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Research Article

Exponential stability for a porous elastic system with fractional-order time delay

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Abstract

This article deals with the solution and asymptotic analysis for a porous-elastic system with fractional-order time delay. Semigroup theory is used. The existence and uniqueness of the solution are obtained by applying the Lumer-Phillips Theorem. Additionally, two results for the asymptotic behavior are presented concerning the (i) strong stability of the C_0-semigroup associated with the system by using a general criterion due to Arendt-Batty and Lyubich-Vũ, (ii) exponential stability by applying Gearhart-Prüss-Huang's theorem.

Keywords: porous-elastic system; fractional-order time delay; asymptotic analysis.

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1. Introduction

Elastic materials with voids, which have good physical properties, have been widely used in engineering, such as vehicles, airplanes, large space structures, etc. Due to their extensive applications, the elasticity problems of these types of materials have become critical issues that attract the attention of many researchers. Since elastic solids with voids provide one of the simple extensions of classical elasticity theory, they allow the treatment of porous solids in which the matrix material is elastic and the interstices are voids of material; see [18, 28] for details.

Consider the following equations of evolution for one-dimensional theories of porous materials:

$$\begin{cases} \rho u_{tt} = T_x, \\ J\phi_{tt} = H_x + G, \end{cases}$$

where T is tension, H is balanced tension, G is balanced body force, q is heat, ρ is the reference mass density, $J = \rho_0 k$, ρ_0 is the mass the density that is assumed positive, k is the equilibrated inertia that is also assumed positive, the variables u and ϕ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are given as:

$$\begin{cases} T = \mu u_x + b\phi, \\ H = \delta\phi_x, \\ G = -bu_x - \xi\phi, \end{cases}$$

where $\mu,\ b,\ \delta,$ and $\xi,$ are the constitutive coefficients, whose physical meanings are well known. The constitutive coefficients in the one-dimensional case satisfy the following relations:

$$\mu > 0, \quad \delta > 0 \ \text{ and } \ b^2 \leqslant \mu\xi. \tag{1}$$

When we introduce the constitutive equations into the evolution equations in the interval $(0, L)$, we get

$$\begin{cases} \rho u_{tt}(x,t) - \mu u_{xx}(x,t) - b\phi_x(x,t) = 0,\ x \in (0,L),\ t \in (0,\infty), \\ J\phi_{tt}(x,t) - \delta\phi_{xx}(x,t) + bu_x(x,t) + \xi\phi(x,t) = 0,\ x \in (0,L),\ t \in (0,\infty). \end{cases} \tag{p}$$

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Since $b^2 \leqslant \mu\xi$, we have

$$\mu|u_x|^2 + 2bu_x\phi + \xi|\phi|^2 = \left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\phi\right)^2 + \left(\mu - \frac{b^2}{\xi}\right)|u_x|^2 \geq 0, \tag{2}$$

and hence the energy of the system is given by

$$E(t) = \frac{1}{2}\int_0^L \left[\rho|u_t|^2 + J|\phi_t|^2 + \delta|\phi_x|^2 + \mu|u_x|^2 + 2bu_x\phi + \xi|\phi|^2\right] dx.$$

A direct calculation leads to $\frac{d}{dt}E(t) = 0$; that is, the system (p) is conservative. For a realistic situation, research on porous elastic systems has been carried out in recent decades considering several stabilization mechanisms; see [4, 8–12, 15, 27, 29, 32, 34, 35]. In the present paper, we are interested in the internal damping of fractional order with time delay.

An essential advantage of fractional differential equations in applications is their non-local property, making fractional calculus more attractive. In [19], the concept of the fractional derivatives, specifically, the Riemann-Liouville, Liouville, Caputo, Weyl, and Riesz versions, are introduced, and the so-called fundamental theorem of fractional calculus is presented and discussed in all these different versions. A new fractional derivative with a non-singular kernel involving exponential and trigonometric functions was proposed in [2]. The suggested fractional operator includes the Caputo-Fabrizio fractional derivative as a particular case.

The Caputo fractional integral of order α, $0 < \alpha < 1$, is defined by

$$I^\alpha w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} w(s)ds, \tag{3}$$

where Γ is the well-known gamma function, $w \in L^1(0, L)$ and $t > 0$.

The Caputo fractional derivative operator of order α is defined by

$$D^\alpha w(t) = I^{1-\alpha}Dw(t) := \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{dw}{ds}(s)ds, \tag{4}$$

with $w \in W^{1,1}(0, L)$, and $t > 0$.

In this work, we use the slightly different versions of (3) and (4), with weight exponential; see [5]. Let $0 < \alpha < 1$ and $\eta \geq 0$. The exponential fractional integral of order α is defined by

$$[I^{\alpha,\,\eta}w](t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}e^{-\eta(t-\tau)}w(\tau)d\tau,\ \ w \in L^1(0, L),\ \ t > 0. \tag{5}$$

The exponential fractional derivative operator of order α, $\ 0 < \alpha < 1$, with respect to the time variable t is defined by

$$\partial_t^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\frac{dw}{ds}(s)ds,\ \ w \in W^{1,1}(0, L),\ \ t > 0. \tag{6}$$

From (5) and (6), we deduce that

$$\partial_t^{\alpha,\eta} w(t) = [I^{1-\alpha,\,\eta}w_t](t). \tag{7}$$

The control of partial differential equations with time delay is an attractive area of research. Time delays often arise in many physical, chemical, biological, and economic phenomena; see [36] and references therein. Whenever energy is physically transmitted from one place to another, a delay is associated with the transmission (see [33]). The central question is that delays source can destabilize an asymptotically stable system in the absence of delays (see [7]). We consider the following model:

$$\begin{cases} \rho u_{tt}(x,t) - \mu u_{xx}(x,t) - b\phi_x(x,t) + a_1\partial_t^{\alpha_1,\eta_1}u(x,t-\tau_1) + a_2u_t(x,t) = 0, \\ J\phi_{tt}(x,t) - \delta\phi_{xx}(x,t) + bu_x(x,t) + \xi\phi(x,t) + \tilde{a_1}\partial_t^{\alpha_2,\eta_2}\phi(x,t-\tau_2) + \tilde{a_2}\phi_t(x,t) = 0, \end{cases} \tag{P}$$

where $\tau_1, \tau_2 > 0$ are the time delays and a_2, $\tilde{a_2}$, a_1, $\tilde{a_1}$ positive parameters. System (P) is completed with initial and boundary conditions:

$$\begin{cases} (u(x,0), \phi(x,0)) = (u_0(x), \phi_0(x)),\ (u_t(x,0), \phi_t(x,0)) = (u_1(x), \phi_1(x)),\ x \in (0, L), \\ u_t(x, t-\tau_1) = f_0(x, t-\tau_1),\ \phi_t(x, t-\tau_2) = g_0(x, t-\tau_2),\ x \in (0, L),\ t \in (0, \tau_i), \\ u(0,t) = u(L,t) = \phi(0,t) = \phi(L,t) = 0, \text{ in } (0, \infty), \end{cases}$$

such that $(u_0, u_1, f_0, \phi_0, \phi_1, g_0)$ belong to a suitable functional spaces.

During the last few years, stabilizing porous elastic systems with different damping have been studied in a significant number of publications. We mention here some of them. In [9], the existence of a global solution and the exponential decay was given for a nonlinear porous elastic system with delay, where a nonlinear source, as well as the delay, acted in the volume fraction equation. A one-dimensional linear porous system with finite memory effective on the equilibrated stress vector was considered in [13] and an energy decay rate was given for which exponential and polynomial rates are special cases. In [4], the exponential stability was proved for a one-dimensional porous-thermoelastic system with two kinds of damping: viscosity and thermal dissipation. A porous-thermoelastic system with Cattaneo's law heat conduction and the energy associated with the solution, not necessarily positive ($b^2 = \mu\xi$), was analyzed in [11]. In [23], it is shown that viscoelasticity and temperature produce slow decay in time; however, when the viscoelasticity is coupled with porous damping or with micro-temperatures, the exponential decay holds. In [29], a porous-elasticity problem with history was studied and it was shown that when the porous viscosity and the elastic dissipation are present, the system lacks analyticity but has exponential decay. A transmission problem for a porous-elastic system with internal dissipation was considered in [32] and it was proved that the semigroup associated with the dissipative system is analytic and consequently exponentially stable. For more results on porous elasticity, see [6, 8, 12, 14–16, 20, 26, 27, 34, 35] and the references therein. As far as we know, introducing a fractional delay term in the internal feedback of the porous elastic system makes our problem different from those previously considered in the literature.

This paper is organized as follows. In Section 2, the problem (P) is reformulated in an augmented system (P'), coupling the (P) system with a suitable diffusion equation. Section 3 shows that the energy functional $E(t)$ associated with the augmented system (P') is dissipative. Section 4 deals with the semigroup setup for the augmented problem. Section 5 establishes the well-posedness of the system (P'). In Section 6, the strong stability of the C_0-semigroup associated with the system is proved using the Arendt-Batty and Lyubich-Vũ's general criterion. In Section 7, exponential stability is proved by applying Gearhart-Prüss-Huang's theorem.

2. Augmented model

Proposition 2.1 (see [24]). *Let ω be a function defined as*

$$\omega(y) = |y|^{\frac{2\alpha-1}{2}}, \qquad y \in (-\infty, +\infty), \;\; 0 < \alpha < 1.$$

Then, the relation between the Input U and the Output $\mathcal{O}$ of the following system

$$\begin{cases} \varphi_t(y,t) + y^2\varphi(y,t) + \eta\varphi(y,t) - U(t)\omega(y) = 0, \;\; \eta \geqslant 0, \; t > 0, \\ \varphi(y,0) = 0, \\ \mathcal{O}(t) = \gamma \displaystyle\int_{-\infty}^{+\infty} \omega(y)\varphi(y,t)dy, \end{cases} \tag{8}$$

where $\gamma = \dfrac{\sin(\alpha\pi)}{\pi} = \dfrac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}$ and $U \in C([0, +\infty))$, is given by

$$\mathcal{O}(t) = I^{1-\alpha,\,\eta}u(t) = D^{\alpha,\,\eta}U(t), \tag{9}$$

where

$$[I^{\alpha,\eta}w] = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}e^{-\eta(t-s)}w(s)ds. \tag{10}$$

Proof. Multiplying the first equation of (8) by $e^{(y^2+\eta)t}$, we get

$$e^{(y^2+\eta)t}\varphi_t(y,t) + \left(y^2+\eta\right)e^{(y^2+\eta)t}\varphi(y,t) = \omega(y)e^{(y^2+\eta)t}U(t),$$

that is,

$$\frac{\partial}{\partial_t}\left(e^{(y^2+\eta)t}\varphi(y,t)\right) = \omega(y)e^{(y^2+\eta)t}U(t). \tag{11}$$

