Braided monoidal categories of coalgebras

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Abstract

Let \( V \) be a braided monoidal category. Given a braided monoidal endofunctor \( F \) on \( V \), it is proved that \( F \)-coalgebras form a braided monoidal category, denoted as \( V_F \). Particularly, if the category \( V \) admits coproducts and if \( F \) is a fully faithful symmetric monoidal endofunctor, then it is proved that \( V_F \) is symmetric monoidal closed whenever \( V \) is symmetric monoidal closed.

Keywords: braided monoidal category; coalgebra; monoidal functor.

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1. Introduction

A braided monoidal category is a monoidal category equipped with a natural isomorphism \( \lambda_{AB} : A \otimes B \rightarrow B \otimes A \), known as the braiding. If \( \lambda_{AB} \circ \lambda_{BA} = 1_{A \otimes B} \), then the braided monoidal category is called a symmetric monoidal category. So, every symmetric monoidal category is braided. On the other hand, not every braided monoidal category is symmetric. For instance, the category of graded modules over a commutative ring is braided for the usual monoidal structure on it; however, this monoidal category is not symmetric (see [4]). A symmetric monoidal category \( V \) is said to be closed if, for each \( B \in \mathcal{V}_0 \), the functor \( \ast \otimes - \) has a right adjoint \( [B, \ast] \).

The notion of braiding occurs naturally in the theoretical context of further multiplication on a monoidal category, has an adequate coherence theorem, and, is as versatile as symmetry [7]. For any monoidal category \( V \), an extra multiplication on \( V \) leads to a braiding; also, each braiding leads to a multiplication. This gives a natural explanation for braidings as opposed to symmetries. Important examples of braidings can be found especially in homotopy and cohomology theories.

A monoidal functor \( F : V \rightarrow W \) between monoidal categories consists of an object \( F(A) \) and a morphism \( F(f) : F(A) \otimes F(B) \rightarrow F(A \otimes B) \) for every pair \( A, B \) of objects of \( V \), which is natural in \( A, B \), and a morphism \( \Phi_o : I \rightarrow FI \) of \( W \), where \( I \) is the unit of \( V \) and \( J \) is the unit of \( W \); all these objects must satisfy the associativity and unit conditions. A braided monoidal functor is a monoidal functor between braided categories together with the coherence condition expressed by the commutativity of the following diagram:

\[
\begin{array}{ccc}
FA \otimes FB & \xrightarrow{F(A \otimes B)} & FB \otimes FA \\
\Phi_{AB} & & \Phi_{BA} \\
F(A \otimes B) & \xrightarrow{F(\lambda_{AB})} & F(B \otimes A)
\end{array}
\]

Particularly, a symmetric monoidal functor is a braided monoidal functor whose domain and codomain are symmetric monoidal categories.

Given a monoidal endofunctor \( F : V \rightarrow V \), a pair \( (A, \tau_A) \) consisting of an object \( A \) in \( V_0 \) and a morphism \( \tau_A : A \rightarrow FA \) is called an \( F \)-coalgebra. The \( V_0 \)-morphism \( \tau_A \) is called the coalgebra structure of \( (A, \tau_A) \). An \( F \)-coalgebra homomorphism \( f : (A, \tau_A) \rightarrow (B, \tau_B) \) is a \( V_0 \)-morphism \( f : A \rightarrow B \) satisfying the condition \( F(f) \circ \tau_A = \tau_B \circ f \). Also, \( F \)-coalgebra homomorphisms are stable under composition. Therefore, \( F \)-coalgebras and their homomorphisms form a category denoted by \( V_{0_F} \).

This paper aims to explore the transferability of a braided monoidal structure on a given category to the category of coalgebras for a braided monoidal endofunctor on this category. First, it is proved that the category \( V_F \) of coalgebras for a monoidal endofunctor \( F : V \rightarrow V \) is monoidal. Then assuming that \( F \) is braided monoidal, it is proved that the category \( V_F \)...
is braided monoidal. As a consequence, the category of $\mathcal{V}_F$ is symmetric monoidal provided that $F$ is symmetric monoidal. Particularly, if the category $\mathcal{V}$ admits coproducts and if $F$ is a fully faithful symmetric monoidal endofunctor then it is proved that $\mathcal{V}_F$ is symmetric monoidal closed whenever $\mathcal{V}$ is symmetric monoidal closed.

2. Preliminaries

In this section, several categorical concepts including definitions and basic properties are recalled, which will be used in the rest of this paper.

