Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **5** (2023) 8–17
DOI: 10.47443/ejm.2022.044

Research Article

Higher gradient bounds for a class of higher-order nonlinear systems

Mahmoud Qafsaoui*

Department of Mathematics, ESTACA – Groupe ISAE, Laval, France

(Received: 7 November 2022. Received in revised form: 8 February 2023. Accepted: 14 February 2023. Published online: 22 February 2023.)

Abstract

In this article, fundamental L^p-estimates and the large-time behavior for higher derivatives of solutions to a generalized class of higher-order nonlinear parabolic systems are studied.

Keywords: higher-order operators; heat kernel estimates; L^p-estimates; asymptotic behavior.

2020 Mathematics Subject Classification: 35B40, 35K25, 35K57, 35G05, 35A08.

1. Statements of the main results

This paper is devoted to the study of solutions to the nonlinear parabolic system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{L}_m u = \mathbf{a}.\nabla^{\theta}\big(f(|\nabla^{\kappa} v|)\, u\big), \\ \dfrac{\partial v}{\partial t} + \mathcal{L}_m v = \mathbf{b}(t).\nabla^{\sigma}(u - v), \\ u_0\,,\ v_0 \in L^1(\mathbb{R}^n), \end{cases} \tag{1}$$

where $t > 0$, $\kappa + \sigma \leq 2m - 1$ and ∇^k denotes the vector $\left(D^{\gamma}\right)_{|\gamma|=k}$ with $\gamma = (\gamma_1, \cdots, \gamma_n) \in \mathbb{N}^n$. The constant vector $\mathbf{a}$ is in $\mathbb{R}^n$ and the vectorial function $\mathbf{b} : t \mapsto \mathbf{b}(t)$ satisfies the inequality

$$|\mathbf{b}(t)| \leq C\, t^{-\frac{4m+\epsilon-\sigma}{4m}}$$

for a small enough $\epsilon > 0$. The nonlinear function f is such that $f \in \mathcal{C}^{\theta}(\mathbb{R})$ and $f(0) = 0$. The class of homogeneous operators $\mathcal{L}_m = \triangle^m \chi \triangle^m$ is of order $4m$ and the function χ is bounded, measurable, positive, and independent of time.

The particular case, when $\mathcal{L}_m = \triangle$, $\mathbf{a} = \mathbf{b} = \mathbf{1}$, $\theta = \kappa = 1$, $\sigma = 0$ and f is the identity function, corresponds to the parabolic system, modeling the well-known chemotaxis biological phenomenon dealing with the movement of an organism in response to a chemical stimulus. The function $(x, t) \longmapsto u(x, t)$ stands for the population density of the organism at position x and time t, and the function $(x, t) \longmapsto v(x, t)$ represents the concentration of the chemical [9]. Also, we cite the Keller-Segel model describing the movement of amoebae with density u, in the presence of a chemoattractant with the concentration v [13].

The literature concerning the existence, uniqueness, large-time behavior of solutions and the blow-up problem is abundant, see [4,6,8–10,13,15]. For example, Nagai et al. in [9] studied the existence and the uniqueness of bounded solutions to the parabolic system in $\mathbb{R}^n$ when $n \geq 2$. Also, the large time behavior was discussed under the condition

$$\sup_{t>0} \left(\|u(t)\|_p + \|v(t)\|_p\right) < \infty$$

for $p = 1, \infty$. In fact, it was shown that for every $1 < p \leq \infty$,

$$\sup_{t>0} \Big((1+t)^{\frac{n}{2}(1-1/p)}\left(\|u(t)\|_p + \|v(t)\|_p\right)\Big) < \infty\,,$$

$$\lim_{t\to\infty} \Big(t^{\frac{n}{2}(1-1/p)}\big\|w(t) - M_0 K(t)\big\|_p\Big) = 0,$$

*E-mail address: mahmoud.qafsaoui@estaca.fr

where $K:(x,t)\longmapsto K(x,t)$ is the heat kernel, $w=u$ or v and

$$M_0=\int_{\mathbb{R}^n}u_0dy.$$

Under the same condition as mentioned before and $n\geq 2$, Nagai and Yamada established in [10] the following decay:

$$\lim_{t\to\infty}\left(t^{\frac{n}{2}(1-1/p)+1/2}\big\|w(t)-\big(M_0K(t)-(E_0+V).\nabla K(t)\big)\big\|_p\right)=0,$$

where

$$E_0=\int_{\mathbb{R}^n}yu_0dy\quad\text{and}\quad V=\int_0^\infty\int_{\mathbb{R}^n}u\nabla vdyds.$$

Also, it is important to mention the reference [15], where sharp asymptotic expansions are obtained under the conditions $(|x|^nu_0)\in L^1(\mathbb{R}^n)$ and

$$\sup_{x\in\mathbb{R}^n\,,\,t>0\,,\,0\leq\mu\leq n}\Big((1+|x|)^{n-\mu}(1+t)^{\mu/2}\big(|u(x,t)|+|v(x,t)|\big)\Big)<\infty.$$

Now, let us turn to the core of our work, which is an extension of the work reported in the article [12]. The class of higher-order operators $\mathcal{L}_m=\triangle^m\chi\triangle^m$, used in the system (1), is homogeneous and of order $4m$. It was shown in [1] that a more general class of operators with non-smooth coefficients of type $T_0=L_0^*aL_0$, where $L_0=\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}D^{\alpha+\beta}$ with constant coefficients $a_{\alpha\beta}$ and $a\in L^\infty(\mathbb{R}^n,\mathbb{C})$ with $\mathcal{R}e(a)\geq\delta>0$, verifies an elliptic condition of De Giorgi type which is equivalent to Gaussian estimates of the kernel and its derivatives. More precisely, the distributional kernel $\mathcal{K}:(x,t)\longmapsto\mathcal{K}(x,t)$ of the semi-group e^{-tT_0} (i.e. the heat kernel associated with T_0) satisfies the following Gaussian estimates:

(i) There are constants $c_0>0$, $c_1>0$ such that for all $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, all $t>0$ and all multi-index $\gamma=(\gamma_1,\cdots,\gamma_n)\in\mathbb{N}^n$ such that $|\gamma|\leq 2m-1$, one has

$$|D_x^\gamma\mathcal{K}(x,t)|\leq c_0\,t^{-\frac{n+|\gamma|}{4m}}\,exp\left(-c_1\left(\frac{|x|}{t^{1/4m}}\right)^{\frac{4m}{4m-1}}\right),\tag{2}$$

where $|\gamma|=\gamma_1+\cdots+\gamma_n$ and $D_x^\gamma=\frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}}\cdots\frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$.