Performing integration on $(0,t)$ in (11), we obtain

$$e^{(y^2+\eta)t}\varphi(y,t) - \varphi(y,0) = \omega(y)\int_0^t e^{(y^2+\eta)s}U(s)\,ds.$$

As $\varphi(y,0)=0$, we have

$$\varphi(t,y)=\omega(y)\int_0^t e^{-(y^2+\eta)(t-s)}U(s)\,ds. \tag{12}$$

By using (12) in the last equation of (8), we get

$$\mathcal{O}(t) \;=\; \gamma\int_{-\infty}^{\infty}\int_0^t [\omega(y)]^2\, e^{-(y^2+\eta)(t-s)}U(s)\,ds\,dy,$$

that is,

$$\mathcal{O}(t) \;=\; \gamma\int_0^t\int_0^{+\infty} 2y^{2\alpha-1}e^{-(y^2+\eta)(t-s)}U(s)\,dy\,ds. \tag{13}$$

Taking $\sigma=y^2(t-s)$, we get $d\sigma=2y(t-s)d\xi$ and

$$\sigma^{\alpha-1}=\frac{y^{2\alpha-1}}{y}\cdot\frac{(t-s)^{\alpha}}{t-s}.$$

Thus, we have

$$\sigma^{\alpha-1}(t-s)^{-\alpha}d\sigma=2y^{2\alpha-1}dy. \tag{14}$$

From (13) and (14), we obtain

$$\begin{aligned}
\mathcal{O}(t) &= \gamma\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\int_0^{+\infty}\sigma^{\alpha-1}e^{-\sigma}\,d\sigma\,U(s)\,ds\\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}\Gamma(\alpha)\,U(s)\,ds\\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}e^{-\eta(t-s)}U(s)\,ds\\
&= I^{1-\alpha,\eta}U(t).
\end{aligned}$$

□

We make the following assumptions about the damping and delay functions:

$$\begin{cases} |a_1|\eta_1^{\alpha_1-1}<a_2,\\ |\tilde{a_1}|\eta_2^{\alpha_2-1}<\tilde{a_2}. \end{cases} \tag{15}$$

Let us introduce the following new variables:

$$\begin{cases} z_1(x,p,t)=u_t(x,t-\tau_1 p), & \text{in }]0,L[\times]0,1[\times]0,+\infty[,\\ z_2(x,p,t)=\phi_t(x,t-\tau_2 p), & \text{in }]0,L[\times]0,1[\times]0,+\infty[. \end{cases} \tag{16}$$

Then, for $j=1,2$, we have

$$\begin{cases} \tau_j z_{jt}(x,p,t)+z_{jp}(x,p,t)=0, \;\text{ in }]0,L[\times]0,1[\times]0,+\infty[,\\ z_1(x,0,t)=u_t(x,t),\;\; z_2(x,0,t)=\phi_t(x,t), \;\text{ in }]0,L[\times]0,+\infty[. \end{cases} \tag{17}$$

The strategy is the elimination of the fractional derivatives in time. To do this, first, we consider the equations given in (16) with $p=1$. Applying Proposition 2.1 with $U_1(t)=a_1z_1(x,1,t)$ and taking into account (7), that is,

$$[I^{1-\alpha,\,\eta}w_t](t)=\partial_t^{\alpha,\eta}w(t),$$

we deduce

$$\begin{aligned}
\gamma\int_{-\infty}^{\infty}\omega_1(y)\varphi_1(y,t)\,dy &=\mathcal{O}(t)\\
&=I^{1-\alpha,\eta}U_1(t)\\
&=I^{1-\alpha,\eta}a_1z_1(x,1,t)\\
&=a_1I^{1-\alpha,\eta}u_t(x,t-\tau_1)=a_1\partial_t^{\alpha,\eta}u(x,t-\tau_1).
\end{aligned} \tag{18}$$

Similarly, considering the second equation of (16) with $p = 1$ and $U_2(t) = \tilde{a_1} z_2(x, 1, t)$, we get

$$\tilde{\gamma} \int_{-\infty}^{\infty} \omega_2(y)\varphi_2(y,t)\, dy = \tilde{a_1} \partial_t^{\alpha,\eta} \phi(x, t-\tau_1). \tag{19}$$

Now, by using (18) and (19), we reformulate system (P) into the augmented model:

$$\begin{cases} \rho u_{tt}(x,t) - \mu u_{xx}(x,t) - b\phi_x(x,t) + \gamma \displaystyle\int_{-\infty}^{+\infty} \omega_1(y)\varphi_1(x,y,t)dy + a_2 u_t(t) = 0, \\ \tau_1 z_{1t}(x,p,t) + z_{1p}(x,p,t) = 0, \\ \varphi_{1t}(x,y,t) + [y^2+\eta_1]\varphi_1(x,y,t) - z_1(x,1,t)\omega_1(y) = 0, \\ J\phi_{tt}(x,t) - \delta\phi_{xx}(x,t) + bu_x(x,t) + \xi\phi(x,t) + \tilde{\gamma} \displaystyle\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y,t)dy + \tilde{a_2}\phi_t(t) = 0, \\ \tau_2 z_{2t}(x,p,t) + z_{2p}(x,p,t) = 0, \\ \varphi_{2t}(x,y,t) + [y^2+\eta]\varphi_2(x,y,t) - z_2(x,1,t)\omega_2(y) = 0, \\ (u(x,0),\phi(x,0)) = (u_0(x),\phi_0(x)), \quad \text{in} \quad (0,L), \\ (u_t(x,0),\phi_t(x,0)) = (u_1(x),\phi_1(x)), \quad \text{in} \quad (0,L), \\ u_t(x,t-\tau_1) = f_0(x,t-\tau_1), \;\; \phi_t(x,t-\tau_2) = g_0(x,t-\tau_2), \;\; t \in (0,\tau_i) \\ u(0,t) = u(L,t) = \phi(0,t) = \phi(L,t) = 0, \quad \text{in} \quad (0,\infty), \\ \varphi_1(y,0) = \varphi_2(y,0) = 0, \end{cases} \tag{P'}$$

where $\gamma = \dfrac{\sin(\alpha_1\pi)a_1}{\pi}$ and $\tilde{\gamma} = \dfrac{\sin(\alpha_2\pi)a_2}{\pi}$, while $\omega_1(y)$ and $\omega_2(y)$ come from Proposition 2.1.

3. Energy of the system

In this section, it is shown that the energy functional $E(t)$ associated with the augmented system (P') is dissipative.

Lemma 3.1. *For all $\lambda \in \mathbb{R}$ and $\eta > 0$, we have*

$$A_1 = \int_{\mathbb{R}^d} \frac{|y|^{2\alpha-d}}{|\lambda| + \eta + y^2} dy = c(|\lambda| + \eta)^{\alpha-1}$$

and

$$A_3 = \left(\int_{\mathbb{R}^d} \frac{|y|^{2\alpha-d}}{\left(|\lambda| + \eta + y^2\right)^2} dy \right)^{\frac{1}{2}} = \tilde{c}(|\lambda| + \eta)^{\frac{\alpha}{2}-1},$$

where, c and $\tilde{c}$ are positive constants given by

$$c = \frac{d\pi^{\frac{d}{2}+1}}{2\Gamma\left(\frac{d}{2}+1\right)\sin(\alpha\pi)} \quad \textit{and} \quad \tilde{c} = \left(\frac{d\pi^{\frac{d}{2}}}{2\Gamma\left(\frac{d}{2}+1\right)} \int_1^{+\infty} \frac{(\xi-1)^\alpha}{\xi^2} d\xi \right)^{\frac{1}{2}}.$$

Proof. See Page 60 in [25]. □

Lemma 3.2. *If $\lambda \in D_\eta = \mathbb{C} \setminus]-\infty, -\eta]$ then*

$$\int_{-\infty}^{+\infty} \frac{\omega^2(y)}{\lambda + \eta + y^2} dy = \frac{\pi}{\sin(\alpha\pi)} (\lambda + \eta)^{\alpha-1}.$$

Proof. See Page 4 in [1]. □

Taking into account (2), the energy associated with the problem (P') is defined by

$$\begin{aligned} E(t) = {} & \frac{\rho}{2}\int_0^L |u_t|^2 dx + \frac{J}{2}\int_0^L |\phi_t|^2 dx + \frac{\delta}{2}\int_0^L |\phi_x|^2 dx + \frac{\mu}{2}\int_0^L |u_x|^2 dx + b\int_0^L u_x\phi dx + \frac{\xi}{2}\int_0^L |\phi|^2 dx \\ & + \frac{\nu_1}{2}\int_0^L\int_0^1 |z_1(x,p)|^2 dp dx + \frac{\nu_2}{2}\int_0^L\int_0^1 |z_2(x,p)|^2 dp dx \\ & + \frac{|\gamma|}{2}\int_0^L\int_{-\infty}^{+\infty} |\varphi_1(y,t)|^2 dy dx + \frac{|\tilde{\gamma}|}{2}\int_0^L\int_{-\infty}^{+\infty} |\varphi_2(y,t)|^2 dy dx, \end{aligned} \tag{20}$$

where

$$\begin{cases} \tau_1|\gamma|\left(\int_{-\infty}^{+\infty}\frac{\omega_1^2(y)}{y^2+\eta_1}dy\right)<\nu_1<\tau_1\left(2a_2-|\gamma|\left(\int_{-\infty}^{+\infty}\frac{\omega_1^2(y)}{y^2+\eta_1}dy\right)\right), \\ \tau_2|\tilde{\gamma}|\left(\int_{-\infty}^{+\infty}\frac{\omega_2^2(y)}{y^2+\eta_2}dy\right)<\nu_2<\tau_2\left(2\tilde{a}_2-|\tilde{\gamma}|\left(\int_{-\infty}^{+\infty}\frac{\omega_2^2(y)}{y^2+\eta_2}dy\right)\right). \end{cases} \tag{21}$$

Remark 3.1. *From Lemma 3.2, the condition (21) leads to*

$$\begin{cases} \tau_1|a_1|\eta_1^{\alpha_1-1}<\nu_1<\tau_1\left(2a_2-|a_1|\eta_1^{\alpha_1-1}\right), \\ \tau_2|\tilde{a}_1|\eta_2^{\alpha_2-1}<\nu_2<\tau_2\left(2\tilde{a}_2-|\tilde{a}_1|\eta_2^{\alpha_2-1}\right). \end{cases}$$

Proposition 3.1. *Let $(u,z_1,\varphi_1,\phi,z_2,\varphi_2)$ be the solution of (P'). Then, there is a positive constant C such that $E(t)$ defined by (20) satisfies*

$$\frac{d}{dt}E(t)=-C\left(\int_0^L(|u_t(x,t)|^2+|z_1(x,1,t)|^2)dx+\int_0^L(|\phi_t(x,t)|^2+|z_2(x,1,t)|^2)dx\right). \tag{22}$$

