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*Research Article*

# General Zagreb indices of composite graphs

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**Abstract**

The first and second general Zagreb indices, $M_1^\alpha$ and $M_2^\alpha$, are the sum of the terms $\delta(u)^\alpha+\delta(v)^\alpha$ and $\delta(u)^\alpha\cdot\delta(v)^\alpha$, respectively, over all pairs of adjacent vertices $u,v$ of a graph, where $\delta(x)$ is the degree of the vertex $x$, and $\alpha$ is a real number. For $\alpha=1$, $M_1^\alpha$ and $M_2^\alpha$ are equal to the ordinary first and second Zagreb indices. For some other values of $\alpha$, $M_1^\alpha$ and $M_2^\alpha$ reduce to a variety of other, earlier considered, topological indices. In this paper, we establish expressions for $M_1^\alpha$ and $M_2^\alpha$ for several types of composite graphs, and give examples pointing at possible applications of these expressions.

**Keywords:** topological indices; general Zagreb indices; composite graphs.

**2020 Mathematics Subject Classification:** 05C07, 05C09, 05C92.

## 1. Introduction

Throughout this paper, we consider simple finite connected graphs. The vertex and edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The degree of the vertex $u\in V(G)$ is the number of edges incident with this vertex and is denoted by $\delta_G(u)$. In the mathematical and chemical literature, numerous graph invariants defined in terms of vertex degrees are being studied [8, 22]. Of these, the oldest and most thoroughly investigated are the first and second Zagreb indices [5, 6, 9, 19], defined as

$$M_1(G)=\sum_{v\in V(G)}\delta_G(v)^2=\sum_{uv\in E(G)}\big[\delta_G(u)+\delta_G(v)\big]$$

and

$$M_2(G)=\sum_{uv\in E(G)}\delta_G(u)\,\delta_G(v)\,,$$

and the Randić connectivity index [16, 21], defined as

$$R(G)=\sum_{uv\in E(G)}\frac{1}{\sqrt{\delta_G(u)\,\delta_G(v)}}\,.$$

All these three indices were invented in the 1970s [11, 12, 20].

In the present work, we are concerned with the generalized version of the first Zagreb index, defined by Li and Zheng [17] as

$$M_1^\alpha(G)=\sum_{v\in V(G)}\delta_G(v)^{\alpha+1}=\sum_{uv\in E(G)}\big[\delta_G(u)^\alpha+\delta_G(v)^\alpha\big] \tag{1}$$

and with the generalized second Zagreb index,

$$M_2^\alpha(G)=\sum_{uv\in E(G)}\big[\delta_G(u)\,\delta_G(v)\big]^\alpha, \tag{2}$$

first considered by Bollobás and Erdős [4]. In formulas (1) and (2), $\alpha$ is a pertinently chosen real number.

Evidently, for $\alpha=1$, the first and second general Zagreb indices reduce, respectively, to the ordinary first and second Zagreb indices. In addition, for $\alpha=2$, $\alpha=3$, and $\alpha=4$, $M_1^\alpha(G)$ coincides with the forgotten topological index [7], the $Y$-index [2, 3], and the $S$-index [18], respectively, that all have been separately investigated in the earlier literature. The

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so-called zeroth-order Randić index is equal to $M_1^\alpha(G)$ for $\alpha = -3/2$ [15]. Therefore, $M_1^\alpha(G)$ is sometimes referred to as the "general zeroth-order Randić index" [10, 15].

The Randić connectivity index is the special case of $M_2^\alpha(G)$ for $\alpha = -1/2$, whereas $M_2^\alpha(G)$ for $\alpha = 2$ is the second hyper Zagreb index [14].

In the subsequent sections, we obtain relations for $M_1^\alpha(G)$ and $M_2^\alpha(G)$ of several types of composite graphs. For particular values of the parameter $\alpha$, these relations are then applicable, as special cases, to all above-mentioned degree-based topological indices.

In the next section, we start by considering a simple composite graph. Before that, we introduce an auxiliary vertex-dependent quantity.

**Definition 1.1.** *Let $G$ be a graph and let "$a$" be a vertex of $G$. If $v_1, v_2, \dots, v_k$, are the vertices of $G$ that are adjacent to "$a$" (see Figure 1), we set*

$$A_G(a)^\alpha = \sum_{i=1}^{k} \delta_G(v_i)^\alpha \,.$$

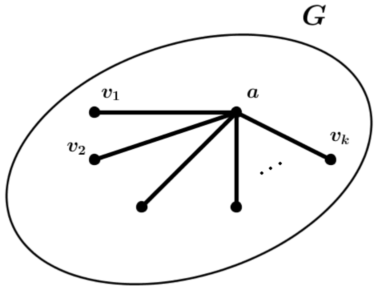


Figure 1: The vertices of the graph $G$, adjacent to the vertex "$a$".

## 2. Gluing of two graphs at a vertex

Let $H_1$ and $H_2$ be two graphs. If $a_1$ and $a_2$ are vertices of $H_1$ and $H_2$, respectively, define a new graph $G$ by gluing $H_1$ and $H_2$ at a vertex "$a$" that corresponds to $a_1$ in $H_1$ and $a_2$ in $H_2$. To simplify our discussion, we keep the same notation "$a$" for both $a_1$ and $a_2$ (see Figure 2), and we set $\sigma = \delta_{H_1}(a)$ and $\tau = \delta_{H_2}(a)$.

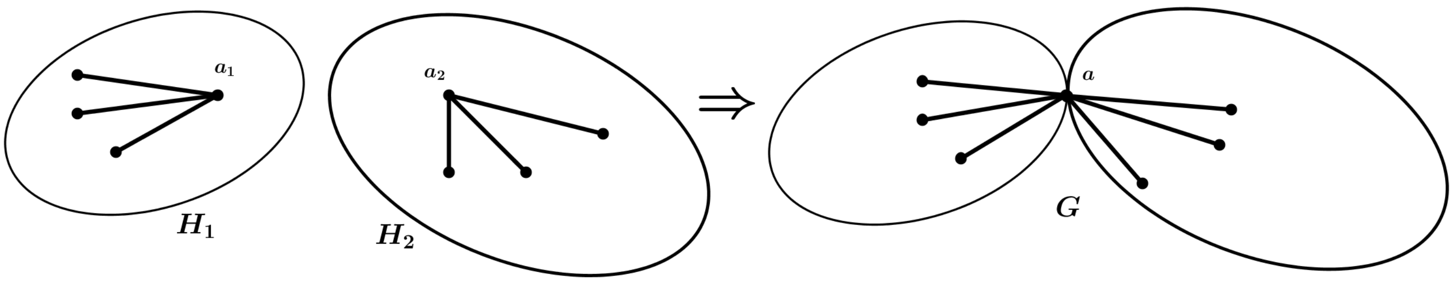


Figure 2: Gluing of two graphs at a vertex.

**Proposition 2.1.** *Let $H_1$ and $H_2$ be two graphs, and let $G$ be the graph obtained by gluing $H_1$ and $H_2$ at a vertex "$a$" that corresponds to $a_1$ in $H_1$ and $a_2$ in $H_2$. Then*

$$M_1^\alpha(G) = M_1^\alpha(H_1) + M_1^\alpha(H_2) + (\sigma+\tau)^{\alpha+1} - \sigma^{\alpha+1} - \tau^{\alpha+1} \,.$$

**Proof.** We have

$$\begin{aligned} M_1^\alpha(G) &= \sum_{v\in V(H_1)\setminus\{a\}} \delta_{H_1}^{\alpha+1}(v) + \sum_{v\in V(H_2)\setminus\{a\}} \delta_{H_2}^{\alpha+1}(v) + (\delta_{H_1}(a)+\delta_{H_2}(a))^{\alpha+1} \\ &= M_1^\alpha(H_1) - \delta_{H_1}^{\alpha+1}(a) + M_1^\alpha(H_2) - \delta_{H_2}^{\alpha+1}(a) + (\delta_{H_1}(a)+\delta_{H_2}(a))^{\alpha+1} \\ &= M_1^\alpha(H_1) + M_1^\alpha(H_2) + (\sigma+\tau)^{\alpha+1} - \sigma^{\alpha+1} - \tau^{\alpha+1}. \end{aligned}$$

□

**Proposition 2.2.** *Let the notation be same as in Proposition 2.1. Then*

$$M_2^\alpha(G) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + (A_{H_1}^\alpha(a) + A_{H_2}^\alpha(a))(\sigma+\tau)^\alpha - A_{H_1}^\alpha(a)\sigma^\alpha - A_{H_2}^\alpha(a)\tau^\alpha .$$