Monoidal categories

A monoidal category $\mathcal{V} = (V_0, \otimes, I, a, l, r)$ consists of a category $V_0$, a bifunctor $\otimes : V_0 \times V_0 \to V_0$ called the tensor product of $V$, an object $I$ of $V_0$ called the unit and natural isomorphisms $a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $l_A : I \otimes A \to A$, $r_A : A \otimes I \to A$, subject to two coherence axioms expressing the commutativity of the following diagrams:

$$
\begin{align*}
(A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{ABC} \otimes 1} A \otimes ((B \otimes C) \otimes D) \\
& \xrightarrow{a} A \otimes (B \otimes (C \otimes D))
\end{align*}
$$

and

$$
\begin{align*}
(A \otimes I) \otimes B & \xrightarrow{a_{AB} \otimes 1} A \otimes (I \otimes B) \\
& \xrightarrow{\lambda} A \otimes (I \otimes B)
\end{align*}
$$

Every category with finite products and a terminal object is a monoidal category (see [2, 10]). Particularly, the category $\mathcal{V}$ of Abelian groups and their homomorphisms becomes a monoidal category when it is provided with the tensor product of representations: if for a representation $A$ of vector spaces over a field $K$, an object $\mathcal{V}$ is a monoidal category equipped with a natural isomorphism $\lambda_{AB} : A \otimes B \to B \otimes A$, known as the braiding, such that the following diagrams commute for all objects involved (called the hexagon identities):

$$
\begin{align*}
(A \otimes B) \otimes C & \xrightarrow{a_{ABC}} A \otimes (B \otimes C) \xrightarrow{\lambda_{ABA}} (B \otimes C) \otimes A \\
& \xrightarrow{\lambda_{BCA}} (B \otimes C) \otimes A
\end{align*}
$$

and

$$
\begin{align*}
A \otimes (B \otimes C) & \xrightarrow{a_{BAC}^{-1}} (A \otimes B) \otimes C \xrightarrow{\lambda_{ABC} \otimes 1} C \otimes (A \otimes B) \\
& \xrightarrow{\lambda_{AC} \otimes 1} C \otimes (A \otimes B)
\end{align*}
$$

If $\lambda_{AB} \circ \lambda_{BA} = 1_{\otimes B}$, then the braided monoidal category is called a symmetric monoidal category. A symmetry is exactly a braiding which also satisfies the property $\lambda_{AB} \circ \lambda_{BA} = 1_{\otimes B}$; but not every braiding is a symmetry (see [7]). For illustration, let $\mathcal{G}Mod_R$ be the category of graded modules over a commutative ring $R$ and graded homomorphisms. Then, it is a monoidal category with the tensor product given by

$$(A \otimes B)_n = \sum_{p+q=n} A_p \otimes_R B_q$$
and the unit object by the trivial graded module $R$ (where $R_n = R$ if $n = 0$ and $R_n = 0$ otherwise). Braiding

$$\lambda_{AB} : A \otimes B \to B \otimes A$$

for this monoidal structure on $GMod_R$ are in bijection with invertible elements $r$ of $R$ via the formula

$$\lambda_{AB}(x \otimes y) = r^{pq}(y \otimes x) \quad \text{where} \quad x \in A_p \text{ and } y \in B_q.$$

Symmetries are in bijection with elements $r$ of $R$ satisfying the condition $r^2 = 1$ (see [4]). That is, every symmetry is a braiding; obviously, the converse does not hold. The monoidal categories $Set$, $Ab$ and $Vect_K$ are symmetric (see [1,9]).

A symmetric monoidal category $\mathcal{V}$ is said to be closed if, for each $B \in \mathcal{V}_0$, the functor $- \otimes B : \mathcal{V}_0 \to \mathcal{V}_0$ has a right adjoint $[B, -]$, so that we have a natural bijection

$$\mathcal{V}_0(A \otimes B, C) \cong \mathcal{V}_0(A, [B, C])$$

with unit and counit (the latter called evaluation) say $d_A : A \to [B, A \otimes B]$ and $ev_{B,C} : [B, C] \otimes B \to C$.


## Enriched categories

An enriched category over a monoidal category $\mathcal{V}$ or a $\mathcal{V}$-category $\mathcal{A}$ consists of a class $ob\mathcal{A}$, a hom-object $\mathcal{A}(A, B) \in \mathcal{V}_0$ for each pair of objects of $\mathcal{A}$, a composition law $C_{ABC} : \mathcal{A}(A, B) \otimes \mathcal{A}(B, C) \to \mathcal{A}(A, C)$ for each triple of objects, and an identity element $j_A : I \to \mathcal{A}(A, A)$ for each object; subject to the associativity axiom expressed by the commutativity of the following diagram:

$$\begin{array}{ccc}
C_{ABC} & \to & \mathcal{A}(A, C) \otimes \mathcal{A}(C, D) \\
\downarrow C_{ACD} & & \downarrow C_{ACD} \\
\mathcal{A}(A, B) \otimes (\mathcal{A}(B, C) \otimes \mathcal{A}(C, D)) & \xrightarrow{1 \otimes C_{BCD}} & \mathcal{A}(A, B) \otimes \mathcal{A}(B, D) \\
\downarrow 1 \otimes C_{BCD} & & \downarrow C_{ABD} \\
\mathcal{A}(A, B) \otimes \mathcal{A}(B, D) & \xrightarrow{C_{ABD}} & \mathcal{A}(A, D)
\end{array}$$

and unit axioms expressed by the identities:

$$C_{ABB} \circ (1_{\mathcal{A}(A,B)} \otimes j_B) = r_{\mathcal{A}(A,B)} \quad \text{and} \quad C_{AAB} \circ (j_A \otimes 1_{\mathcal{A}(A,B)}) = l_{\mathcal{A}(A,B)}.$$

Every symmetric monoidal closed $\mathcal{V}$ can be provided with the structure of a $\mathcal{V}$-category (see [2,9]). More precisely, there is a $\mathcal{V}$-category, whose objects are those of $\mathcal{V}_0$, and whose hom-object $\mathcal{V}(A, B)$ is $[A, B]$. Its composition law

$$C_{ABC} : [A, B] \otimes [B, C] \to [A, C]$$

corresponds under adjunction to the composite morphism

$$\begin{array}{ccc}
([A, B] \otimes [B, C]) \otimes A & \xrightarrow{s_{[A,B],[B,C]} \otimes 1} & ([B, C] \otimes [A, B]) \otimes A \\
\downarrow q_{[B,C],[A,B],A} & & \downarrow q_{[B,C],[A,B],A} \\
[B, C] \otimes ([A, B] \otimes A) & \xrightarrow{1 \otimes ev_{AB}} & [B, C] \otimes B \\
\downarrow 1 \otimes ev_{AB} & & \downarrow ev_{BC} \\
C & \xrightarrow{ev_{BC}} & C
\end{array}$$

and its identity element $j_A : I \to [A, A]$ corresponds under adjunction to $l_A : I \otimes A \to A$.


## Monoidal functors

A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ between monoidal categories consists of a functor $F : \mathcal{V}_0 \to \mathcal{W}_0$ together with a morphism $\phi_{AB} : FA \otimes FB \to F(A \otimes B)$ for every pair $A, B$ of objects of $\mathcal{V}_0$, which is natural in $A, B$, and a morphism $\phi_0 : J \to FI$ of
\( W \), where \( I \) stands for the unit of \( V \) and \( J \) is the unit of \( W \). These data must satisfy the associativity condition expressed by the commutativity of the following diagram:

\[
\begin{array}{cccc}
(F_A \otimes FB) \otimes FC & \xrightarrow{\alpha_{FABFC}} & FA \otimes (FB \otimes FC) \\
\phi_{AB} \otimes 1 & & 1 \otimes \phi_{BC} \\
F(A \otimes B) \otimes FC & \xrightarrow{\phi_{AB,C}} & FA \otimes (FB \otimes FC) \\
\phi_{A,B,C} & & \phi_{A,B \otimes C} \\
F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{ABC})} & F(A \otimes (B \otimes C))
\end{array}
\]

and the unit conditions expressed by the commutativity of the following diagrams:

\[
\begin{array}{cccc}
J \otimes FA & \xrightarrow{\lambda_{FA}} & FI \otimes FA & \\
\phi_{o} \otimes 1 & & \phi_{IA} & \\
F(I \otimes A) & \xrightarrow{F(l_{A})} & FA & \\
\phi_{o} & & \phi_{IA} & \\
F(A \otimes I) & \xrightarrow{F(r_{A})} & FA & \\
\end{array}
\]

For instance, the forgetful functor \( U : (Ab, \otimes, \mathbb{Z}) \rightarrow (Set, \times, \{\star\}) \) from the category of Abelian groups to the category of sets is monoidal. In this case, the mapping \( \phi_{AB} : U(A) \times U(B) \rightarrow U(A \otimes B) \) sends \((a, b)\) to \( a \otimes b \) and the mapping \( \phi_{o} : \{\star\} \rightarrow \mathbb{Z} \) sends \( \star \) to \( 1 \). Modules over a commutative ring \( R \) and their homomorphisms form a monoidal category (see [1,2]). The monoidal product is given by the tensor product of modules and the unit object of this category is \( R \). If \( f : R \rightarrow S \) is a commutative ring homomorphism, then the restriction functor \((\text{Mod}_S, \otimes, S) \rightarrow (\text{Mod}_R, \otimes, R)\) is monoidal (see [3]).