Proof. Multiplying $(P')_1$ by u_t, $(P')_4$ by ϕ_t, and making integration by parts, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L\left[\rho|u_t|^2+\mu|u_x|^2+J|\phi_t|^2+\delta|\phi_x|^2+2bu_x\phi+\xi|\phi|^2\right]dx = &-\gamma\int_0^L u_t\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx \\ &-\tilde{\gamma}\int_0^L\phi_t\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx \\ &-a_2\int_0^L|u_t(t)|^2dx-\tilde{a}_2\int_0^L|\phi_t(t)|^2dx. \end{aligned} \tag{23}$$

Multiplying $(P')_3$ by $|\gamma|\varphi_1$, $(P')_6$ by $|\tilde{\gamma}|\varphi_2$, and integrating on $(0,L)\times(-\infty,+\infty)$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L\int_{-\infty}^{+\infty}|\gamma||\varphi_1(x,y,t)|^2dydx+\frac{1}{2}\frac{d}{dt}\int_0^L\int_{-\infty}^{+\infty}|\tilde{\gamma}||\varphi_2(x,y,t)|^2dydx = &-|\gamma|\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_1]|\varphi_1(x,y,t)|^2dydx \\ &-|\tilde{\gamma}|\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_2]|\varphi_2(x,y,t)|^2dydx \\ &+|\gamma|\int_0^L z_1(x,1,t)\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx \\ &+|\tilde{\gamma}|\int_0^L z_2(x,1,t)\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx. \end{aligned} \tag{24}$$

Multiplying $(P')_2$ and $(P')_5$ by $\nu_1 z_1$ and $\nu_2 z_2$, respectively, and integrating on $(0,L)\times(0,1)$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left\{\nu_1\int_0^L\int_0^1|z_1(x,p,t)|^2dpdx+\nu_2\int_0^L\int_0^1|z_2(x,p,t)|^2dpdx\right\} = &-\frac{\nu_1\tau_1^{-1}}{2}\int_0^L(|z_1(x,1,t)|^2-|u_t(x,t)|^2)dx \\ &-\frac{\nu_2\tau_2^{-1}}{2}\int_0^L(|z_2(x,1,t)|^2-|\phi_t(x,t)|^2)dx. \end{aligned} \tag{25}$$

From (23)–(25), we obtain

$$\begin{aligned} \frac{d}{dt}E(t)= &-a_2\int_0^L|u_t(t)|^2dx-\tilde{a}_2\int_0^L|\phi_t(t)|^2dx-\gamma\int_0^L u_t\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx \\ &-\tilde{\gamma}\int_0^L\phi_t\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx-|\gamma|\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_1]|\varphi_1(x,y,t)|^2dydx \\ &-|\tilde{\gamma}|\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_2]|\varphi_2(x,y,t)|^2dydx+|\gamma|\int_0^L z_1(x,1,t)\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx \\ &+|\tilde{\gamma}|\int_0^L z_2(x,1,t)\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx+\frac{\nu_1\tau_1^{-1}}{2}\int_0^L|u_t(x,t)|^2dx \\ &-\frac{\nu_1\tau_1^{-1}}{2}\int_0^L|z_1(x,1,t)|^2dx+\frac{\nu_2\tau_2^{-1}}{2}\int_0^L|\phi_t(x,t)|^2dx-\frac{\nu_2\tau_2^{-1}}{2}\int_0^L|z_2(x,1,t)|^2dx. \end{aligned} \tag{26}$$

Using Cauchy-Schwarz inequality, we get

$$\left|\int_{-\infty}^{+\infty}\omega_j(y)\varphi_j(x,y,t)dy\right| \leqslant \left(\int_{-\infty}^{+\infty}\frac{\omega_j^2(y)}{y^2+\eta_j}dy\right)^{\frac{1}{2}}\left(\int_{-\infty}^{+\infty}(y^2+\eta_j)|\varphi_j(x,y,t)|^2dy\right)^{\frac{1}{2}}.$$

Thus,

$$\left|\int_0^L z_j(x,1,t)\int_{-\infty}^{+\infty}\omega_j(y)\varphi_j(x,y,t)dydx\right| \leqslant$$
$$\leqslant \left(\int_{-\infty}^{+\infty}\frac{\omega_j^2(y)}{y^2+\eta_j}dy\right)^{\frac{1}{2}}\left(\int_0^L|z_j(x,1,t)|^2dx\right)^{\frac{1}{2}}\left(\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_j)|\varphi_j(x,y,t)|^2dydx\right)^{\frac{1}{2}},$$

$$\left|\int_0^L u_t(x,t)\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx\right| \leqslant$$
$$\leqslant \left(\int_{-\infty}^{+\infty}\frac{\omega_1^2(y)}{y^2+\eta_1}dy\right)^{\frac{1}{2}}\left(\int_0^L|u_t(x,t)|^2dx\right)^{\frac{1}{2}}\left(\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_1)|\varphi_1(x,y,t)|^2dydx\right)^{\frac{1}{2}},$$

$$\left|\int_0^L \phi_t(x,t)\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx\right| \leqslant$$
$$\leqslant \left(\int_{-\infty}^{+\infty}\frac{\omega_2^2(y)}{y^2+\eta_2}dy\right)^{\frac{1}{2}}\left(\int_0^L|\phi_t(x,t)|^2dx\right)^{\frac{1}{2}}\left(\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_2)|\varphi_2(x,y,t)|^2dydx\right)^{\frac{1}{2}}.$$

Using the Cauchy-Schwarz and Young inequalities in (26) and the estimates above, we get

$$\frac{d}{dt}E(t) \leqslant \left(-a_2+\frac{|\gamma|I_1^2}{2}+\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L|u_t(x,t)|^2dx+\left(\frac{I_1^2|\gamma|}{2}-\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L|z_1(x,1,t)|^2dx$$
$$+\left(-\tilde{a}_2+\frac{|\tilde{\gamma}|I_2^2}{2}+\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L|\phi_t(x,t)|^2dx+\left(\frac{I_2^2|\tilde{\gamma}|}{2}-\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L|z_2(x,1,t)|^2dx,$$

which implies that

$$\frac{d}{dt}E(t) \leqslant -C_1\int_0^L(|u_t(x,t)|^2+|z_1(x,1,t)|^2)dx - C_2\int_0^L(|\phi_t(x,t)|^2+|z_2(x,1,t)|^2)dx,$$

with

$$\begin{cases} C_1 = \min\left\{\left(-a_2+\dfrac{|\gamma|I_1^2}{2}+\dfrac{\nu_1\tau_1^{-1}}{2}\right),\ \left(\dfrac{I_1^2|\gamma|}{2}-\dfrac{\nu_1\tau_1^{-1}}{2}\right)\right\}, \\ C_2 = \min\left\{\left(-\tilde{a}_2+\dfrac{|\tilde{\gamma}|I_2^2}{2}+\dfrac{\nu_2\tau_2^{-1}}{2}\right),\ \left(\dfrac{I_2^2|\tilde{\gamma}|}{2}-\dfrac{\nu_2\tau_2^{-1}}{2}\right)\right\}. \end{cases} \tag{27}$$

As ν_j is chosen to satisfy the assumption (21), the constants C_1 and C_2 are positive. The proof of Proposition (3.1) is complete. □

4. Semigroup setup

In this section, we present the semigroup formulation for the augmented problem. Let $U = (u, v, z_1, \varphi_1, \phi, \psi, z_2, \varphi_2)^T$, where $v = u_t$ and $\psi = \phi_t$. The system (P') can be written as a Cauchy evolution problem

$$\begin{aligned} U_t &= \mathcal{A}U, \\ U(0) &= (u_0, u_1, \phi_{01}, \phi_0, \psi_1, \varphi_{02})^T, \end{aligned} \tag{28}$$

where the operator $\mathcal{A} : D(\mathcal{A}) \subseteq \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathcal{A}U := \begin{pmatrix} v \\ \frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x - \frac{\gamma}{\rho}\int_{-\infty}^{+\infty} \omega_1(y)\varphi_1(x,y,t)dy - \frac{a_2}{\rho}v \\ -\frac{z_{1p}}{\tau_1}(x,p,t) \\ -(y^2+\eta_1)\varphi_1 + z_1(x,1,t)\omega_1(y) \\ \psi \\ \frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi - \frac{\tilde{\gamma}}{J}\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y,t)dy - \frac{\tilde{a_2}}{J}\psi \\ -\frac{z_{2p}}{\tau_2}(x,p,t) \\ -(y^2+\eta_2)\varphi_2 + z_2(x,1,t)\omega_2(y) \end{pmatrix}, \tag{29}$$

with domain

$$D(\mathcal{A}) = \left\{ \mathcal{U} \in \mathcal{H} \;\middle|\; \begin{array}{l} v,\ \psi \in H_0^1(0,L),\ u,\phi \in H^2(0,L) \cap H_0^1(0,L), \\ -(y^2+\eta_1)\varphi_1 + z_1(x,1)\omega_1(y) \in L^2\left((0,L)\times(-\infty,+\infty)\right), \\ -(y^2+\eta_2)\varphi_2 + z_2(x,1)\omega_2(y) \in L^2\left((0,L)\times(-\infty,+\infty)\right), \\ z_1 \in L^2((0,L)\times H^1(0,L)),\ z_2 \in L^2((0,L)\times H^1(0,L)), \\ |y|\varphi_1,\ |y|\varphi_2 \in L^2\left((0,L)\times(-\infty,+\infty)\right), \\ (v,\psi) = (z_1,z_2)(\,\cdot\,,0) \text{ on } (0,L). \end{array} \right\}$$

and the phase space $\mathcal{H}$ is given by

$$\mathcal{H} = \left(H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1))\times L^2((0,L)\times(-\infty,+\infty))\right)^2. \tag{30}$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Take $U = (u,v,z_1,\varphi_1,\phi,\psi,z_2,\varphi_2)^T$ and $\overline{U} = (\overline{u},\overline{v},\overline{z_1},\overline{\varphi_1},\overline{\phi},\overline{\psi},\overline{z_2},\overline{\varphi_2})^T$. The inner product in $\mathcal{H}$ is defined by

$$\begin{aligned} \langle U,\overline{U}\rangle_{\mathcal{H}} = & \int_0^L \left[\rho v\overline{v} + J\psi\overline{\psi} + \mu u_x\overline{u_x} + \delta\phi_x\overline{\phi_x} + b(u_x\overline{\phi} + \overline{u_x}\phi) + \xi\phi\overline{\phi}\right]dx \\ & + \nu_1\int_0^L\int_0^1 z_1(x,p)\overline{z_1}(x,p)dpdx + \nu_2\int_0^L\int_0^1 z_2(x,p)\overline{z_2}(x,p)dpdx \\ & + |\gamma|\int_0^L\int_{-\infty}^{+\infty} \varphi_1\overline{\varphi_1}dydx + |\tilde{\gamma}|\int_0^L\int_{-\infty}^{+\infty} \varphi_2\overline{\varphi_2}dydx. \end{aligned} \tag{31}$$

From (2) we recall that $\mu|u_x|^2 + 2bu_x\phi + \xi|\phi|^2 \geq 0$, and therefore

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 = \langle U,U\rangle_{\mathcal{H}} = & \int_0^L [\rho|v|^2 + J|\psi|^2 + \mu|u_x|^2 + \delta|\phi_x|^2 + 2bu_x\phi + \xi|\phi|^2]dx \\ & + \nu_1\int_0^L\int_0^1 |z_1(x,p)|^2dpdx + \nu_2\int_0^L\int_0^1 |z_2(x,p)|^2dpdx \\ & + |\gamma|\int_0^L\int_{-\infty}^{+\infty} |\varphi_1|^2dydx + |\tilde{\gamma}|\int_0^L\int_{-\infty}^{+\infty} |\varphi_2|^2dydx, \end{aligned} \tag{32}$$

is a norm in $\mathcal{H}$.

