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Research Article

Ramsey chains in graphs

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Abstract

Let G be a graph with a red-blue coloring c of the edges of G. A Ramsey chain in G with respect to c is a sequence G_1, G_2, …, G_k of pairwise edge-disjoint subgraphs of G such that each subgraph G_i $(1 \le i \le k)$ is monochromatic of size i and G_i is isomorphic to a subgraph of G_{i+1} $(1 \le i \le k-1)$. The Ramsey index $AR_c(G)$ of G with respect to c is the maximum length of a Ramsey chain in G with respect to c. The Ramsey index $AR(G)$ of G is the minimum value of $AR_c(G)$ among all red-blue colorings c of G. A Ramsey chain with respect to c is maximal if it cannot be extended to one of greater length. The lower Ramsey index $AR_c^-(G)$ of G with respect to c is the minimum length of a maximal Ramsey chain in G with respect to c. The lower Ramsey index $AR^-(G)$ of G is the minimum value of $AR_c^-(G)$ among all red-blue colorings c of G. Ramsey chains and maximal Ramsey chains are investigated for stars, matchings, and cycles. It is shown that (1) for every two integers p and q with $2 \le p < q$, there exists a graph with a red-blue coloring possessing a maximal Ramsey chain of length p and a maximum Ramsey chain of length q and (2) for every positive integer k, there exists a graph with a red-blue coloring possessing at least k maximal Ramsey chains of distinct lengths with prescribed conditions. A conjecture and additional results are also presented.

Keywords: red-blue edge coloring; Ramsey chain; Ramsey index.

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1. Introduction

In 1987 a conjecture was stated that has drawn the interest of many researchers. When the famous mathematician Paul Erdős first learned of it, he immediately doubted its truth. Soon afterward, Erdős offered a cash reward for a counterexample or a proof if it were true (as was common for Erdős). This conjecture appeared in a book [4, p.72] containing a list of graph theory problems that are associated with Erdős. Now, more than 35 years later, the conjecture has neither been proved nor disproved. Let us describe this conjecture.

If G is a nonempty graph of size m (without isolated vertices), then there is a unique positive integer k such that $\binom{k+1}{2} \le m < \binom{k+2}{2}$. The graph G is said to have an *ascending subgraph decomposition* $\{G_1, G_2, \ldots, G_k\}$ into k (pairwise edge-disjoint) subgraphs of G if G_i is isomorphic to a proper subgraph of G_{i+1} for $i = 1, 2, \ldots, k-1$. The following conjecture was stated in [1].

The Ascending Subgraph Decomposition Conjecture. *Every nonempty graph has an ascending subgraph decomposition.*

If this conjecture was shown to be false, then the question occurred of determining the maximum length ℓ of a sequence G_1, G_2, …, G_ℓ of ℓ pairwise edge-disjoint subgraphs (without isolated vertices) of G such that

(1) G_i has size i for each $i \in [\ell] = \{1, 2, \ldots, \ell\}$ and

(2) G_i is isomorphic to a subgraph of G_{i+1} for each $i \in [\ell - 1]$.

A sequence with properties (1) and (2) is called an *ascending subgraph sequence* of the graph G and the maximum length of such a sequence is the *ascending subgraph index* of G, denoted by $AS(G)$. The following conjecture is therefore equivalent to the Ascending Subgraph Decomposition Conjecture.

The Ascending Subgraph Index Conjecture. *Let G be a nonempty graph of size m. Then $AS(G) = k$ if and only if*

$$\binom{k+1}{2} \le m < \binom{k+2}{2}.$$

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While the truth of the Ascending Subgraph Decomposition Conjecture remains an open question, this conjecture has been verified for many classes of graphs, including all regular graphs [5].

We now turn briefly to a different topic. A well-known area within graph theory is Ramsey theory and a well-known concept in this theory is Ramsey numbers. Let G be a graph without isolated vertices and let each edge of G be assigned one of two given colors (a 2-*edge coloring* of G). Typically, these colors are chosen to be red or blue (or 1 or 2). In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H (without isolated vertices), the *Ramsey number* $R(F, H)$ is the minimum positive integer n for which every red-blue coloring of the complete graph K_n of order n results in either a subgraph of K_n isomorphic to F all of whose edges are colored red (a red F) or a subgraph of K_n isomorphic to H all of whose edges are colored blue (a blue H). It is a consequence of a theorem of Ramsey [7] that the number $R(F, H)$ exists for every two graphs F and H. If $F \cong H$, then $R(F, H) = R(F, F)$ is the minimum positive integer n such that every red-blue coloring of K_n results in a monochromatic F. If F and H are both complete graphs, then $R(F, H)$ is called a *classical Ramsey number*. For example, it is well known that $R(K_3, K_3) = 6$, $R(K_4, K_4) = 18$, and $R(K_5, K_5)$ is unknown. Many variations of Ramsey numbers have been studied, such as considering classes of graphs different from complete graphs and allowing the edges of the graphs in question to be colored with more than two colors (see [6] for example).

In [2], a concept was introduced that involves both ascending subgraph decompositions and a Ramsey-type coloring problem. Let G be a graph (without isolated vertices) of size m with a red-blue edge coloring c. A *Ramsey chain* of G *with respect to* c is a sequence G_1, G_2, …, G_ℓ of pairwise edge-disjoint subgraphs of G such that each subgraph G_i $(1 \le i \le \ell)$ is monochromatic of size i and G_i is isomorphic to a subgraph of G_{i+1} for $1 \le i \le \ell - 1$. Each subgraph G_i $(1 \le i \le \ell)$ in a Ramsey chain is called a *link* of this chain. The maximum length of a Ramsey chain of G with respect to c is the (ascending) *Ramsey index* $AR_c(G)$ of G. The (ascending) *Ramsey index* $AR(G)$ of G is defined by

$$AR(G) = \min\{AR_c(G) : \ c \text{ is a red-blue edge coloring of } G\}.$$

These concepts were introduced in [2, 3], using somewhat different technology.

2. Ramsey chains in stars and matchings

Among the observations presented in [3] is the following.

Observation 2.1. *If G is a graph of size m where $2 \le m < \binom{k+2}{2}$ for a positive integer k, then $AR(G) \le k$.*

On the other hand, if G is a graph of size m such that $AR(G) \ge k$, then $m \ge \binom{k+1}{2}$. The following result presents a class of graphs G for which $AR(G) = k$ in terms of the size of G.

Theorem 2.1. *Let G be a graph of size $m \ge 2$ without isolated vertices such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$. Then $AR(G) = k$ if and only if*

$$\binom{k+1}{2} \le m < \binom{k+2}{2}.$$

Proof. First, we verify the following claim.

> **Claim.** Let G be a graph of size $m \ge 2$ without isolated vertices such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$. If $m \ge \binom{k+1}{2}$ for some positive integer k, then $AR(G) \ge k$.

We proceed by induction on k. The truth of the claim is immediate if $k = 1$ or $k = 2$. Assume for an integer $k \ge 2$ that a graph G' without isolated vertices has $AR(G') \ge k$ if G' has size $m' \ge \binom{k+1}{2}$ such that for every two subgraphs F' and H' of G' without isolated vertices, $|E(F')| < |E(H')|$ implies $F' \subseteq H'$. Let G be a graph without isolated vertices having size $m \ge \binom{k+2}{2}$ such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$. We show that $AR(G) \ge k + 1$. Let there be given a red-blue coloring c of G. Since $k \ge 2$, it follows that $\frac{1}{2}\binom{k+2}{2} \ge k + 1$. Thus, G has a monochromatic subgraph G_{k+1} of size $k + 1$. Let $G' = G - E(G_{k+1})$, where G' has size $m' = m - (k + 1)$. Then the restriction c' of c to G' is a red-blue coloring of G'. Since $m \ge \binom{k+2}{2}$, it follows that

$$m' = m - (k + 1) \ge \binom{k+1}{2}.$$

By the induction hypothesis, G' has a Ramsey chain $(G_1, G_2, \ldots, G_k)$ of length k with respect to c'. Then $(G_1, G_2, \ldots, G_k, G_{k+1})$ is a Ramsey chain of length $k + 1$ in G. Thus, the claim holds.

Now, let G be a graph of size $m \geq 2$ without isolated vertices such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$. First, assume that

$$\binom{k+1}{2} \leq m < \binom{k+2}{2}.$$

Since $m < \binom{k+2}{2}$, it follows by Observation 2.1 that $AR(G) \leq k$. Since $m \geq \binom{k+1}{2}$, it follows by the claim that $AR(G) \geq k$. Therefore, $AR(G) = k$.