**Proof.** Let $\{u_1, u_2, \ldots, u_r\}$ be the set of vertices of $H_1$ that are adjacent to $a$ in $H_1$ and let $\{v_1, v_2, \ldots, v_s\}$ be the set of vertices of $H_2$ that are adjacent to $a$ in $H_2$. We have

$$\begin{aligned} M_2^\alpha(G) &= \sum_{uv\in E(H_1\setminus\{a\})} \delta_{H_1}^\alpha(u)\,\delta_{H_1}^\alpha(v) + \sum_{uv\in E(H_2\setminus\{a\})} \delta_{H_2}^\alpha(u)\,\delta_{H_2}^\alpha(v) \\ &+ \left(\sum_{i=1}^r \delta_{H_1}^\alpha(u_i) + \sum_{i=1}^s \delta_{H_2}^\alpha(v_i)\right)(\delta_{H_1}(a)+\delta_{H_2}(a))^\alpha \\ &= M_2^\alpha(H_1) - \left(\sum_{i=1}^r \delta_{H_1}^\alpha(u_i)\right)\delta_{H_1}^\alpha(a) + M_2^\alpha(H_2) - \left(\sum_{i=1}^s \delta_{H_2}^\alpha(v_i)\right)\delta_{H_2}^\alpha(a) \\ &+ \left(\sum_{i=1}^r \delta_{H_1}^\alpha(u_i) + \sum_{i=1}^s \delta_{H_2}^\alpha(v_i)\right)(\delta_{H_1}(a)+\delta_{H_2}(a))^\alpha \\ &= M_2^\alpha(H_1) + M_2^\alpha(H_2) + (A_{H_1}^\alpha(a) + A_{H_2}^\alpha(a))(\sigma+\tau)^\alpha - A_{H_1}^\alpha(a)\sigma^\alpha - A_{H_2}^\alpha(a)\tau^\alpha. \end{aligned}$$

□

**Corollary 2.1.** *Let $H$ be a graph and $C_m$ be a cycle, and let $G$ be the graph obtained by gluing $H$ and $C_m$ at a vertex "$a$" that corresponds to $a_1$ in $H$ and $a_2$ in $C_m$ (see Figure 3). Then*

$$\begin{aligned} (1)\qquad M_1^\alpha(G) &= M_1^\alpha(H) + (m-1)2^{\alpha+1} + (\sigma+2)^{\alpha+1} - \sigma^{\alpha+1}. \\ (2)\qquad M_2^\alpha(G) &= M_2^\alpha(H) + (m-2)4^\alpha + (A_H^\alpha(a) + 2^{\alpha+1})(\sigma+2)^\alpha - A_H^\alpha(a)\sigma^\alpha. \end{aligned}$$

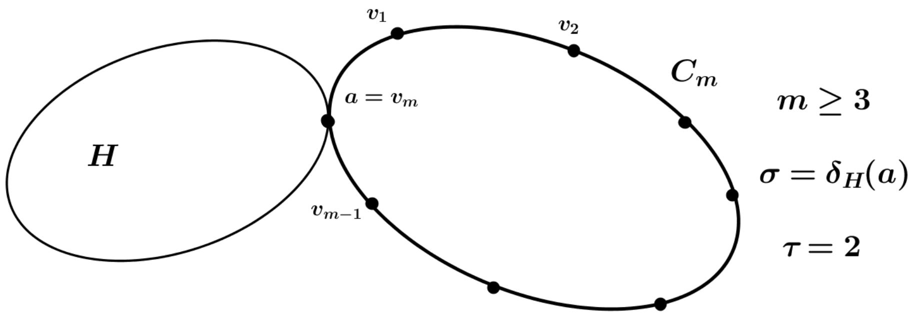


Figure 3: A graph by gluing a graph $H$ and a cycle $C_m$.

**Corollary 2.2.** *(a) Let $C_n$ and $C_m$ be two cycles, and let $G$ be the graph obtained by gluing $C_n$ and $C_m$ at a vertex "$a$" (see Figure 4). Then*

$$M_1^\alpha(G) = (n+m-2)2^{\alpha+1} + 4^{\alpha+1} \quad \textit{and} \quad M_2^\alpha(G) = (n+m-4)4^\alpha + 4\cdot 8^\alpha.$$

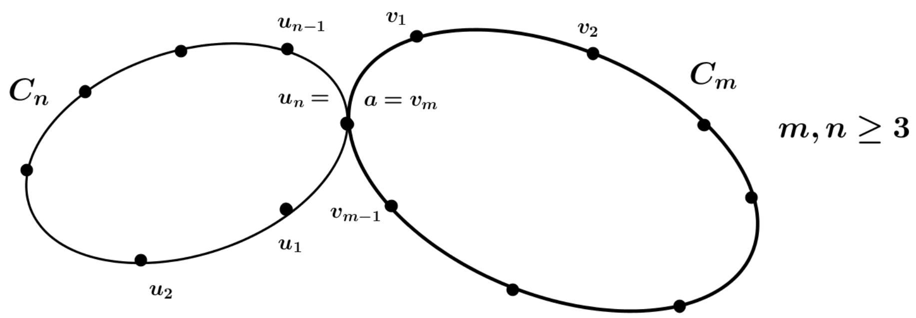


Figure 4: A graph obtained by gluing two cycles.

*(b) Let $C_{n_i}$, $i = 1, 2, 3, \ldots, p$, be cycles, and let $G$ be the graph resulting by gluing them at a common vertex $v$. Set $n = \sum_{i=1}^{p} n_i$. Then*

$$M_1^\alpha(G) = (n - p + p^{\alpha+1})2^{\alpha+1} \quad \textit{and} \quad M_2^\alpha(G) = (n - 2p + 2p^{\alpha+1})4^\alpha \,.$$

If we take $n_1, n_2, \ldots, n_p = m$ in Corollary 2.2(b), then the general Zagreb indices of the Dutch windmill graph $D_m^p$ [1,13] are given by:

$$M_1^\alpha(D_m^p) = p(m - 1 + p^\alpha)2^{\alpha+1} \quad \text{and} \quad M_2^\alpha(D_m^p) = p(m - 2 + 2p^\alpha)4^\alpha \,.$$

**Corollary 2.3.** *If $G_n$ is the chain graph obtained by gluing $n \geq 2$ copies of a regular graph $H$ (see Figure 5), then*

$$(1) \quad M_1^\alpha(G_n) = nM_1^\alpha(H) + 2(n-1)(2^\alpha - 1)\sigma^{\alpha+1} \quad \textit{and} \quad (2) \quad M_2^\alpha(G_n) = nM_2^\alpha(H) + 2(n-1)(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha \,.$$

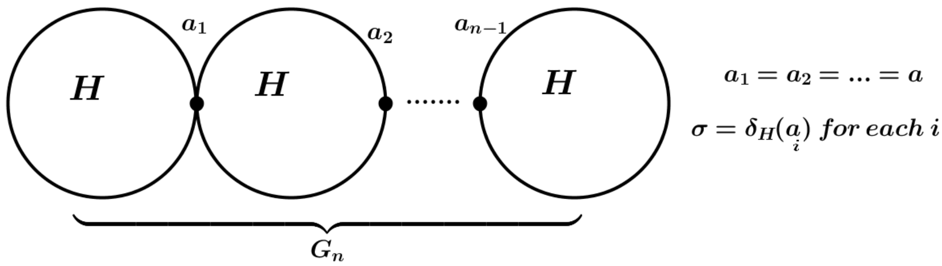


Figure 5: The chain graph $G_n$ obtained by gluing $n \geq 2$ copies of a regular graph $H$.

**Proof.** (1) If $n = 2$, then

$$\begin{aligned} M_1^\alpha(G_2) &= 2M_1^\alpha(H) + (2\sigma)^{\alpha+1} - \sigma^{\alpha+1} - \sigma^{\alpha+1} \\ &= 2M_1^\alpha(H) - (2^{\alpha+1} - 2)\sigma^{\alpha+1} \\ &= 2M_1^\alpha(H) - 2(2^\alpha - 1)\sigma^{\alpha+1}. \end{aligned}$$

Suppose that $M_1^\alpha(G_{n-1}) = (n-1)M_1^\alpha(H) + 2(n-2)(2^\alpha - 1)\sigma^{\alpha+1}$ and let us prove it for $n$.

$$\begin{aligned} M_1^\alpha(G_n) &= M_1^\alpha(G_{n-1}) + M_1^\alpha(H) + (2\sigma)^{\alpha+1} - 2\sigma^{\alpha+1} \\ &= (n-1)M_1^\alpha(H) + 2(n-2)(2^\alpha - 1)\sigma^{\alpha+1} + M_1^\alpha(H) + (2^{\alpha+1} - 2)\sigma^{\alpha+1} \\ &= nM_1^\alpha(H) + 2(n-1)(2^\alpha - 1)\sigma^{\alpha+1}. \end{aligned}$$

(2) If $n = 2$, then

$$\begin{aligned} M_2^\alpha(G_2) &= 2M_2^\alpha(H) + 2A_H^\alpha(a)(2\sigma)^\alpha - 2A_H^\alpha(a)\sigma^\alpha \\ &= 2M_2^\alpha(H) + 2(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha. \end{aligned}$$