**Definition 2.1.** A braided monoidal functor is a monoidal functor between braided categories such that the following diagram commutes:

\[
\begin{array}{cccc}
FA \otimes FB & \xrightarrow{\lambda_{FAB}} & FB \otimes FA & \\
\phi_{AB} & & \phi_{BA} & \\
F(A \otimes B) & \xrightarrow{F(\lambda_{AB})} & F(B \otimes A) & \\
\end{array}
\]

Notice that every monoidal functor which preserves the tensor product is braided monoidal. In particular, every \( Set \)-endofunctor preserving finite products is braided monoidal.

**Definition 2.2.** A symmetric monoidal functor is a braided monoidal functor whose domain and codomain are symmetric monoidal categories.

For instance, the functor \((Set, \times, \{\star\}) \rightarrow (\text{Vect}_K, \otimes, K)\) which takes a set to a vector space by taking the elements of the set as a basis is symmetric monoidal. Also, the forgetful functor \( U : (Ab, \otimes, \mathbb{Z}) \rightarrow (Set, \times, \{\star\}) \) is symmetric monoidal.

**Proposition 2.1** (see [2]). Let \( F : \mathbb{V} \rightarrow \mathbb{W} \) be a symmetric monoidal functor, where \( \mathbb{V} \) and \( \mathbb{W} \) are in fact symmetric monoidal closed categories. There exist morphisms in \( \mathbb{W} \)

\[
\sigma_{AB} : F[A, B] \rightarrow [FA, FB]
\]

for every pair \( A, B \), of objects of \( \mathbb{V} \) and these morphisms satisfy the following conditions:

(1) The morphisms \( \sigma_{AB} \) are natural in \( A \) and \( B \).
(2). The diagram

\[
\begin{array}{c}
F[[A, B] \otimes [B, C]] \xrightarrow{F_{ABC}} F[A, C] \\
\phi_{[A, B], [B, C]} \downarrow \downarrow \sigma_{AC}
\end{array}
\]

\[
F[A, B] \otimes F[B, C] \xrightarrow{\sigma_{AB} \otimes \sigma_{BC}} [FA, FB] \otimes [FB, FC] \\
\xrightarrow{\sigma_{AB} \otimes \sigma_{BC}} [FA, FC]
\]

commutes for all objects \(A, B, C\) of \(\mathcal{V}\).

(3). The diagram

\[
\begin{array}{c}
FI \xrightarrow{F_{JA}} F[A, A] \\
\phi_o \downarrow \downarrow \sigma_{AA}
\end{array}
\]

\[
J \xrightarrow{j_{FA}} [FA, FA]
\]

commutes for every object \(A\) of \(\mathcal{V}\).

3. Coalgebras for a braided monoidal functor

Let \(F\) denote a monoidal endofunctor on a monoidal category \(\mathcal{V}\). A pair \((A, \tau_A)\) consisting of an object \(A\) in \(\mathcal{V}_0\) and a morphism \(\tau_A : A \to FA\) is called an \(F\)-coalgebra. We call \(\tau_A\) the coalgebra structure of \((A, \tau_A)\). Given \(F\)-coalgebras \((A, \tau_A)\) and \((B, \tau_B)\), by a homomorphism we mean a \(\mathcal{V}_0\)-morphism \(f : A \to B\) for which the following diagram commutes:

\[
\begin{array}{c}
A \xrightarrow{f} B \\
\tau_A \downarrow \downarrow \tau_B \\
FA \xrightarrow{F(f)} FB
\end{array}
\]

This definition turns the class of \(F\)-coalgebras and their homomorphisms into a category, denoted as \(\mathcal{V}_{F}\).

**Proposition 3.1.** Let \(F\) be a monoidal endofunctor on a monoidal category \(\mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r)\). The category \(\mathcal{V}_F = (\mathcal{V}_0, \otimes_F, I, a, l, r)\) of \(F\)-coalgebras is monoidal.

**Proof.** Consider the correspondence \(\otimes_F : \mathcal{V}_F \times \mathcal{V}_F \to \mathcal{V}_F\) defined on each pair of \(F\)-coalgebras \((A, \tau_A)\) and \((B, \tau_B)\) as

\[
(A, \tau_A) \otimes_F (B, \tau_B) = (A \otimes B, \phi_{AB} \circ (\tau_A \otimes \tau_B)).
\]

Also, \(\otimes_F\) extends to a bifunctor as \(\otimes\) is bifunctor and the morphism \(\phi_{AB} : FA \otimes FB \to F(A \otimes B)\) is natural in \(A\) and \(B\). Take the unit to be the pair \((I, \phi_o : I \to FI)\). First, we show that \(a, l, \) and \(r\) are natural \(F\)-coalgebra isomorphisms. Since \(a, l,\) and \(r\) are natural isomorphisms in \(\mathcal{V}_0\), it suffices to prove that they are homomorphisms: a \(\mathcal{V}_0\)-isomorphism is an \(F\)-coalgebra isomorphism if and only if it is a homomorphism (see [6]). Moreover a natural \(\mathcal{V}_0\)-morphism is a natural homomorphism provided that it is a homomorphism.