(ii) The Gaussian estimates (2) hold for the kernel of $\left(t\,\frac{d}{dt}\left(e^{-tT_0}\right)\right)$ and as a consequence, the following L^p-estimates are obtained on the heat kernel $\mathcal{K}$ and its time derivatives:

$$\|D_x^\gamma\mathcal{K}(t)\|_p\leq C(p,m)\,t^{-\frac{n+q|\gamma|}{4mq}},\tag{3}$$

$$\|D_x^\gamma\mathcal{K}(t+1)-D_x^\gamma\mathcal{K}(t)\|_p\leq C(p,m)\,t^{-\frac{n+q|\gamma|}{4mq}-1},\tag{4}$$

for all $p\in[1,\infty]$ $\big($with $1/p+1/q=1\big)$ and all $\gamma\in\mathbb{N}^n$ such that $|\gamma|\leq 2m-1$.

On the other hand, by using the estimates (3), among others, it was shown in [7] that the solution w of the heat equation $\frac{\partial w}{\partial t}+\mathcal{L}_mw=0$, with $w(0)=w_0$, satisfies

$$\lim_{t\to\infty}\left(t^{\frac{n+q|\gamma|}{4mq}}\,\|D_x^\gamma w(t)-W_0D_x^\gamma\mathcal{K}(t)\|_p\right)=0,\tag{5}$$

where

$$W_0=\int_{\mathbb{R}^n}w_0(x)dx$$

while p and γ are the same as defined before.

The main objective of this article is to establish L^p-estimates for the higher-order gradient of solutions to the system (1) and then to derive an asymptotic behavior for these solutions under some conditions on the nonlinearity f. The solutions of our interest are mild solutions. More precisely, by a solution (u,v) to the system (1), we mean functions u, $v\in\mathcal{C}\big([0,\infty)\,,\,L^1(\mathbb{R}^n)\big)$ satisfying the Duhamel integral formula

$$\begin{cases}u(t)=e^{-t\mathcal{L}_m}u_0+\displaystyle\int_0^te^{-(t-s)\mathcal{L}_m}\Big(\mathbf{a}.\nabla^\theta\big(u(s)\,f\big(|\nabla^\mu v(s)|\big)\Big)ds\\ v(t)=e^{-t\mathcal{L}_m}v_0+\displaystyle\int_0^te^{-(t-s)\mathcal{L}_m}\Big(\mathbf{b}(s).\nabla^\sigma\big(u-v)(s)\Big)ds\,,\end{cases}\tag{6}$$

or by using the heat kernel $\mathcal{K}:(x,t)\longmapsto\mathcal{K}(x,t)$ associated with $\mathcal{L}_m$,

$$\begin{cases} u(t)=\mathcal{K}(t)*u_0+\displaystyle\int_0^t\mathcal{K}(t-s)*\Big(\mathbf{a}.\nabla^\theta\big(u(s)\,f\big(|\nabla^\mu v(s)|\big)\big)\Big)ds\\ v(t)=\mathcal{K}(t)*v_0+\displaystyle\int_0^t\mathcal{K}(t-s)*\Big(\mathbf{b}(s).\nabla^\sigma\big(u-v\big)(s)\Big)ds\,,\end{cases} \tag{7}$$

where the symbol $*$ stands for the Laplace convolution operator.

We prove that the solutions and their derivatives decay to 0 when $t\to\infty$ and behave like those of the heat kernel. More precisely, we show that for all multi-index $\lambda\in\mathbb{N}^n$ such that $|\lambda|+\theta\le 2m-1$, we have

$$||D_x^\lambda\phi(t)||_p\le C\,t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}}=C\,t^{-\frac{n+q|\lambda|}{4mq}},$$

when the function f is dominated by power functions depending on m, n, θ and κ. Alternatively,

$$\lim_{t\to\infty}\Big(t^{\frac{n+q|\lambda|}{4mq}}||D_x^\lambda\phi(t)-\Phi_0\,D_x^\lambda\mathcal{K}(t)||_p\Big)=0,$$

where ϕ stands for u or v, $\Phi_0=\displaystyle\int_{\mathbb{R}^n}\phi_0(x)\,dx$ and $\phi_0\in L^1(\mathbb{R}^n)$.
To obtain these estimates, we rely, in particular, on the following useful result giving L^p-estimates of ϕ:

$$||\phi(t)||_p\le C\,t^{-\frac{n}{4mq}},$$

which holds for all $p\in[1,\infty]$ and all $t>0$. The constant C depends on n, m and p, and $p+q=pq$.
Also, we mention that by using the arguments employed in the forthcoming analysis and the L^p-contraction property,

$$||\phi(t)||_p\le||\phi_0||_p\,, \tag{8}$$

valid for all $p\in[1,\infty]$, it is possible to handle a more general system of the form:

$$\begin{cases}\dfrac{\partial u}{\partial t}+\mathcal{L}_m u=\mathbf{a}.\nabla^\theta\Big(u\,f\big(|\nabla^\kappa v|\big)\Big)\\ \dfrac{\partial v}{\partial t}+\mathcal{L}_m v=\mathbf{b}(t).\nabla^\sigma\big(g(u-v)\big)\\ u_0\,,\;v_0\in L^1(\mathbb{R}^n),\end{cases}$$

with $|g(t)|\le C\,t^N$. The adaptation and details will be left to the reader.

