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Research Article

Braided monoidal categories of coalgebras

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Abstract

Let $\mathcal{V}$ be a braided monoidal category. Given a braided monoidal endofunctor F on $\mathcal{V}$, it is proved that F-coalgebras form a braided monoidal category, denoted as $\mathcal{V}_F$. Particularly, if the category $\mathcal{V}$ admits coproducts and if F is a fully faithful symmetric monoidal endofunctor, then it is proved that $\mathcal{V}_F$ is symmetric monoidal closed whenever $\mathcal{V}$ is symmetric monoidal closed.

Keywords: braided monoidal category; coalgebra; monoidal functor.

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1. Introduction

A braided monoidal category is a monoidal category equipped with a natural isomorphism $\lambda_{AB} : A \otimes B \to B \otimes A$, known as the braiding. If $\lambda_{AB} \circ \lambda_{BA} = 1_{A\otimes B}$, then the braided monoidal category is called a symmetric monoidal category. So, every symmetric monoidal category is braided. On the other hand, not every braided monoidal category is symmetric. For instance, the category of graded modules over a commutative ring is braided for the usual monoidal structure on it; however, this monoidal category is not symmetric (see [4]). A symmetric monoidal category $\mathcal{V}$ is said to be closed if, for each $B \in \mathcal{V}_0$, the functor $- \otimes B$ has a right adjoint $[B, -]$.

The notion of braiding occurs naturally in the theoretical context of further multiplication on a monoidal category, has an adequate coherence theorem, and, is as versatile as symmety [7]. For any monoidal category $\mathcal{V}$, an extra multiplication on $\mathcal{V}$ leads to a braiding; also, each braiding leads to a multiplication. This gives a natural explanation for braidings as opposed to symmetries. Important examples of braidings can be found especially in homotopy and cohomology theories.

A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ between monoidal categories consists of a functor $F : \mathcal{V}_0 \to \mathcal{W}_0$ together with a morphism $\phi_{AB} : FA\otimes FB \to F(A\otimes B)$ defined for every pair A, B of objects of $\mathcal{V}_0$, which is natural in A, B, and a morphism $\phi_o : J \to FI$ of $\mathcal{W}$, where I is the unit of $\mathcal{V}$ and J is the unit of $\mathcal{W}$; all these objects must satisfy the associativity and unit conditions. A braided monoidal functor is a monoidal functor between braided categories together with the coherence condition expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} FA\otimes FB & \xrightarrow{\lambda_{FAFB}} & FB\otimes FA \\ {\scriptstyle \phi_{AB}}\downarrow & & \downarrow{\scriptstyle \phi_{BA}} \\ F(A\otimes B) & \xrightarrow[F(\lambda_{AB})]{} & F(B\otimes A) \end{array}$$

Particularly, a symmetric monoidal functor is a braided monoidal functor whose domain and codomain are symmetric monoidal categories.

Given a monoidal endofunctor $F : \mathcal{V} \to \mathcal{V}$, a pair (A, τ_A) consisting of an object A in $\mathcal{V}_0$ and a morphism $\tau_A : A \to FA$ is called an F-coalgebra. The $\mathcal{V}_0$-morphism τ_A is called the coalgebra structure of (A, τ_A). An F-coalgebra homomorphism $f : (A, \tau_A) \to (B, \tau_B)$ is a $\mathcal{V}_0$-morphism $f : A \to B$ satisfying the condition $F(f) \circ \tau_A = \tau_B \circ f$. Also, F-coalgebra homomorphisms are stable under composition. Therefore, F-coalgebras and their homomorphisms form a category denoted by $\mathcal{V}_{0_F}$.

This paper aims to explore the transferability of a braided monoidal structure on a given category to the category of coalgebras for a braided monoidal endofunctor on this category. First, it is proved that the category $\mathcal{V}_F$ of coalgebras for a monoidal endofunctor $F : \mathcal{V} \to \mathcal{V}$ is monoidal. Then assuming that F is braided monoidal, it is proved that the category $\mathcal{V}_F$

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is braided monoidal. As a consequence, the category of $\mathcal{V}_F$ is symmetric monoidal provided that F is symmetric monoidal. Particularly, if the category $\mathcal{V}$ admits coproducts and if F is a fully faithful symmetric monoidal endofunctor then it is proved that $\mathcal{V}_F$ is symmetric monoidal closed whenever $\mathcal{V}$ is symmetric monoidal closed.

2. Preliminaries

In this section, several categorical concepts including definitions and basic properties are recalled, which will be used in the rest of this paper.