Given \(F\)-coalgebras \((A, \tau_A), (B, \tau_B),\) and \((C, \tau_C)\), the \(\mathcal{V}_0\)-objects \((A \otimes B) \otimes C\) and \(A \otimes (B \otimes C)\) are respectively equipped with the coalgebra structures:

\[
(A \otimes B) \otimes C \\
\xrightarrow{\tau_A \otimes \tau_B \otimes \tau_C}
\]

\[
(FA \otimes FB) \otimes FC \\
\xrightarrow{\phi_{AB} \otimes 1}
\]

\[
F((A \otimes B) \otimes C)
\]

\[
\xrightarrow{\phi_{AB} \otimes \sigma BC}
\]

\[
F((A \otimes B) \otimes C)
\]

\[
\xrightarrow{\phi_{AB} \otimes 1}
\]

\[
F((A \otimes B) \otimes C)
\]

\[
\xrightarrow{\phi_{AB} \otimes \sigma BC}
\]

\[
F((A \otimes B) \otimes C)
\]
and

\[
\begin{array}{c}
A \otimes (B \otimes C) \\
\downarrow \tau_A \otimes (\tau_B \otimes \tau_C) \\
FA \otimes (FB \otimes FC) \\
\downarrow 1 \otimes \phi_{BC} \\
FA \otimes F(B \otimes C) \\
\downarrow \phi_{A,B:BC} \\
F(A \otimes (B \otimes C))
\end{array}
\]

Besides the equality \(\tau_A \otimes (\tau_B \otimes \tau_C) \circ a_{ABC} = a_{FABFC} \circ (\tau_A \otimes \tau_B) \otimes \tau_C\) holds by naturality of \(a\). The associativity condition yields that the following diagram commutes:

\[
\begin{array}{c}
(A \otimes B) \otimes C \\
\downarrow (\tau_A \otimes \tau_B) \otimes \tau_C \\
FA \otimes (FB \otimes FC) \\
\downarrow \phi_{AB} \otimes 1 \\
F(A \otimes B) \otimes FC \\
\downarrow \phi_{A,B:BC} \\
F((A \otimes B) \otimes C)
\end{array} \xrightarrow{\phi_{A,B:BC}}
\begin{array}{c}
(A \otimes B) \otimes C \\
\downarrow \tau_A \otimes (\tau_B \otimes \tau_C) \\
FA \otimes (FB \otimes FC) \\
\downarrow 1 \otimes \phi_{BC} \\
FA \otimes F(B \otimes C) \\
\downarrow \phi_{A,B:BC} \\
F(A \otimes (B \otimes C))
\end{array}
\]

This proves that \(a_{ABC}\) is a homomorphism. Also, \(l_A\) is a homomorphism for every \(F\)-coalgebra \((A, \tau_A)\) because the following diagram commutes due to both the left unit condition and the naturality of \(l\):

\[
\begin{array}{c}
I \otimes A \\
\downarrow 1 \otimes \tau_A \\
I \otimes FA \\
\downarrow \phi_{A} \otimes 1 \\
FI \otimes FA \\
\downarrow \phi_{A} \\
F(I \otimes A) \\
\downarrow F(l_A) \\
FA
\end{array} \xrightarrow{l_A} 
\begin{array}{c}
A \\
\downarrow \tau_A \\
FA
\end{array}
\]

Similarly, one proves that \(r_A\) is a homomorphism. Therefore \(a\), \(l\), and \(r\) are natural \(F\)-coalgebra isomorphisms. It is not difficult to check the coherence conditions. Thus, \(V_F = (V_0_F, \otimes_F, I, a, l, r)\) is a monoidal category.

\[\square\]

**Corollary 3.1.** Let \(F\) be a braided monoidal endofunctor on a braided monoidal category \(V = (V_0, \otimes, I, a, l, r)\). The category \(V_F = (V_0_F, \otimes_F, I, a, l, r)\) of \(F\)-coalgebras is braided monoidal.