For the converse, assume that $AR(G) = k$. Since $AR(G) = k$, there is a Ramsey chain of length k for every red-blue coloring of G. Thus, $m \geq \binom{k+1}{2}$. By the claim, if $m \geq \binom{k+2}{2}$, then $AR(G) \geq k+1$. Since $AR(G) = k$, it follows that $m < \binom{k+2}{2}$. Consequently, $\binom{k+1}{2} \leq m < \binom{k+2}{2}$. □

For example, the 5-cycle C_5 satisfies the hypothesis of Theorem 2.1 and so $AR(C_5) = 2$. The 6-cycle C_6 does not satisfy the hypothesis of Theorem 2.1, however, since both P_3 and $3K_2$ are subgraphs of C_6 but $P_3 \not\subseteq 3K_2$. In fact, there are only two classes of graphs of size 6 or more that satisfy the hypothesis of Theorem 2.1.

Proposition 2.1. *Let G be a graph of size at least 6 without isolated vertices such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$. Then G is either a star or a matching.*

Proof. Assume, to the contrary, that G is neither a star nor a matching. Thus, G contains two adjacent edges and two nonadjacent edges. If G contains a vertex of degree at least 3 or a matching of size at least 3, then G contains subgraphs F and H where F has a smaller size than H and F is not isomorphic to a subgraph of H. Therefore, we may assume that $\Delta(G) = 2$ and $2K_2$ is a maximum matching in G. The graph G contains no triangle since $2K_2 \not\subseteq K_3$. If G contains a k-cycle C_k where $k \geq 4$, then C_k is a component of G and so G contains a matching of size 3, a contradiction. Hence, G is a linear forest with two components. Since the size of G is at least 6, it follows that G contains a matching of size 3, a contradiction. □

As a consequence of Theorem 2.1 and Proposition 2.1, we have the following result.

Corollary 2.1. [3] *Let $k \geq 2$ be an integer and let G be the star $K_{1,m}$ or the matching mK_2. Then $AR(G) = k$ if and only if*

$$\binom{k+1}{2} \leq m < \binom{k+2}{2}.$$

3. Ramsey chains in cycles

One question that arises is whether there is a familiar class of graphs different from stars and matchings such that every graph G of size m in this class has the property that $AR(G) = k$ if and only if $\binom{k+1}{2} \leq m < \binom{k+2}{2}$. While this question has not been answered, there is a class of graphs of small size for which this is the case, namely the cycles C_m of order and size m. Every proper subgraph of C_m is a linear forest (where each component is a path). In order to verify this, we first present some observations and preliminary results.

Observation 3.1. *Let G be a graph of size $m \geq 2$.*

(*a*) *If $m = 2$, then $AR(G) = 1$ and if $m > 2$, then $AR(G) \geq 2$.*

(*b*) *If $m = 3, 4, 5$, then $AR(G) = 2$.*

(*c*) *If $6 \leq m \leq 8$, then $AR(G) \in \{2, 3\}$. Furthermore, if $m \geq 6$ and c is a 2-edge coloring of G such that (i) there is a monochromatic subgraph F of G where $F \in \{P_4, P_3 + K_2\}$ and (ii) $G - E(F)$ has a monochromatic subgraph of size 2, then $AR_c(G) \geq 3$.*

If G is a graph of size 8, then we only know that $AR(G) = 2$ or $AR(G) = 3$. The situation is clearer if G has size 9.

Proposition 3.1. *If G is a graph of size 9, then $AR(G) = 3$.*

Proof. Let G be a graph of size 9. By Observation 2.1, $AR(G) \leq 3$. It remains to show that $AR(G) \geq 3$. Let there be given a red-blue coloring c of G, where G_r is the red subgraph and G_b is the blue subgraph. Let m_r be the size of G_r and m_b the size of G_b. Thus, $m_r + m_b = 9$. We may assume that $m_r > m_b$ and so $m_r \geq 5$. If G_r is a star or a matching of size $m_r \geq 5$, then G has a Ramsey chain of length 3 and so $AR_c(G) \geq 3$. If G_r is neither a star nor a matching, then either $P_3 + K_2 \subseteq G_r$ or $P_4 \subseteq G_r$ and so $AR_c(G) \geq 3$ by Observation 3.1(c). Therefore, $AR(G) \geq 3$ and so $AR(G) = 3$. □

For the following results, it is convenient to refer the colors in a 2-edge coloring of a graph as 1 and 2.

Observation 3.2. *Let c be a 2-edge coloring of the cycle $H = C_m$ of size $m \geq 3$. For $i = 1, 2$, let H_i be the subgraph of size m_i in H induced by the set of edges colored i.*

(a) *If $m_i \geq 3$ where $i \in \{1, 2\}$, then $2K_2 \subseteq H_i$. Thus, if $m \geq 6$ and such that $2K_2 \subseteq H_i$ and $m_j \geq 3$, where $\{i, j\} = \{1, 2\}$, then $AR_c(C_m) \geq 3$.*

(b) *If $m_1, m_2 \geq 3$, then $AR_c(C_m) \geq 3$.*

(c) *If $m_i \geq 5$ for some $i \in \{1, 2\}$, then $AR_c(C_m) \geq 3$.*

In order to present the next result, we first present the following observation.

Observation 3.3. *For every 2-edge coloring of the cycle C_m of size $m \geq 3$, either (i) the colors of every two edges at distance 2 in C_m are the same or (ii) there exists an edge in C_m whose neighboring edges are colored differently.*

The following result will be useful to us.

Theorem 3.1. *For each integer $m \geq 3$, $AR(C_m) \leq AR(C_{m+1})$.*

Proof. Let c be a 2-edge coloring of $C_{m+1} = (v_1, v_2, \ldots, v_{m+1}, v_1)$ such that $AR_c(C_{m+1}) = AR(C_{m+1}) = k$. For $i = 1, 2$, let H_i be the subgraph of size m_i induced by the set of edges colored i in C_{m+1}. By Observation 3.3, either (i) $H_i = m_i K_2$ for $i = 1, 2$ where $m_1 = m_2 = (m + 1)/2$ or (ii) there are three consecutive edges f_1, f_2, f_3 in C_{m+1} such that $c(f_1) \neq c(f_3)$ and the color $c(f_2)$ is assigned to at least two edges of C_{m+1}. We consider these two cases.

Case 1. *$H_i = m_i K_2$ for $i = 1, 2$ where $m_1 = m_2 = (m + 1)/2$.* Thus, $m + 1 = 2\ell$ for some integer $\ell \geq 2$ and $H_1 = H_2 = \ell K_2$. We may assume that $c(v_{m+1}v_1) = 2$ and so $c(v_m v_{m+1}) = c(v_1 v_2) = 1$. By contracting the edge $v_{m+1}v_1$ in C_{m+1} and labeling the identified vertices v_{m+1} and v_1 by v_1, we obtain the cycle $C_m = (v_1, v_2, \ldots, v_m, v_1)$ and a 2-edge coloring c' of C_m defined by $c'(e) = c(e)$ for each edge $e \in E(C_m) - \{v_1 v_m\}$ and $c'(v_m v_1) = c(v_m v_{m+1}) = 1$. Let H_1' and H_2' be the resulting subgraphs of C_m such that the edges of H_i' are colored i by c' for $i = 1, 2$. Then $H_1' = (\ell - 2)K_2 + P_3$ and $H_2' = (\ell - 1)K_2 \subset H_2$ in C_{m+1}. We claim that there is no ascending Ramsey sequence of length $k + 1$ in C_m with respect to c'. Assume, to the contrary, that there is a Ramsey chain $(G_1, G_2, \ldots, G_{k+1})$ of length $k + 1$ in C_m with respect to c'. We may assume that $|E(G_j)| = j$ for $1 \leq j \leq k + 1$. Hence, $G_1, G_2, \ldots, G_{k+1}$ are pairwise edge-disjoint subgraphs of C_m such that

(1) G_j is monochromatic for $1 \leq j \leq k + 1$,

(2) G_j is isomorphic to a proper subgraph of G_{j+1} in for $1 \leq j \leq k$, and

(3) $G_j = jK_2$ for $1 \leq j \leq k$ and $G_{k+1} \in \{(k + 1)K_2, (k - 1)P_3 + K_2\}$.

For $1 \leq j \leq k$, if $v_m v_1 \notin E(G_j)$, then G_j is a subgraph of C_{m+1}; while if $v_m v_1 \in E(G_j)$, then G_j can be considered as a subgraph of C_{m+1} by replacing $v_m v_1$ by $v_m v_{m+1}$. Thus, each G_j is a subgraph of C_{m+1} for $1 \leq j \leq k$, where $v_m v_1$ is replaced by $v_m v_{m+1}$ if necessary.

- ⋆ If $G_{k+1} = (k + 1)K_2$, then G_{k+1} is also a subgraph of C_{m+1}, where $v_m v_1$ is replaced by $v_m v_{m+1}$ if necessary. Hence, $(G_1, G_2, \ldots, G_{k+1})$ is a Ramsey chain of length $k + 1$ in C_{m+1}, which is impossible.