Monoidal categories

A *monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$ consists of a category $\mathcal{V}_0$, a bifunctor $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$ called the tensor product of $\mathcal{V}$, an object I of $\mathcal{V}_0$ called the unit and natural isomorphisms $a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : I \otimes A \to A$, $r_A : A \otimes I \to A$, subject to two coherence axioms expressing the commutativity of the following diagrams:

$$\begin{array}{ccccc}
((A \otimes B \otimes C) \otimes D) & \xrightarrow{a} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a} & A \otimes (B \otimes (C \otimes D)) \\
{\scriptstyle a\otimes 1}\downarrow & & & & \uparrow{\scriptstyle 1\otimes a} \\
(A \otimes (B \otimes C)) \otimes D & & \xrightarrow[a]{\qquad\qquad} & & A \otimes ((B \otimes C) \otimes D)
\end{array}$$

and

$$\begin{array}{ccc}
(A \otimes I) \otimes B & \xrightarrow{a_{AIB}} & A \otimes (I \otimes B) \\
& {\scriptstyle r_A \otimes 1}\searrow \quad \swarrow{\scriptstyle 1\otimes l_B} & \\
& A \otimes B &
\end{array}$$

Every category with finite products and a terminal object is a monoidal category (see [2, 10]). Particularly, the category Set of sets and mappings is a monoidal category where the tensor product is the Cartesian product and the unit object is a one-element set. The category Ab of Abelian groups and their homomorphisms becomes a monoidal category when it is provided with the tensor product of Abelian groups, denoted as $\otimes_{\mathbb{Z}}$, where $\mathbb{Z}$ is the Abelian group of integers. Let K be any field. The category $Vect_K$ of vector spaces over K and K-linear mappings is a monoidal category with the tensor product being the usual tensor product of vector spaces, denoted by $\otimes_K$, and the one-dimensional vector space K as the unit (see [1, 10]). The category $Rep_K(G)$ of all representations of group G on a vector space V over a field K is a monoidal category with $\otimes$ being the tensor product of representations: if for a representation V one denotes by $\varphi_V : G \to GL(V)$ the corresponding mapping, then

$$\varphi_{V\otimes W}(g) = \varphi_V(g) \otimes \varphi_W(g).$$

The unit object in this category is the trivial representation $1 = K$. A similar statement holds for the category of finite dimensional representations of G (see [5]).

A *braided monoidal category* is a monoidal category equipped with a natural isomorphism $\lambda_{AB} : A \otimes B \to B \otimes A$, known as the *braiding*, such that the following diagrams commute for all objects involved (called the *hexagon identities*):

$$\begin{array}{ccccc}
(A \otimes B) \otimes C & \xrightarrow{a_{ABC}} & A \otimes (B \otimes C) & \xrightarrow{\lambda_{A,B\otimes C}} & (B \otimes C) \otimes A \\
{\scriptstyle \lambda_{AB}\otimes 1}\downarrow & & & & \downarrow{\scriptstyle a_{BCA}} \\
(B \otimes A) \otimes C & \xrightarrow[a_{BAC}]{} & B \otimes (A \otimes C) & \xrightarrow[1\otimes\lambda_{AC}]{} & B \otimes (C \otimes A)
\end{array}$$

and

$$\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{a^{-1}_{ABC}} & (A \otimes B) \otimes C & \xrightarrow{\lambda_{A\otimes B,C}} & C \otimes (A \otimes B) \\
{\scriptstyle 1\otimes\lambda_{BC}}\downarrow & & & & \downarrow{\scriptstyle a^{-1}_{CAB}} \\
A \otimes (C \otimes B) & \xrightarrow[a^{-1}_{ACB}]{} & (A \otimes C) \otimes B & \xrightarrow[\lambda_{AC}\otimes 1]{} & (C \otimes A) \otimes B
\end{array}$$

If $\lambda_{AB} \circ \lambda_{BA} = 1_{A\otimes B}$, then the braided monoidal category is called a *symmetric monoidal category*. A symmetry is exactly a braiding which also satisfies the property $\lambda_{AB} \circ \lambda_{BA} = 1_{A\otimes B}$; but not every braiding is a symmetry (see [7]). For illustration, let $GMod_R$ be the category of graded modules over a commutative ring R and graded homomorphisms. Then, it is a monoidal category with the tensor product given by

$$(A \otimes B)_n = \sum_{p+q=n} A_p \otimes_R B_q$$

and the unit object by the trivial graded module R (where $R_n = R$ if $n = 0$ and $R_n = 0$ otherwise). Braidings

$$\lambda_{AB} : A \otimes B \to B \otimes A$$

for this monoidal structure on $GMod_R$ are in bijection with invertible elements r of R via the formula

$$\lambda_{AB}(x \otimes y) = r^{pq}(y \otimes x) \quad \text{where } x \in A_P \text{ and } y \in B_q.$$

Symmetries are in bijection with elements r of R satisfying the condition $r^2 = 1$ (see [4]). That is, every symmetry is a braiding; obviously, the converse does not hold. The monoidal categries Set, Ab and $Vect_K$ are symmetric (see [1, 9]).

A symmetric monoidal category $\mathcal{V}$ is said to be *closed* if, for each $B \in \mathcal{V}_0$, the functor $- \otimes B : \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[B, -]$, so that we have a natural bijection

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$$

with *unit* and *counit* (the latter called evaluation) say $d_A : A \to [B, A \otimes B]$ and $ev_{B,C} : [B, C] \otimes B \to C$.