The present work is organized as follows. Section 2 is dedicated to fundamental L^p-estimates of higher derivatives of solutions to (1). For the reader's comfort, the computations are divided into three steps. We start by establishing the classical conservation of mass and L^p-contraction properties, which are very useful for the sequel. The second step focuses on the estimates of the concentration function v, which are essential for the derivation of estimates on the density function u in the last step. The first part of Section 3 is concerned with the asymptotic behavior of higher-order derivatives using the heat kernel as a regulator factor. The second part of of Section 3 deals with the rapid decay when the initial data ϕ_0 belongs to $L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. More precisely, we prove that for $\lambda\in\mathbb{N}^n$ such that $|\lambda|+\theta<2m-1$, we have

$$||D_x^\lambda\phi(t)-\Phi_0\,D_x^\lambda\mathcal{K}(t)||_p\le C\begin{cases}t^{-\frac{n+q|(\lambda|+1)}{4mq}}\text{ or}\\ t^{-\frac{n+q|(\lambda|+1)}{4mq}}\ln(1+t)\text{ or}\\ t^{\mu/4m}(1+t)^{-n/4m}\end{cases}$$

depending on the power of the dominant function associated with f while μ depends on n, m, θ and κ.

2. Fundamental L^p-estimates

The objective of this section is to prove that, for $1\le p\le\infty$ and under conditions on f and the parameters of the system (1), there exists a constant $C>0$ such that the coupled solution (u,v) verifies

$$||D_x^\lambda u(t)||_p+||D_x^\lambda v(t)||_p\le C\,t^{-\frac{1}{4m}\big(n(1-1/p)+|\lambda|\big)}, \tag{9}$$

for all $t>0$ and all $\lambda\in\mathbb{N}^n$ such that $\max\big(|\lambda|+\theta,|\lambda|+\sigma,\kappa+\sigma\big)\le 2m-1$. The first step to prove the estimates (9) concerns essentially some well-known conservation and L^1-contraction properties.

Conservation and contraction properties

Proposition 2.1. *Let $u_0 \in L^1(\mathbb{R}^n)$ and $v_0 \in L^1(\mathbb{R}^n)$. Then the couple (u,v) satisfies, for all $t \geq 0$,*

(i) $\int_{\mathbb{R}^n} u(x,t)dx = \int_{\mathbb{R}^n} u_0(x)dx$ *and* $\int_{\mathbb{R}^n} v(x,t)dx = \int_{\mathbb{R}^n} v_0(x)dx$.

(ii) $||u(t)||_1 \leq ||u_0||_1$ *and* $||v(t)||_1 \leq ||v_0||_1$.

Proof. To obtain the conservation property, we integrate equations in (1) with respect to the space variable x. Firstly, we have

$$\int_{\mathbb{R}^n} \mathbf{a}.\nabla^\theta\big(u\, f\big(|\nabla^\kappa v|\big)\big)dx = \int_{\mathbb{R}^n} \mathbf{b}(t).\nabla^\sigma\big(u-v\big)dx = 0$$

and by using the Fourier transform

$$\int_{\mathbb{R}^n} \mathcal{L}_m\phi(x,t)dx = \widehat{\mathcal{L}_m\phi}(0,t) = 0\,,$$

where ϕ stands for u or v. This leads to

$$\frac{d}{dt}\int_{\mathbb{R}^n} \phi(x,t)dx = 0$$

and the conservation property (i).
For the proof of the contraction property, a straightforward adaptation of the classical arguments used (for example) in [3] (Theorem 3.1) and [5] (Proposition 1), allows to derive (ii). □

It is worth mentioning that we can use the arguments mentioned in Proposition 2.1 to derive the L^p-contraction property (8) valid for all $p \in [1,\infty]$. Now, let us state the main results of this section concerning higher-order derivatives estimates for the couple (u,v) solution to the system (1). We derive the estimates of $D_x^\lambda v$, u, and $D_x^\lambda u$.

Higher Gradient estimates of the concentration function

We start with the L^p-estimates of $D_x^\lambda v(t)$.

Proposition 2.2. *Let $1 \leq p\,,q \leq \infty$ be such that $p+q = pq$. Then there is a constant $C = C\big(p,n,m,\|u_0\|_1,\|v_0\|_1\big) > 0$ such that the function v of (1) satisfies*

$$||D_x^\lambda v(t)||_p \leq C\, t^{-\frac{n+q|\lambda|}{4mq}}\,, \tag{10}$$

for all $t > 0$ and all $\lambda \in \mathbb{N}^n$, provided that

$$|\lambda| + \sigma \leq 2m-1.$$

Proof. Let $\lambda \in \mathbb{N}^n$ be such that $|\lambda| + \sigma \leq 2m-1$. In all the sequel, D^λ stands for D_x^λ and the constant C may change from line to line.
The second integral equation of (6) involves

$$D^\lambda v(t) = D^\lambda e^{-t\mathcal{L}_m} v_0 + \int_0^t D^\lambda e^{-(t-s)\mathcal{L}_m}\Big(\mathbf{b}(s).\nabla^\sigma\big((u-v)(s)\big)\Big)ds\,. \tag{11}$$

Then, by using successively Young's inequality $\big(||e^{-t\mathcal{L}_m}\varphi||_r \leq ||e^{-t\mathcal{L}_m}||_\tau\, ||\varphi||_{\tau'}$ with $1+1/r = 1/\tau + 1/\tau'$, and $r,\tau,\tau' \geq 1\big)$, the estimates (3) on the heat kernel and the contraction property (ii) of Proposition 2.1, we obtain