Remark 4.1. *The conditions* $|y|\varphi_1,\ |y|\varphi_2 \in L^2\left((0,L)\times(-\infty,+\infty)\right)$ *are essential to ensure that* $\int_{-\infty}^{+\infty} |y|^{\frac{2\alpha-1}{2}}\varphi_i dy \in L^2(0,L)$, *as we have considered the integral in the sense of Lebesgue. In other words, the absolute value of* $\omega(y)\varphi_i$ *must also be integrable over the real line.*

5. Existence and uniqueness of the solution

In this section, we study the well-posedness of the system (P') using the semigroup theory of linear operators. We use the Lumer-Phillips Theorem (see Theorem 4.3 on Page 13 in [30]).

Lemma 5.1. *The operator $\mathcal{A}$ is dissipative.*

Proof. Take $U=(u,v,\varphi_1,\phi,\psi,\varphi_2)^T\in D(\mathcal{A})$. Straightforward calculation leads to

$$\begin{aligned}\mathcal{R}e\langle \mathcal{A}U,U\rangle_{\mathcal{H}}=&-a_2\int_0^L|v|^2dx-\tilde{a_2}\int_0^L|\psi|^2dx\\&-\gamma\mathcal{R}e\int_0^L\overline{v}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y,t)dydx-\tilde{\gamma}\mathcal{R}e\int_0^L\overline{\psi}\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y,t)dydx\\&-\frac{\nu_1}{\tau_1}\mathcal{R}e\int_0^L\int_0^1 z_{1p}(x,p)\overline{z_1}(x,p)dpdx-\frac{\nu_2}{\tau_2}\mathcal{R}e\int_0^L\int_0^1 z_{2p}(x,p)\overline{z_2}(x,p)dpdx\\&-|\gamma|\mathcal{R}e\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_1]|\varphi_1(x,y)|^2dydx-|\tilde{\gamma}|\mathcal{R}e\int_0^L\int_{-\infty}^{+\infty}[y^2+\eta_2]|\varphi_2(x,y)|^2dydx\\&+|\gamma|\mathcal{R}e\int_0^L z_1(x,1)\int_{-\infty}^{+\infty}\omega_1(y)\overline{\varphi_1}(x,y)dydx+|\tilde{\gamma}|\mathcal{R}e\int_0^L z_2(x,1)\int_{-\infty}^{+\infty}\omega_2(y)\overline{\varphi_2}(x,y,t)dydx.\end{aligned}\tag{33}$$

Using $z_1(x,0)=v(x)$ and $z_2(x,0)=\psi(x)$, we note that

$$\begin{aligned}&\frac{\nu_1}{\tau_1}\mathcal{R}e\int_0^L\int_0^1 z_{1p}(x,p)\overline{z_1}(x,p)dpdx+\frac{\nu_2}{\tau_2}\mathcal{R}e\int_0^L\int_0^1 z_{2p}(x,p)\overline{z_2}(x,p)dpdx\\&\qquad=\frac{\nu_1}{\tau_1}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial p}|z_1(x,p)|^2dpdx+\frac{\nu_2}{\tau_2}\int_0^L\int_0^1\frac{1}{2}\frac{\partial}{\partial p}|z_2(x,p)|^2dpdx\\&\qquad=\frac{\nu_1}{2\tau_1}\int_0^L\{|z_1(x,1)|^2-|v(x)|^2\}dx+\frac{\nu_2}{2\tau}\int_0^L\{|z_2(x,1)|^2-|\psi(x)|^2\}dx.\end{aligned}\tag{34}$$

Applying the Cauchy-Schwarz and Young's inequalities, and then proceeding in the same way as in the proof of Proposition 3.1, we obtain

$$\begin{aligned}\mathcal{R}e\langle \mathcal{A}U,U\rangle_{\mathcal{H}}\leqslant&\left(-a_2+\frac{|\gamma|I_1^2}{2}+\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L|v(x)|^2dx+\left(\frac{I_1^2|\gamma|}{2}-\frac{\nu_1\tau_1^{-1}}{2}\right)\int_0^L|z_1(x,1)|^2dx\\&+\left(-\tilde{a_2}+\frac{|\tilde{\gamma}|I_2^2}{2}+\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L|\psi(x)|^2dx+\left(\frac{I_2^2|\tilde{\gamma}|}{2}-\frac{\nu_2\tau_2^{-1}}{2}\right)\int_0^L|z_2(x,1)|^2dx\\&\leqslant -C_1\int_0^L(|v(x)|^2+|z_1(x,1)|^2)dx-C_2\int_0^L(|\psi(x)|^2+|z_2(x,1)|^2)dx\leqslant 0,\end{aligned}\tag{35}$$

where C_1 and C_2 are positive constants and $I_j=\displaystyle\int_{-\infty}^{+\infty}\frac{\omega_j^2(y)}{y^2+\eta_j}dy,\ j=1,2.$ □

Theorem 5.1. *The operator $\mathcal{A}$, defined by (29), is the infinitesimal generator of a C_0-semigroup of contractions $S(t),\ t\geqslant 0$, on $\mathcal{H}$.*

Proof. As $\mathcal{A}$ is densely defined and dissipative, it is enough to prove that $\mathcal{A}$ is maximal. Given $F=(f_1,f_2,\cdots,f_7,f_8)^T\in\mathcal{H}$, consider the resolving equation in $\mathcal{H}$:

$$(I-\mathcal{A})U=F.\tag{36}$$

We need to show that the solution $U=(u,v,\varphi_1,\phi,\psi,\varphi_2)^T\in\mathcal{H}$ of (36) is in $D(\mathcal{A})$. The resolving equation in terms of its components is given by

$$u-v=f_1\ \in\ H_0^1(0,L),\tag{37}$$

$$v-\frac{\mu}{\rho}u_{xx}-\frac{b}{\rho}\phi_x+\frac{\gamma}{\rho}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dy-\frac{a_2}{\rho}v=f_2\ \in L^2(0,L),\tag{38}$$

$$z_1+\frac{z_{1p}}{\tau_1}=f_3\ \in L^2\left((0,L)\times(0,1)\right),\tag{39}$$

$$\varphi_1 + (y^2+\eta_1)\varphi_1 - z_1(x,1)\omega_1(y) = f_4 \ \in L^2\left((0,L)\times(-\infty,+\infty)\right), \tag{40}$$

$$\phi - \psi = f_5 \ \in \ H_0^1(0,L), \tag{41}$$

$$\psi - \frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi + \int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dy - \frac{\tilde{a_2}}{J}\psi = f_6 \ \in L^2(0,L), \tag{42}$$

$$z_2 + \frac{z_{2p}}{\tau_2} = f_7 \ \in L^2\left((0,L)\times(0,1)\right), \tag{43}$$

$$\varphi_2 + (y^2+\eta_2)\varphi_2 - z_2(x,1)\omega_2(y) = f_8 \ \in \ L^2\left((0,L)\times(-\infty,+\infty)\right). \tag{44}$$

From (37) and (41), we get

$$\begin{cases} v = u - f_1 & \in \ H_0^1(0,L), \\ \psi = \phi - f_5 & \in \ H_0^1(0,L). \end{cases} \tag{45}$$

By using (40) and (44), we find φ_i $(i=1,2)$ such that

$$\begin{cases} \varphi_1 = \dfrac{f_4(x,y) + z_1(x,1)\omega_1(y)}{y^2+\eta_1+1}, \\[2ex] \varphi_2 = \dfrac{f_8(x,y) + z_2(x,1)\omega_2(y)}{y^2+\eta_2+1}. \end{cases} \tag{46}$$

From (39) and (43) and $(z_1,z_2)(x,0) = (v,\psi)(x)$, it follows that

$$\begin{cases} z_1(x,p) = u(x)e^{-\tau_1 p} - f_1(x)e^{-\tau_1 p} + \tau_1 e^{-\tau_1 p}\displaystyle\int_0^p e^{\tau_1 s} f_3(x,s)ds, \\[2ex] z_2(x,p) = \phi(x)e^{-\tau_2 p} - f_5(x)e^{-\tau_2 p} + \tau_2 e^{-\tau_2 p}\displaystyle\int_0^p e^{\tau_2 s} f_7(x,s)ds. \end{cases} \tag{47}$$

Using

$$\begin{cases} (z_1)_0(x) = -f_1(x)e^{-\tau_1} + \tau_1 e^{-\tau_1}\displaystyle\int_0^1 e^{\tau_1 s} f_3(x,s)ds, \\[2ex] (z_2)_0(x) = -f_5(x)e^{-\tau_2} + \tau_2 e^{-\tau_2}\displaystyle\int_0^1 e^{\tau_2 s} f_7(x,s)ds, \end{cases}$$

and (47), we deduce that

$$\begin{cases} z_1(x,1) = u(x)e^{-\tau_1} + (z_1)_0(x), \\ z_2(x,1) = \phi(x)e^{-\tau_2 p} + (z_2)_0(x). \end{cases} \tag{48}$$

By using (45) in (38) and (42), we obtain u and ϕ, which satisfy the following system:

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x + \gamma\displaystyle\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dy + a_2 u = \rho(f_1+f_2) + a_2 f_1, \\[2ex] J\phi - \delta\phi_{xx} + bu_x + \xi\phi + \tilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dy + \tilde{a_2}\phi = J(f_5+f_6) + \tilde{a_2}f_5. \end{cases} \tag{49}$$