Enriched categories

An *enriched category* over a monoidal category $\mathcal{V}$ or a $\mathcal{V}$-category $\mathcal{A}$ consists of a class $ob\mathcal{A}$, a hom-object $\mathcal{A}(A, B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a composition law $C_{ABC} : \mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object; subject to the associativity axiom expressed by the commutativity of the following diagram:

$$\begin{array}{ccc}
(\mathcal{A}(A,B) \otimes \mathcal{A}(B,C)) \otimes \mathcal{A}(C,D) & \xrightarrow{C_{ABC}\otimes 1} & \mathcal{A}(A,C) \otimes \mathcal{A}(C,D) \\
\downarrow{\scriptstyle a_{\mathcal{A}(A,B),\mathcal{A}(B,C),\mathcal{A}(C,D)}} & & \\
\mathcal{A}(A,B) \otimes (\mathcal{A}(B,C) \otimes \mathcal{A}(C,D)) & & \downarrow{\scriptstyle C_{ACD}} \\
\downarrow{\scriptstyle 1\otimes C_{BCD}} & & \\
\mathcal{A}(A,B) \otimes \mathcal{A}(B,D) & \xrightarrow[C_{ABD}]{} & \mathcal{A}(A,D)
\end{array}$$

and unit axioms expressed by the identities:

$$C_{ABB} \circ (1_{\mathcal{A}(A,B)} \otimes j_B) = r_{\mathcal{A}(A,B)} \text{ and } C_{AAB} \circ (j_A \otimes 1_{\mathcal{A}(A,B)}) = l_{\mathcal{A}(A,B)}.$$

Every symmetric monoidal closed $\mathcal{V}$ can be provided with the structure of a $\mathcal{V}$-category (see [2, 9]). More precisely, there is a $\mathcal{V}$-category, whose objects are those of $\mathcal{V}_0$, and whose hom-object $\mathcal{V}(A, B)$ is $[A, B]$. Its composition law

$$C_{ABC} : [A, B] \otimes [B, C] \to [A, C]$$

corresponds under adjunction to the composite morphism

$$\begin{array}{c}
([A,B] \otimes [B,C]) \otimes A \\
\downarrow{\scriptstyle s_{[A,B],[B,C]}\otimes 1} \\
([B,C] \otimes [A,B]) \otimes A \\
\downarrow{\scriptstyle a_{[B,C],[A,B],A}} \\
[B,C] \otimes ([A,B] \otimes A) \\
\downarrow{\scriptstyle 1\otimes ev_{AB}} \\
[B,C] \otimes B \\
\downarrow{\scriptstyle ev_{BC}} \\
C
\end{array}$$

and its identity element $j_A : I \to [A, A]$ corresponds under adjunction to $l_A : I \otimes A \to A$.

Monoidal functors

A *monoidal functor* $F : \mathcal{V} \to \mathcal{W}$ between monoidal categories consists of a functor $F : \mathcal{V}_0 \to \mathcal{W}_0$ together with a morphism $\phi_{AB} : FA \otimes FB \to F(A \otimes B)$ for every pair A, B of objects of $\mathcal{V}_0$, which is natural in A, B, and a morphism $\phi_o : J \to FI$ of

$\mathcal{W}$, where I stands for the unit of $\mathcal{V}$ and J is the unit of $\mathcal{W}$. These data must satisfy the associativity condition expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} (FA \otimes FB) \otimes FC & \xrightarrow{a_{FAFBFC}} & FA \otimes (FB \otimes FC) \\ \downarrow{\scriptstyle \phi_{AB}\otimes 1} & & \downarrow{\scriptstyle 1\otimes\phi_{BC}} \\ F(A \otimes B) \otimes FC & & FA \otimes (FB \otimes FC) \\ \downarrow{\scriptstyle \phi_{A\otimes B,C}} & & \downarrow{\scriptstyle \phi_{A,B\otimes C}} \\ F((A \otimes B) \otimes C) & \xrightarrow[F(a_{ABC})]{} & F(A \otimes (B \otimes C)) \end{array}$$

and the unit conditions expressed by the commutativity of the following diagrams:

$$\begin{array}{ccc} J \otimes FA & & \\ \downarrow{\scriptstyle \phi_o\otimes 1} & \searrow{\scriptstyle l_{FA}} & \\ FI \otimes FA & & \\ \downarrow{\scriptstyle \phi_{IA}} & & \\ F(I \otimes A) & \xrightarrow[F(l_A)]{} & FA \end{array}$$

and

$$\begin{array}{ccc} FA \otimes J & & \\ \downarrow{\scriptstyle 1\otimes\phi_o} & \searrow{\scriptstyle r_{FA}} & \\ FA \otimes FI & & \\ \downarrow{\scriptstyle \phi_{AI}} & & \\ F(A \otimes I) & \xrightarrow[F(r_A)]{} & FA \end{array}$$

For instance, the forgetful functor $U : (Ab, \otimes_{\mathbb{Z}}, \mathbb{Z}) \to (Set, \times, \{\star\})$ from the category of Abelian groups to the category of sets is monoidal. In this case, the mapping $\phi_{AB} : U(A) \times U(B) \to U(A \otimes_{\mathbb{Z}} B)$ sends (a, b) to $a \otimes_{\mathbb{Z}} b$ and the mapping $\phi_o : \{\star\} \to \mathbb{Z}$ sends $\star$ to 1. Modules over a commutative ring R and their homomorphisms form a monoidal category (see [1, 2]). The monoidal product is given by the tensor product of modules and the unit object of this category is R. If $f : R \to S$ is a commutative ring homomorphism, then the restriction functor $(Mod_S, \otimes_S, S) \to (Mod_R, \otimes_R, R)$ is monoidal (see [3]).