- ⋆ If $G_{k+1} = (k - 1)P_3 + K_2$, then $G_{k+1} \subseteq H_1'$ and $P_3 = (v_m, v_1, v_2)$. Thus, $v_m v_1 \notin E(G_j)$ for $1 \leq j \leq k$ and so G_j is a subgraph of C_{m+1} for $1 \leq j \leq k$. Furthermore, the subgraph $(k - 1)K_2$ of G_{k+1} is also a subgraph of C_{m+1}. We define a sequence $F_1, F_2, \ldots, F_{k+1}$ of $k + 1$ subgraphs of C_{m+1} by $F_j = G_j = jK_2 \subseteq C_{m+1}$ for $1 \leq j \leq k$ and

$$F_{k+1} = (G_{k+1} - v_m v_1) + v_m v_{m+1} = (k + 1)K_2 \subseteq H_1,$$

  where $P_3 = (v_m, v_1, v_2) \subseteq C_m$ in G_{k+1} is replaced by $2K_2$ (whose edge set is $\{v_m v_{m+1}, v_1 v_2\}$) in C_{m+1}. Thus, $F_1, F_2, \ldots, F_{k+1}$ is a sequence of $k+1$ pairwise edge-disjoint subgraphs of C_{m+1} such that F_j is monochromatic for $1 \leq j \leq k+1$ and F_j is isomorphic to a proper subgraph of F_{j+1} for $1 \leq j \leq k$. Hence, $(F_1, F_2, \ldots, F_{k+1})$ is a Ramsey chain of length $k + 1$ in C_{m+1}, which is impossible.

Therefore, $AR(C_m) \leq AR_{c'}(C_m) \leq k = AR(C_{m+1})$.

Case 2. *There are three consecutive edges* f_1, f_2, f_3 *in* C_{m+1} *such that* $c(f_1) \neq c(f_3)$. We may assume that $f_1 = v_m v_{m+1}$, $f_2 = v_{m+1}v_1$ and $f_3 = v_1 v_2$ such that $c(v_m v_{m+1}) = c(v_{m+1}v_1) = 1$ and $c(v_1v_2) = 2$. By contracting the edge $v_{m+1}v_1$ in C_{m+1} and labeling the identified vertices v_{m+1} and v_1 by v_1, we obtain the cycle $C_m = (v_1, v_2, \ldots, v_m, v_1)$. The 2-edge coloring c of C_{m+1} gives rise to a 2-edge coloring c' of C_m defined by $c'(e) = c(e)$ for each edge $e \in E(C_m) - \{v_1v_m\}$ and $c'(v_mv_1) = c(v_mv_{m+1}) = 1$. Let H_1' and H_2' be the resulting subgraphs of C_m such that the edges of H_i' are colored i by c' for $i = 1, 2$. Thus, $H_1' = H_1 - v_1 \subseteq C_{m+1}$ where v_mv_{m+1} in H_1 in C_{m+1} is replaced by v_mv_1 in H_1' in C_m and $H_2' = H_2$. We claim that there is no a Ramsey chain of length $k+1$ in C_m with respect to c'. Assume, to the contrary, that there is a Ramsey chain $(G_1, G_2, \ldots, G_{k+1})$ of length $k+1$ in C_m with respect to c'. Hence, $G_1, G_2, \ldots, G_{k+1}$ are pairwise edge-disjoint subgraphs of C_m such that $G_j \subseteq H_1'$ or $G_j \subseteq H_2'$ for each integer j with $1 \le j \le k+1$ and G_j is isomorphic to a proper subgraph of G_{j+1} in C_m for $1 \le j \le k$. By the defining property of C_{m+1} and the coloring c, it follows that $(G_1, G_2, \ldots, G_{k+1})$ is a Ramsey chain of length $k+1$ in C_{m+1}, which is impossible. Therefore, $AR(C_m) \le AR_{c'}(C_m) \le k = AR(C_{m+1})$. □

Not only is $AR(C_{m+1}) \ge AR(C_m)$ for $m \ge 3$, but even more can be said.

Theorem 3.2. $\lim_{m\to\infty} AR(C_m) = \infty$.

Proof. We show, for every positive integer k, that there is a positive integer m such that $AR(C_m) \ge k$. Let

$$m = \binom{k+1}{2} - 1 = 2\left[\binom{k+1}{2} - 1\right] + 1$$

and let c be any red-blue coloring of C_m. We show that $AR_c(C_m) \ge k$. Let H_r be the red subgraph of size m_r induced by the set of red edges and let H_b be the blue subgraph of size m_b induced by the set of blue edges, where $m_r \le m_b$. Since

$$m_b \ge \left\lceil \frac{2\left[\binom{k+1}{2} - 1\right] + 1}{2} \right\rceil = \binom{k+1}{2},$$

it follows that H_b contains edge-disjoint copies of $K_2, 2K_2, \ldots, kK_2$ and so $AR_c(C_m) \ge k$. Therefore, $AR(C_m) \ge k$. It then follows by Theorem 3.1 that $\lim_{m\to\infty} AR(C_m) = \infty$. □

We are now prepared to determine the Ramsey indices of all cycles C_m for $3 \le m \le 20$.

Proposition 3.2. *The Ramsey index of* C_m *for* $3 \le m \le 20$ *is given as follows*

$$AR(C_m) = \begin{cases} 2 & \text{if } 3 \le m \le 5 \\ 3 & \text{if } 6 \le m \le 9 \\ 4 & \text{if } 10 \le m \le 14 \\ 5 & \text{if } 15 \le m \le 20. \end{cases}$$

Proof. Since the proof is rather lengthy and the reasoning technique is similar, we only show that $AR(C_m) = 5$ for $15 \le m \le 20$. To do this, it suffices to show that $AR(C_{15}) = 5$. Since $20 < \binom{6+1}{2}$, once it has been verified that $AR(C_{15}) = 5$, it follows by Theorem 3.1 that $AR(C_m) = 5$ for $15 \le m \le 20$. Since the size of C_{15} is $15 = \binom{5+1}{2}$, it follows that $AR(C_{15}) \le 5$. It therefore suffices to show that $AR(C_{15}) \ge 5$. Let c be a red-blue edge coloring of $H = C_{15}$ using the colors 1 and 2. We show that there is a Ramsey chain $R_c = (G_1, G_2, G_3, G_4, G_5)$ of length 5 in H with respect to c. Since the size of C_{15} is 15, it follows that $\{G_1, G_2, G_3, G_4, G_5\}$ is a decomposition of C_{15}. Let H_a denote the subgraph of H of size a induced by the set of red edges of H and let H_b denote the subgraph of H of size b induced by the set of blue edges of H. We may assume that $a < b$. Then $1 \le a \le 7$ and $a + b = 15$. Hence, $(a, b) = (i, 15 - i)$ for $i = 1, 2, \ldots, 7$. Furthermore, H_a and H_b have the same number κ of components. Then $1 \le \kappa \le a \le 7$. We consider these seven cases. For convenience, let $Q_q = P_{q+1}$ denote a path of size $q \ge 1$.

Case 1. $\kappa = 1$. Then H_a is the path of size a where $1 \le a \le 7$ and H_b is the path of size b where $8 \le b \le 14$ and $a+b = 15$. First, observe that C_{15} can be decomposed into *five consecutive paths* Q_1, Q_2, Q_3, Q_4, Q_5. If $1 \le a \le 5$, let $H_a = Q_a$. Then $H_b = Q_{15-a}$ can be decomposed into the remaining four paths. If $a = 6$, then H_a can be decomposed into Q_1, Q_2, Q_3 and H_b can be decomposed into Q_4 and Q_5. If $a = 7$, then H_a can be decomposed into Q_3 and Q_4 and H_b can be decomposed into Q_5, Q_1, and Q_2. Therefore, R_c =(Q_1, Q_2, Q_3, Q_4, Q_5).

Case 2. $\kappa = 2$. Then $H_a = Q_{a_1} + Q_{a_2}$, and $H_b = Q_{b_1} + Q_{b_2}$ where $2 \le a = a_1 + a_2 \le 7$ with $a_1 \ge a_2$ and $b = b_1 + b_2$ with $b_1 \ge b_2$.

⋆ If $a = 2$, then $H_a = 2K_2$ and $H_b \in \{P_{13} + K_2, P_{12} + P_3, P_{11} + P_4, P_{10} + P_5, P_9 + P_6, P_8 + P_7\}$. The graph H_b can be decomposed into $K_2, 3K_2, 4K_2, 5K_2$, as shown in Figure 1, where an edge labeled i belongs to iK_2. Therefore, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

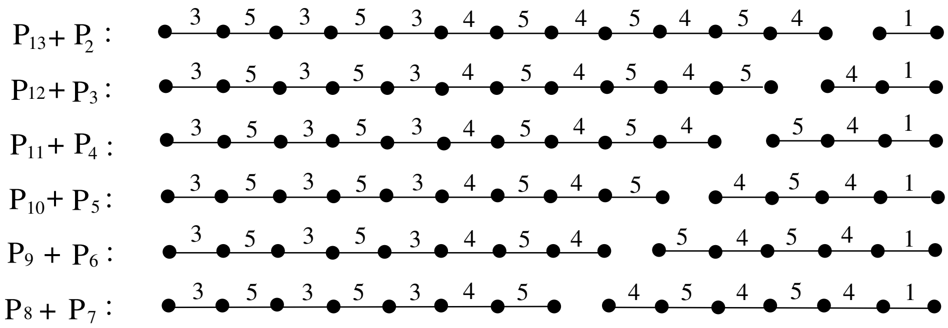

Figure 1: Decompositions of H_b when $b = 13$.

