u_{r-s}^{n+2} w_{mn+t} + u_{r-s}^{n+1} u_{r-m} w_{mn+m+t-s}}{u_{r-s}^2 + q^{m-s} u_{r-m}^2 + u_{r-s} u_{r-m} v_{m-s}}\\ &\quad + \frac{(-1)^n u_{r-m}^{n+1}\left(q^{(m-s)(n+1)+m} u_{r-s} w_{sn+s+t-m} + q^{(m-s)(n+2)+s} u_{r-m} w_{sn+t}\right)}{q^m u_{r-s}^2 + q^{2m-s} u_{r-m}^2 + q^m u_{r-s} u_{r-m} v_{m-s}}.\end{aligned}$$

Proof. Set $(x,y)=(\tau^m u_{r-s},-q^{m-s}\tau^s u_{r-m})$ and $(x,y)=(\sigma^m u_{r-s},-q^{m-s}\sigma^s u_{r-m})$, in turn, in (35). Multiply through the τ equation by τ^t and the σ equation by σ^t. Use the Binet formula and Lemma 5.5. □

Corollary 5.4. *Let m, n, r, s and t be integers such that t is non-negative and $r\geq m\geq s\geq 0$. Let*

$$X=X(m,n,r,s,t):=q^m u_{r-s}^2+q^{2m-s}u_{r-m}^2+q^m u_{r-s}u_{r-m}v_{m-s}$$

and

$$\begin{aligned}Y=Y(m,n,r,s,t):=\;&q^m u_{r-s}^{n+2}w_{mn+t}+q^m u_{r-s}^{n+1}u_{r-m}w_{mn+m+t-s}\\&+(-1)^n u_{r-m}^{n+1}\left(q^{(m-s)(n+1)+m}u_{r-s}w_{sn+s+t-m}+q^{(m-s)(n+2)+s}u_{r-m}w_{sn+t}\right).\end{aligned}$$

Then $X\mid Y$, provided that the Horadam sequence parameters p, q, a and b are integers.

The next set of results generalizes Theorem 2.2.

Theorem 5.3. *If n, r and t are any integers, then*

$$\sum_{j=0}^{2n}(-1)^j q^{rj}w_{2r(n-j)+t}=\frac{w_t}{2}\sum_{j=0}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j v_{2r(n-j)}+\frac{w_t}{u_r}\sum_{j=1}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j u_{r(2n-2j+1)}=\frac{w_t v_{r(2n+1)}}{v_r}$$

and

$$\begin{aligned}\sum_{j=0}^{2n-1}(-1)^j q^{rj}w_{r(2n-1-2j)+t}&=\frac{w_{t+1}-qw_{t-1}}{2}\sum_{j=0}^{n-1}\left(\frac{u_r^2\Delta^2}{4}\right)^j u_{r(2n-2j-1)}+\frac{w_{t+1}-qw_{t-1}}{u_r\Delta^2}\sum_{j=1}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j v_{r(2n-2j)}\\&=\frac{w_{t+2rn}-q^{2rn}w_{t-2rn}}{v_r}.\end{aligned}\tag{59}$$

Proof. As the steps in the proofs are clear, we omit the details. Choose $(x,y)=(\alpha^r,-\beta^r)$ and $(x,y)=(\beta^r,-\alpha^r)$, in turn, and multiply through by α^t and β^t, respectively. Combine, using the Binet formula and Lemma 5.4. The proof of (59) is similar. □

Corollary 5.5. *Let m be any positive odd integer and n any positive even integer. Let r and t be any non-negative integers. Then*

$$v_r\mid v_{rm},$$
$$v_r\mid w_{t+rn}-q^{rn}w_{t-rn};$$

provided that the Horadam sequence parameters p, q, a and b are integers. In particular, $v_r\mid u_{rn}$.

Corollary 5.6. *If r, n and t are any integer, then*

$$\sum_{j=0}^{2n}(-1)^j q^{rj}u_{2r(n-j)}=0,$$

$$\sum_{j=0}^{2n-1}(-1)^j q^{rj}v_{r(2n-1-2j)}=0,$$

$$\sum_{j=0}^{2n-1}(-1)^j q^{rj}u_{r(2n-1-2j)+t}=\sum_{j=0}^{n-1}\left(\frac{u_r^2\Delta^2}{4}\right)^j u_{r(2n-2j-1)}+\frac{2}{u_r\Delta^2}\sum_{j=1}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j v_{r(2n-2j)}=2\frac{u_{2rn}}{v_r},$$

and

$$\sum_{j=0}^{2n}(-1)^j q^{rj}v_{2r(n-j)}=\sum_{j=0}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j v_{2r(n-j)}+\frac{2}{u_r}\sum_{j=1}^{n}\left(\frac{u_r^2\Delta^2}{4}\right)^j u_{r(2n-2j+1)}=\frac{2v_{r(2n+1)}}{v_r}.$$

6. Conclusion

Using a variant of Sury's polynomial identity, we derived several new expressions for finite Fibonacci and Lucas sums. These sum relations have been shown to be very useful in establishing and discovering divisibility properties of Fibonacci and Lucas numbers. We extended our results to Fibonacci and Chebyshev polynomials, and also to Horadam sequences. The findings may offer valuable insights that extend the understanding of these prominent (polynomial) sequences.

Acknowledgement

We thank the three referees for careful reading and for providing useful suggestions which helped to improve the presentation of this paper.

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