Definition 2.1. *A braided monoidal functor is a monoidal functor between braided categories such that the following diagram commutes:*

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\lambda_{FAFB}} & FB \otimes FA \\ \downarrow{\scriptstyle \phi_{AB}} & & \downarrow{\scriptstyle \phi_{BA}} \\ F(A \otimes B) & \xrightarrow[F(\lambda_{AB})]{} & F(B \otimes A) \end{array}$$

Notice that every monoidal functor which preserves the tensor product is braided monoidal. In particular, every Set-endofunctor preserving finite products is braided monoidal.

Definition 2.2. *A symmetric monoidal functor is a braided monoidal functor whose domain and codomain are symmetric monoidal categories.*

For instance, the functor $(Set, \times, \{\star\}) \to (Vect_K, \otimes_K, K)$ which takes a set to a vector space by taking the elements of the set as a basis is symmetric monoidal. Also, the forgetful functor $U : (Ab, \otimes_{\mathbb{Z}}, \mathbb{Z}) \to (Set, \times, \{\star\})$ is symmetric monoidal.

Proposition 2.1 (see [2])**.** *Let $F : \mathcal{V} \to \mathcal{W}$ be a symmetric monoidal functor, where $\mathcal{V}$ and $\mathcal{W}$ are in fact symmetric monoidal closed categories. There exist morphisms in $\mathcal{W}$*

$$\sigma_{AB} : F[A, B] \longrightarrow [FA, FB]$$

for every pair $A, B,$ of objects of $\mathcal{V}$ and these morphisms satisfy the following conditions:

(1). *The morphisms σ_{AB} are natural in A and B.*

(2). *The diagram*

$$\begin{CD} F[[A,B]\otimes[B,C]] @>F(C_{ABC})>> F[A,C] \\ @A\phi_{[A,B],[B,C]}AA @. \\ F[A,B]\otimes F[B,C] @. @VV\sigma_{AC}V \\ @V\sigma_{AB}\otimes\sigma_{BC}VV @. \\ [FA,FB]\otimes[FB,FC] @>>C_{FAFBFC}> [FA,FC] \end{CD}$$

commutes for all objects A, B, C *of* $\mathcal{V}$.

(3). *The diagram*

$$\begin{CD} FI @>Fj_A>> F[A,A] \\ @A\phi_oAA @VV\sigma_{AA}V \\ J @>>j_{FA}> [FA,FA] \end{CD}$$

commutes for every object A *of* $\mathcal{V}$.

3. Coalgebras for a braided monoidal functor

Let F denote a monoidal endofunctor on a monoidal category $\mathcal{V}$. A pair (A,τ_A) consisting of an object A in $\mathcal{V}_0$ and a morphism $\tau_A : A \to FA$ is called an F-coalgebra. We call τ_A the coalgebra structure of (A,τ_A). Given F-coalgebras (A,τ_A) and (B,τ_B), by a homomorphism we mean a $\mathcal{V}_0$-morphism $f : A \to B$ for which the following diagram commutes:

$$\begin{CD} A @>f>> B \\ @V\tau_AVV @VV\tau_BV \\ FA @>>F(f)> FB \end{CD}$$

This definition turns the class of F-coalgebras and their homomorphisms into a category, denoted as $\mathcal{V}_{0_F}$.

Proposition 3.1. *Let* F *be a monoidal endofunctor on a monoidal category* $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$. *The category* $\mathcal{V}_F = (\mathcal{V}_{0_F}, \otimes_F, I, a, l, r)$ *of* F*-coalgebras is monoidal.*

Proof. Consider the correspondence $\otimes_F : \mathcal{V}_{0_F} \times \mathcal{V}_{0_F} \to \mathcal{V}_{0_F}$ defined on each pair of F-coalgebras (A,τ_A) and (B,τ_B) as

$$(A,\tau_A) \otimes_F (B,\tau_B) = (A \otimes B, \phi_{AB} \circ (\tau_A \otimes \tau_B)).$$

Also, $\otimes_F$ extends to a bifunctor as $\otimes$ is bifunctor and the morphism $\phi_{AB} : FA \otimes FB \to F(A \otimes B)$ is natural in A and B. Take the unit to be the pair $(I, \phi_o : I \to FI)$. First, we show that a, l, and r are natural F-coalgebra isomorphisms. Since a, l, and r are natural isomorphisms in $\mathcal{V}_0$, it suffices to prove that they are homomorphisms: a $\mathcal{V}_0$-isomorphism is an F-coalgebra isomorphism if and only if it is a homomorphism (see [6]). Moreover a natural $\mathcal{V}_0$-morphism is a natural homomorphism provided that it is a homomorphism.