$$\begin{aligned}
||D^\lambda v(t)||_1 &\leq ||D^\lambda e^{-t\mathcal{L}_m} v_0||_1 + \int_0^t \Big\|\mathbf{b}(s).\nabla^\sigma\big(D^\lambda e^{-(t-s)\mathcal{L}_m}\big)\big(u-v\big)(s)\Big\|_1 ds \\
&\leq C\,||D^\lambda \mathcal{K}(t)||_1\, ||v_0||_1 + \int_0^t ||\mathbf{b}(s).\nabla^\sigma(D^\lambda\mathcal{K}(t-s))||_1||(u-v)(s)||_1\, ds \\
&\leq C\left(t^{-\frac{|\lambda|}{4m}}\, ||v_0||_1 + \big(||u_0||_1 + ||v_0||_1\big)\int_0^t (t-s)^{-\frac{|\lambda|+\sigma}{4m}}\, s^{-(1+(\epsilon-\sigma)/4m)}\, ds\right) \\
&\leq C\, \max\left(t^{-\frac{|\lambda|}{4m}}\,,\, t^{-\frac{|\lambda|+\epsilon}{4m}}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned} ||D^\lambda v(t)||_\infty &\le ||D^\lambda e^{-t\mathcal{L}_m} v_0||_\infty + \int_0^t \left\|\mathbf{b}(s).\nabla^\sigma\left(D^\lambda e^{-(t-s)\mathcal{L}_m}\right)(u-v)(s)\right\|_\infty ds \\ &\le C\,||D^\lambda \mathcal{K}(t)||_\infty\,||v_0||_1 + \int_0^t ||\mathbf{b}(s).\nabla^\sigma(D^\lambda \mathcal{K}(t-s))||_\infty ||(u-v)(s)||_1\, ds \\ &\le C\left(t^{-\frac{n}{4m}-\frac{|\lambda|}{4m}}\,||v_0||_1 + (||u_0||_1 + ||v_0||_1)\int_0^t (t-s)^{-\frac{n}{4m}-\frac{\sigma+|\lambda|}{4m}}\, s^{-(1+(\epsilon-\sigma)/4m)}\, ds\right) \\ &\le C\,\max\left(t^{-\frac{n+|\lambda|}{4m}}, t^{-\frac{n+|\lambda|+\epsilon}{4m}}\right). \end{aligned}$$

Eventually, by interpolation we obtain

$$||D^\lambda v(t)||_p \le ||D^\lambda v(t)||_1^{1/p}\,||D^\lambda v(t)||_\infty^{1-1/p} \le C\,t^{-\frac{|\lambda|}{4mp}}\,t^{-\frac{n+|\lambda|}{4m}(1-1/p)}.$$

That is, for all $\lambda \in \mathbb{N}^n$ such that $|\lambda| + \sigma \le 2m-1$, we have the estimates

$$||D^\lambda v(t)||_p \le C\,t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}},$$

which corresponds to (10). □

Higher Gradient estimates related to the population density function

Now, we move towards the L^p-estimates related to the density function u.

Proposition 2.3. *Let $1 \le p\,, q \le \infty$ be such that $p+q = pq$, $M = \frac{4m-\theta}{n+\kappa}$ and suppose that for all $t \in \{s \in \mathbb{R}/|s| \le 1\}$,*

$$|f(t)| \le C|t|^M.$$

Then there is a constant $C > 0$ such that the function u of (1) verifies the following estimates

$$||u(t)||_p \le C\,t^{-\frac{n}{4mq}}, \tag{12}$$

for all $t > 0$ and all $\lambda \in \mathbb{N}^n$ such that $\max(\theta\,, \kappa+\sigma) \le 2m-1$.

Proof. The arguments used in more detail above and the inequality (10) lead to the following estimates

$$\begin{aligned} ||u(t)||_\infty &\le ||e^{-t\mathcal{L}_m} u_0||_\infty + \int_0^t \left\|e^{-(t-s)\mathcal{L}_m}\Big(\mathbf{a}.\nabla^\theta\big(u(s)\,f(|\nabla^\kappa v(s)|)\big)\Big)\right\|_\infty ds \\ &\le C\Big(t^{-\frac{n}{4m}}||u_0||_1 + \int_0^t \left\|\nabla^\theta \mathcal{K}(t-s)\right\|_\infty \left\|u(s)\,f(|\nabla^\kappa v(s)|)\right\|_1 ds\Big) \\ &\le C\Big(t^{-\frac{n}{4m}}||u_0||_1 + \int_0^t (t-s)^{-\frac{n+\theta}{4m}}\,||u(s)||_1\,||f(|\nabla^\kappa v(s)|)||_\infty\, ds\Big) \\ &\le C\,(t/2)^{-\frac{n}{4m}}. \end{aligned}$$

As before, it follows by using the contraction property and the interpolation inequality that

$$||u(t)||_p \le ||u(t)||_1^{1/p}\,||u(t)||_\infty^{1-1/p} \le C||u_0||_1^{1/p}\,t^{-\frac{n}{4m}(1-\frac{1}{p})},$$

which corresponds to the estimates (12). □

Finally, in order to complete the proof of the estimates (9), we extend the above estimates to the higher derivatives $D^\lambda_x u(t)$.

Proposition 2.4. *Let $p\,, q$ and f be as in Proposition 2.3. Then there is a constant $C = C\big(p, n, m, \|u_0\|_1, \|v_0\|_1\big) > 0$ such that the function u of (1) satisfies*

$$||D^\lambda_x u(t)||_p \le C\,t^{-\frac{n+q|\lambda|}{4mq}}, \tag{13}$$

for all $t > 0$ and all $\lambda \in \mathbb{N}^n$, provided $\max\big(|\lambda| + \theta\,, \kappa+\sigma\big) \le 2m-1$.

Proof. Let $\lambda \in \mathbb{N}^n$ be such that $\max\big(|\lambda| + \theta\,,\, \kappa + \sigma\big) \le 2m - 1$. According to (7), we have

$$D^{\lambda}u(t) = \big(D^{\lambda}\mathcal{K}(t) * u_0\big) + \int_0^t D^{\lambda}\mathcal{K}(t-s) * \Big(\mathbf{a}.\nabla^{\theta}\big(u(s)f\big(|\nabla^{\kappa}v(s)|\big)\big)\Big)ds\,. \tag{14}$$

Using successively Young's inequality and the estimates (3) on the kernel $\mathcal{K}$, we obtain

$$\begin{aligned}||D^{\lambda}u(t)||_p &\le ||D^{\lambda}\mathcal{K}(t) * u_0||_p + \int_0^t ||\mathbf{a}.\nabla^{\theta}(D^{\lambda}\mathcal{K}(t-s)) * \big(u(s)\,f\big(|\nabla^{\kappa}v(s)|\big)\big)||_p ds\\ &\le ||D^{\lambda}\mathcal{K}(t)||_p||u_0||_1 + |\mathbf{a}|\underbrace{\int_0^t ||\nabla^{\theta}(D^{\lambda}\mathcal{K}(t-s)) * \big(u(s)\,f\big(|\nabla^{\kappa}v(s)|\big)\big)||_p ds}_{I}\\ &\le C\,t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}} + |\mathbf{a}|\,I.\end{aligned}$$