Combining (46) and (49), we get

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x + \gamma\displaystyle\int_{-\infty}^{+\infty} z_1(x,1)\frac{\omega_1^2(y)}{y^2+\eta_1+1}dy + a_2 u \\[2ex] \qquad = \rho(f_1+f_2) + a_2 f_1 - \gamma\displaystyle\int_{-\infty}^{+\infty}\frac{\omega_1(y)f_4(x,y)}{y^2+\eta_1+1}dy, \\[2ex] J\phi - \delta\phi_{xx} + bu_x + \xi\phi + \tilde{\gamma}\displaystyle\int_{-\infty}^{+\infty} z_2(x,1)\frac{\omega_2^2(y)}{y^2+\eta_2+1}dy + \tilde{a_2}\phi \\[2ex] \qquad = J(f_5+f_6) + \tilde{a_2}f_5 - \tilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\frac{\omega_2(y)}{y^2+\eta_2+1}f_8(x,y)dy. \end{cases} \tag{50}$$

Using (48) in (50), and recalling the fact that

$$\int_{-\infty}^{+\infty}\frac{\omega_j^2(y)}{y^2+\eta_j+1}dy = \frac{\pi}{\sin(\alpha_j\pi)}(1+\eta_j)^{\alpha_j-1} = \frac{a_j}{\gamma}(1+\eta_j)^{\alpha_j-1}, \ \ j=1,2,$$

we obtain

$$\begin{cases} \rho u - \mu u_{xx} - b\phi_x + u\left(a_1(1+\eta_1)^{\alpha_1-1}e^{-\tau_1} + a_2\right) \\ \quad = \rho(f_1+f_2) - a_1(1+\eta_1)^{\alpha_1-1}(z_1)_0(x) \\ \quad +a_2 f_1 - \gamma\displaystyle\int_{-\infty}^{+\infty}\frac{\omega_1(y)}{y^2+\eta_1+1}f_4(x,y)dy, \\ J\phi - \delta\phi_{xx} + bu_x + \xi\phi + \phi\left(a_2(1+\eta_2)^{\alpha_2-1}e^{-\tau_2} + \tilde{a_2}\right) \\ \quad = J(f_5+f_6) + a_2(1+\eta_2)^{\alpha_2-1}(z_2)_0(x) \\ \quad +\tilde{a_2}f_5 - \tilde{\gamma}\displaystyle\int_{-\infty}^{+\infty}\frac{\omega_2(y)}{y^2+\eta_2+1}f_8(x,y)dy. \end{cases} \tag{51}$$

Solving the system (51) is equivalent to finding $(u,\ \phi)\in H_0^1(0,L)\times H_0^1(0,L)$ such that

$$\rho\int_0^L u\chi dx - \mu\int_0^L u_{xx}\chi dx - b\int_0^L \phi_x\chi dx + \left(a_1(1+\eta_1)^{\alpha_1-1}e^{-\tau_1}+a_2\right)\int_0^L u\chi dx$$

$$= \rho\int_0^L (f_1+f_2)\chi dx - a_1(1+\eta_1)^{\alpha_1-1}\int_0^1 (z_1)_0(x)\chi dx + a_2\int_0^L f_1\chi dx - \gamma\int_0^L \chi\int_{-\infty}^{+\infty}\frac{\omega_1(y)}{y^2+\eta_1+1}f_4(x,y)dydx,$$

$$J\int_0^L \phi\zeta dx - \delta\int_0^L \phi_{xx}\zeta dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx + \left(a_2(1+\eta_2)^{\alpha_2-1}e^{-\tau_2}+\tilde{a_2}\right)\int_0^L \phi\zeta dx$$

$$= J\int_0^L (f_5+f_6)\zeta dx - a_2(1+\eta_2)^{\alpha_2-1}\int_0^1 (z_2)_0(x)\zeta dx + \tilde{a_2}\int_0^L f_5\zeta dx - \tilde{\gamma}\int_0^L \zeta\int_{-\infty}^{+\infty}\frac{\omega_2(y)}{y^2+\eta_2+1}f_8(x,y)dydx$$

for all $(\chi,\ \zeta)\in H_0^1(0,L)\times H_0^1(0,L)$. Making integration by parts, we get

$$\rho\int_0^L u\chi dx + \mu\int_0^L u_x\chi_x dx + b\int_0^L \phi\chi_x dx + \left(a_1(1+\eta_1)^{\alpha_1-1}e^{-\tau_1}+a_2\right)\int_0^L u\chi dx$$

$$= \rho\int_0^L (f_1+f_2)\chi dx - a_1(1+\eta_1)^{\alpha_1-1}\int_0^1 (z_1)_0(x)\chi dx + a_2\int_0^L f_1\chi dx - \gamma\int_0^L \chi\int_{-\infty}^{+\infty}\frac{\omega_1(y)}{y^2+\eta_1+1}f_4(x,y)dydx,$$

$$J\int_0^L \phi\zeta dx + \delta\int_0^L \phi_x\zeta_x dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx + \left(a_2(1+\eta_2)^{\alpha_2-1}e^{-\tau_2}+\tilde{a_2}\right)\int_0^L \phi\zeta dx$$

$$= J\int_0^L (f_5+f_6)\zeta dx - a_2(1+\eta_2)^{\alpha_2-1}\int_0^1 (z_2)_0(x)\zeta dx + \tilde{a_2}\int_0^L f_5\zeta dx - \tilde{\gamma}\int_0^L \zeta\int_{-\infty}^{+\infty}\frac{\omega_2(y)}{y^2+\eta_2+1}f_8(x,y)dydx.$$

The problem given in the above systems is equivalent to finding $(u,\phi)\in\mathcal{V}$ for the following variational problem:

$$\mathbb{B}\big((u,\phi),\ (\chi,\zeta)\big) = \mathbb{L}((\chi,\zeta)),\quad \forall\ (\chi,\zeta)\in\mathcal{V}, \tag{52}$$

with $\mathbb{B}:\ \mathcal{V}\times\mathcal{V}\longrightarrow\ \mathbb{C}$ and $\mathbb{L}:\mathcal{V}\longrightarrow\mathbb{C}$, where $\mathcal{V}=H_0^1(0,L)\times H_0^1(0,L)$ is a Hilbert space equipped with the following norm

$$\|(u,\phi)\|_{\mathcal{V}}^2 = \rho\int_0^L |u|^2dx + \delta\int_0^L |\phi_x|^2dx + J\int_0^L |\phi|^2dx + \mu\int_0^L |u_x|^2dx + 2b\int_0^L u_x\phi dx + \xi\int_0^L |\phi|^2dx$$

$$+\left(a_1(1+\eta_1)^{\alpha_1-1}e^{-\tau_1}+a_2\right)\int_0^L |u|^2dx + \left(a_2(1+\eta_2)^{\alpha_2-1}e^{-\tau_2}+\tilde{a_2}\right)\int_0^L |\phi|^2dx,$$

and

$$\mathbb{B}\big((u,\phi),\ (\chi,\zeta)\big) = \rho\int_0^L u\chi dx + \delta\int_0^L \phi_x\zeta_x dx + J\int_0^L \phi\zeta dx + \mu\int_0^L u_x\chi_x dx + b\int_0^L \phi\chi_x dx + b\int_0^L u_x\zeta dx + \xi\int_0^L \phi\zeta dx$$

$$+\left(a_1(1+\eta_1)^{\alpha_1-1}e^{-\tau_1}+a_2\right)\int_0^L u\chi dx + \left(a_2(1+\eta_2)^{\alpha_2-1}e^{-\tau_2}+\tilde{a_2}\right)\int_0^L \phi\zeta dx,$$

$$\mathbb{L}((\chi,\zeta)) = \rho\int_0^L (f_1+f_2)\chi dx + a_2\int_0^L f_1\chi dx + \tilde{a_2}\int_0^L f_5\zeta dx + J\int_0^L (f_5+f_6)\zeta dx$$
$$- a_1(1+\eta_1)^{\alpha_1-1}\int_0^1 (z_1)_0(x)\chi dx - a_2(1+\eta_2)^{\alpha_2-1}\int_0^1 (z_2)_0(x)\zeta dx$$
$$- \gamma\int_0^L \chi\int_{-\infty}^{+\infty} \frac{\omega_1(y)}{y^2+\eta_1+1} f_4(x,y)dydx - \tilde{\gamma}\int_0^L \zeta\int_{-\infty}^{+\infty} \frac{\omega_2(y)}{y^2+\eta_2+1} f_8(x,y)dydx.$$

A straightforward calculation shows that $\mathbb{B}$ is a sesquilinear, bounded, and coercive form on the Hilbert space $\mathcal{V}\times\mathcal{V}$ and that $\mathbb{L}$ is a linear and continuous form on $\mathcal{V}$. By applying the Lax-Milgram Lemma, we deduce that for all $(\chi,\zeta)\in H_0^1(0,L)\times H_0^1(0,L)$, variational problem (52) has a unique solution $(u,\phi)\in H_0^1(0,L)\times H_0^1(0,L)$. Because of the standard elliptic regularity, it follows from (49) that $(u,\phi)\in H^2(0,L)\times H^2(0,L)$, and thus

$$(u,\phi)\in [H_0^1(0,L)\cap H^2(0,L)]\times[H_0^1(0,L)\cap H^2(0,L)].$$

Clearly, $v,\psi\in H_0^1(0,L)$. Finally, to complete the proof that $(u,v,\phi,\psi)\in D(\mathcal{A})$, note that the last condition $|y|\varphi_1,\ |y|\varphi_2\in L^2\left((0,L)\times(-\infty,+\infty)\right)$ imposed by the domain of $\mathcal{A}$ is satisfied, because $0<\alpha_j<1,\ j=1,2$. Therefore, we conclude that $\mathcal{A}$ is maximal. □

The existence and uniqueness of the solution are given by the next theorem.

Theorem 5.2. *(a) If $U_0\in\mathcal{H}$, then the system* (28) *has a unique weak solution*

$$U\in C^0(\mathbb{R}_+,\mathcal{H}).$$

(b) If $U_0\in D(\mathcal{A})$, then the system (28) *has a unique strong solution*

$$U\in C^0(\mathbb{R}_+,D(\mathcal{A}))\cap C^1(\mathbb{R}_+,\mathcal{H}).$$

Proof. Note that $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup of contractions $S(t)=e^{\mathcal{A}t}$, where $t\geqslant 0$. Define $U(t)=e^{\mathcal{A}t}U(0)$. By the general theory of semigroups of linear operators, $U(t)$ is a unique solution of (28) satisfying the conditions (a) and (b). □

6. Asymptotic stability

In this section, we use a general criterion due to Arendt-Batty [3] and Lyubich-Vũ [22] to show the asymptotic stability of the C_0-semigroup $S(t)=e^{t\mathcal{A}}$ associated with the system (P').