Given F-coalgebras (A,τ_A), (B,τ_B), and (C,τ_C), the $\mathcal{V}_0$-objects $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ are respectively equipped with the coalgebra structures:

$$\begin{CD} (A\otimes B)\otimes C \\ @V(\tau_A\otimes\tau_B)\otimes\tau_CVV \\ (FA\otimes FB)\otimes FC \\ @V\phi_{AB}\otimes 1VV \\ F(A\otimes B)\otimes FC \\ @V\phi_{A\otimes B,C}VV \\ F((A\otimes B)\otimes C) \end{CD}$$

and

$$\begin{array}{c}
A\otimes(B\otimes C)\\
\Big\downarrow{\scriptstyle \tau_A\otimes(\tau_B\otimes\tau_C)}\\
FA\otimes(FB\otimes FC)\\
\Big\downarrow{\scriptstyle 1\otimes\phi_{BC}}\\
FA\otimes F(B\otimes C)\\
\Big\downarrow{\scriptstyle \phi_{A,B\otimes C}}\\
F(A\otimes(B\otimes C))
\end{array}$$

Besides the equality $\tau_A\otimes(\tau_B\otimes\tau_C)\circ a_{ABC}=a_{FAFBFC}\circ(\tau_A\otimes\tau_B)\otimes\tau_C$ holds by naturality of a. The associativity condition yields that the following diagram commutes:

$$\begin{array}{ccc}
(A\otimes B)\otimes C & \xrightarrow{a_{ABC}} & A\otimes(B\otimes C)\\
\Big\downarrow{\scriptstyle (\tau_A\otimes\tau_B)\otimes\tau_C} & & \Big\downarrow{\scriptstyle \tau_A\otimes(\tau_B\otimes\tau_C)}\\
(FA\otimes FB)\otimes FC & \xrightarrow{a_{FAFBFC}} & FA\otimes(FB\otimes FC)\\
\Big\downarrow{\scriptstyle \phi_{AB}\otimes 1} & & \Big\downarrow{\scriptstyle 1\otimes\phi_{BC}}\\
F(A\otimes B)\otimes FC & & FA\otimes F(B\otimes C)\\
\Big\downarrow{\scriptstyle \phi_{A\otimes B,C}} & & \Big\downarrow{\scriptstyle \phi_{A,B\otimes C}}\\
F((A\otimes B)\otimes C) & \xrightarrow[F(a_{ABC})]{} & F(A\otimes(B\otimes C))
\end{array}$$

This proves that a_{ABC} is a homomorphism. Also, l_A is a homomorphism for every F-coalgebra (A,τ_A) because the following diagram commutes due to both the left unit condition and the naturality of l:

$$\begin{array}{ccc}
I\otimes A & \xrightarrow{l_A} & A\\
\Big\downarrow{\scriptstyle 1\otimes\tau_A} & & \Big|\\
I\otimes FA & & \Big|\\
\Big\downarrow{\scriptstyle \phi_o\otimes 1} & \searrow{\scriptstyle l_{FA}} & \Big\downarrow{\scriptstyle \tau_A}\\
FI\otimes FA & & \Big|\\
\Big\downarrow{\scriptstyle \phi_{IA}} & & \Big\downarrow\\
F(I\otimes A) & \xrightarrow[F(l_A)]{} & FA
\end{array}$$

Similarly, one proves that r_A is a homomorphism. Therefore a, l, and r are natural F-coalgebra isomorphisms. It is not difficult to check the coherence conditions. Thus, $\mathcal{V}_F=(\mathcal{V}_{0_F},\otimes_F,I,a,l,r)$ is a monoidal category. □

Corollary 3.1. *Let F be a braided monoidal endofunctor on a braided monoidal category $\mathcal{V}=(\mathcal{V}_0,\otimes,I,a,l,r)$. The category $\mathcal{V}_F=(\mathcal{V}_{0_F},\otimes_F,I,a,l,r)$ of F-coalgebras is braided monoidal.*

Proof. First, the category $\mathcal{V}_F$ is monoidal due to Proposition 3.1. Next, we prove that the category $\mathcal{V}_F$ has braidings. Let (A,τ_A) and (B,τ_B) be F-coalgebras. From the naturality of the braiding $\lambda_{AB}:A\otimes B\to B\otimes A$, it follows that

$$\lambda_{FAFB}\circ(\tau_A\otimes\tau_B)=(\tau_B\otimes\tau_A)\circ\lambda_{AB}.$$

In addition, $\phi_{BA}\circ\lambda_{FAFB}=F(\lambda_{AB})\circ\phi_{AB}$ because F is a braided monoidal endofunctor. As a result, the following diagram commutes:

$$\begin{array}{ccc}
A\otimes B & \xrightarrow{\lambda_{AB}} & B\otimes A\\
\Big\downarrow{\scriptstyle \tau_A\otimes\tau_B} & & \Big\downarrow{\scriptstyle \tau_B\otimes\tau_A}\\
FA\otimes FB & \xrightarrow{\lambda_{FAFB}} & FB\otimes FA\\
\Big\downarrow{\scriptstyle \phi_{AB}} & & \Big\downarrow{\scriptstyle \phi_{BA}}\\
F(A\otimes B) & \xrightarrow[F(\lambda_{AB})]{} & F(B\otimes A)
\end{array}$$