Suppose that

$$M_2^\alpha(G_{n-1}) = (n-1)M_2^\alpha(H) + 2(n-2)(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha$$

and let us prove it for $n$.

$$\begin{aligned} M_2^\alpha(G_n) &= M_2^\alpha(G_{n-1}) + M_2^\alpha(H) + 2A_H^\alpha(a)(2\sigma)^\alpha - 2A_H^\alpha(a)\sigma^\alpha \\ &= (n-1)M_2^\alpha(H) + 2(n-2)(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha + M_2^\alpha(H) + 2(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha \\ &= nM_2^\alpha(H) + 2(n-1)(2^\alpha - 1)A_H^\alpha(a)\sigma^\alpha. \end{aligned}$$

□

If $H$ is a cycle in Corollary 2.3, then

$$M_1^\alpha(G_n) = (nm - 2n + 2)2^{\alpha+1} + (n-1)4^{\alpha+1} \quad \text{and} \quad M_2^\alpha(G_n) = (nm - 4n + 4)4^\alpha + 4(n-1)8^\alpha.$$

## 3. Joining of two graphs by a path

**Definition 3.1.** *Let $G$ be a graph and let "$a$" be a vertex of $G$. Then for convenience, we set*

$$\Delta_G^\alpha(a) = (\delta_G(a) + 1)^\alpha - \delta_G^\alpha(a)\,.$$

*Let $H_1$ and $H_2$ be two graphs. Let $a$ and $b$ be vertices of $H_1$ and $H_2$, respectively. Define a new graph $G$ by joining $H_1$ and $H_2$ by via a path with the starting vertex "$a$" of $H_1$ and the ending vertex "$b$" of $H_2$, (see Figure 6).*

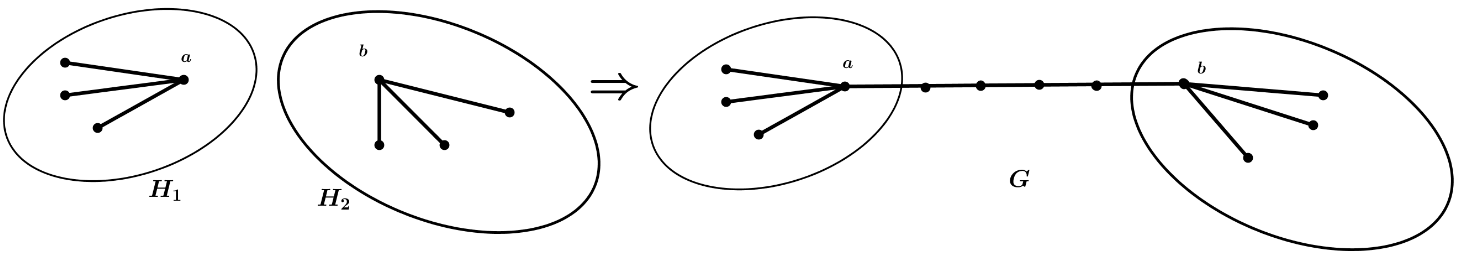


Figure 6: Joining two graphs by a path.

**Proposition 3.1.** *Let $H_1$ and $H_2$ be two graphs, and let $G$ be the graph obtained by joining $H_1$ and $H_2$ by a path with a starting vertex $a$ and an ending vertex $b$ (see Figure 6). Then*

$$M_1^\alpha(G) = M_1^\alpha(H_1) + M_1^\alpha(H_2) + \Delta_{H_1}^{\alpha+1}(a) + \Delta_{H_2}^{\alpha+1}(b) + (d_G(a,b) - 1)2^{\alpha+1}\,.$$

**Proof.** Let $m = d(a,b) \geq 1$. Then there is a path $(a, v_1, v_2, \ldots, v_m = b)$ from $a$ to $b$ in $G$. Set $\sigma = \delta_{H_1}(a)$ and $\tau = \delta_{H_2}(b)$, (see Figure 7). Let $H_1'$ be the graph that results by gluing $H_1$ to the path $(a, v_1, v_2, \ldots, v_m)$ into $a$, (see Figure 8). In view of Proposition 2.1, we have

$$M_1^\alpha(H_1') = M_1^\alpha(H_1) + (m-1)2^{\alpha+1} + 1^{\alpha+1} + 1^{\alpha+1} + (\sigma+1)^{\alpha+1} - \sigma^{\alpha+1} - 1^{\alpha+1}\,.$$

Notice that $G$ is the graph that results by gluing $H_1'$ and $H_2$ into $b = v_m$. Once again, by Proposition 2.1, we get

$$M_1^\alpha(G) = M_1^\alpha(H_1') + M_1^\alpha(H_2) + (\tau+1)^{\alpha+1} - \tau^{\alpha+1} - 1^{\alpha+1}\,.$$

Thus,

$$\begin{aligned} M_1^\alpha(G) &= M_1^\alpha(H_1) + (m-1)2^{\alpha+1} + 1^{\alpha+1} + (\sigma+1)^{\alpha+1} - \sigma^{\alpha+1} + M_1^\alpha(H_2) + (\tau+1)^{\alpha+1} - \tau^{\alpha+1} - 1^{\alpha+1} \\ &= M_1^\alpha(H_1) + M_1^\alpha(H_2) + \Delta_{H_1}^{\alpha+1}(a) + \Delta_{H_2}^{\alpha+1}(b) + (m-1)2^{\alpha+1}\,. \end{aligned}$$

□

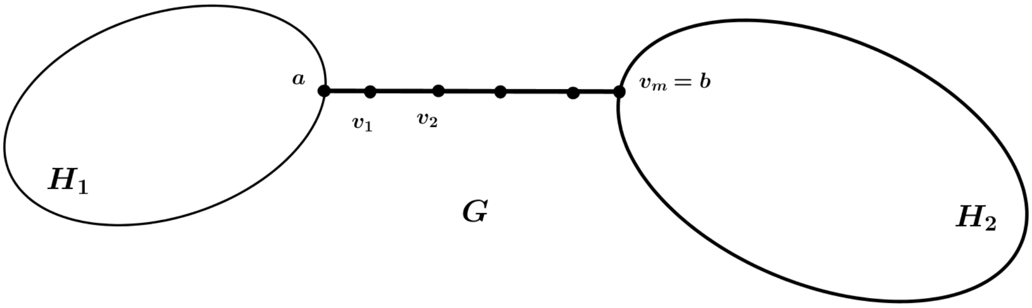


Figure 7: A graph used in Proposition 3.1.

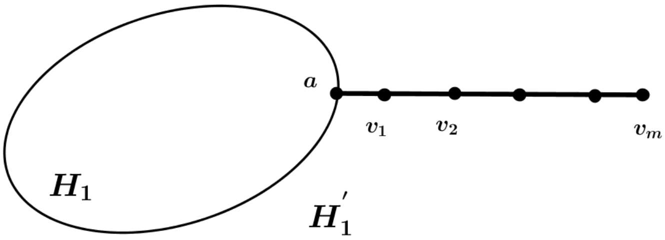


Figure 8: Gluing of a graph and a path by one vertex.

**Proposition 3.2.** *Let $H_1$ and $H_2$ be two graphs, and let $G$ be a graph obtained by joining $H_1$ and $H_2$ by a path with a starting vertex $a$ and an ending vertex $b$ (see Figure 6). Set $\sigma = \delta_{H_1}(a)$ and $\tau = \delta_{H_2}(b)$.*

(1) *If $d_G(a,b) = 1$, then*

$$M_2^\alpha(G) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + A_{H_1}^\alpha(a)\Delta_{H_1}^\alpha(a) + A_{H_2}^\alpha(b)\Delta_{H_2}^\alpha(b) + (\sigma+1)^\alpha(\tau+1)^\alpha\,.$$

(2) *If $d_G(a,b) > 1$, then*

$$M_2^\alpha(G) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + A_{H_1}^\alpha(a)\Delta_{H_1}^\alpha(a) + A_{H_2}^\alpha(b)\Delta_{H_2}^\alpha(b) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (d_G(a,b)-2)4^\alpha\,.$$

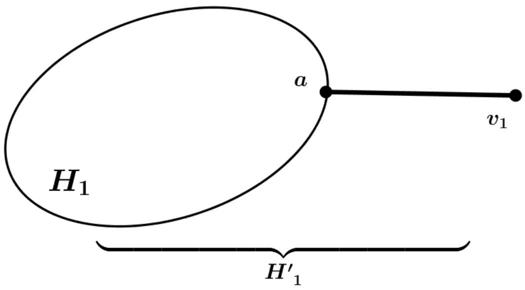


Figure 9: Gluing of a graph $H$ and an edge at the vertex "$a$".