Theorem 6.1 (Stability Theorem: see Page 837 in [3]). *Let $\mathcal{A}$ be the generator of a bounded C_0-semigroup $\{S(t)\}_{t\geqslant 0}$ over a Hilbert space $\mathcal{H}$. If no eigenvalue of $\mathcal{A}$ lies on the imaginary axis $i\mathbb{R}$ and if $\sigma(\mathcal{A})\cap i\mathbb{R}$ is countable, then $\{S(t)\}_{t\geqslant 0}$ is asymptotically stable. That is, $\lim_{t\to\infty}\|S(t)x\|_{\mathcal{H}}=0$ for all $x\in\mathcal{H}$.*

Lemma 6.1. *The generator $\mathcal{A}$ has no eigenvalue on $i\mathbb{R}$.*

Proof. We consider two cases according to $i\lambda\neq 0$ and $i\lambda=0$. Here,

$$\mathcal{H}=\left(H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1))\times L^2((0,L)\times(-\infty,+\infty))\right)^2. \tag{53}$$

Case 1: $i\lambda\neq 0$. Let us argue by contradiction. Suppose that there exists $\lambda\in\mathbb{R},\quad \lambda\neq 0$ and $U\neq 0$ such that $\mathcal{A}U=i\lambda U$; we have the following equations in terms of its components:

$$i\lambda u - v = 0 \text{ in } H_0^1(0,L), \tag{54}$$

$$i\lambda v - \frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\phi_x + \frac{\gamma}{\rho}\int_{-\infty}^{+\infty}\omega_1(x,y)\varphi_1(x,y)dy + \frac{a_2}{\rho}v = 0 \text{ in } L^2(0,L), \tag{55}$$

$$i\lambda z_1 + \frac{z_{1p}}{\tau_1} = 0 \text{ in } L^2((0,L)\times(0,1)), \tag{56}$$

$$i\lambda\varphi_1 + (y^2+\eta_1)\varphi_1 - z_1(x,1)\omega_1(x,y) = 0 \text{ in } L^2((0,L)\times(-\infty,+\infty)), \tag{57}$$

$$i\lambda\phi - \psi = 0 \text{ in } H_0^1(0,L), \tag{58}$$

$$i\lambda\psi - \frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi + \frac{\tilde{\gamma}}{J}\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dy + \frac{\tilde{a_2}}{J}\psi = 0 \text{ in } L^2(0,L), \tag{59}$$

$$i\lambda z_2 + \frac{z_{2p}}{\tau_2} = 0 \text{ in } L^2((0,L)\times(0,1)), \tag{60}$$

$$i\lambda\varphi_2 + (y^2+\eta_2)\varphi_2 - z_2(x,1)\omega_2(y) = 0 \text{ in } L^2((0,L)\times(-\infty,+\infty)). \tag{61}$$

Consider $U \in D(\mathcal{A})$ with the unit norm $\|U\|_{\mathcal{H}} = 1$. By making the inner product of U with $\mathcal{A}U$ in the resolvent equation, taking the real part, and using (35), we get

$$C_1\int_0^L \left(|v|^2 + |z(x,1)|^2\right)dx + C_2\int_0^L \left(|\psi|^2 + |z_2(x,1)|^2\right)dx = 0,\ C_1, C_2 > 0,$$

and thus,

$$v = 0,\quad \psi = 0,\quad z_j(x,1) = 0,\quad j = 1,2, \text{ a.e. in } L^2((0,L). \tag{62}$$

From (54) and (58), we obtain

$$u = \phi = 0, \text{ a.e. in } L^2((0,L). \tag{63}$$

From (57) and (61), we deduce that

$$\varphi_1, \varphi_2 = 0, \text{ a.e. in } L^2((0,L). \tag{64}$$

Using (62), (63), (64), we get

$$\begin{cases} -\frac{\mu}{\rho}u_{xx} - \frac{b}{\rho}\phi_x &= 0,\\ -\frac{\delta}{J}\phi_{xx} + \frac{b}{J}u_x + \frac{\xi}{J}\phi &= 0,\\ u(0) = \phi(0) = u(L) = \phi(L) &= 0. \end{cases} \tag{65}$$

Multiplying $(65)_{1,2}$ respectively by $\rho\overline{u}$, $J\overline{\phi}$ and integrating on $(0,L)$, we obtain

$$\begin{cases} -\mu\int_0^L u_{xx}\overline{u}dx - b\int_0^L \phi_x\overline{u}dx &= 0,\\ -\delta\int_0^L \phi_{xx}\overline{\phi}dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L \phi\overline{\phi}dx &= 0. \end{cases} \tag{66}$$

Integrating by parts, we get

$$\mu\int_0^L |u_x|^2dx + b\int_0^L \phi\overline{u_x}dx = 0 \quad\text{and}\quad \delta\int_0^L |\phi_x|^2dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2dx = 0.$$

Adding the above equations and taking the real part, we obtain

$$\mu\int_0^L |u_x|^2dx + 2b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2dx + \delta\int_0^L |\phi_x|^2dx = 0.$$

Using (2), we have

$$\int_0^L\left(\frac{b}{\sqrt{\xi}}u_x + \sqrt{\xi}\,\overline{\phi}\right)^2dx + \left(\mu - \frac{b^2}{\xi}\right)\int_0^L |u_x|^2dx + \delta\int_0^L |\phi_x|^2dx = 0.$$

By (1), we have $\mu - \frac{b^2}{\xi} \geq 0$, and thus

$$u_x, \phi_x = 0, \text{ a.e. in } L^2((0,L). \tag{67}$$

From (63) and (67), we deduce that

$$u, \phi = 0, \text{ a.e. in } H_0^1((0,L), \tag{68}$$

and thereby $\|U\|_{\mathcal{H}} = 0$, and we have a contradiction. Therefore, $\lambda \neq 0$ is not an eigenvalue of $\mathcal{A}$.

Case 2: $i\lambda = 0$. Taking $\lambda = 0$ in the resolvent equation, we get $v = 0, \psi = 0, z_j(x,1) = 0, \varphi_1 = 0, \varphi_2 = 0$, a.e. in $L^2((0,L)$ and so, $-\mathcal{A}U = 0$ leads to (65). Proceeding in the same way as in Case 1, we deduce that $\|U\|_{\mathcal{H}} = 0$ and thus we have a contradiction. Therefore, $\lambda = 0$ is not an eigenvalue of $\mathcal{A}$. □

Remark 6.1. *Note that Lemma 6.1 is related to the first condition of Theorem 6.1. The second condition of Theorem 6.1 will be satisfied if we show that $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is at most a countable set. The proof of this fact will be obtained as an immediate consequence of the next proposition.*

Proposition 6.1. *$i\mathbb{R} \subset \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$.*

Proof. If $i\mathbb{R} \subset \rho(\mathcal{A})$ is not valid, there would be a $\lambda \in \mathbb{R}$ such that $i\lambda \in \sigma(\mathcal{A})$, the spectrum of $\mathcal{A}$, which contradicts the fact that there are no eigenvalues of $\mathcal{A}$ on the imaginary axis $i\mathbb{R}$. □

Lemma 6.2. *$\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable.*

Proof. As $i\mathbb{R} \subset \rho(\mathcal{A})$ we have $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\,\}$. □

The main result of this section is the next theorem.

Theorem 6.2. *The C_0-semigroup $S(t) = e^{t\mathcal{A}}$ is asymptotically stable; that is, for $U(0) \in D(\mathcal{A})$, the solution of (28) satisfies*

$$\lim_{t\to\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

Proof. As $\mathcal{A}$ has no eigenvalue in $i\mathbb{R}$ and $\sigma(\mathcal{A}) \cap i\mathbb{R}$ is countable, by Theorem 6.1, the C_0-semigroup $S(t) = e^{t\mathcal{A}}$, $t \geq 0$, is asymptotically stable on $\mathcal{H}$ □

7. Exponential stability

In order to prove the exponential stability, we use the following result, which is due to Gearhart [17] (see also [21,31]):

Theorem 7.1. *Let $S(t) = e^{\mathcal{A}t}$ be a C_0-semigroup of contractions on a Hilbert space $\mathcal{H}$. Then, $S(t)$ is exponentially stable, if and only if,*

$$\rho(\mathcal{A}) \supseteq \{i\beta\,;\ \ \beta \in \mathbb{R}\} = i\mathbb{R} \tag{69}$$

and

$$\overline{\lim_{|\beta|\to\infty}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{70}$$

Theorem 7.2. *The C_0-semigroup $S(t) = e^{t\mathcal{A}}$, $t \geq 0$, is exponentially stable on $\mathcal{H}$.*

Proof. The resolvent equation

$$(i\lambda I - \mathcal{A})U = F,\ \ \lambda \in \mathbb{R}, \tag{71}$$

where $U = (u, v, z_1, \varphi_1, \phi, \psi, z_2, \varphi_2)^T \in D(\mathcal{A})$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, leads to

$$\left\{\begin{array}{rl}
i\lambda u - v & = f_1,\\
i\lambda v - \dfrac{\mu}{\rho} u_{xx} - \dfrac{b}{\rho}\phi_x + \dfrac{\gamma}{\rho}\displaystyle\int_{-\infty}^{+\infty} \omega_1(x,y)\varphi_1(x,y)dy + \dfrac{a_2}{\rho} v & = f_2,\\
i\lambda z_1 + \dfrac{z_{1p}}{\tau_1} & = f_3,\\
i\lambda\varphi_1 + (y^2+\eta_1)\varphi_1 - z_1(x,1)\omega_1(x,y) & = f_4,\\
i\lambda\phi - \psi & = f_5,\\
i\lambda\psi - \dfrac{\delta}{J}\phi_{xx} + \dfrac{b}{J} u_x + \dfrac{\xi}{J}\phi + \dfrac{\tilde{\gamma}}{J}\displaystyle\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y)dy + \dfrac{\tilde{a_2}}{J}\psi & = f_6,\\
i\lambda z_2 + \dfrac{z_{2p}}{\tau_2} & = f_7,\\
i\lambda\varphi_2 + (y^2+\eta_2)\varphi_2 - z_2(x,1)\omega_2(y) & = f_8.
\end{array}\right. \tag{72}$$

Making the inner product of U with F in (71) and taking the real part, we get

$$|Re\,\langle \mathcal{A}U, U\rangle_{\mathcal{H}}| \leqslant \|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{73}$$

From (35), we deduce

$$\int_0^L |v(x)|^2 dx,\ \ \int_0^L |\psi|^2 dx,\ \ \int_0^L |z_j(x,1)|^2 dx\ \leqslant C\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}}. \tag{74}$$