Hence, λ_{AB} is a homomorphism, that is a natural F-coalgebra isomorphism because it is a natural $\mathcal{V}_0$-isomorphism by definition. Therefore, λ_{AB} is a braiding for the category $\mathcal{V}_F$. Consequently, the category $\mathcal{V}_F = (\mathcal{V}_{0_F}, \otimes_F, I, a, l, r)$ of F-coalgebras is braided monoidal. □

Example 3.1. *Let K be a field and let M be a monoid. Denote by Vec_M the category of M-graded finite dimensional vector spaces over K consisting of vector spaces with a decomposition*

$$V = \bigoplus_{m \in M} V_m,$$

where the morphisms are linear mappings which preserves the grading. It is a monoidal category with the tensor product defined by

$$(V \otimes W)_m = \bigoplus_{x, x' \in M : xx' = m} V_x \otimes W_{x'}$$

and the unit object given by

$$1_m = \begin{cases} K, & \text{if } m = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Let M be a monoid and let $f : M \to M$ be a monoid homomorphism. We then have a functor

$$F : Vec_M \to Vec_M$$

defined on objects as follows: for an M-graded finite dimensional vector space,

$$F(V) = F\left(\bigoplus_{m \in M} V_m\right) = \bigoplus_{f(m) : m \in M} V_{f(m)}$$

and for each linear mapping $q : V \to W$,

$$F(q)\left(\bigoplus_{f(m) : m \in M} v_{f(m)}\right) = \bigoplus_{f(m) : m \in M} q\left(v_{f(m)}\right)$$

The endofunctor F is monoidal; see [10]. Hence, the category of F-coalgebras is monoidal by Proposition 3.1.

Example 3.2. *Given a commutative ring R, the homology functor is braided monoidal as*

$$H_\star : (GMod_R, \otimes, R) \to (GMod_R, \otimes, R)$$

via the mapping $H_\star(C_1) \otimes H_\star(C_2) \to H_\star(C_1 \otimes C_2)$; $[x_1] \otimes [x_2] \mapsto [x_1 \otimes x_2]$; see [8]. Then, the category of $H_\star$-coalgebras is braided monoidal due to Corollary 3.1.

Recall that every symmetric monoidal category is braided monoidal.

Corollary 3.2. *Let F be a braided monoidal endofunctor on a symmetric monoidal category $\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)$. The category $\mathcal{V}_F = (\mathcal{V}_{0_F}, \otimes_F, I, a, l, r)$ of F-coalgebras is symmetric monoidal.*

Proof. By Corollary 3.1, the category $\mathcal{V}_F$ is braided monoidal. Then, for any F-coalgebras (A, τ_A) and (B, τ_B), the braidings λ_{AB} and λ_{BA} are homomorphisms. Also, $\lambda_{AB} \circ \lambda_{BA} = 1_{A \otimes B}$ as the category $\mathcal{V}$ is symmetric monoidal. Hence, the category $\mathcal{V}_F$ of F-coalgebras is symmetric monoidal. □

Example 3.3. *Let $F : (Set, \times, \{\star\}) \to (Set, \times, \{\star\})$ be a Set-endofunctor which preserves finite products. Then F is braided monoidal. But, Set is a symmetric monoidal category. As a consequence of Corollary 3.1, the category of F-coalgebras is symmetric monoidal, that is braided monoidal.*

More generally, the category of coalgebras for a braided monoidal endofunctor F is braided monoidal provided that F preserves the tensor product.

Example 3.4. *Consider the covariant power set functor*

$$\mathcal{P} : (Set, \times, \{\star\}) \to (Set, \times, \{\star\}),$$

which maps every set to its power set and every function $f : A \to B$ to the mapping $\mathcal{P}(f)$, which sends $U \in \mathcal{P}(A)$ to its image $f(U) \in \mathcal{P}(B)$. It is a symmetric monoidal functor. The coherence maps are the mapping $\phi_o : \{\star\} \to \mathcal{P}(\{\star\})$ which sends $\star$ to $\{\star\}$ and, the mapping $\phi_{AB} : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \times B)$ which sends (U, V) to $U \times V$. Also, the symmetry condition holds. Therefore, the category of $\mathcal{P}$-coalgebras is symmetric monoidal due to Corollary 3.2.

Every category $\mathcal{V}$ admitting finite coproducts is a symmetric monoidal category. The tensor product is defined for every pair $A, B,$ of objects of $\mathcal{V}$ by

$$A \otimes B = A \coprod B$$

and the unit I as the initial object of $\mathcal{V}$. If an endofunctor $F : \mathcal{V} \to \mathcal{V}$ is given, then F is a symmetric monoidal functor. It suffices to define the coherence maps $\phi_o : I \to FI$ as the unique arrow from I to FI and for every pair $A, B,$ of objects of $\mathcal{V}$,

$$\phi_{AB} : FA \coprod FB \to F\left(A \coprod B\right)$$

as the unique arrow arising from the universality of the coproduct. Corollary 3.2 implies that $\mathcal{V}_F$ is a symmetric monoidal category.