**Proof.** First, the category \(V_F\) is monoidal due to Proposition 3.1. Next, we prove that the category \(V_F\) has braidings. Let \((A, \tau_A)\) and \((B, \tau_B)\) be \(F\)-coalgebras. From the naturality of the braiding \(\lambda_{AB} : A \otimes B \to B \otimes A\), it follows that

\[
\lambda_{FABF} \circ (\tau_A \otimes \tau_B) = (\tau_B \otimes \tau_A) \circ \lambda_{AB}.
\]

In addition, \(\phi_{BA} \circ \lambda_{FABF} = F(\lambda_{AB}) \circ \phi_{AB}\) because \(F\) is a braided monoidal endofunctor. As a result, the following diagram commutes:

\[
\begin{array}{c}
A \otimes B \\
\downarrow \tau_A \otimes \tau_B \\
FA \otimes FB \\
\downarrow \phi_{AB} \\
F(A \otimes B)
\end{array} \xrightarrow{\lambda_{AB}}
\begin{array}{c}
B \otimes A \\
\downarrow \tau_B \otimes \tau_A \\
FB \otimes FA \\
\downarrow \phi_{BA} \\
F(B \otimes A)
\end{array}
\]
Hence, \( \lambda_{AB} \) is a homomorphism, that is a natural \( F \)-coalgebra isomorphism because it is a natural \( \mathcal{V}_0 \)-isomorphism by definition. Therefore, \( \lambda_{AB} \) is a braiding for the category \( \mathcal{V}_F \). Consequently, the category \( \mathcal{V}_F = (\mathcal{V}_0, \otimes, I, a, l, r) \) of \( F \)-coalgebras is braided monoidal.

**Example 3.1.** Let \( K \) be a field and let \( M \) be a monoid. Denote by \( \text{Vec}_M \) the category of \( M \)-graded finite dimensional vector spaces over \( K \) consisting of vector spaces with a decomposition

\[
V = \bigoplus_{m \in M} V_m,
\]

where the morphisms are linear mappings which preserve the grading. It is a monoidal category with the tensor product defined by

\[
(V \otimes W)_m = \bigoplus_{x,x' \in M; xx' = m} V_x \otimes W_{x'}
\]

and the unit object given by

\[
1_m = \begin{cases} K, & \text{if } m = 1 \\ 0, & \text{otherwise}. \end{cases}
\]

Let \( M \) be a monoid and let \( f : M \to M \) be a monoid homomorphism. We then have a functor

\[
F : \text{Vec}_M \to \text{Vec}_M
\]

defined on objects as follows: for an \( M \)-graded finite dimensional vector space,

\[
F(V) = F \left( \bigoplus_{m \in M} V_m \right) = \bigoplus_{f(m) \in M} V_{f(m)}
\]

and for each linear mapping \( q : V \to W \),

\[
F(q) \left( \bigoplus_{f(m) \in M} V_{f(m)} \right) = \bigoplus_{f(m) \in M} q \left( V_{f(m)} \right)
\]

The endofunctor \( F \) is monoidal; see [10]. Hence, the category of \( F \)-coalgebras is monoidal by Proposition 3.1.

**Example 3.2.** Given a commutative ring \( R \), the homology functor is braided monoidal as

\[
H_* : (\text{GMod}_R, \otimes, R) \to (\text{GMod}_R, \otimes, R)
\]

via the mapping \( H_*(C_1) \otimes H_*(C_2) \to H_*(C_1 \otimes C_2); [x_1] \otimes [x_2] \mapsto [x_1 \otimes x_2] \); see [8]. Then, the category of \( H_* \)-coalgebras is braided monoidal due to Corollary 3.1.

Recall that every symmetric monoidal category is braided monoidal.

**Corollary 3.2.** Let \( F \) be a braided monoidal endofunctor on a symmetric monoidal category \( \mathcal{V} = (\mathcal{V}_0, \otimes, I, a, l, r) \). The category \( \mathcal{V}_F = (\mathcal{V}_0, \otimes_F, I, a, l, r) \) of \( F \)-coalgebras is symmetric monoidal.

**Proof.** By Corollary 3.1, the category \( \mathcal{V}_F \) is braided monoidal. Then, for any \( F \)-coalgebras \( (A, \tau_A) \) and \( (B, \tau_B) \), the braidings \( \lambda_{AB} \) and \( \lambda_{BA} \) are homomorphisms. Also, \( \lambda_{AB} \circ \lambda_{BA} = 1_{A \otimes B} \) as the category \( \mathcal{V} \) is symmetric monoidal. Hence, the category \( \mathcal{V}_F \) of \( F \)-coalgebras is symmetric monoidal.