⋆ If $a = 3$, then $H_a = P_3 + K_2$ and so H_a can be decomposed into K_2 and $2K_2$. Furthermore,

$$H_b \in \{P_{12} + K_2, P_{11} + P_3, P_{10} + P_4, P_9 + P_5, P_8 + P_6, P_7 + P_7\}.$$

Hence, H_b can be decomposed into $3K_2, 4K_2, 5K_2$, as shown in Figure 2, where an edge labeled i belongs to iK_2. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

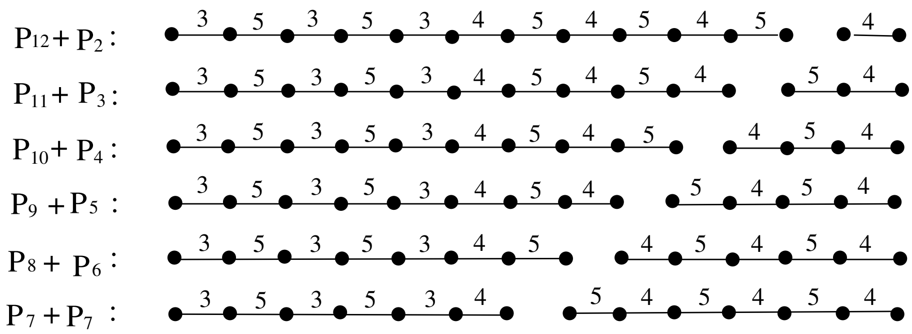

Figure 2: Decompositions of H_b when $b = 12$.

⋆ If $a = 4$, then $H_a \in \{P_4 + K_2, 2P_3\}$ and $H_b \in \{P_{11} + K_2, P_{10} + P_3, P_9 + P_4, P_8 + P_5, P_7 + P_6\}$.

◦ If $H_a = P_4 + K_2$, then H_a can be decomposed into K_2 and $3K_2$ and H_b can be decomposed into $2K_2, 4K_2, 5K_2$, as shown in Figure 3, where an edge labeled i belongs to iK_2. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

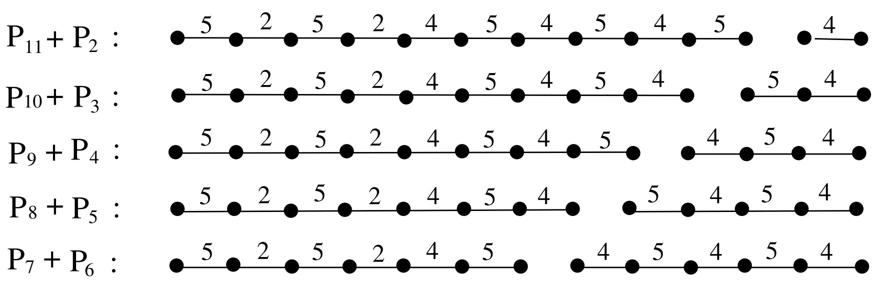

Figure 3: Decompositions of H_b when $b = 11$ and when $H_a = P_4 + K_2$.

◦ If $H_a = 2P_3$, then H_a can be decomposed into $G_1 = K_2$ and $G_3 = P_3 + K_2$ and H_b can be decomposed into $G_2 = P_3, G_4 = P_3 + 2K_2, G_5 = P_3 + 3K_2$, where an edge labeled i belongs to G_i for $i = 2, 4, 5$. Therefore, $R_c = (K_2, P_3, P_3 + K_2, P_3 + 2K_2, P_3 + 3K_2)$.

⋆ If $a = 5$, then $H_a \in \{P_5 + K_2, P_4 + P_3\}$ and $H_b \in \{P_{10} + K_2, P_9 + P_3, P_8 + P_4, P_7 + P_5, P_6 + P_6\}$. Then H_a can be decomposed into $2K_2$ and $3K_2$ and H_b can be decomposed into $K_2, 4K_2, 5K_2$, as shown in Figure 4, where an edge labeled i belongs to iK_2. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

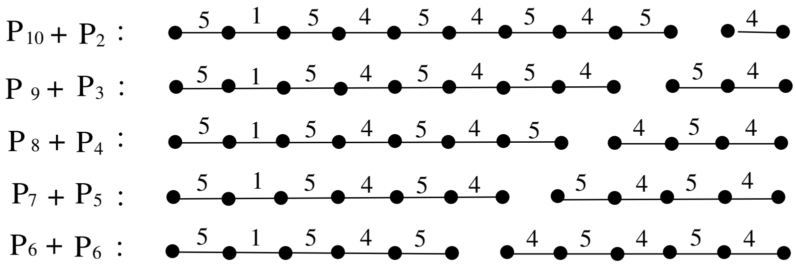

Figure 4: Decompositions of H_b when $b = 10$.

⋆ If $a = 6$, then $H_a \in \{P_6 + K_2, P_5 + P_3, 2P_4\}$ and $H_b \in \{P_9 + K_2, P_8 + P_3, P_7 + P_4, P_6 + P_5\}$. Then H_a can be decomposed into K_2, $2K_2$, $3K_2$ and H_b can be decomposed into $4K_2$ and $5K_2$, as shown in Figure 5, where an edge labeled i belongs to iK_2. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

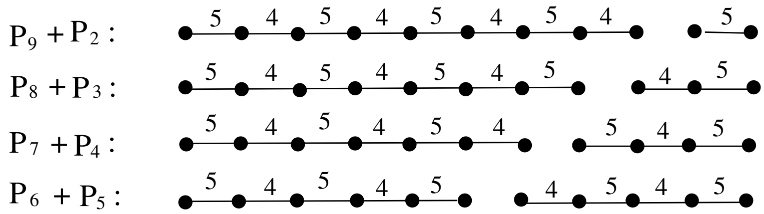

Figure 5: Decompositions of H_b when $b = 9$ and when $H_a \in \{P_6 + K_2, P_5 + P_3, 2P_4\}$.

⋆ If $a = 7$, then $H_a \in \{P_7 + K_2, P_6 + P_3, P_5 + P_4\}$ and $H_b \in \{P_8 + K_2, P_7 + P_3, P_6 + P_4, 2P_5\}$. Then H_a can be decomposed into $G_3 = P_3 + K_2$ and $G_4 = P_3 + 2K_2$ and H_b can be decomposed into $G_1 = K_2, G_2 = P_3, G_5 = P_4 + 2K_2$, as shown in Figure 6, where an edge labeled i belongs to G_i. Thus, $R_c = (K_2, P_3, P_3 + K_2, P_3 + 2K_2, P_4 + 2K_2)$.

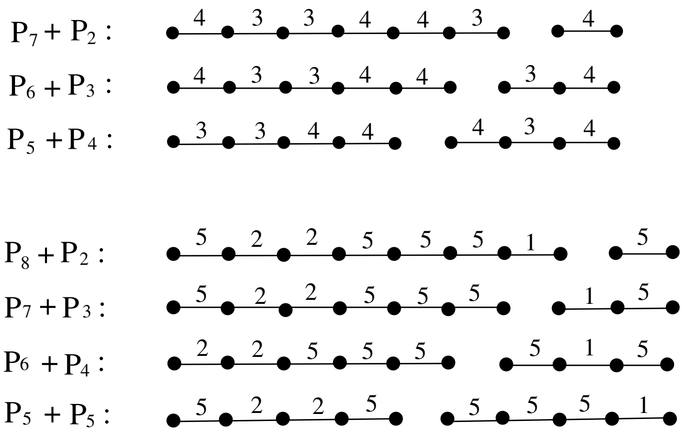

Figure 6: Decompositions of H_a and H_b when $a = 7$.