Lemma 3.1. *Suppose that $\mathcal{V}$ is a symmetric monoidal closed category admitting coproducts. Let $F : \mathcal{V} \to \mathcal{V}$ be a fully faithful symmetric monoidal endofunctor. For every pair $A, B,$ of objects of $\mathcal{V}$, the morphisms*

$$\sigma_{AB} : F[A, B] \longrightarrow [FA, FB]$$

are invertible.

Proof. For every pair $A, B,$ of objects of $\mathcal{V}_0$, consider the mapping

$$\varphi : \mathcal{V}_0(I, F[A, B]) \longrightarrow \mathcal{V}_0(I, [FA, FB]) : u \mapsto \sigma_{AB} \circ u$$

Let $v : I \to [FA, FB]$ be a $\mathcal{V}_0$-morphism. Since F is full, there is a $\mathcal{V}_0$-morphism $w : I \to [A, B]$ such that

$$\sigma_{AB} \circ (F(w) \circ \phi_o) = v.$$

Also, the composite $F(w) \circ \phi_o$ is the only $\mathcal{V}_0$-morphism with this property as F is faithful. Thus, one deduces a mapping

$$\psi : \mathcal{V}_0(I, [FA, FB]) \longrightarrow \mathcal{V}_0(I, F[A, B]) : v \mapsto F(w) \circ \phi_o.$$

We say that φ and ψ are inverse of each other. Indeed, for every $\mathcal{V}_0$-morphism $u : I \to F[A, B]$, we have that

$$(\psi \circ \varphi)(u) = \psi(\varphi(u)) = \psi(\sigma_{AB} \circ u) = u.$$

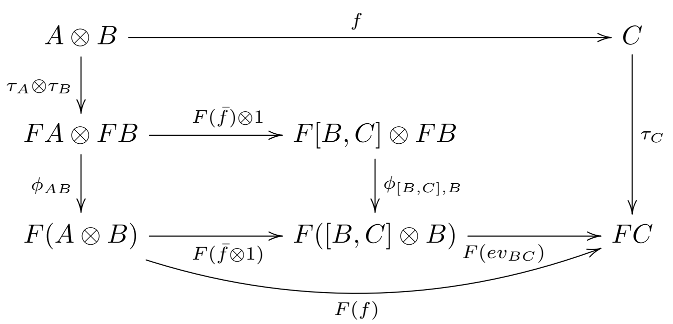
Conversely, for every $\mathcal{V}_0$-morphsm $v : I \to [FA, FB]$, we have that

$$(\varphi \circ \psi)(v) = \varphi(\psi(v)) = \varphi(F(w) \circ \phi_o) = \sigma_{AB} \circ (F(w) \circ \phi_o) = v.$$

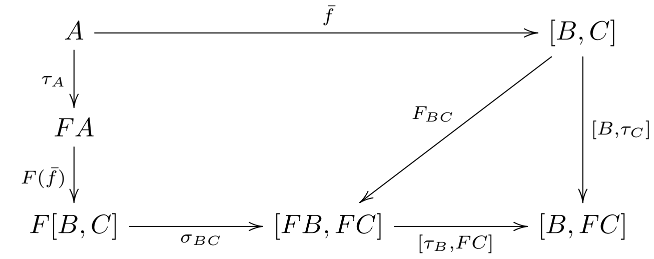
As a result, the mappings φ and ψ are inverse of each other. Subsequently, σ_{AB} is invertible as the functor $\mathcal{V}_0(I, -) : \mathcal{V} \to Set$ has a left adjoint (see [2]). □

Proposition 3.2. *Let $\mathcal{V}$ be a symmetric monoidal closed category admitting coproducts. Let $F : \mathcal{V} \to \mathcal{V}$ be a fully faithful symmetric monoidal endofunctor. The category $\mathcal{V}_F$ is symmetric monoidal closed.*

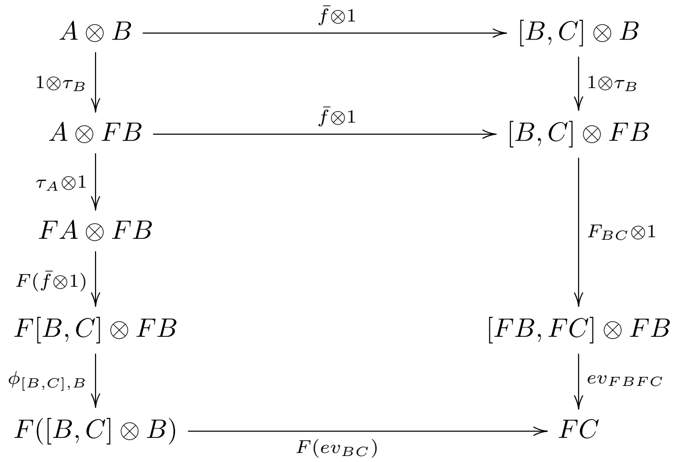
Proof. By Corollary 3.2, the category $\mathcal{V}_F$ is symmetric monoidal. Next, we prove that $\mathcal{V}_F$ is also closed. For given F-coalgebras (A, τ_A), (B, τ_B) and (C, τ_C), let $f : (A, \tau_A) \otimes_F (B, \tau_B) \to (C, \tau_C)$ be a homomorphism. Since $\mathcal{V}$ is closed, the $\mathcal{V}_0$-morphism $f : A \otimes B \to C$ corresponds under adjunction with an arrow $\bar{f} : A \to [B, C]$. Also, ϕ_{AB} is natural in A, B. Then the following commutative diagram:



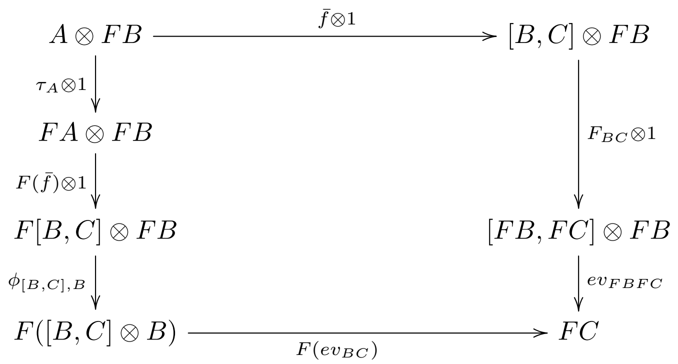
corresponds under adjunction with the following commutative diagram:



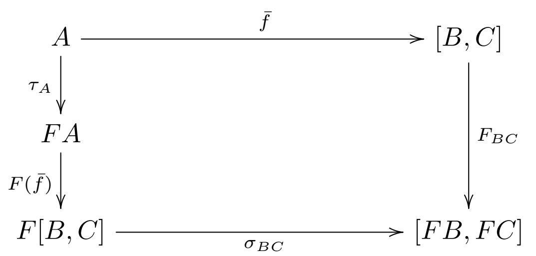
which in turn corresponds under adjunction with the following commutative diagram:



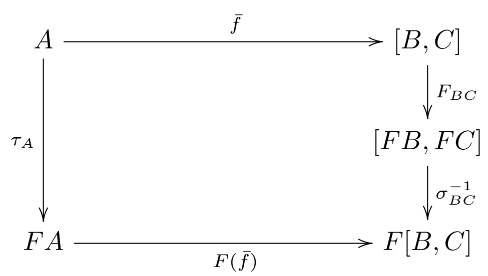
As $\otimes$ is bifunctor, the following diagram commutes:



and it corresponds under adjunction with the following commutative diagram:



Furthermore, σ_{BC} is invertible due to Lemma 3.1. One therefore deduces that the following diagram commutes:



Hence, the $\mathcal{V}_0$-morphism $\bar{f}$ is homomorphism. Denote by $[(B,\tau_B),(C,\tau_C)]_F$ the pair $\big([B,C],\sigma_{BC}^{-1}\circ F_{BC}\big)$. We then have a natural bijection:

$$\mathcal{V}_{0_F}((A,\tau_A)\otimes_F(B,\tau_B),(C,\tau_C))\cong\mathcal{V}_{0_F}((A,\tau_A),[(B,\tau_B),(C,\tau_C)]_F)$$

That is, the functor $-\otimes_F(B,\tau_B):\mathcal{V}_{0_F}\to\mathcal{V}_{0_F}$ has a right adjoint $[(B,\tau_B),-]_F$. Consequently, $\mathcal{V}_F$ is closed. □

References

[1] D. Balacu, S. Caenepeel, F. Panaite, F. V. Oystaeyen, *Quasi-Hopf Algebras: A Categorical Approach*, Cambridge University Press, Cambridge, 2019.

[2] F. Borceux, *Handbook of Categorical Algebra*, Encyclopedia of Mathematics and its Applications **2**, Cambridge University Press, Cambridge, 1994.

[3] D. Dummit, R. Foote, *Abstract Algebra*, 3rd Edition, Wiley, Hoboken, 2004.

[4] S. Eilenberg, G. M. Kelly, *Closed categories*, In: S. Eilenberg, D. K. Harrison, S. MacLane, H. Röhrl (Eds.), *Proceedings of the Conference on Categorical Algebra*, Springer, Berlin, 1966, 421–562.

[5] P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik, *Tensor Categories*, Mathematcal Surveys and Monographs **205**, American Mathematical Society, Providence, 2015.

[6] H. P. Gumm, *Elements of the General Theory of Coalgebras*, LUATCS 99, Rand Afrikaans University, Johanesbourg, 1999.

[7] A. Joyal, R. Street, *Braided Monoidal Categories*, Macquarie Mathematics Reports, Macquarie University, North Ryde, 1986, #860081.

[8] G. M. Kelly, Doctrinal adjunction, In: G. M. Kelly (Ed.), *Category Seminar*, Lecture Notes in Mathematics **420**, Springer, Berlin, 1974, 257–280.

[9] G. M. Kelly, Basic concepts of enriched category theory, *Repr. Theory Appl. Categ.* **10** (2005) 1–136.

[10] I. Matsson, *Algebra in Monoidal Categories*, U.U.D.M. Project Report, Uppsala University, Uppsala, 2021, #17.