**Example 3.3.** Let \( F : (\text{Set}, \times, \{\ast\}) \to (\text{Set}, \times, \{\ast\}) \) be a \( \text{Set} \)-endofunctor which preserves finite products. Then \( F \) is braided monoidal. But, \( \text{Set} \) is a symmetric monoidal category. As a consequence of Corollary 3.1, the category of \( F \)-coalgebras is symmetric monoidal, that is braided monoidal.

More generally, the category of coalgebras for a braided monoidal endofunctor \( F \) is braided monoidal provided that \( F \) preserves the tensor product.

**Example 3.4.** Consider the covariant power set functor

\[
P : (\text{Set}, \times, \{\ast\}) \to (\text{Set}, \times, \{\ast\}),
\]

which maps every set to its power set and every function \( f : A \to B \) to the mapping \( P(f) \), which sends \( U \in P(A) \) to its image \( f(U) \in P(B) \). It is a symmetric monoidal functor. The coherence maps are the mapping \( \phi_n : \{\ast\} \to P(\{\ast\}) \) which sends \( \ast \) to \( \{\ast\} \) and, the mapping \( \phi_{AB} : P(A) \times P(B) \to P(A \times B) \) which sends \( (U, V) \) to \( U \times V \). Also, the symmetry condition holds. Therefore, the category of \( P \)-coalgebras is symmetric monoidal due to Corollary 3.2.
Every category $\mathcal{V}$ admitting finite coproducts is a symmetric monoidal category. The tensor product is defined for every pair $A, B$, of objects of $\mathcal{V}$ by

$$A \otimes B = A \coprod B$$

and the unit $I$ as the initial object of $\mathcal{V}$. If an endofunctor $F : \mathcal{V} \to \mathcal{V}$ is given, then $F$ is a symmetric monoidal functor. It suffices to define the coherence maps $\phi_o : I \to FI$ as the unique arrow from $I$ to $FI$ and for every pair $A, B$, of objects of $\mathcal{V}$,

$$\phi_{AB} : FA \coprod FB \to F(A \coprod B)$$

as the unique arrow arising from the universality of the coproduct. Corollary 3.2 implies that $\mathcal{V}_F$ is a symmetric monoidal category.

**Lemma 3.1.** Suppose that $\mathcal{V}$ is a symmetric monoidal closed category admitting coproducts. Let $F : \mathcal{V} \to \mathcal{V}$ be a fully faithful symmetric monoidal endofunctor. For every pair $A, B$, of objects of $\mathcal{V}$, the morphisms

$$\sigma_{AB} : F[A, B] \to [FA, FB]$$

are invertible.

**Proof.** For every pair $A, B$, of objects of $\mathcal{V}_0$, consider the mapping

$$\varphi : \mathcal{V}_0(I, F[A, B]) \to \mathcal{V}_0(I, [FA, FB]) : u \mapsto \sigma_{AB} \circ u$$

Let $v : I \to [FA, FB]$ be a $\mathcal{V}_0$-morphism. Since $F$ is full, there is a $\mathcal{V}_0$-morphism $w : I \to [A, B]$ such that

$$\sigma_{AB} \circ (F(w) \circ \phi_o) = v.$$

Also, the composite $F(w) \circ \phi_o$ is the only $\mathcal{V}_0$-morphism with this property as $F$ is faithful. Thus, one deduces a mapping

$$\psi : \mathcal{V}_0(I, [FA, FB]) \to \mathcal{V}_0(I, [F[A, B]]) : v \mapsto F(w) \circ \phi_o.$$

We say that $\varphi$ and $\psi$ are inverse of each other. Indeed, for every $\mathcal{V}_0$-morphism $u : I \to F[A, B]$, we have that

$$(\psi \circ \varphi)(u) = \psi(\varphi(u)) = \psi(\sigma_{AB} \circ u) = u.$$

Conversely, for every $\mathcal{V}_0$-morphism $v : I \to [FA, FB]$, we have that

$$(\varphi \circ \psi)(v) = \varphi(\psi(v)) = \varphi(F(w) \circ \phi_o) = \sigma_{AB} \circ (F(w) \circ \phi_o) = v.$$

As a result, the mappings $\varphi$ and $\psi$ are inverse of each other. Subsequently, $\sigma_{AB}$ is invertible as the functor $\mathcal{V}_0(I, -) : \mathcal{V} \to \Set$ has a left adjoint (see [2]).

**Proposition 3.2.** Let $\mathcal{V}$ be a symmetric monoidal closed category admitting coproducts. Let $F : \mathcal{V} \to \mathcal{V}$ be a fully faithful symmetric monoidal endofunctor. The category $\mathcal{V}_F$ is symmetric monoidal closed.