Case 3. $\kappa = 3$. Then $H_a = Q_{a_1} + Q_{a_2} + Q_{a_3}$, and $H_b = Q_{b_1} + Q_{b_2} + Q_{b_3}$ where $3 \leq a = a_1 + a_2 + a_3 \leq 7$ with $a_1 \geq a_2 \geq a_3$ and $b = b_1 + b_2 + b_3$ with $b_1 \geq b_2 \geq b_3$. If $a = 3$, then $H_a = 3K_2$ and so H_a can be decomposed into K_2 and $2K_2$. The graph H_b is a linear forest of size 12 with three components. It can be shown that H_b can be decomposed into $3K_2, 4K_2, 5K_2$ (see Figure 2 where H_a is decomposed into K_2 and $2K_2$ and H_b has *two* components and $b = 12$). Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 4$, then $H_a = P_3 + 2K_2$ and then H_a can be decomposed into K_2 and $3K_2$ The graph H_b is a linear forest of size 11 with three components. It can be shown that H_b can be decomposed into $2K_2, 4K_2, 5K_2$, (see Figure 3 where H_a is decomposed into K_2 and $3K_2$ and H_b has *two* components and $b = 11$). Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 5$, then $H_a \in \{P_4 + 2K_2, 2P_3 + K_2\}$ and so H_a can be decomposed into $2K_2$ and $3K_2$. The graph H_b is a linear forest of size 10 with three components. It can be shown that H_b can be decomposed into $K_2, 4K_2, 5K_2$, (see Figure 4 where H_a is decomposed into $2K_2$ and $3K_2$ and H_b has *two* components and $b = 10$). Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 6$, then $H_a \in \{P_5 + 2K_2, P_4 + P_3 + K_2, 3P_3\}$ and so H_a can be decomposed into K_2, $2K_2$, $3K_2$. The graph H_b is a linear forest of size 9 with three components. It can be shown that H_b can be decomposed into $4K_2$ and $5K_2$, (see Figure 5, where H_a is decomposed into K_2, $2K_2$, $3K_2$ and H_b has *two* components and $b = 9$). Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 7$, then $H_a \in \{P_6 + 2K_2, P_5 + P_3 + K_2, 2P_4 + K_2, P_4 + 2P_3\}$ and so H_a can be decomposed into $3K_2$, $4K_2$. The graph H_b is a linear forest of size 8 with three components and can be decomposed into K_2, $2K_2$, $5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

Case 4. $\kappa = 4$. Then $H_a = Q_{a_1} + Q_{a_2} + Q_{a_3} + Q_{a_4}$, and $H_b = Q_{b_1} + Q_{b_2} + Q_{b_3} + Q_{b_4}$ where $4 \le a = a_1 + a_2 + a_3 + a_4 \le 7$ with $a_1 \ge a_2 \ge a_3 \ge a_4$ and $b = b_1 + b_2 + b_3 + b_4$ with $b_1 \ge b_2 \ge b_3 \ge b_4$. If $a = 4$, then $H_a = 4K_2$ and so H_a can be decomposed into K_2, $3K_2$. The graph H_b is a linear forest of size 11 with four components. It can be shown that H_b can be decomposed into $2K_2, 4K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 5$, then $H_a = P_3 + 3K_2$ and so H_a can be decomposed into K_2, $4K_2$. The graph H_b is a linear forest of size 10 with four components. It can be shown that H_b can be decomposed into $2K_2, 3K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 6$, then $H_a \in \{P_4 + 3K_2, 2P_3 + 2K_2\}$ and so H_a can be decomposed into $2K_2$, $4K_2$. The graph H_b is a linear forest of size 9 with four components and can be decomposed into $K_2, 3K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 7$, then $H_a \in \{P_5 + 3K_2, P_4 + P_3 + 2K_2, 3P_3 + K_2\}$ and so H_a can be decomposed into $3K_2$, $4K_2$. The graph H_b is a linear forest of size 8 with four components and can be decomposed into $K_2, 2K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

Case 5. $\kappa = 5$. Then $H_a = Q_{a_1} + Q_{a_2} + Q_{a_3} + Q_{a_4} + Q_{a_5}$ and $H_b = Q_{b_1} + Q_{b_2} + Q_{b_3} + Q_{b_4} + Q_{b_5}$ where $5 \le a = a_1 + a_2 + a_3 + a_4 + a_5 \le 7$ with $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5$ and $b = b_1 + b_2 + b_3 + b_4 + b_5$ with $b_1 \ge b_2 \ge b_3 \ge b_4 \ge b_5$. If $a = 5$, then $H_a = 5K_2$ and so H_a can be decomposed into K_2, $4K_2$. The graph H_b is a linear forest of size 10 with five components and can be decomposed into $2K_2, 3K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 6$, then $H_a = P_3 + 4K_2$ and so H_a can be decomposed into K_2, $5K_2$. The graph H_b is a linear forest of size 9 with five components and can be decomposed into $2K_2, 3K_2, 4K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 7$, then $H_a \in \{P_4 + 4K_2, 2P_3 + 3K_2\}$ and so H_a can be decomposed into $2K_2$, $5K_2$. The graph H_b is a linear forest of size 8 with five components and can be decomposed into $K_2, 3K_2, 4K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

Case 6. $\kappa = 6$. Then $H_a = Q_{a_1} + Q_{a_2} + \cdots + Q_{a_6}$ and $H_b = Q_{b_1} + Q_{b_2} + \cdots + Q_{b_6}$ where $6 \le a = a_1 + a_2 + \cdots + a_6 \le 7$ with $a_1 \ge a_2 \ge \cdots \ge a_6$ and $b = b_1 + b_2 + \cdots + b_6$ with $b_1 \ge b_2 \ge \cdots \ge b_6$. If $a = 6$, then $H_a = 6K_2$ and so H_a can be decomposed into K_2, $5K_2$. The graph H_b is a linear forest of size 9 with five components and can be decomposed into $2K_2, 3K_2, 4K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$. If $a = 7$, then $H_a = P_3 + 5K_2$ and so H_a can be decomposed into $2K_2$, $5K_2$. The graph H_b is a linear forest of size 8 with five components and can be decomposed into $K_2, 3K_2, 4K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

Case 7. $\kappa = 7$. Then (1) $a = 7$ and $H_a = 7K_2$ and (2) $b = 8$ and $H_b = P_3 + 6K_2$. The graph H_a can be decomposed into $3K_2, 4K_2$ and the graph H_b can be decomposed into $K_2, 3K_2, 5K_2$. Thus, $R_c = (K_2, 2K_2, 3K_2, 4K_2, 5K_2)$.

Therefore, $AR(C_{15}) = 5$ and so $AR(C_m) = 5$ for $15 \le m \le 20$. □

Proposition 3.2 can therefore be stated as below.

Proposition 3.3. *Let k be an integer such that $2 \le k \le 5$. Then $AR(C_m) = k$ if and only if*

$$\binom{k+1}{2} \le m < \binom{k+2}{2}.$$

Consequently, for cycles of size 20 or less, we have the same result as stated in Corollary 2.1 for stars and matchings. There is reason to believe that Corollary 2.1 holds for all cycles as well as all stars and matchings.

Conjecture 3.1. *For every integer $m \ge 3$, $AR(C_m) = k$ if and only if*

$$\binom{k+1}{2} \le m < \binom{k+2}{2}.$$

4. Maximal Ramsey chains

Let c be a red-blue edge coloring of a graph G. A Ramsey chain $(G_1, G_2, \ldots, G_k)$ of G with respect to c is *maximal* if the chain cannot be extended to one of greater length. The minimum length of a maximal Ramsey chain in G with respect to c is referred to as the *lower Ramsey index* $AR_c^-(G)$ of G *with respect to* c. The *lower Ramsey index* $AR^-(G)$ of G is

$$AR^-(G) = \min\{AR_c^-(G) : c \text{ is a red-blue coloring of } G\}.$$

Thus, $AR^-(G) \le AR(G)$ for every graph G. We now investigate this inequality.

By Corollary 2.1, if G is the star $K_{1,m}$ or the matching mK_2 of size $m \ge 3$, then $AR(G) = k$ for an integer $k \ge 2$ if and only if $\binom{k+1}{2} \le m < \binom{k+2}{2}$. We now determine the lower Ramsey indices of stars and matchings. To do this, we return to a class of graphs we encountered in Theorem 2.1. For a Ramsey chain R in a graph, we write $E(R)$ for the union of the edge sets of the links in R.

Theorem 4.1. *Let G be a graph of size $m \geq 6$ without isolated vertices such that for every two subgraphs F and H of G without isolated vertices, $|E(F)| < |E(H)|$ implies $F \subseteq H$.*

(1) *If $\binom{k+1}{2} \leq m \leq \binom{k+2}{2} - 3$, then $AR^-(G) = k-1$.*

(2) *If $\binom{k+2}{2} - 2 \leq m \leq \binom{k+2}{2} - 1$, then $AR^-(G) = k$.*

Proof. To verify (1), we first show that if $m = \binom{k+1}{2}$, then $AR^-G) = k-1$. Let c be the red-blue coloring of G that assigns red to all edges of G except one and let s_c be a Ramsey chain of length $k-1$ consisting of $k-1$ red subgraphs of G. Then s_c is maximal and so $AR^-(G) \leq k-1$. Next, we show that

$$AR^-(G) \geq k-1.$$

We claim that every Ramsey chain $R_j = (G_1, G_2, \ldots, G_j)$ of length j, where $j \leq k-2$, can be extended to a Ramsey chain $(G_1, G_2, \ldots, G_j, G_{j+1})$ of length $j+1$. Observe that

$$|E(G) - E(R_j)| \quad = \quad \binom{k+1}{2} - \binom{j+1}{2} = (j+1) + (j+2) + \cdots + k \geq 2k-1$$

and so

$$\left\lceil \frac{\binom{k+1}{2} - \binom{j+1}{2}}{2} \right\rceil \geq k-1.$$

Hence, $G - E(R_j)$ contains a monochromatic subgraph of size at least $k-1$. Since the required size of G_{j+1} is $j+1$ and $j+1 \leq k-1$, the chain R_j can be extended to $(G_1, G_2, \ldots, G_j, G_{j+1})$. Thus, $AR^-(G) \geq k-1$ and so $AR^-(G) = k-1$.