From $(72)_4$ and $(72)_8$, we find φ_i, $i = 1, 2$, such that

$$\begin{cases} \varphi_1 = \dfrac{f_4(x,y) + z_1(x,1)\omega_1(y)}{y^2+\eta_1+1}, \\ \varphi_2 = \dfrac{f_8(x,y) + z_2(x,1)\omega_2(y)}{y^2+\eta_2+1}. \end{cases} \tag{75}$$

Applying Young's inequality, we obtain

$$\begin{aligned} \|\varphi_1\|_{L^2((0,L)\times(-\infty,+\infty))} &\leqslant \left\|\frac{\omega_1(y)}{y^2+\eta_1+i\lambda}\right\|_{L^2(-\infty,+\infty)} \|z_1(x,1)\|_{L^2(0,L)} + \left\|\frac{f_4}{y^2+\eta_1+i\lambda}\right\|_{L^2(0,L)\times(-\infty,+\infty)} \\ &\leqslant \left(2(1-\alpha_1)\frac{\pi}{\sin(\alpha_1\pi)}(|\lambda|+\eta_1)^{\alpha_1-2}\right)^{\frac{1}{2}} \|z_1(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{|\lambda|+\eta_1}\|f_4\|_{L^2((0,L)\times(-\infty,+\infty))} \end{aligned} \tag{76}$$

and

$$\begin{aligned} \|y\varphi_1\|_{L^2((0,L)\times(-\infty,+\infty))} &\leqslant \left\|\frac{y\omega_1(y)}{y^2+\eta_1+i\lambda}\right\|_{L^2(-\infty,+\infty)} \|z_1(x,1)\|_{L^2(0,L)} + \left\|\frac{yf_4}{y^2+\eta_1+i\lambda}\right\|_{L^2(0,L)\times(-\infty,+\infty)} \\ &\leqslant \left(2\alpha_1\frac{\pi}{\sin(\alpha_1\pi)}(|\lambda|+\eta_1)^{\alpha_1-1}\right)^{\frac{1}{2}} \|z_1(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{\sqrt{|\lambda|+\eta_1}}\|f_4\|_{L^2((0,L)\times(-\infty,+\infty))}. \end{aligned} \tag{77}$$

Analogously, we have

$$\begin{aligned} \|\varphi_2\|_{L^2((0,L)\times(-\infty,+\infty))} &\leqslant \left\|\frac{\omega_2(y)}{y^2+\eta_2+i\lambda}\right\|_{L^2(-\infty,+\infty)} \|z_2(x,1)\|_{L^2(0,L)} + \left\|\frac{f_8}{y^2+\eta_2+i\lambda}\right\|_{L^2(0,L)\times(-\infty,+\infty)} \\ &\leqslant \left(2(1-\alpha_2)\frac{\pi}{\sin(\alpha_2\pi)}(|\lambda|+\eta_2)^{\alpha_2-2}\right)^{\frac{1}{2}} \|z_2(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{|\lambda|+\eta_2}\|f_4\|_{L^2((0,L)\times(-\infty,+\infty))} \end{aligned} \tag{78}$$

and

$$\begin{aligned} \|y\varphi_2\|_{L^2((0,L)\times(-\infty,+\infty))} &\leqslant \left\|\frac{y\omega_2(y)}{y^2+\eta_2+i\lambda}\right\|_{L^2(-\infty,+\infty)} \|z_2(x,1)\|_{L^2(0,L)} + \left\|\frac{yf_8}{y^2+\eta_2+i\lambda}\right\|_{L^2(0,L)\times(-\infty,+\infty)} \\ &\leqslant \left(2\alpha_2\frac{\pi}{\sin(\alpha_2\pi)}(|\lambda|+\eta_2)^{\alpha_2-1}\right)^{\frac{1}{2}} \|z_2(x,1)\|_{L^2(0,L)} + \frac{\sqrt{2}}{\sqrt{|\lambda|+\eta_2}}\|f_8\|_{L^2((0,L)\times(-\infty,+\infty))}. \end{aligned} \tag{79}$$

Multiplying $(72)_6$ by $J\overline{\phi}$ and integrating on $(0, L)$, we get

$$\begin{aligned} &i\lambda J\int_0^L \psi\overline{\phi}dx - \delta\int_0^L \phi_{xx}\overline{\phi}dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L \phi\overline{\phi}dx \\ &\quad + \tilde{\gamma}\int_0^L \overline{\phi}\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y)dydx + \tilde{a_2}\int_0^L \psi\overline{\phi}dx = J\int_0^L f_6\overline{\phi}dx. \end{aligned} \tag{80}$$

By $(72)_5$ we get

$$\begin{aligned} &-\delta\int_0^L \phi_{xx}\overline{\phi}dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2dx \\ &= J\int_0^L |\psi|^2dx - \tilde{\gamma}\int_0^L \overline{\phi}\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y)dydx - \tilde{a_2}\int_0^L \psi\overline{\phi}dx + J\int_0^L f_6\overline{\phi}dx + J\int_0^L \psi\overline{f_5}dx, \end{aligned}$$

and integrating by parts, we obtain

$$\begin{aligned} &\delta\int_0^L |\phi_x|^2dx + b\int_0^L u_x\overline{\phi}dx + \xi\int_0^L |\phi|^2dx = J\int_0^L |\psi|^2dx - \tilde{a_2}\int_0^L \psi\overline{\phi}dx \\ &\quad - \tilde{\gamma}\int_0^L \overline{\phi}\int_{-\infty}^{+\infty} \omega_2(y)\varphi_2(x,y)dydx + J\int_0^L f_6\overline{\phi}dx + J\int_0^L \psi\overline{f_5}dx, \end{aligned}$$

and thus

$$\delta\int_0^L|\phi_x|^2dx+b\int_0^L u_x\overline{\phi}dx+\xi\int_0^L|\phi|^2dx = J\int_0^L|\psi|^2dx+\tilde{a_2}\left|\int_0^L\psi\overline{\phi}dx\right| + \tilde{\gamma}\left|\int_0^L\overline{\phi}\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dydx\right|+J\int_0^L|f_6\overline{\phi}+\psi\overline{f_5}|dx. \tag{81}$$

Multiplying $(72)_2$ by $\rho\overline{u}$ and integrating on $(0,L)$, we get

$$i\lambda\rho\int_0^L v\overline{u}dx-\mu\int_0^L u_{xx}\overline{u}dx-b\int_0^L\phi_x\overline{u}dx+\gamma\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx+a_2\int_0^L v\overline{u}dx=\rho\int_0^L f_2\overline{u}dx,$$

and using $(72)_1$, we have

$$-\mu\int_0^L u_{xx}\overline{u}dx-b\int_0^L\phi_x\overline{u}dx=\rho\int_0^L|v|^2dx-a_2\int_0^L v\overline{u}dx-\gamma\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx+\rho\int_0^L f_2\overline{u}dx+\rho\int_0^L v\overline{f_1}dx.$$

Making integration by parts, we obtain

$$\mu\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx=\rho\int_0^L|v|^2dx-a_2\int_0^L v\overline{u}dx-\gamma\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx+\rho\int_0^L(f_2\overline{u}+v\overline{f_1})dx,$$

and thus

$$\mu\int_0^L|u_x|^2dx+b\int_0^L\phi\overline{u_x}dx\leqslant\rho\int_0^L|v|^2dx+\gamma\left|\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx\right|+a_2\left|\int_0^L v\overline{u}dx\right|+\rho\int_0^L|f_2\overline{u}+v\overline{f_1}|dx. \tag{82}$$

By adding (81) and (82), we get

$$\begin{aligned}&\mu\int_0^L|u_x|^2dx+b\int_0^L(\phi\overline{u_x}+u_x\overline{\phi})dx+\xi\int_0^L|\phi|^2dx+\delta\int_0^L|\phi_x|^2dx\\ &\leqslant\rho\int_0^L|v|^2dx+J\int_0^L|\psi|^2dx+\gamma\left|\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx\right|+\tilde{\gamma}\left|\int_0^L\overline{\phi}\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dydx\right|\\ &+a_2\left|\int_0^L v\overline{u}dx\right|+\tilde{a_2}\left|\int_0^L\psi\overline{\phi}\right|+\rho\int_0^L|f_2\overline{u}+v\overline{f_1}|dx+J\int_0^L|f_6\overline{\phi}+\psi\overline{f_5}|dx.\end{aligned} \tag{83}$$

Applying the Cauchy-Schwarz inequality, we get

$$\left|\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx\right|\leqslant\|u\|_{L^2(0,L)}\left(\int_{-\infty}^{+\infty}\frac{|\omega_1(y)|^2}{y^2+\eta_1}dy\right)^{\frac12}\left(\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_1)|\varphi_1(x,y)|^2dydx\right)^{\frac12}.$$

Using Young's inequality, we have

$$\left|\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx\right|\leqslant\frac{\epsilon}{2}\left(\int_{-\infty}^{+\infty}\frac{|\omega_1(y)|^2}{y^2+\eta_1}dy\right)\|u\|^2_{L^2(0,L)}+\frac{1}{2\epsilon}\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_1)|\varphi_1(x,y)|^2dydx.$$

Using Poincaré's inequality, we obtain

$$\left|\int_0^L\overline{u}\int_{-\infty}^{+\infty}\omega_1(y)\varphi_1(x,y)dydx\right|\leqslant\frac{\epsilon}{2}C_p\left(\int_{-\infty}^{+\infty}\frac{|\omega_1(y)|^2}{y^2+\eta_1}dy\right)\|u_x\|^2_{L^2(0,L)}+\frac{1}{2\epsilon}\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_1)|\varphi_1(x,y)|^2dydx. \tag{84}$$

Analogously, we find

$$\left|\int_0^L\overline{\phi}\int_{-\infty}^{+\infty}\omega_2(y)\varphi_2(x,y)dydx\right|\leqslant\frac{\epsilon}{2}C_p\left(\int_{-\infty}^{+\infty}\frac{|\omega_2(y)|^2}{y^2+\eta_2}dy\right)\|\phi_x\|^2_{L^2(0,L)}+\frac{1}{2\epsilon}\int_0^L\int_{-\infty}^{+\infty}(y^2+\eta_2)|\varphi_2(x,y)|^2dydx. \tag{85}$$