**Proof.** By Corollary 3.2, the category $\mathcal{V}_F$ is symmetric monoidal. Next, we prove that $\mathcal{V}_F$ is also closed. For given $F$-coalgebras $(A, \tau_A)$, $(B, \tau_B)$ and $(C, \tau_C)$, let $f : (A, \tau_A) \otimes_F (B, \tau_B) \to (C, \tau_C)$ be a homomorphism. Since $\mathcal{V}$ is closed, the $\mathcal{V}_0$-morphism $f : A \otimes B \to C$ corresponds under adjunction with an arrow $\bar{f} : A \to [B, C]$. Also, $\phi_{AB}$ is natural in $A, B$. Then the following commutative diagram:

```
\begin{array}{ccc}
A \otimes B & \xrightarrow{f} & C \\
\downarrow^{\tau_A \otimes \tau_B} & & \\
FA \otimes FB & \xrightarrow{F(f) \otimes 1} & F[B, C] \otimes FB \\
\downarrow^{\phi_{AB}} & & \downarrow^{\phi_{[B, C], u}} \\
F(A \otimes B) & \xrightarrow{F(f) \otimes 1} & F([B, C] \otimes B) \\
\downarrow^{F(f)} & & \downarrow^{F(ev_{BC})} \\
& & FC \\
\end{array}
```
corresponds under adjunction with the following commutative diagram:

\[
\begin{array}{c}
A \xrightarrow{f} [B, C] \\
\downarrow \tau_A \\
F(A) \xrightarrow{F(f)} F([B, C]) \\
\end{array}
\]

which in turn corresponds under adjunction with the following commutative diagram:

\[
\begin{array}{c}
A \otimes B \xrightarrow{f \otimes 1} [B, C] \otimes B \\
\downarrow 1 \otimes \tau_B \\
A \otimes FB \xrightarrow{f \otimes 1} [B, C] \otimes FB \\
\downarrow \tau_A \otimes 1 \\
FA \otimes FB \xrightarrow{F(f) \otimes 1} F([B, C] \otimes B) \\
\downarrow \phi_{[B,C],B} \\
F([B, C] \otimes B) \xrightarrow{F(\sigma_{BC})} FC \\
\end{array}
\]

As $\otimes$ is bifunctor, the following diagram commutes:

\[
\begin{array}{c}
A \otimes FB \xrightarrow{f \otimes 1} [B, C] \otimes FB \\
\downarrow \tau_A \otimes 1 \\
FA \otimes FB \xrightarrow{F(f) \otimes 1} F([B, C] \otimes B) \\
\downarrow \phi_{[B,C],B} \\
F([B, C] \otimes B) \xrightarrow{F(\sigma_{BC})} FC \\
\end{array}
\]

and it corresponds under adjunction with the following commutative diagram:

\[
\begin{array}{c}
A \xrightarrow{f} [B, C] \\
\downarrow \tau_A \\
F(A) \xrightarrow{F(f)} F([B, C]) \\
\downarrow \sigma_{BC} \\
F([B, C]) \xrightarrow{[FB, FC]} [FB, FC] \\
\end{array}
\]

Furthermore, $\sigma_{BC}$ is invertible due to Lemma 3.1. One therefore deduces that the following diagram commutes:

\[
\begin{array}{c}
A \xrightarrow{f} [B, C] \\
\downarrow \tau_A \\
F(A) \xrightarrow{F(f)} F([B, C]) \\
\downarrow \sigma_{BC}^{-1} \\
F([B, C]) \xrightarrow{F(f)} F([B, C]) \\
\end{array}
\]
Hence, the $\mathcal{V}_0$-morphism $\bar{f}$ is homomorphism. Denote by $\left[\left(\mathcal{B}, \tau_B\right), \left(\mathcal{C}, \tau_C\right)\right]_F$ the pair $\left[\mathcal{B}, \mathcal{C}\right], \sigma_{BC}^{-1} \circ F_{BC}$. We then have a natural bijection:

$$\mathcal{V}_0\left(\left(\mathcal{A}, \tau_A\right) \otimes_F \left(\mathcal{B}, \tau_B\right), \left(\mathcal{C}, \tau_C\right)\right) \cong \mathcal{V}_0\left(\left(\mathcal{A}, \tau_A\right), \left[\left(\mathcal{B}, \tau_B\right), \left(\mathcal{C}, \tau_C\right)\right]_F\right)$$

That is, the functor $- \otimes_F \left(\mathcal{B}, \tau_B\right): \mathcal{V}_0 \rightarrow \mathcal{V}_0$ has a right adjoint $\left[\left(\mathcal{B}, \tau_B\right), -\right]_F$. Consequently, $\mathcal{V}_F$ is closed.

References