Now, write

$$I = \int_0^{t/2}\cdots + \int_{t/2}^{t}\cdots := I_1 + I_2.$$

By applying Young's inequality and the estimates (3), (10), (12) to I_1 and I_2, we derive the following estimates.

$$\begin{aligned}I_1 &\le \int_0^{t/2}\big\|\nabla^{\theta}(D^{\lambda}\mathcal{K}(t-s))\big\|_p\,\big\|u(s)\,f\big(|\nabla^{\kappa}v(s)|\big)\big\|_1\,ds\\ &\le C\int_0^{t/2}(t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|+\theta}{4m}}\big\|u(s)\big\|_{\infty}\,\big\|f\big(|\nabla^{\kappa}v(s)|\big)\big\|_1\,ds\\ &\le C\int_0^{t/2}(t-s)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|+\theta}{4m}}s^{-\frac{n}{4m}}||\nabla^{\kappa}v(s)||_M^M\,ds \le C(t/2)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}}\,.\end{aligned}$$

Similar computations on I_2 yield

$$I_2 \le \int_{t/2}^{t}\big\|\nabla^{\theta}(D^{\lambda}\mathcal{K}(t-s))\big\|_1\,\big\|u(s)\,f\big(|\nabla^{\kappa}v(s)|\big)\big\|_p\,ds \le C(t/2)^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}}\,.$$

Finally, for all $p \in [1,\infty]$ and all λ such that $\max\big(\theta + |\lambda|\,,\,\kappa+\sigma\big) \le 2m-1$,

$$||D^{\lambda}u(t)||_p \le C\,t^{-\frac{n}{4m}(1-\frac{1}{p})-\frac{|\lambda|}{4m}}\,.$$

□

With all the above estimates, the main estimates (9) are completely now proven.

3. Large time behavior

Asymptotic behavior for higher derivatives of solutions

Under some conditions on θ, κ, σ, and the nonlinearity of f, we show that the higher derivatives of u and v behave like those of the heat kernel. More precisely, we have the following result.

Theorem 3.1. *Suppose that* $\lim_{t\to 0}\big(|t|^{-M}f(t)\big) = 0$ *for* $M = \frac{4m-\theta}{n+\kappa}$. *Then the solution* (u,v) *of (1) verifies for all* $p \in [1,\infty]$ *and all* $\lambda \in \mathbb{N}^n$ *such that* $\max\big(|\lambda|+\theta\,,\,|\lambda|+\sigma\,,\,\kappa+\sigma\big) \le 2m-1$,

$$\begin{cases}(i)\ \lim\limits_{t\to\infty}\Big(t^{\frac{n+q|\lambda|}{4mq}}||D_x^{\lambda}u(x,t) - \mathcal{U}_0\,D_x^{\lambda}\mathcal{K}(x,t)||_p\Big) &= 0\,,\\ (ii)\ \lim\limits_{t\to\infty}\Big(t^{\frac{n+q|\lambda|}{4mq}}||D_x^{\lambda}v(x,t) - \mathcal{V}_0\,D_x^{\lambda}\mathcal{K}(x,t)||_p\Big) &= 0\,,\end{cases} \tag{15}$$

where $1/p + 1/q = 1$,

$$\mathcal{U}_0 = \int_{\mathbb{R}^n} u_0(x)\,dx$$

and

$$\mathcal{V}_0 = \int_{\mathbb{R}^n} v_0(x)\,dx.$$

Proof. Let $\lambda \in \mathbb{N}^n$ with $|\lambda| + \theta \leq 2m-1$ and $p\,,q \in [1,\infty]$ be such that $p+q = pq$.

(i) Behavior of higher derivatives of u. According to (14), write

$$D^\lambda u(t+1) - \mathcal{U}_0\, D^\lambda \mathcal{K}(t+1) = \big(D^\lambda u(t+1) - \mathcal{U}_0 D^\lambda \mathcal{K}(t)\big) - \mathcal{U}_0\big(D^\lambda \mathcal{K}(t+1) - D^\lambda \mathcal{K}(t)\big)$$

and

$$\begin{aligned}&D^\lambda u(t+1) - \mathcal{U}_0 D^\lambda \mathcal{K}(t+1) = \\ &\Big(D^\lambda \mathcal{K}(t) * u(1) - \mathcal{U}_0 D^\lambda \mathcal{K}(t)\Big) - \mathcal{U}_0\Big(D^\lambda \mathcal{K}(t+1) - D^\lambda \mathcal{K}(t)\Big) + \int_0^t D^\lambda \mathcal{K}(t-s) * \Big(\mathbf{a}.\nabla^\theta\big(u(s+1)\, f(|\nabla^\kappa v(s+1)|)\big)\Big) ds\,.\end{aligned}$$

Using $\displaystyle\int_{\mathbb{R}^n} u(x,1)dx = \int_{\mathbb{R}^n} u_0(x)dx = \mathcal{U}_0$ (Proposition 2.1) and (5) leads to

$$\lim_{t\to\infty} \Big(t^{\frac{n+q|\lambda|}{4mq}} ||D^\lambda \mathcal{K}(t) * u(1) - \mathcal{U}_0 D^\lambda \mathcal{K}(t)||_p\Big) = 0.$$

According to (4) , we have

$$||D^\lambda \mathcal{K}(t+1) - D^\lambda \mathcal{K}(t)||_p \leq C t^{-\frac{n+q|\lambda|}{4mq}-1}$$

and then

$$\lim_{t\to\infty} \Big(t^{\frac{n+q|\lambda|}{4mq}} ||D^\lambda \mathcal{K}(t+1) - D^\lambda \mathcal{K}(t)||_p\Big) = 0.$$

To complete our estimates, we need to show that

$$\lim_{t\to\infty} \Big(t^{\frac{n+q|\lambda|}{4mq}} \underbrace{\Big\| \int_0^t D^\lambda \mathcal{K}(t-s) * \Big(\mathbf{a}.\nabla^\theta\big(u(s+1)\, f(|\nabla^\kappa v(s+1)|)\big)\Big) ds\Big\|_p}_{I}\Big) = 0\,. \tag{16}$$

Observe that

$$I \leq \int_0^{t/2} ||\cdots||_p\, ds + \int_{t/2}^t ||\cdots||_p\, ds := I_1 + I_2\,.$$