We now show that if $m = \binom{k+1}{2} - 3$, then $AR^-(G) = k-1$. Since $k \geq 2$, it follows that

$$\binom{k+2}{2} - 3 \geq \binom{k+1}{2}$$

and so

$$AR^-(G) \geq AR^-\left(\binom{k+1}{2} K_2\right) = k-1.$$

Thus, it remains to show that $AR^-(G) \leq k-1$. Let c be the red-blue coloring of G that assigns blue to $k-1$ edges of G and assigns red to the remaining $\binom{k+2}{2} - 3 - (k-1)$ edges of G. Since

$$\binom{k+2}{2} - 3 - (k-1) = \binom{k+1}{2} - 1,$$

there is a Ramsey chain $R_c = (G_1, G_2, \ldots, G_{k-1})$ of length $k-1$ where each subgraph G_i $(1 \leq i \leq k-1)$ is a red subgraph of G but no such sequence of length k where all subgraphs are red. Since there are only $k-1$ blue edges, there is no blue subgraph of size k. Thus, s_c cannot be extended and so s_c is maximal. Therefore, $AR^-(G) \leq k-1$ and so $AR^-(G) = k-1$.

If $\binom{k+1}{2} \leq m \leq \binom{k+2}{2} - 3$, then

$$AR^-\left(\binom{k+1}{2} K_2\right) \leq AR^-(G) \leq AR^-\left(\left[\binom{k+2}{2} - 3\right] K_2\right)$$

and so $AR^-(G) = k-1$.

Next, we verify (2). Let $m = \binom{k+2}{2} - 2$. Since $AR^-(G) \leq AR(G) \leq k$ by Corollary 2.1, it remains to show that $AR^-(G) \geq k$. We claim that every Ramsey chain $R_j =(G_1, G_2, \ldots, G_j)$ of length j, where $j \leq k-1$, can be extended to a Ramsey chain $(G_1, G_2, \ldots, G_j, G_{j+1})$ of length $j+1$. Observe that

$$|E(G) - E(R_j)| = \left[\binom{k+2}{2} - 2\right] - \binom{j+1}{2} = [(j+1) + (j+2) + \cdots + k + (k+1)] - 2 \geq 2k-1$$

and so

$$\left\lceil \frac{\left[\binom{k+2}{2} - 2\right] - \binom{j+1}{2}}{2} \right\rceil \geq k.$$

Hence, $G - E(R_j)$ contains a monochromatic subgraph of size at least k. Since the required size of G_{j+1} is $j+1$ and $j+1 \leq k$, the chain R_j can be extended to a Ramsey chain $(G_1, G_2, \ldots, G_j, G_{j+1})$. Thus, $AR^-(G) \geq k$. Therefore, $AR^-(G) = k$. □

The following is a consequence of Proposition 2.1 and Theorem 4.1.

Corollary 4.1. *Let $k \geq 2$ be an integer and let G be the star $K_{1,m}$ or the matching mK_2.*

(1) *If $\binom{k+1}{2} \leq m \leq \binom{k+2}{2} - 3$, then $AR^-(G) = k-1$.*

(2) *If $\binom{k+2}{2} - 2 \leq m \leq \binom{k+2}{2} - 1$, then $AR^-(G) = k$.*

5. Comparing two Ramsey indices

We have seen that $AR^{-}(G) \le AR(G)$ for every graph G. By Theorems 2.1 and 4.1, if $G \in \{K_{1,m}, mK_2\}$ where

$$\binom{k+2}{2} - 2 \le m \le \binom{k+2}{2} - 1,$$

then $AR^{-1}(G) = AR(G) = k$; while if $\binom{k+1}{2} \le m < \binom{k+2}{2} - 3$, then $AR^{-1}(G) = k-1$ and $AR(G) = k$. Therefore, there are graphs G for which $AR^{-1}(G) = AR(G)$ and graphs G for which $AR(G) = AR^{-1}(G) + 1$. This brings up the question as to how large the number $AR(G) - AR^{-}(G)$ may be for some graph G. In order to answer this question, we first present a lemma.

Lemma 5.1. *Let $q \ge 3$ be an integer. For each integer m with $\frac{1}{2}\binom{q+1}{2} < m < \binom{q+1}{2}$, there exist integers $k_1, k_2, \ldots, k_t$ with $1 \le k_1 < k_2 < \cdots < k_t = q$ such that $\sum_{i=1}^{t} k_i = m$.*

Proof. We proceed by induction on q. If $q = 3$, then $\binom{q+1}{2} = \binom{4}{2} = 6$. If m is an integer such that $3 < m < 6$, then $m = 4$ or $m = 5$. If $m = 4$, then $1 + 3 = 4$; while if $m = 5$, then $2 + 3 = 5$. Thus, the statement is true for $m = 3$. Assume that the statement is true for an integer q where $q \ge 3$. We show that the statement is true for $q+1$. Let m be an integer such that $\frac{1}{2}\binom{q+2}{2} < m < \binom{q+2}{2}$. Since $q+1 \ge 4$, it follows that $q+1 < \frac{1}{2}\binom{q+2}{2}$. Let $m' = m - (q+1)$. Then

$$\frac{1}{2}\binom{q+2}{2} - (q+1) < m' < \binom{q+2}{2} - (q+1).$$

Hence,

$$q \le \frac{1}{2}\binom{q+1}{2} < m' < \binom{q+1}{2}$$

for each integer $q \ge 3$. By the induction hypothesis, there exists integers $k_1, k_2, \ldots, k_t$ with $1 \le k_1 < k_2 < \cdots < k_t = q$ such that $\sum_{i=1}^{t} k_i = m'$. Letting $k_{t+1} = q+1$, we obtain $\sum_{i=1}^{t+1} k_i = m$. □

With the aid of Lemma 5.1 and Ramsey chains of cycles, we now show that $AR(G) - AR^{-}(G)$ can be arbitrarily large.

Theorem 5.1. *For every two integers p and q with $2 \le p < q$, there exists a cycle with a red-blue coloring possessing a maximal Ramsey chain of length p and a maximum Ramsey chain of length q.*

Proof. If $\binom{p+1}{2}$ and $\binom{q+1}{2}$ are of opposite parity, let $n = \binom{q+1}{2}$; while if $\binom{p+1}{2}$ and $\binom{q+1}{2}$ are of the same parity, let $n = \binom{q+1}{2}+1$. Let $G = C_n$ where the n consecutive edges of G are denoted by $e_1, e_2, \ldots, e_n$. We now define a red-blue coloring of G where e_i is colored red if $1 \le i \le \binom{p+1}{2}$, e_i is colored blue if $i = \binom{p+1}{2}+1, \binom{p+1}{2}+3, \cdots, n$, and all remaining edges of G are colored red. Therefore, the red subgraph of G is

$$G_r = Q_{\binom{p+1}{2}} + \left\lfloor \frac{n - \binom{p+1}{2}}{2} \right\rfloor K_2,$$

where $Q_{\binom{p+1}{2}}$ is a path of size $\binom{p+1}{2}$ in G_r, and the blue subgraph of G is

$$G_b = \left\lceil \frac{n - \binom{p+1}{2}}{2} \right\rceil K_2.$$

Let $m_r = \binom{p+1}{2} + \left\lfloor \frac{n-\binom{p+1}{2}}{2} \right\rfloor$ be the number of red edges of G and let $m_b = \left\lceil \frac{n-\binom{p+1}{2}}{2} \right\rceil$ be the number of blue edges of G. Then $m_r > m_b$ and $m_r + m_b = n$.