**Proof.** (1) Suppose that $d_G(a,b) = 1$ (see Figure 9). Let $H_1'$ be the graph obtained by gluing the graph $H_1$ and the edge $av_1$ at the vertex "$a$". By Proposition 2.2, we have

$$M_2^\alpha(H_1') = M_2^\alpha(H_1) + 1^\alpha\, 1^\alpha + (A_{H_1}^\alpha(a) + 1^\alpha)(\sigma+1)^\alpha - A_{H_1}^\alpha(a)\sigma^\alpha - 1^\alpha\, 1^\alpha,$$

that is,

$$M_2^\alpha(H_1') = M_2^\alpha(H_1) + A_{H_1}^\alpha(a)\Delta_{H_1}^\alpha(a) + (\sigma+1)^\alpha\,.$$

Let $G$ be the graph obtained by gluing the graphs $H_1'$ and $H_2$ via $v_1 = b$ (see Figure 10). According to Proposition 2.2, and because of $A^{\alpha}_{H'_1}(v_1) = (\sigma+1)^{\alpha}$, we have

$$\begin{aligned} M_2^{\alpha}(G) &= M_2^{\alpha}(H_1') + M_2^{\alpha}(H_2) + ((\sigma+1)^{\alpha} + A^{\alpha}_{H_2}(b))(1+\tau)^{\alpha} - (\sigma+1)^{\alpha} - A^{\alpha}_{H_2}(b)\tau^{\alpha} \\ &= M_2^{\alpha}(H_1) + M_2^{\alpha}(H_2) + A^{\alpha}_{H_1}(a)\Delta^{\alpha}_{H_1}(a) + A^{\alpha}_{H_2}(b)\Delta^{\alpha}_{H_2}(b) + (\sigma+1)^{\alpha}(\tau+1)^{\alpha}\,. \end{aligned}$$

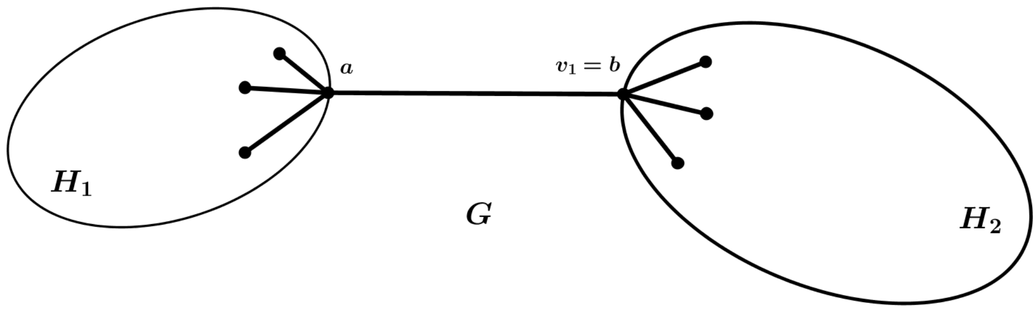


Figure 10: Joining two graphs by an edge.

(2) Let $H_1'$ be the graph obtained by gluing $H$ to the path $a, v_1, v_2, \ldots, v_m$ at the vertex "$a$", where $m = d_G(a,b) \geq 2$, (see Figures 11 and 12). Let $G$ be the graph obtained by gluing the graphs $H_1'$ and $H_2$ at the vertex $b$. Then

$$M_2^{\alpha}(H'_1) = M_2^{\alpha}(H_1) + 1\cdot 2^{\alpha} + 1\cdot 2^{\alpha} + (m-2)2^{\alpha}\cdot 2^{\alpha} + (A^{\alpha}_{H_1}(a) + 2^{\alpha})(\sigma+1)^{\alpha} - A^{\alpha}_{H_1}(a)\sigma^{\alpha} - 2^{\alpha}\cdot 1,$$

that is,

$$M_2^{\alpha}(H_1') = M_2^{\alpha}(H_1) + A^{\alpha}_{H_1}(a)\Delta^{\alpha}_{H_1}(a) + 2^{\alpha}(\sigma+1)^{\alpha} + 2^{\alpha} + (m-2)4^{\alpha}\,.$$

It follows that

$$\begin{aligned} M_2^{\alpha}(G) &= M_2^{\alpha}(H'_1) + M_2^{\alpha}(H_2) + ((2^{\alpha} + A^{\alpha}_{H_2}(b))(1+\tau)^{\alpha} - 2^{\alpha}\cdot 1 - A^{\alpha}_{H_2}(b)\tau^{\alpha} \\ &= M_2^{\alpha}(H_1) + A^{\alpha}_{H_1}(a)\Delta^{\alpha}_{H_1}(a) + 2^{\alpha}(\sigma+1)^{\alpha} + 2^{\alpha} + (m-2)4^{\alpha} + 2^{\alpha}(\tau+1)^{\alpha} + A^{\alpha}_{H_2}(b)\Delta^{\alpha}_{H_2}(b) - 2^{\alpha} + M_2^{\alpha}(H_2) \\ &= M_2^{\alpha}(H_1) + M_2^{\alpha}(H_2) + A^{\alpha}_{H_1}(a)\Delta^{\alpha}_{H_1}(a) + A^{\alpha}_{H_2}(b)\Delta^{\alpha}_{H_2}(b) + 2^{\alpha}[(\sigma+1)^{\alpha} + (\tau+1)^{\alpha}] + (m-2)4^{\alpha}\,. \end{aligned}$$

□

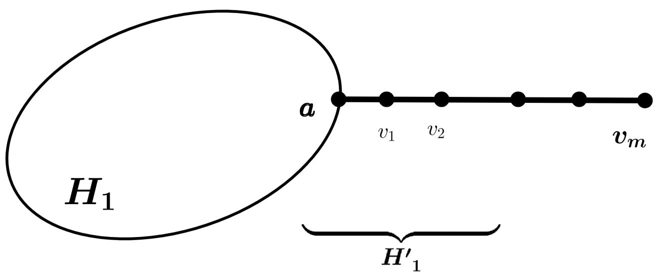


Figure 11: Gluing of a graph and a path at the vertex "$a$".

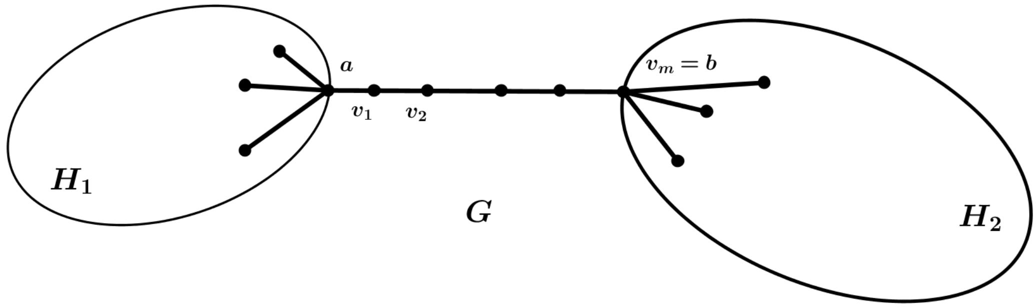


Figure 12: Joining two graphs by a path.

**Corollary 3.1.** *Let $H$ and $C_m$ be a graph and a cycle, respectively. Let $G$ be the graph obtained by joining $H$ and $C_m$ via a path (see Figure 13). Then*

(1) $M_1^\alpha(G) = M_1^\alpha(H) + \Delta_H^{\alpha+1}(a) + (d_G(a,b) + m - 2)2^{\alpha+1} + 3^{\alpha+1}$.

(2) *If* $d_G(a,b) = 1$*, then*

$$M_2^\alpha(G) = M_2^\alpha(H) + (m-2)4^\alpha + A_H^\alpha(a)\Delta_H^\alpha(a) + 3^\alpha[(\sigma+1)^\alpha + 2^{\alpha+1}].$$

*If* $d_G(a,b) > 1$*, then*

$$M_2^\alpha(G) = M_2^\alpha(H) + (d(a,b) + m - 4)4^\alpha + A_H^\alpha(a)\Delta_H^\alpha(a) + 3 \cdot 6^\alpha + 2^\alpha(\sigma+1)^\alpha.$$

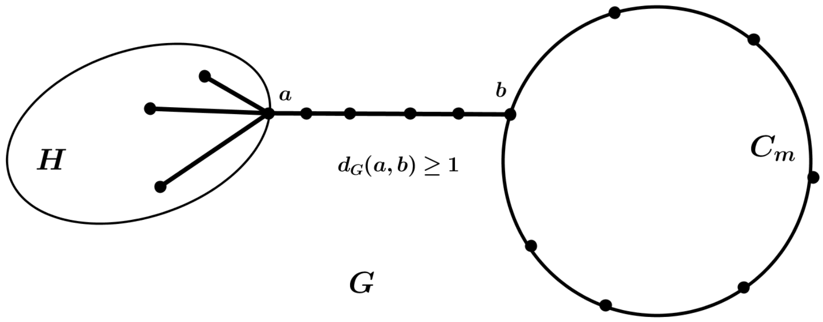


Figure 13: Joining of a graph and a cycle by a path.