On the other hand, if we put $h(s) := |s|^{-M} f(s)$ then

$$||f(|\nabla^\kappa v(s)|)||_p \leq ||h(|\nabla^\kappa v(s)|)||_\infty\, ||\nabla^\kappa v(s)||_{pM}^M\,. \tag{17}$$

It follows from (3), (10), (12) and (17) that

$$\begin{aligned} I_1 &\leq C \int_0^{t/2} ||\nabla^\theta (D^\lambda \mathcal{K}(t-s))||_p\, ||u(s+1)\, f(|\nabla^\kappa v(s+1)|)||_1\, ds \\ &\leq C \int_0^{t/2} (t-s)^{-\frac{n+q(|\lambda|+\theta)}{4mq}}\, (s+1)^{-\frac{n}{4m}}\, ||h(|\nabla^\kappa v(s+1)|)||_\infty\, ||\nabla^\kappa v(s+1)||_M^M\, ds \\ &\leq C \int_0^{t/2} (t-s)^{-\frac{n+q(|\lambda|+\theta)}{4mq}} (s+1)^{-(1-\frac{\theta}{4m})}\, ||h(|\nabla^\kappa v(s+1)|)||_\infty\, ds \\ &\leq C t^{-\frac{n+q|\lambda|}{4mq}} \underbrace{\Big(t^{-\frac{\theta}{4m}} \int_0^{t/2} (s+1)^{-(1-\frac{\theta}{4m})}\, ||h(|\nabla^\kappa v(s+1)|)||_\infty\, ds\Big)}_{\tau(t)}. \end{aligned}$$

Since $\lim\limits_{t\to\infty} ||h(|\nabla^\kappa v(t)|)||_\infty = 0$ then for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$||h(|\nabla^\kappa v(s+1)|)||_\infty \leq \varepsilon$$

for all $s \geq \delta$ and then

$$t^{-\frac{\theta}{4m}} \int_\delta^{t/2} (s+1)^{-(1-\frac{\theta}{4m})}\, ||h(|\nabla^\kappa v(s+1)|)||_\infty\, ds \leq C\varepsilon\,,$$

where C is a constant depending on m and θ. Also, as

$$\lim_{t\to\infty} \Big(t^{-\frac{\theta}{4m}} \int_0^\delta (s+1)^{-(1-\frac{\theta}{4m})}\, ||h(|\nabla^\kappa v(s+1)|)||_\infty\, ds\Big) = 0,$$

then we obtain $\lim_{t\to\infty} \big(\tau(t)\big) = 0$. Similar computations on I_2 provide

$$\begin{aligned} I_2 &\le C \int_{t/2}^{t} ||\nabla^{\theta}(D^{\lambda}\mathcal{K}(t-s))||_1 \, ||u(s+1)\, f(|\nabla^{\kappa} v(s+1)|)||_p \, ds \\ &\le C \int_{t/2}^{t} (t-s)^{-\frac{|\lambda|+\theta}{4m}} \, (s+1)^{-\frac{n}{4m}(1-\frac{1}{p})-(1-\frac{\theta}{4m})} \, ||h(|\nabla^{\kappa} v(s+1)|)||_\infty \, ds \\ &\le C \sup_{s\ge t/2+1} ||h(|\nabla^{\kappa} v(s)|)||_\infty \, t^{-\frac{n+q|\lambda|}{4mq}} . \end{aligned}$$

Since

$$\lim_{t\to\infty} \Big(\sup_{s\ge t/2+1} ||h(|\nabla^{\kappa} v(s)|)||_\infty \Big) = 0,$$

we have

$$\lim_{t\to\infty} \big(t^{\frac{n+q|\lambda|}{4mq}} \, I_2\big) = 0$$

and this ends the proof of (i) of (15).

(ii) Behavior of higher derivatives of v. According to (11), the same decomposition and the estimates used above for $D^{\lambda}u(t)$, it is sufficient to show that

$$\lim_{t\to\infty} \Big(t^{\frac{n+q|\lambda|}{4mq}} \underbrace{\Big\| \int_0^t \mathbf{b}(s+1).\nabla^{\sigma}\big(D^{\lambda}\mathcal{K}(t-s)\big) * (u-v)(s+1)\, ds \Big\|_p}_{J} \Big) = 0 \,. \tag{18}$$

Observe that the decomposition $J \le J_1 + J_2$ as above, gives

$$\begin{aligned} J_1 &\le \int_0^{t/2} ||\mathbf{b}(s+1).\nabla^{\sigma}(D^{\lambda}\mathcal{K}(t-s))||_p \, ||(u-v)(s+1)||_1 \, ds \\ &\le C \big(||u_0||_1 + ||v_0||_1\big) \, (t/2)^{-\frac{n+q|\lambda|}{4mq} - \frac{\epsilon}{4m}} \end{aligned}$$

and

$$\begin{aligned} J_2 &\le C \int_{t/2}^{t} ||\mathbf{b}(s+1).\nabla^{\sigma}(D^{\lambda}\mathcal{K}(t-s))||_1 \, \big(||u(s+1)||_p + ||v(s+1)||_p\big) \, ds \\ &\le C\,(t/2)^{-\frac{n}{4mq}} \int_{t/2}^{t} (t-s)^{-\frac{|\lambda|+\sigma}{4m}} \, (s+1)^{-(1+(\epsilon-\sigma)/4m)} \, ds \le C\,(t/2)^{-\frac{n+q|\lambda|}{4mq} - \frac{\epsilon}{4m}} . \end{aligned}$$

This implies

$$\lim_{t\to\infty} \Big(t^{\frac{n+q|\lambda|}{4mq}} \, J \Big) = 0,$$

which completes the proof of (ii) of (15) and hence the proof of Theorem 3.1 is completed. □

Sharp large time behavior

Suppose that for all $p \in [1,\infty]$, there is a constant $C > 0$ such that the solution (u,v) of (1) satisfies for all $t > 0$,

$$||u(t)||_p \, + \, ||v(t)||_p \le C\,(1+t)^{-\frac{n}{4mq}}, \tag{19}$$

where $1/p + 1/q = 1$. This condition seems to be valid when u_0 and v_0 are in $L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ (see for example, Corollary 3.2 and its extension in [3]).