First, we show that there is a maximal Ramsey chain of length p in G. The subgraph $Q_{\binom{p+1}{2}}$ of G_r can be decomposed into $\{Q_1, Q_2, \ldots, Q_p\}$ where Q_i is a path of size i for $1 \le i \le p$. Thus, $R = (Q_1, Q_2, \ldots, Q_p)$ is a Ramsey chain in G_r and in G. Since $G - E(R)$ contains no monochromatic subgraph isomorphic to either Q_{p+1} or $Q_p + K_2$, the chain R is a maximal Ramsey chain of length p in G. Next, we show that there is a maximum Ramsey chain of length q in G. Define a sequence $\mathcal{S}$ of the m_r red edges of G_r as follows:

- ⋆ If $\binom{p+1}{2}$ is even, then let $\mathcal{S} = \left(\underline{e_1, e_3, \ldots, e_{\binom{p+1}{2}-1}}, \ \underline{e_{\binom{p+1}{2}+2}, e_{\binom{p+1}{2}+4}, \ldots, e_{n-1}}, \ \underline{e_2, e_4, \ldots, e_{\binom{p+1}{2}}}\right)$.
- ⋆ If $\binom{p+1}{2}$ is odd, the let $\mathcal{S} = \left(\underline{e_1, e_3, \ldots, e_{\binom{p+1}{2}}}, \ \underline{e_{\binom{p+1}{2}+2}, e_{\binom{p+1}{2}+4}, \ldots, e_{n-1}}, \ \underline{e_2, e_4, \ldots, e_{\binom{p+1}{2}-1}}\right)$.

Then no two consecutive edges in $\mathcal{S}$ are adjacent. Denote the sequence $\mathcal{S}$ by $(f_1, f_2, \ldots, f_{m_r})$, where then $f_i f_{i+1} \notin E(G)$ for $1 \le i \le m_r - 1$. To construct a maximum Ramsey chain of length q in G, we consider two cases, according to whether $n = \binom{q+1}{2}$ or $n = \binom{q+1}{2} + 1$.

Case 1. $n = \binom{q+1}{2}$. Since $\frac{1}{2}\binom{q+1}{2} < m_r < \binom{q+1}{2}$, it follows by Lemma 5.1 that there exist integers $a_1, a_2, \ldots, a_t$ with $1 \le a_1 < a_2 < \cdots < a_t = q$ such that $\sum_{i=1}^{t} a_i = m_r$. Define a labeling ℓ of $\mathcal{S}$ by

$$\ell(f_i) = \begin{cases} t & \text{if } 1 \le i \le a_t = q \\ t-1 & \text{if } a_t + 1 \le i \le a_t + a_{t-1} \\ \vdots & \vdots \\ 1 & \text{if } a_t + a_{t-1} + \cdots + a_2 + 1 \le i \le m_r. \end{cases}$$

Since $q \le \frac{1}{2}\binom{q+1}{2} < \frac{1}{2}m_r$, it follows that for every pair i, j of distinct integers with $1 \le i, j \le t$, if $\ell(f_i) = \ell(f_j)$, then f_i and f_j are not adjacent. Thus, for $1 \le i \le t$, the a_i edges labeled i form the matching $a_i K_2$ and so G_r can be decomposed into the matchings $a_1 K_2$, $a_2 K_2$, …, $a_t K_2 = qK_2$. Since

$$\left(\sum_{i=1}^{t} a_i\right) + m_b = \binom{q+1}{2} = \sum_{i=1}^{q} i,$$

it follows that there exist t' distinct integers $b_1, b_2, \ldots, b_{t'}$, where $t' = q - t$ and $1 \le b_1 < b_2 < \cdots < b_{t'} \le q - 1$ such that (i) $\sum_{i=1}^{t'} b_i = m_b$ and (ii) $a_i \ne b_j$ for every pair i, j of integers with $1 \le i \le t$ and $1 \le j \le t'$. That is,

$$\{a_1, a_2, \ldots, a_t\} \cup \{b_1, b_2, \ldots, b_{t'}\} = \{1, 2, \ldots, q\}.$$

The blue subgraph $G_b = m_b K_2$ can be decomposed into the matchings $b_1 K_2$, $b_2 K_2$, …, $b_{t'} K_2$. Consequently,

$$(K_2, 2K_2, 3K_3, \ldots, qK_2)$$

is a maximum Ramsey chain of length q in G.

Case 2. $n = \binom{q+1}{2} + 1$. Thus, $\binom{p+1}{2}$ and $\binom{q+1}{2}$ are of the same parity. Then

$$m_r = \binom{p+1}{2} + \left\lfloor \frac{\binom{q+1}{2} + 1 - \binom{p+1}{2}}{2} \right\rfloor = \frac{1}{2}\left[\binom{q+1}{2} + \binom{p+1}{2}\right].$$

Since $3 \le p < q$, it follows that

$$\frac{1}{2}\binom{q+1}{2} < \frac{1}{2}\left[\binom{q+1}{2} + \binom{p+1}{2}\right] < \binom{q+1}{2}$$

and so $\frac{1}{2}\binom{q+1}{2} < m_r < \binom{q+1}{2}$. By the argument in Case 1, there exist integers $a_1, a_2, \ldots, a_t$ with $1 \le a_1 < a_2 < \cdots < a_t = q$ such that $\sum_{i=1}^{t} a_i = m_r$ and the red subgraph G_r can be decomposed into the matchings $a_1 K_2$, $a_2 K_2$, …, $a_t K_2$. In this case, $m_r + (m_b - 1) = \binom{q+1}{2}$. By the argument in Case 1, there exist t' distinct integers $b_1, b_2, \ldots, b_{t'}$, where $t' = q - t$ and $1 \le b_1 < b_2 < \cdots < b_{t'} \le q - 1$ such that (i) $\sum_{i=1}^{t'} b_i = m_b - 1$ and (ii) $a_i \ne b_j$ for every pair i, j of integers with $1 \le i \le t$ and $1 \le j \le t'$. The blue subgraph $(m_b - 1)K_2 \subseteq G_b$ can be decomposed into the matchings $b_1 K_2$, $b_2 K_2$, …, $b_{t'} K_2$. Since the size of G is $n = \binom{q+1}{2} + 1 < \binom{q+2}{2}$, there is no Ramsey chain of length $q+1$ and so $(K_2, 2K_2, 3K_3, \ldots, qK_2)$ is a maximum Ramsey chain of G. □

The following is therefore a consequence of Theorem 5.1.

Corollary 5.1. *For each positive integer N, there is a graph G such that $AR(G) - AR^{-}(G) > N$.*

6. Alternating Ramsey chains

In the proof of Theorem 5.1, every link of both the maximal Ramsey chain and the maximum Ramsey chain has the same color, namely red. We now show that Corollary 5.1 can be obtained without all the links having the same color.

An *alternating Ramsey chain* in a graph with a red-blue coloring is a Ramsey chain in which the colors of every two consecutive links are distinct. For integers p and q with $1 \le p < q$, we write $Q_q(p)$ to denote the subpath of length p obtained by selecting the first p edges (in clockwise direction) from a path Q_q of length q in a cycle. We now show that there is a red-blue coloring of a cycle that produces arbitrarily many maximal alternating Ramsey chains of distinct lengths.

Theorem 6.1. *For every positive integer k, there exists a cycle with a red-blue coloring possessing at least k maximal alternating Ramsey chains of distinct lengths.*

Proof. The statement is true trivially for $k = 1$ and so we may assume that $k \geq 2$. Let $G = C_n$ where

$$n = \begin{cases} \dfrac{11k^2 - 5k}{2} & \text{if } k \text{ is odd} \\[2ex] \dfrac{11k^2 - 5k + 2}{2} & \text{if } k \text{ is even.} \end{cases}$$

We now describe a red-blue coloring of G as follows.