**Corollary 3.2.** *Let $C_n$ and $C_m$ be two cycles. Let $G$ be the graph obtained by joining $C_n$ and $C_m$ via a path (see Figure 14).*

(1) $M_1^\alpha(G) = (n + m + d_G(a,b) - 3)2^{\alpha+1} + 2 \cdot 3^{\alpha+1}$.

(2) *If* $d_G(a,b) = 1$*, then* $M_2^\alpha(G) = (n+m-4)4^\alpha + 4 \cdot 6^\alpha + 9^\alpha$*. If* $d_G(a,b) > 1$*, then* $M_2^\alpha(G) = (n+m+d_G(a,b)-6)4^\alpha + 6 \cdot 6^\alpha$.

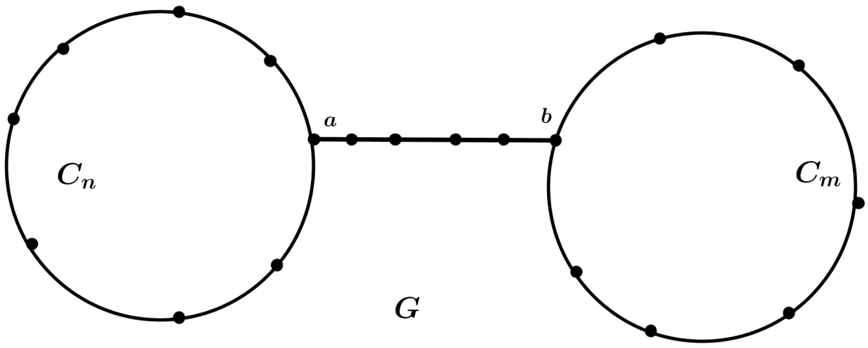


Figure 14: Joining two cycles by a path.

**Corollary 3.3.** *Let $G_n$ be the graph obtained by joining $n \geq 2$ copies of a regular graph $H$ (see Figure 15). Concerning the graph $G_n$ (shown in Figure 15), define the following notation: $a_i = a, b_i = b, \sigma = \delta_H(a_i), \tau = \delta_H(b_i)$ for each $i$, $A_H^\alpha(a) = \lambda, A_H^\alpha(b) = \mu$ and $d_{G_n}(a_i, b_i) = d$ for each $i$.*

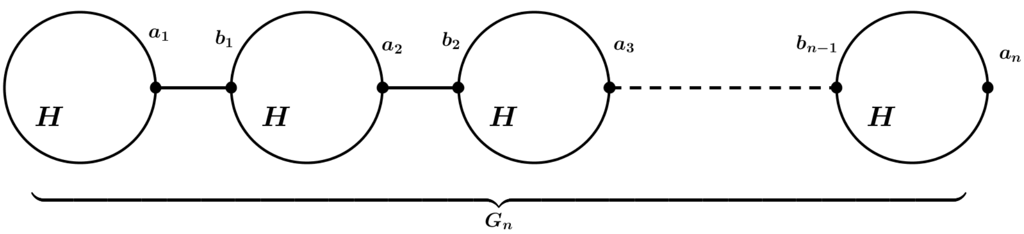


Figure 15: The graph $G_n$ obtained by joining $n \geq 2$ copies of a regular graph $H$.

(1) $M_1^\alpha(G_n) = n\,M_1^\alpha(H) + (n-1)[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + (n-1)[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (n-1)(d-1)2^{\alpha+1}$.

(2) *If* $d = 1$, *then*

$$M_2^\alpha(G_n) = n\,M_2^\alpha(H) + (n-1)\lambda[(\sigma+1)^\alpha - \sigma^\alpha] + (n-1)\mu[(\tau+1)^\alpha - \tau^\alpha] + (n-1)(\sigma+1)^\alpha(\tau+1)^\alpha.$$

*If* $d > 1$, *then*

$$M_2^\alpha(G_n) = (n-1)\lambda[(\sigma+1)^\alpha - \sigma^\alpha] + (n-1)\mu[(\tau+1)^\alpha - \tau^\alpha] + 2^\alpha(n-1)[(\sigma+1)^\alpha + (\tau+1)^\alpha] + nM_2^\alpha(H) + (n-1)(d-2)4^\alpha\,.$$

**Proof.** (1) For $n = 2$, Proposition 3.1 ensures that

$$\begin{aligned} M_1^\alpha(G_2) &= 2M_1^\alpha(H) + \Delta_H^{\alpha+1}(a) + \Delta_H^{\alpha+1}(b) + (d-1)2^{\alpha+1} \\ &= 2M_1^\alpha(H) + [(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + [(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (d-1)2^{\alpha+1}\,. \end{aligned}$$

Suppose that

$$M_1^\alpha(G_{n-1}) = (n-1)M_1^\alpha(H) + (n-2)[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + (n-2)[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (n-2)(d-1)2^{\alpha+1}\,.$$

Now, we evaluate $M_1^\alpha(G_n)$ as follows:

$$\begin{aligned} M_1^\alpha(G_n) &= M_1^\alpha(G_{n-1}) + M_1^\alpha(H) + [(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + [(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (d-1)2^{\alpha+1} \\ &= nM_1^\alpha(H) + (n-1)[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + (n-1)[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (n-1)(d-1)2^{\alpha+1}\,. \end{aligned}$$

By using Proposition 3.2 and induction on $n$, we prove part (2). □

## 4. Inserting a path between two vertices of a graph

Let $H$ be a graph and let $a, b$ be two of its non-adjacent vertices. We define a new graph $G$ by inserting a new path $(a = v_1, v_2, \ldots, v_m = b)$ between $a$ and $b$ (see Figure 16).

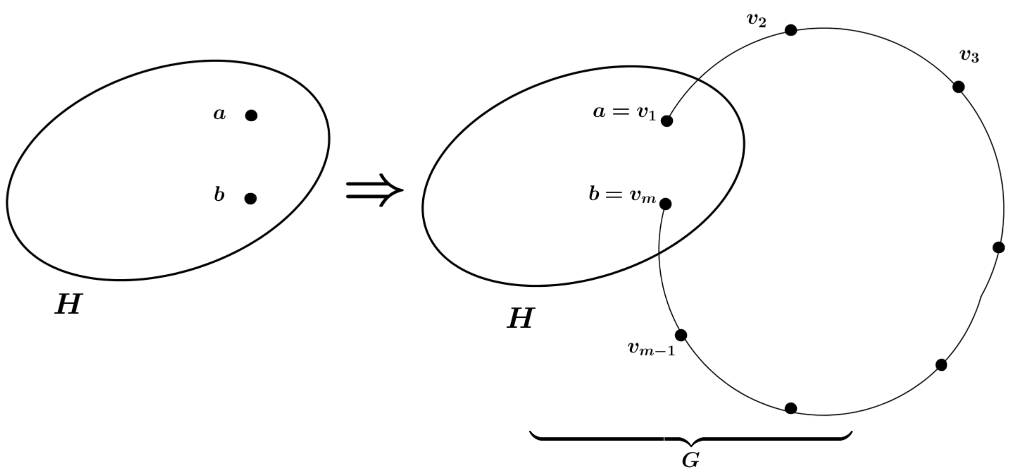


Figure 16: Inserting a path between non-adjacent vertices $a, b$ of the graph $H$.

**Proposition 4.1.** *Let $G$ be the graph obtained by inserting the path $(a = v_1, v_2, \ldots, v_m = b)$ between the vertices $a, b$ of a graph $H$. Then*

$$M_1^\alpha(G) = M_1^\alpha(H) + \Delta_H^{\alpha+1}(a) + \Delta_H^{\alpha+1}(b) + (m-2)2^{\alpha+1}.$$

**Proof.** Since $\delta_G^{\alpha+1}(a) = (\delta_H(a)+1)^{\alpha+1}$ and $\delta_G^{\alpha+1}(b) = (\delta_H(b)+1)^{\alpha+1}$, we have

$$\begin{aligned} M_1^\alpha(G) &= \sum_{v \in V(H)\setminus\{a,b\}} \delta_H^{\alpha+1}(v) + \delta_G^{\alpha+1}(a) + \delta_G^{\alpha+1}(b) + (m-2)2^{\alpha+1} \\ &= M_1^\alpha(H) + \delta_G^{\alpha+1}(a) - \delta_H^{\alpha+1}(a) + \delta_G^{\alpha+1}(b) - \delta_H^{\alpha+1}(b) + (m-2)2^{\alpha+1} \\ &= M_1^\alpha(H) + \Delta_H^{\alpha+1}(a) + \Delta_H^{\alpha+1}(b) + (m-2)2^{\alpha+1}. \end{aligned}$$

□

**Proposition 4.2.** *Let $G$ be the same graph as in Proposition 4.1.*

(1) *If $d_H(a,b) > 1$ and $m \geq 3$, then*

$$M_2^\alpha(G) = M_2^\alpha(H) + A_H^\alpha(a)\Delta_H^\alpha(a) + A_H^\alpha(b)\Delta_H^\alpha(b) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (m-3)4^\alpha\,.$$