Applying the inequalities of Cauchy-Schwarz, Young, and Poincaré, we get

$$\left|\int_0^L v\overline{u}dx\right|\leqslant\int_0^L|\overline{u}||v|dx\leqslant\|u\|_{L^2(0,L)}\|v\|_{L^2(0,L)}\leqslant\frac{\epsilon}{2}\|u\|^2_{L^2(0,L)}+\frac{1}{2\epsilon}\|v\|^2_{L^2(0,L)}\leqslant\frac{\epsilon}{2}C_p\|u_x\|^2_{L^2(0,L)}+\frac{1}{2\epsilon}\|v\|^2_{L^2(0,L)}. \tag{86}$$

$$\left|\int_0^L \overline{u} f_2 dx\right| \leqslant \int_0^L |\overline{u}||f_2|dx \leqslant \|u\|_{L^2(0,L)}\|f_2\|_{L^2(0,L)} \leqslant \frac{\epsilon}{2}\|u\|^2_{L^2(0,L)} + \frac{1}{2\epsilon}\|f_2\|^2_{L^2(0,L)} \leqslant \frac{\epsilon}{2}C_p\|u_x\|^2_{L^2(0,L)} + \frac{1}{2\epsilon}\|f_2\|^2_{L^2(0,L)}. \tag{87}$$

$$\left|\int_0^L \overline{\phi}\psi dx\right| \leqslant \|\phi\|_{L^2(0,L)}\|\psi\|_{L^2(0,L)} \leqslant \frac{\epsilon}{2}C_p\|\phi_x\|^2_{L^2(0,L)} + \frac{1}{2\epsilon}\|\psi\|^2_{L^2(0,L)}. \tag{88}$$

$$\left|\int_0^L \overline{\phi} f_6 dx\right| \leqslant \frac{\epsilon}{2}C_p\|\phi_x\|^2_{L^2(0,L)} + \frac{1}{2\epsilon}\|f_6\|^2_{L^2(0,L)}. \tag{89}$$

$$\left|\int_0^L \psi \overline{f_5} dx\right| \leqslant \frac{\epsilon}{2}\|\psi\|^2_{L^2(0,L)} + \frac{1}{2\epsilon}\|f_5\|^2_{L^2(0,L)}. \tag{90}$$

Choosing $\epsilon > 0$ small enough and using the inequalities (74) and (84)–(90), we deduce from the inequality (83) that

$$\mu\int_0^L |u_x|^2dx + b\int_0^L (\phi\overline{u_x} + u_x\overline{\phi})dx + \xi\int_0^L |\phi|^2dx + \delta\int_0^L |\phi_x|^2dx \leqslant C\|F\|^2_{\mathcal{H}} + C'\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}}. \tag{91}$$

From $(72)_{3,7}$, it follows that

$$z_1(x,p) = e^{-i\lambda\tau_1 p}z_1(x,0) + \tau_1 e^{-i\lambda\tau_1 p}\int_0^p e^{i\lambda\tau_1 s}f_3(x,s)ds = e^{-i\lambda\tau_1 p}v(x) + \tau_1 e^{-i\lambda\tau_1 p}\int_0^p e^{i\lambda\tau_1 s}f_3(x,s)ds$$

and

$$z_2(x,p) = e^{-i\lambda\tau_2 p}z_2(x,0) + \tau_2 e^{-i\lambda\tau_2 p}\int_0^p e^{i\lambda\tau_2 s}f_7(x,s)ds = e^{-i\lambda\tau_2 p}\psi(x) + \tau_2 e^{-i\lambda\tau_2 p}\int_0^p e^{i\lambda\tau_2 s}f_7(x,s)ds,$$

where we have

$$\|z_1(x,p)\|_{L^2((0,L)\times(0,1))} \leqslant \|v(x)\|_{L^2(0,L)} + \tau_1\|f_3(x,p)\|_{L^2((0,L)\times(0,1))} \tag{92}$$

and

$$\|z_2(x,p)\|_{L^2((0,L)\times(0,1))} \leqslant \|\psi(x)\|_{L^2(0,L)} + \tau_2\|f_7(x,p)\|_{L^2((0,L)\times(0,1))}. \tag{93}$$

By (74), (76), (78), (91)–(93), we conclude that

$$\|U\|^2_{\mathcal{H}} \leqslant C'\|U\|_{\mathcal{H}}\|F\|_{\mathcal{H}} + C\|F\|^2_{\mathcal{H}}, \quad C', C > 0.$$

Applying Young's inequality, we arrive at

$$\|U\|^2_{\mathcal{H}} \leqslant C\|F\|^2_{\mathcal{H}}, \; C > 0,$$

and thus

$$\|U\|_{\mathcal{H}} \leqslant C\|F\|_{\mathcal{H}}, \quad \forall\, U \in D(\mathcal{A}). \tag{94}$$

Finally, (94) leads to

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leqslant C$$

and hence (70) is proved. By Theorem 7.1, the C_0-semigroup $S(t) = e^{t\mathcal{A}}$, $t \geq 0$, is exponentially stable on $\mathcal{H}$. □

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References

[1] Z. Achouri, N. Amroun, A. Benaissa, The Euler-Bernoulli beam equation with boundary dissipation of fractional derivative type, *Math. Methods Appl. Sci.* **40** (2017) 3837–3854.

[2] A. Alshabanat, M. Jleli, S. Kumar, B. Samet, Generalization of Caputo-Fabrizio fractional derivative and applications to electrical circuits, *Front. Phys.* **8** (2020) #64.

[3] W. Arendt, C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.* **306** (1988) 837–852.

[4] P. S. Casas, R. Quintanilla, Exponential decay in one-dimensional porous-thermo-elasticity, *Mech. Res. Commun.* **32** (2005) 652–658.

[5] J. Choi, R. Maccamy, Fractional order Volterra equations with applications to elasticity, *J. Math. Anal. Appl.* **139** (1989) 448–464.

[6] S. C. Cowin, The viscoelastic behavior of linear elastic materials with voids, *J. Elasticity* **15** (1985) 185–191.

[7] R. Datko, J. Lagnese, M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.* **24** (1986) 152–156.

[8] A. Djebabla, A. Choucha, D. Ouchenane, K. Zennir, Explicit stability for a porous thermoelastic system with second sound and distributed delay term, *Int. J. Appl. Comput. Math.* **7** (2021) #50.

[9] M. J. Dos Santos, C. A. Raposo, L. G. R. Miranda, B. Feng, Exponential stability for a nonlinear porous-elastic system with delay, *Commun. Math.* **31** (2023) 359–379.

[10] H. Dridi, A. Djebabla, On the stabilization of linear porous elastic materials by microtemperature effect and porous damping, *Ann. Univ. Ferrara* **66** (2020) 13–25.

[11] A. Fareh and S. Messaoudi, Energy decay for a porous thermoelastic system with thermoelasticity of second sound and with a non-necessary positive definite energy, *Appl. Math. Comput.* **293** (2017) 493–507.

[12] B. Feng, Uniform decay of energy for a porous thermoelasticity system with past history, *Appl. Anal.* **97** (2018) 210–229.

[13] B. Feng, T. A. Apalara, Optimal decay for a porous elasticity system with memory, *J. Math. Anal. Appl.* **470** (2019) 1108–1128.

[14] B. Feng, M. Yin, Decay of solutions for a one-dimensional porous elasticity system with memory: the case of non-equal wave speeds, *Math. Mech. Solids* **24** (2019) 2361–2373.

[15] E. B. Filho, M. L. Santos, On porous-elastic system with a time-varying delay term in the internal feedbacks, *J. Appl. Math. Mech.* **100** (2020) #e201800247.

[16] F. Foughali, S. Zitouni, L. Bouzettouta, H. E. Khochemane, Well-posedness and general decay for a porous-elastic system with microtemperatures effects and time-varying delay term, *Z. Angew. Math. Phys.* **73** (2022) #183.

[17] L. Gearhart, Spectral theory for contraction semigroups on Hilbert space, *Trans. Amer. Math. Soc.* **236** (1978) 385–394.

[18] M. A. Goodman, S. C. Cowin, A continuum theory for granular materials, *Arch. Ration. Mech. Anal.* **44** (1972) 249–266.

[19] E. C. Grigoletto, E. C. Oliveira, Fractional versions of the fundamental theorem of calculus, *Appl. Math.* **4** (2013) 23–33.

[20] F. Hebhoub, L. Bouzettouta, K. Ghennam, K. Kibech, Stabilization of a microtemperature porous-elastic system with distributed delay-time, *Mediterr. J. Math.* **19** (2022) #222.

[21] F. L. Huang, Characteristic conditions for exponential stability of linear dynamical system in Hilbert spaces, *Ann. Differential Equations* **1** (1985) 43–56.

[22] I. Y. Lyubich, Q. P. Vũ, Asymptotic stability of linear differential equations in Banach spaces, *Stud. Math.* **88** (1988) 37–42.

[23] A. Magaña, R. Quintanilla, On the time decay of solutions in one-dimensional theories of porous materials, *Int. J. Solids Struct.* **43** (2006) 3414–3427.

[24] B. Mbodje, Wave energy decay under fractional derivative controls, *IMA J. Math. Control Inf.* **23** (2006) 237–257.

[25] A. Mohammad, *Some Problems of Direct and Indirect Stabilization of Wave Equations with Locally Boundary Fractional Damping or with Llocalised Kelvin-Voigh*, Ph.D. Thesis, Université de Limoges, 2017.

[26] J. E. Muñoz Rivera, R. Quintanilla, On the time polynomial decay in elastic solids with voids, *J. Math. Anal. Appl.* **338** (2008) 1296–1309.

[27] Z. Nid, A. Fareh, T. A. Apalara, On the decay of a porous thermoelasticity type III with constant delay, *Rev. R. Acad. Cienc. Exactas Fiś. Nat. Ser. A Mat.* **117** (2023) #67.

[28] J. W. Nunziato, S. C. Cowin, A nonlinear theory of elastic materials with voids, *Arch. Ration. Mech. Anal.* **72** (1979) 175–201.

[29] P. X. Pamplona, J. E. Muñoz Rivera, R Quintanilha, On the decay of solutions for porous-elastic systems with history, *J. Math. Anal. Appl.* **379** (2011) 682–705.

[30] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[31] J. Prüss, On the Spectrum of C_0-Semigroups, *Trans. Amer. Math. Soc.* **284** (1984) 847–857.

[32] C. A. Raposo, T. A. Apalara, J. O. Ribeiro, Analyticity to transmission problem with delay in porous-elasticity, *J. Math. Anal. Appl.* **466** (2018) 819–834.

[33] F. G. Shinskey, *Process Control Systems*, McGraw-Hill, New York, 1967.

[34] A. Soufyane, Energy decay for porous-thermo-elasticity systems of memory type, *Appl. Anal.* **87** (2008) 451–464.

[35] A. Soufyane, M. Afilal, T. Aouam, M. Chacha, General decay of solutions of a linear one-dimensional porous-thermoelasticity system with a boundary control of memory type, *Nonlinear Anal.* **72** (2010) 3903–3910.

[36] I. H. Suh, Z. Bien, Use of time delay action in the controller design, *IEEE Trans. Automat. Control* **25** (1980) 600–603.