- ⋆ Select an arbitrary edge of G and color it red. This is a red Q_1, which we denote by F_1. As we proceed clockwise about G, the next two edges are colored blue. This results in a blue Q_2, which we denote by F_2. The next three edges are colored red, resulting in a red Q_3, which we denote by F_3. We continue this procedure until arriving at a blue Q_{2k}, denoted by F_{2k}. Thus, the sequence $(F_1, F_2, \ldots, F_{2k}) = (Q_1, Q_2, \ldots, Q_{2k})$ appears (in clockwise direction) on G where $F_i = Q_i$ for $1 \leq i \leq 2k$ and
$$\sum_{i=1}^{2k} |E(F_i)| = \sum_{i=1}^{2k} |E(Q_i)| = \sum_{i=1}^{2k} i = \binom{2k+1}{2}.$$
- ⋆ The next $2k - 1$ edges following F_{2k} are colored red, resulting in a red Q_{2k-1}, denoted by H_1. The next $2k - 2$ edges following H_1 are colored blue, resulting in a red Q_{2k-2}, denoted by H_2. We continue this procedure until arriving at Q_{k+1}, denoted by H_{k-1}. If k is odd, then H_{k-1} is a blue Q_{k+1}; while if k is even, then H_{k-1} is a red Q_{k+1}. Thus, the sequence $(H_1, H_2, \ldots, H_{k-1}) = (Q_{2k-1}, Q_{2k-2}, \ldots, Q_{k+1})$ appears (after F_{2k} in clockwise direction) on G, where $H_i = Q_{2k-i}$ for $1 \leq i \leq k - 1$ and
$$\sum_{i=1}^{k-1} |E(H_i)| = \sum_{i=1}^{k-1} |E(Q_{2k-i})| = \sum_{i=1}^{k-1} (2k - i) = \binom{2k}{2} - \binom{k+1}{2}.$$
- ⋆ Let X be the set consisting of the remaining edges of G, namely
$$X = E(G) - [E(F_1) \cup E(F_2) \cup \cdots \cup E(F_{2k}) \cup E(H_1) \cup E(H_2) \cup \cdots \cup E(H_{k-1})].$$
Then
$$\begin{aligned} |X| &= n - \left[\binom{2k+1}{2} + \binom{2k}{2} - \binom{k+1}{2}\right] = n - \frac{7k^2 - k}{2} \\ &= \begin{cases} \dfrac{11k^2 - 5k}{2} - \dfrac{7k^2 - k}{2} = 2k(k-1) & \text{if } k \text{ is odd} \\[2ex] \dfrac{11k^2 - 5k + 2}{2} - \dfrac{7k^2 - k}{2} = 2k(k-1) + 1 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$
The edges in X are alternately colored red and blue such that the edge following H_{k-1} is colored differently than the edges of H_{k-1} and the edge preceding F_1 is blue. Hence,
  - ◦ if k is odd, then H_{k-1} is blue and so the edge following H_{k-1} is colored red. Since $|X| = 2k(k-1)$ is even, the edge preceding F_1 is blue as required;
  - ◦ if k is even, then H_{k-1} is red and so the edge following H_{k-1} is colored blue. Since $|X| = 2k(k-1) + 1$ is odd, the edge preceding F_1 is blue.

  Consequently, if $|X| = t$, then $G[X] = (f_1, f_2, \ldots, f_t)$ is a subpath Q_t of size t where f_1 is the edge following H_{k-1} and f_t is the edge preceding F_1. The edges of $G[X]$ are alternately colored red and blue such that f_1 is colored differently than the edges of H_{k-1} and f_t is blue.

We now have the following.

Observation. *No red edge in X is adjacent to any red edge in G and no blue edge in X is adjacent to any blue edge in G. Thus, each edge in X is a monochromatic component Q_1 either in the red subgraph G_r of G or in the blue subgraph G_b of G.*

If k is odd, then the red subgraph G_r and the blue subgraph G_b of G are

$$\begin{aligned} G_r &= F_1 + F_3 + \cdots + F_{2k-1} + H_1 + H_3 + \cdots + H_{k-2} + k(k-1)K_2 \\ &= Q_1 + Q_3 + \cdots + Q_{2k-1} + Q_{2k-1} + Q_{2k-3} + \cdots + Q_{k+2} + k(k-1)K_2 \\ G_b &= F_2 + F_4 + \cdots + F_{2k} + H_2 + H_4 + \cdots + H_{k-1} + k(k-1)K_2 \\ &= Q_2 + Q_4 + \cdots + Q_{2k} + Q_{2k-2} + Q_{2k-4} + \cdots + Q_{k+1} + k(k-1)K_2. \end{aligned}$$

If k is even, then the red subgraph G_r and the blue subgraph G_b of G are

$$\begin{aligned} G_r &= F_1 + F_3 + \cdots + F_{2k-1} + H_1 + H_3 + \cdots + H_{k-1} + k(k-1)K_2 \\ &= Q_1 + Q_3 + \cdots + Q_{2k-1} + Q_{2k-1} + Q_{2k-3} + \cdots + Q_{k+1} + k(k-1)K_2 \\ G_b &= F_2 + F_4 + \cdots + F_{2k} + H_2 + H_4 + \cdots + H_{k-2} + [k(k-1)+1]K_2 \\ &= Q_2 + Q_4 + \cdots + Q_{2k} + Q_{2k-2} + Q_{2k-4} + \cdots + Q_{k+2} + [k(k-1)+1]K_2. \end{aligned}$$

We claim that G possesses k maximal alternating Ramsey chains $R_1, R_2, \ldots, R_k$ where R_i has length $2k-1+i$ for $1 \le i \le k$.

- First, $R_1 = (F_1, F_2, \ldots, F_{2k})$ is an alternating Ramsey chain of length $2k$ in G. Since $G - E(R_1)$ contains no monochromatic subgraph isomorphic to $F_{2k} = Q_{2k}$, it follows that R_1 is maximal.
- Next, let $R_2 = (F_1, F_2, \ldots, F_{2k-1}, F_{2k}(2k-1) + K_2, H_1 + 2K_2)$, where the edges of K_2 and $2K_2$ are taken from X such that each link in R_2 is monochromatic. Then R_2 is an alternating Ramsey chain of length $2k+1$ in G. Since $G - E(R_2)$ contains no monochromatic subgraph isomorphic to Q_{2k-1}, it follows that R_2 is maximal.
- Next, let $R_3 = (F_1, F_2, \ldots, F_{2k-2}, F_{2k-1}(2k-2) + K_2, F_{2k}(2k-2) + 2K_2, H_1(2k-2) + 3K_2, H_2 + 4K_2)$, where the edges of $K_2, 2K_2, 3K_2, 4K_2$ are taken from X such that each link in R_3 is monochromatic. Then R_3 is an alternating Ramsey chain of length $2k+2$ in G. Since $G - E(R_3)$ contains no monochromatic subgraph isomorphic to Q_{2k-2}, it follows that R_3 is maximal.
- In general, for $2 \le i \le k$, let

  $$\begin{aligned} R_i = \; & (F_1, F_2, \ldots, F_{2k+1-i}, \quad F_{2k+2-i}(2k+1-i) + K_2, \\ & F_{2k+3-i}(2k+1-i) + 2K_2, \; \ldots, \; F_{2k}(2k+1-i) + (i-1)K_2, \\ & H_1(2k+1-i) + iK_2, \quad H_2(2k+1-i) + (i+1)K_2, \; \ldots, \; H_{i-1} + 2(i-1)K_2) \end{aligned}$$

  where the edges of $K_2, 2K_2, \ldots, 2(i-1)K_2$ are taken from X such that each link in R_i is monochromatic. Thus, R_i is an alternating Ramsey chain of length $2k-1+i$ in G. Since $G - E(R_i)$ contains no monochromatic subgraph isomorphic to Q_{2k+1-i}, it follows that R_i is maximal. In particular,

  $$\begin{aligned} R_k = \; & (F_1, F_2, \ldots, F_{k+1}, \quad F_{k+2}(k+1) + K_2, F_{k+3}(k+1) + 2K_2, \; \ldots, \; F_{2k}(k+1) + (k-1)K_2, \\ & H_1(k+1) + kK_2, \quad H_2(k+1) + (k+1)K_2, \; \ldots, \; H_{k-1} + 2(k-1)K_2) \end{aligned}$$

  where the edges of $K_2, 2K_2, \ldots, 2(k-1)K_2$ are taken from X such that each link in R_k is monochromatic. Thus, R_k is an alternating Ramsey chain of length $3k-1$ in G. Since $G - E(R_k)$ contains no monochromatic subgraph isomorphic to Q_{k+1}, it follows that R_k is maximal.

Finally, we show that each of the k maximal alternating Ramsey chains $R_1, R_2, \ldots, R_k$ of distinct lengths in G can be constructed as described. Of the k alternating Ramsey chains R_1, R_2, …, R_k in G, the longest chain R_k among them takes the maximum number of edges from X. This maximum number is $1 + 2 + \cdots + 2(k-1) = \binom{2k-1}{2} = k(2k-1)$.

- If k is odd, then the link $F_{k+2}(k+1) + K_2$ in R_k is red and so the number of red components Q_1 required in R_k from X is $1 + 3 + \cdots + (2k-3) = (k-1)^2$ and the number of blue components Q_1 in R_k is

  $$2 + 4 + \cdots + 2(k-1) = 2[1 + 2 + \cdots + (k-1)] = 2\binom{k}{2} = k(k-1).$$

  Since $|X| = 2k(k-1)$, where $k(k-1)$ edges are red and $k(k-1)$ edges are blue, it follows by the observation that k such maximal alternating Ramsey chains $R_1, R_2, \ldots, R_k$ of distinct lengths in G can be constructed.

- If k is even, then the link $F_{k+2}(k+1)+K_2$ in R_k is blue and so the number of blue components Q_1 required in R_k is $1+3+\cdots+(2k-3)=(k-1)^2$ and the number of red components Q_1 from X is

$$2+4+\cdots+2(k-1)=2[1+2+\cdots+(k-1)]=2\binom{k}{2}=k(k-1).$$

  Since $|X|=2k(k-1)+1$, where $k(k-1)$ edges are red and $k(k-1)+1$ edges are blue, it follows by the observation that such k maximal alternating Ramsey chains $R_1, R_2, \ldots, R_k$ of distinct lengths in G can be constructed.

This completes the proof. □

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