(2) *If $d_H(a,b) = 1$ and $m \geq 3$, then*

$$M_2^\alpha(G) = M_2^\alpha(H) + (A_H^\alpha(a) - \tau^\alpha)\Delta_H^\alpha(a) + (A_H^\alpha(b) - \sigma^\alpha)\Delta_H^\alpha(b) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (m-3)4^\alpha + (\sigma+1)^\alpha(\tau+1)^\alpha - \sigma^\alpha\tau^\alpha\,.$$

(3) *If $d_H(a,b) > 1$ and $m = 2$, then*

$$M_2^\alpha(G) = M_2^\alpha(H) + A_H^\alpha(a)\Delta_H^\alpha(a) + A_H^\alpha(b)\Delta_H^\alpha(b) + (\sigma+1)^\alpha(\tau+1)^\alpha\,.$$

**Proof.** (1) Suppose that $d_H(a,b) > 1$ and $m \geq 3$. Then

$$\begin{aligned} M_2^\alpha(G) &= \sum_{uv\in E(H);u,v\notin\{a,b\}} \delta_H^\alpha(u)\,\delta_H^\alpha(v) + A_H^\alpha(a)(\sigma+1)^\alpha + A_H^\alpha(b)(\tau+1)^\alpha + (m-3)2^\alpha 2^\alpha + 2^\alpha(\sigma+1)^\alpha + 2^\alpha(\tau+1)^\alpha \\ &= M_2^\alpha(H) + A_H^\alpha(a)(\sigma+1)^\alpha + A_H^\alpha(b)(\tau+1)^\alpha + 2^\alpha(\sigma+1)^\alpha + 2^\alpha(\tau+1)^\alpha - A_H^\alpha(a)\sigma^\alpha - A_H^\alpha(b)\tau^\alpha + (m-3)4^\alpha \\ &= M_2^\alpha(H) + A_H^\alpha(a)\Delta_H^\alpha(a) + A_H^\alpha(b)\Delta_H^\alpha(b) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (m-3)4^\alpha. \end{aligned}$$

(2) Suppose that $d_H(a,b) = 1$ and $m \geq 3$. Then

$$\begin{aligned} M_2^\alpha(G) = &\sum_{uv\in E(H);u,v\notin\{a,b\}} \delta_H^\alpha(u)\,\delta_H^\alpha(v) + (A_H^\alpha(a) - \tau^\alpha)(\sigma+1)^\alpha + (A_H^\alpha(b) - \sigma^\alpha)(\tau+1)^\alpha \\ &+ (\sigma+1)^\alpha(\tau+1)^\alpha + (m-3)2^\alpha 2^\alpha + 2^\alpha(\sigma+1)^\alpha + 2^\alpha(\tau+1)^\alpha \\ &= M_2^\alpha(H) + (A_H^\alpha(a) - \tau^\alpha)(\sigma+1)^\alpha + (A_H^\alpha(b) - \sigma^\alpha)(\tau+1)^\alpha + (m-3)4^\alpha + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] \\ &+ (\sigma+1)^\alpha(\tau+1)^\alpha - (A_H^\alpha(a) - \tau^\alpha)(\sigma)^\alpha - (A_H^\alpha(b) - \sigma^\alpha)(\tau)^\alpha - \sigma^\alpha\tau^\alpha\,. \end{aligned}$$

Thus,

$$M_2^\alpha(G) = M_2^\alpha(H) + (A_H^\alpha(a) - \tau^\alpha)\Delta_H^\alpha(a) + (A_H^\alpha(b) - \sigma^\alpha)\Delta_H^\alpha(b) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (m-3)4^\alpha + (\sigma+1)^\alpha(\tau+1)^\alpha - \sigma^\alpha\tau^\alpha\,.$$

(3) If $d_H(a,b) > 1$ and $m = 2$ (see Figure 17), then

$$\begin{aligned} M_2^\alpha(G) &= \sum_{u,v\notin\{a,b\}} \delta_H^\alpha(u)\,\delta_H^\alpha(v) + A_H^\alpha(a)(\sigma+1)^\alpha + A_H^\alpha(b)(\tau+1)^\alpha + (\sigma+1)^\alpha(\tau+1)^\alpha \\ &= M_2^\alpha(H) + A_H^\alpha(a)(\sigma+1)^\alpha + A_H^\alpha(b)(\tau+1)^\alpha + (\sigma+1)^\alpha(\tau+1)^\alpha - A_H^\alpha(a)\sigma^\alpha - A_H^\alpha(b)\tau^\alpha, \end{aligned}$$

which implies that

$$M_2^\alpha(G) = M_2^\alpha(H) + A_H^\alpha(a)\Delta_H^\alpha(a) + A_H^\alpha(b)\Delta_H^\alpha(b) + (\sigma+1)^\alpha(\tau+1)^\alpha\,.$$

□

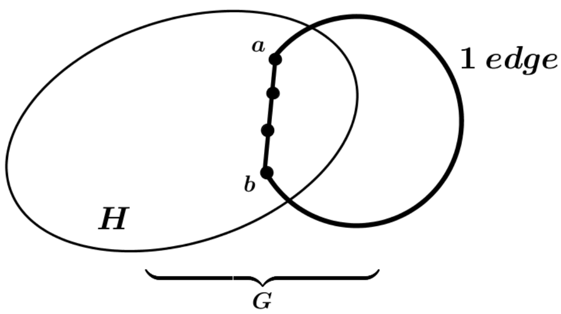


Figure 17: The case $d_H(a,b) > 1$ and $m = 2$.

**Corollary 4.1.** *Let $G$ be the graph obtained by inserting the path $(a = v_1, v_2, \ldots, v_m = b)$ between two vertices $a, b$ of a cycle $C_n$, where $m \geq 3, n \geq 4$, or $m = 2$ if $d(a, b) > 1$ (see Figure 18).*

(1) $M_1^\alpha(G) = (m + n - 4)2^{\alpha+1} + 2 \cdot 3^{\alpha+1}$.

(2) *If $d(a, b) > 1$ and $m \geq 3$, then $M_2^\alpha(G) = (n+m-7)4^\alpha + 6^{\alpha+1}$. Also, if $d(a, b) = 1$, then $M_2^\alpha(G) = (m+n-6)4^\alpha + 4 \cdot 6^\alpha + 9^\alpha$. Moreover, if $d(a, b) > 1$ and $m = 2$, then $M_2^\alpha(G) = (n - 4)4^\alpha + 4 \cdot 6^\alpha + 9^\alpha$.*

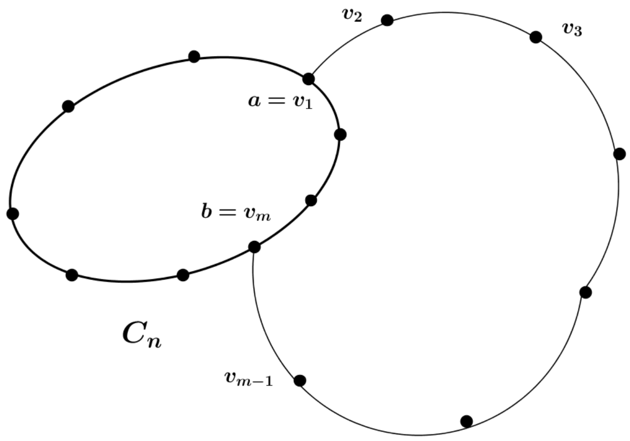


Figure 18: The graph used in Corollary 4.1.

## 5. Joining two graphs by several paths

Let $H_1$ and $H_2$ be two graphs. Let $\{a_1, a_2, \ldots, a_n\}$ be the set of vertices of $H_1$ such that $\delta_{H_1}(a_i) = \sigma$ for each $i \in \{1, 2, \ldots, n\}$, and let $\{b_1, b_2, \ldots, b_n\}$ be the set of vertices of $H_2$ such that $\delta_{H_2}(b_i) = \tau$ for each $i \in \{1, 2, \ldots, n\}$. Let $G_n$ be the graph obtained from $H_1$ and $H_2$ by joining each pair of vertices $(a_i, b_i)$ via a path of length $d = d_G(a_i, b_i)$ (see Figure 19).

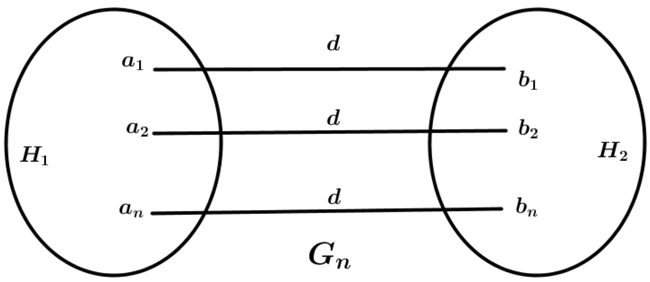


Figure 19: Joining two graphs by several paths.