Theorem 3.2. *Suppose that* $|f(t)| \le C|t|^{\alpha}$ *for all* $t \in \{s \in \mathbb{R} \,/\, |s| \le 1\}$, *with* $\alpha < \frac{n+4m}{n+\kappa}$. *Then the solution* (u,v) *of the system (1) verifies for all* $t > 0$, $p \in [1,\infty]$ *and all* λ *such that* $\max\big\{|\lambda| + \theta \,,\, |\lambda| + \sigma \,,\, \kappa + \sigma\big\} < 2m - 1$,

$$\begin{cases} ||D_x^{\lambda} u(x,t) - \mathcal{U}_0 \, D_x^{\lambda}\mathcal{K}(x,t)||_p \le C\, t^{-\frac{n+q|\lambda|}{4mq}} \times \begin{cases} \max\big(t^{-\frac{1}{4m}} \,,\, t^{-\frac{\theta}{4m}}\big) \text{ if } \frac{4m}{n+\kappa} < \alpha < \frac{n+4m}{n+\kappa} \,, \\ \max\big(t^{-\frac{1}{4m}} \,,\, t^{-\frac{\theta}{4m}}\big) \ln(1+t) \text{ if } \alpha = \frac{4m}{n+\kappa} \,, \\ t^{\frac{\mu}{4m}} (1+t)^{-\frac{n}{4m}} \text{ if } \alpha < \frac{4m}{n+\kappa} \,, \end{cases} \\ \\ ||D_x^{\lambda} v(x,t) - \mathcal{V}_0 \, D_x^{\lambda}\mathcal{K}(x,t)||_p \le C\, t^{-\frac{n+q|\lambda|}{4mq}} \, \max\big(t^{-\frac{1}{4m}} \,,\, t^{-\frac{\epsilon}{4m}}\big) \,, \end{cases}$$

where $\mu = 4m + n - (n+\kappa)\alpha - \theta$ *and with* $1/p + 1/q = 1$, $\mathcal{U}_0 = \int_{\mathbb{R}^n} u_0(x)\, dx$ *and* $\mathcal{V}_0 = \int_{\mathbb{R}^n} v_0(x)\, dx$.

Proof. It suffices to prove the results for u_0 and v_0 in $\mathcal{A} = L^1\big(\mathbb{R}^n; 1+|x|\big)$ knowing that the weighted space

$$\mathcal{A} = \left\{\varphi \in L^1(\mathbb{R}^n), \int_{\mathbb{R}^n} |\varphi(x)|(1+|x|)dx < \infty\right\},$$

with the norm

$$||\varphi||_{L^1(\mathbb{R}^n,|x|)} = \int_{\mathbb{R}^n} |\varphi(x)||x|dx,$$

is dense into $L^1(\mathbb{R}^n)$. Before proceeding for the estimates on $D^\lambda u$, let us recall the following technical result (see, for example, Lemma 2.3. in [2]) which is useful for the sequel.

Lemma 3.1. *Let $\lambda < 1$, $\nu > 0$ and $\zeta < 1$. Then, there is a constant $C > 0$ such that*

$$\int_0^t (t-s)^{-\lambda}\,(1+s)^{-\nu}\,s^{-\zeta}\,ds \le C \begin{cases} t^{-\lambda} & \text{if } \nu+\zeta > 1\,, \\ t^{-\lambda}\ln(1+t) & \text{if } \nu+\zeta = 1\,, \\ t^{1-\lambda-\zeta}\,(1+t)^{-\nu} & \text{if } \nu+\zeta < 1\,. \end{cases}$$

(i) Behavior of derivatives of u. By (7), we have for all $\lambda \in \mathbb{N}^n$,

$$D^\lambda u(t) - \mathcal{U}_0\, D^\lambda \mathcal{K}(t) = \Big(D^\lambda \mathcal{K}(t) * u_0 - \mathcal{U}_0\, D^\lambda \mathcal{K}(t)\Big) + \underbrace{\int_0^t D^\lambda \mathcal{K}(t-s) * \Big(\mathbf{a}.\nabla^\theta\big(u(s)\, f(|\nabla^\kappa v(s)|)\big)\Big) ds}_{\mathcal{P}(t)}.$$

Recall that it was shown in [7] (Theorem 7) that for all $t > 0$, all $p \in [1,\infty]$ and all $\lambda \in \mathbb{N}^n$ such that $|\lambda| < 2m-1$,

$$||D^\lambda \mathcal{K}(t) * u_0 - \mathcal{U}_0 D^\lambda \mathcal{K}(t)||_p \le C t^{-\frac{n+q(|\lambda|+1)}{4mq}} ||u_0||_{L^1(\mathbb{R}^n;|x|)},$$

when $u_0 \in L^1(\mathbb{R}^n; 1+|x|)$ and $p+q = pq$.
Applying the same arguments used in Section 2 and (10)-(19), we obtain

$$\begin{aligned} ||\mathcal{P}(t)||_p &\le \int_0^{t/2} ||\nabla^\theta(D^\lambda \mathcal{K}(t-s))||_p\, ||u(s)\, f(|\nabla^\kappa v(s)|)||_1\, ds + \int_{t/2}^t ||\nabla^\theta(D^\lambda \mathcal{K}(t-s))||_1\, ||u(s)\, f(|\nabla^\kappa v(s)|)||_p\, ds \\ &:= \mathcal{P}_1(t) + \mathcal{P}_2(t) \end{aligned}$$

with the following estimates

$$\begin{cases} \mathcal{P}_1(t) & \le C \displaystyle\int_0^{t/2} (t-s)^{-\frac{n}{4mq}-\frac{|\lambda|+\theta}{4m}} ||u(s)||_\infty\, ||\nabla^\kappa v(s)||_\alpha^\alpha\, ds \\ & \le C\,(t/2)^{-\frac{n}{4mq}-\frac{|\lambda|+\theta}{4m}} \displaystyle\int_0^{t/2} (1+s)^{-\frac{n}{4m}}\, s^{-\frac{1}{4m}((n+\kappa)\alpha-n)}\, ds \\ \mathcal{P}_2(t) & \le C\,(t/2)^{-\frac{n}{4mq}} \displaystyle\int_{t/2}^t (t-s)^{-\frac{|\lambda|+\theta}{4m}}\, (1+s)^{-\frac{n}{4m}}\, s^{-\frac{1}{4m}((n+\kappa)\alpha-n)}\, ds\,. \end{cases}$$