**Proposition 5.1.** *If $G_n$ is the graph defined at the start of this section (see Figure 19), then*

$$M_1^\alpha(G_n) = M_1^\alpha(H_1) + M_1^\alpha(H_2) + n[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + n[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + n(d-1)2^{\alpha+1}.$$

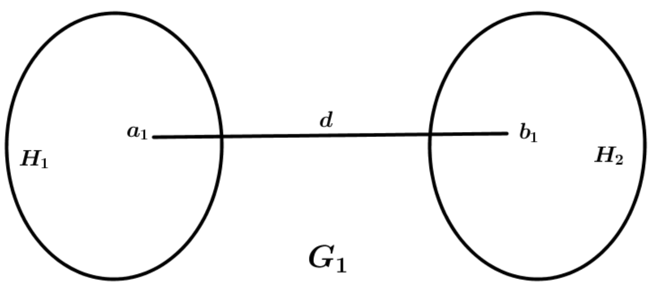


Figure 20: Joining two graphs by a single path.

**Proof.** We use the induction on the number $n$ of paths and utilize Proposition 3.1. If $n = 1$ (see Figure 20), then we have

$$M_1^\alpha(G_1) = M_1^\alpha(H_1) + M_1^\alpha(H_2) + [(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + [(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (d-1)2^{\alpha+1}\,.$$

Suppose that

$$M_1^\alpha(G_{n-1}) = M_1^\alpha(H_1) + M_1^\alpha(H_2) + (n-1)[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + (n-1)[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (n-1)(d-1)2^{\alpha+1}\,.$$

Now, we prove the formula for $n$. By Proposition 4.1, we have

$$M_1^\alpha(G_n) = M_1^\alpha(G_{n-1}) + [(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + [(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (d-1)2^{\alpha+1}$$

because $G_n$ is the result of joining $a_n$ and $b_n$ in $G_{n-1}$ by a path of length $d$. Hence,

$$\begin{aligned} M_1^\alpha(G_n) &= M_1^\alpha(H_1) + M_1^\alpha(H_2) + (n-1)[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + (n-1)[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] \\ &+ (n-1)(d-1)2^{\alpha+1} + [(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + [(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + (d-1)2^{\alpha+1} \\ &= M_1^\alpha(H_1) + M_1^\alpha(H_2) + n[(\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}] + n[(\tau+1)^{\alpha+1} - \tau^{\alpha+1}] + n(d-1)2^{\alpha+1}\,. \end{aligned}$$

Suppose that $H_1$ and $H_2$ satisfy the hypotheses of Proposition 5.1. In addition, suppose that $d(a_i, a_{i+1}) = d(b_i, b_{i+1}) = 1$ and $d(a_i, a_j) > 1, d(b_i, b_j) > 1$ if $i$ and $j$ are not successive (see Figure 21). Since $\delta_{H_1}(a_i) = \sigma$ and $\delta_{H_2}(b_i) = \tau$ for each $i$, we have $\Delta^\alpha_{H_1}(a_i) = (\sigma+1)^\alpha - \sigma^\alpha$ and $\Delta^\alpha_{H_2}(b_i) = (\tau+1)^\alpha - \tau^\alpha$. □

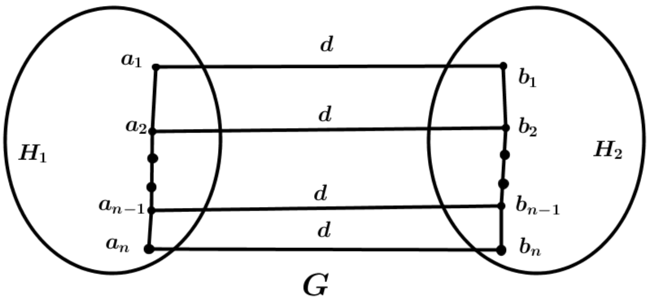


Figure 21: Joining two graphs by $n$ paths according to the proof of Proposition 5.1.

**Proposition 5.2.** *Let $G_n$ be the same graph as in Proposition 5.1. Set $\lambda = (\sigma+1)^{\alpha+1} - \sigma^{\alpha+1}$ and $\mu = (\tau+1)^{\alpha+1} - \tau^{\alpha+1}$.*

(i) *If $d > 1$, then*

$$M_2^\alpha(G_n) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^n A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^n A^\alpha_{H_2}(b_i)\right) + n2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + n(d-2)4^\alpha + (n-1)(\lambda^2+\mu^2)\,.$$

(ii) *If $d = 1$, then*

$$M_2^\alpha(G_n) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^n A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^n A^\alpha_{H_2}(b_i)\right) + (n-1)(\lambda^2+\mu^2) + n(\sigma+1)^\alpha(\tau+1)^\alpha\,.$$

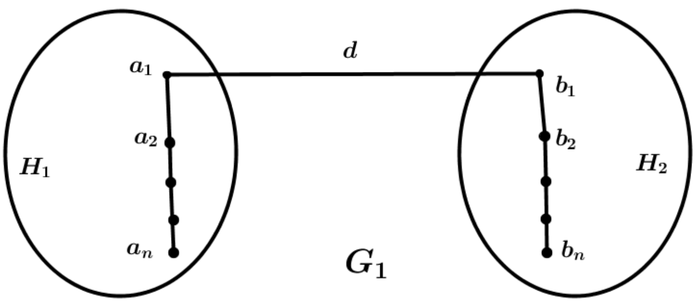


Figure 22: Joining two graphs by a path.

**Proof.** (i) Suppose that $d > 1$ (see Figure 22). By Proposition 3.2(2), we have

$$M_2^\alpha(G_1) = M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda A^\alpha_{H_1}(a_1) + \mu A^\alpha_{H_2}(b_1) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (d-2)4^\alpha\,.$$

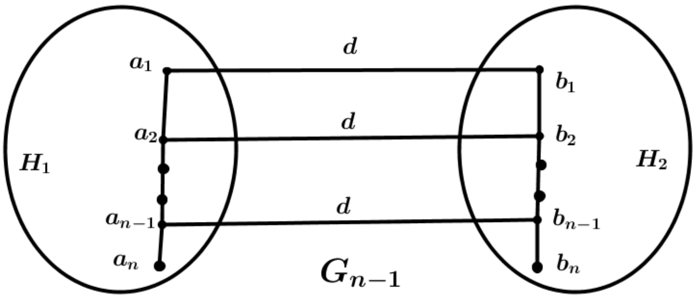


Figure 23: Joining two graphs by $n-1$ paths.

Suppose that for $G_{n-1}$ (see Figure 23), it holds that

$$\begin{aligned} M_2^\alpha(G_{n-1}) &= M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^{n-1} A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^{n-1} A^\alpha_{H_2}(b_i)\right) \\ &+ (n-1)2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (n-1)(d-2)4^\alpha + (n-2)(\lambda^2+\mu^2)\,. \end{aligned}$$

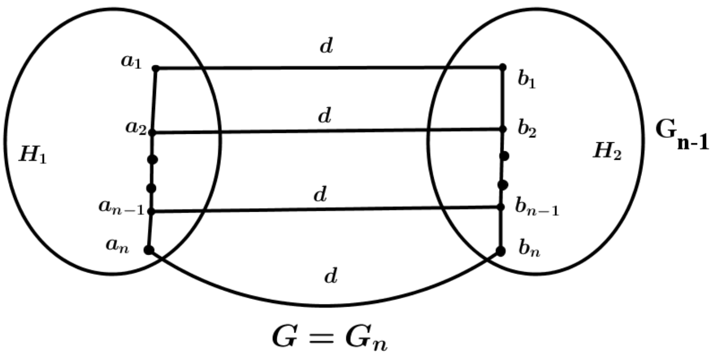


Figure 24: Joining two graphs by $n$ paths.

Let $G_n$ be the graph obtained from $G_{n-1}$ by joining $a_n$ to $b_n$ by a path (see Figure 24). By Proposition 4.2(1), we have

$$M_2^\alpha(G_n) = M_2^\alpha(G) = M_2^\alpha(G_{n-1}) + A^\alpha_{G_{n-1}}(a_n)\Delta^\alpha_{G_{n-1}}(a_n) + A^\alpha_{G_{n-1}}(b_n)\Delta^\alpha_{G_{n-1}}(b_n) + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (d-2)4^\alpha,$$

where $\Delta^\alpha_{G_{n-1}}(a_n) = (\sigma+1)^\alpha - \sigma^\alpha = \lambda$, $\Delta^\alpha_{G_{n-1}}(b_n) = (\tau+1)^\alpha - \tau^\alpha = \mu$, $A^\alpha_{G_{n-1}}(a_n) = A^\alpha_{H_1}(a_n) + (\sigma+1)^\alpha - \sigma^\alpha = A^\alpha_{H_1}(a_n) + \lambda$, and $A^\alpha_{G_{n-1}}(b_n) = A^\alpha_{H_2}(b_n) + (\tau+1)^\alpha - \tau^\alpha = A^\alpha_{H_2}(b_n) + \mu$. Therefore,

$$\begin{aligned} M_2^\alpha(G_n) &= M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^{n-1} A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^{n-1} A^\alpha_{H_2}(b_i)\right) \\ &+ (n-1)2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (n-1)(d-2)4^\alpha + (n-2)(\lambda^2+\mu^2) \\ &+ (A^\alpha_{H_1}(a_n) + \lambda)\lambda + (A^\alpha_{H_2}(b_n) + \mu)\mu + 2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + (d-2)4^\alpha \\ &= M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^{n} A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^{n} A^\alpha_{H_2}(b_i)\right) \\ &+ n2^\alpha[(\sigma+1)^\alpha + (\tau+1)^\alpha] + n(d-2)4^\alpha + (n-1)(\lambda^2+\mu^2)\,. \end{aligned}$$

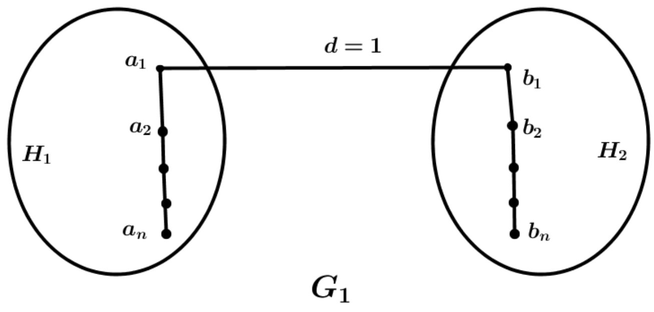


Figure 25: Joining two graphs by an edge.

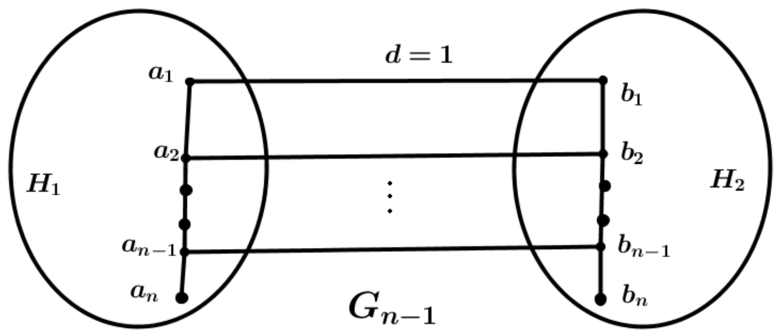


Figure 26: Joining two graphs by $n-1$ edges.

(ii) Suppose that $d=1$ (see Figure 25). By Proposition 3.2(1),

$$M_2^{\alpha}(G_1) = M_2^{\alpha}(H_1) + M_2^{\alpha}(H_2) + \lambda A_{H_1}^{\alpha}(a_1) + \mu A_{H_2}^{\alpha}(b_1) + (\sigma+1)^{\alpha}\,(\tau+1)^{\alpha}\,.$$

Suppose that for $G_{n-1}$ (see Figure 26), we have

$$\begin{aligned} M_2^{\alpha}(G_{n-1}) &= M_2^{\alpha}(H_1) + M_2^{\alpha}(H_2) + \lambda\left(\sum_{i=1}^{n-1} A_{H_1}^{\alpha}(a_i)\right) + \mu\left(\sum_{i=1}^{n-1} A_{H_2}^{\alpha}(b_i)\right) \\ &+ (n-1)(\sigma+1)^{\alpha}\,(\tau+1)^{\alpha} + (n-2)(\lambda^2+\mu^2)\,. \end{aligned}$$

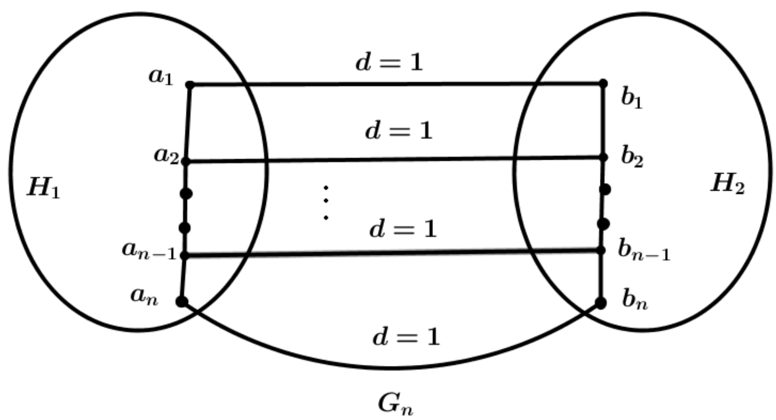


Figure 27: Joining two graphs by $n$ edges.

Let $G_n$ be the graph obtained from $G_{n-1}$ by joining $a_n$ to $b_n$ via an edge (see Figure 27). By Proposition 4.2(3), we have

$$\begin{aligned}
M_2^\alpha(G_n) &= M_2^\alpha(G_{n-1}) + A^\alpha_{G_{n-1}}(a_n)\Delta^\alpha_{G_{n-1}}(a_n) + A^\alpha_{G_{n-1}}(b_n)\Delta^\alpha_{G_{n-1}}(b_n) + (\sigma+1)^\alpha(\tau+1)^\alpha \\
&= M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^{n-1} A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^{n-1} A^\alpha_{H_2}(b_i)\right) \\
&+ (n-2)(\lambda^2+\mu^2) + (n-1)(\sigma+1)^\alpha(\tau+1)^\alpha + (\sigma+1)^\alpha(\tau+1)^\alpha + \lambda(A^\alpha_{H_1}(a_n)+\lambda) + \mu(A^\alpha_{H_2}(b_n)+\mu) \\
&= M_2^\alpha(H_1) + M_2^\alpha(H_2) + \lambda\left(\sum_{i=1}^{n} A^\alpha_{H_1}(a_i)\right) + \mu\left(\sum_{i=1}^{n} A^\alpha_{H_2}(b_i)\right) + (n-1)(\lambda^2+\mu^2) + n(\sigma+1)^\alpha(\tau+1)^\alpha .
\end{aligned}$$

□

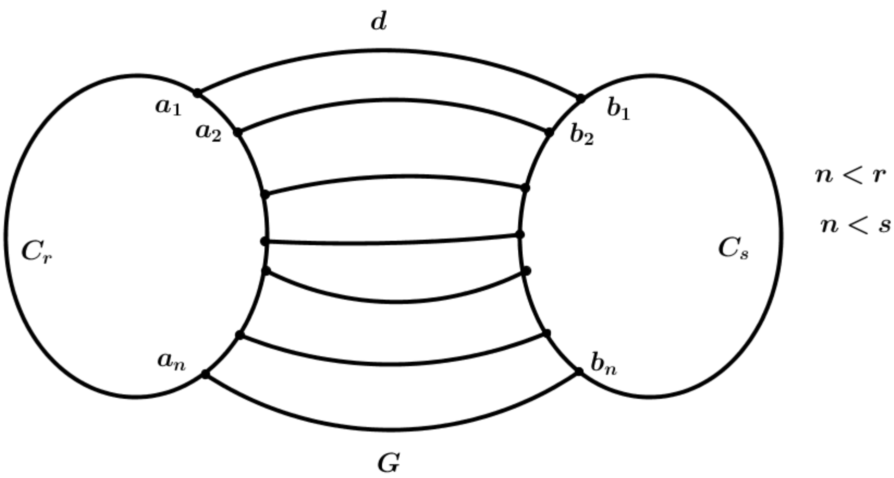


Figure 28: Joining two cycles by several paths.

**Corollary 5.1.** *Let $G$ be the graph obtained by joining two cycles $C_r$ and $C_s$ through $n$ paths, where $n < r$ and $n < s$ (see Figure 28).*

(1) $M_1^\alpha(G) = (r+s-n-n+nd-n)2^{\alpha+1} + 2n\,3^{\alpha+1} = (r+s+nd-3n)2^{\alpha+1} + 2n\,3^{\alpha+1}$.

(2) *If $d > 1$, then $M_2^\alpha(G) = (r+s+nd-4n-2)4^\alpha + 2(n+2)6^\alpha + 2(n-1)9^\alpha$. If $d = 1$, then*

$$M_2^\alpha(G) = (r+s-2n-2)4^\alpha + 4\cdot 6^\alpha + (3n-2)9^\alpha.$$

## 6. Concluding remarks

In the previous four sections, formulas for calculating general first and second Zagreb indices, $M_1^\alpha$ and $M_2^\alpha$, are established for a variety of composite graphs. For $\alpha = 1$, these formulas reduce to the ordinary first and second Zagreb indices; whereas, for some other values of $\alpha$, the obtained formulas yield several other, earlier proposed, topological indices including the Randić index, the zeroth-order Randić index, the hyper Zagreb index, the forgotten index, the $Y$-index, and the $S$-index. Corollaries illustrating the applicability of derived formulas show that sometimes chemically interesting results can be obtained; that is, the considered topological indices of chemically relevant (molecular) graphs can be calculated.

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