It then follows, from Lemma 3.1, that

$$\max\big\{\mathcal{P}_1(t)\,,\, \mathcal{P}_2(t)\big\} \le C\, t^{-\frac{n+q(|\lambda|+\theta)}{4mq}}\, \xi(t),$$

$$\text{where } \xi(t) = \begin{cases} 1 \quad \text{if } \frac{4m}{n+\kappa} < \alpha < \frac{n+4m}{n+\kappa}\,, \\ \ln(1+t) \quad \text{if } \alpha = \frac{4m}{n+\kappa}\,, \\ t^{\frac{1}{4m}\left(n+4m-(n+\kappa)\alpha\right)}\,(1+t)^{-\frac{n}{4m}} \quad \text{if } \alpha < \frac{4m}{n+\kappa}\,. \end{cases}$$

(ii) Behavior of derivatives of v. As for the function u, we have

$$D^\lambda v(t) - \mathcal{V}_0\, D^\lambda \mathcal{K}(t) = \Big(D^\lambda \mathcal{K}(t) * v_0 - \mathcal{V}_0\, D^\lambda \mathcal{K}(t)\Big) + \underbrace{\int_0^t \mathbf{b}(s).\nabla^\theta\big(D^\lambda \mathcal{K}(t-s)\big) * (u-v)(s)\, ds}_{\mathcal{Q}(t)}.$$

First, for $v_0 \in L^1(\mathbb{R}^n; 1+|x|)$ and $p+q = pq$, we have

$$||D^\lambda \mathcal{K}(t) * v_0 - \mathcal{V}_0 D^\lambda \mathcal{K}(t)||_p \le C t^{-\frac{n+q(|\lambda|+1)}{4mq}} ||u_0||_{L^1(\mathbb{R}^n;|x|)}.$$

For $\mathcal{Q}$, as for $\mathcal{P}$, write $||\mathcal{Q}(t)||_p \leq \mathcal{Q}_1(t) + \mathcal{Q}_2(t)$ and derive the following estimates,

$$\triangleright\ \mathcal{Q}_1(t) \leq C \int_0^{t/2} ||\mathbf{b}(s).\nabla^\sigma(D^\lambda \mathcal{K}(t-s))||_p\, ||(u-v)(s)||_1\, ds$$
$$\leq C\big(||u_0||_1 + ||v_0||_1\big) \int_0^{t/2} (t-s)^{-\frac{n}{4mq} - \frac{|\lambda|+\sigma}{4m}} s^{-(1+(\epsilon-\sigma)/4m)} ds \leq C(t/2)^{-\frac{n+q(|\lambda|+\epsilon)}{4mq}}.$$

$$\triangleright\ \mathcal{Q}_2(t) \leq C \int_{t/2}^{t} ||\mathbf{b}(s).\nabla^\sigma(D^\lambda \mathcal{K}(t-s))||_1 ||(u-v)(s)||_p\, ds$$
$$\leq C \int_{t/2}^{t} (t-s)^{-\frac{|\lambda|+\sigma}{4m}} (1+s)^{-\frac{n}{4mq}} s^{-(1+(\epsilon-\sigma)/4m)} ds \leq C(1+t/2)^{-\frac{n}{4mq}} (t/2)^{-\frac{|\lambda|+\epsilon}{4m}}.$$

This ends the proof of Theorem 3.2. □

Acknowledgement

The author is grateful to the referees for their remarks and suggestions.

References

[1] P. Auscher, M. Qafsaoui, Equivalence between regularity theorems and heat kernel estimates for higher order elliptic operators and systems under divergence form, *J. Funct. Anal.* **177** (2000) 310–364.

[2] H. O. Bae, B. J. Jin, Asymptotic behavior for the Navier-Stokes equations in 2D exterior domains, *J. Funct. Anal.* **240** (2006) 508–529.

[3] P. Biler, G. Karch, W. A. Woyczyński, Asymptotics for conservation laws involving Lévy diffusion generators, *Studia Math.* **148** (2001) 171–192.

[4] J. I. Diaz, T. Nagai, J. M. Rakotoson, Symmetrization techniques on unbounded domains: Application to a chemotaxis system on $\mathbb{R}^n$, *J. Differential Equations* **145** (1998) 156–183.

[5] M. Escobedo, E. Zuazua, Large time behavior for convection-diffusion equations in $\mathbb{R}^n$, *J. Funct. Anal.* **100** (1991) 119–161.

[6] M. A. Herrero, J. J. L. Velázquez, A blow-up mechanism for a chemotaxis model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **24** (1997) 633–683.

[7] M. Kirane, M. Qafsaoui, On the asymptotic behavior for convection-diffusion equations associated to higher order elliptic operators under divergence form, *Rev. Mat. Complut.* **15** (2002) 585–598.

[8] T. Nagai, Blow-up of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, *J. Inequal. Appl.* **6** (2001) 37–55.

[9] T. Nagai, R. Syukuinn, M. Umesako, Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in $\mathbb{R}^n$, *Funkcial. Ekvac.* **46** (2003) 383–407.

[10] T. Nagai, T. Yamada, Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space, *J. Math. Anal. Appl.* **336** (2007) 704–726.

[11] M. Qafsaoui, On a class of higher order operators with complex coefficients, elliptic in the sense of Gårding inequality, *Differential Integral Equations* **16** (2003) 1223–1248.

[12] M. Qafsaoui, Large time behavior for nonlinear higher order convection-diffusion equations, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **23** (2006) 911–927.

[13] A. Raczyński, Diffusion-dominated asymptotics of solution to chemotaxis model, *J. Evol. Equ.* **11** (2011) 509–529.

[14] M. P. Tran, T. N. Nguyen, Pointwise gradient bounds for a class of very singular quasilinear elliptic equations, *Discrete Contin. Dyn. Syst.* **41** (2021) 4461–4476.

[15] T. Yamada, Higher-order asymptotic expansions for a parabolic system modeling chemotaxis in the whole space, *Hiroshima Math. J.* **39** (2009) 363–420.