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*Research Article*

# Weakly $(m, n)$–semiprime submodules

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**Abstract**

This article gives several properties of a new type of submodules, namely *weakly* $(m, n)$*-semiprime submodules* where $m$ and $n$ are positive integers satisfying $m > n$. The primary objectives of the present article are to characterize weakly $(m, n)$-semiprime submodules and to provide a new characterization of the von Neumann regular modules in terms of weakly $(m, n)$-semiprime submodules.

**Keywords:** von Neumann regular module; weakly $(m, n)$-semiprime submodule; duplication modules.

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## 1. Introduction

In this paper, all the considered rings are commutative and unitary. Also, all the modules studied in this paper are unitary. Let $A$ be a commutative ring and consider a proper ideal $I$ of $A$. According to [10], $I$ is said to be weakly semiprime ideal if the following property holds: whenever $0 \neq r^2 \in I$ for some $r \in R$ then $r \in I$. Also, according to [5], $I$ is the $(m, n)$-closed ideal if $x^m \in I$ implies $x^n \in I$ for each $x \in A$. A weak version of the $(m, n)$-closed ideal, namely the weakly $(m, n)$-closed ideal, was introduced and studied in [6] by Fahid et al., who generalized the concept of weakly semiprime ideal. The ideal $I$ is said to be weakly $(m, n)$-closed ideal for some positive integers $m$ and $n$, with $m > n$, if the following property holds: for each $x \in A$ with $0 \neq x^m \in I$, it holds that $x^n \in I$. Mostafanasab and Darani [21] studied the concept of quasi $n$-absorbing ideal; the ideal $I$ is said to be quasi $n$-absorbing ideal if $r^n x \in I$ for $r, x \in R$ implies $r^n \in I$ or $r^{n-1}x \in I$, where $n$ is a positive integer. Notice that a semiprime ideal is exactly a quasi 2-absorbing ideal.

In [24], Saraç studied the properties of semiprime submodules. According to [24], a proper submodule $N$ of an $A$-module $M$ is said to be semiprime submodule if whenever $a^2x \in N$ for some $a \in A$ and $x \in M$, then $ax \in N$. The concept of 2-absorbing (also, weakly 2-absorbing) submodules was introduced and investigated in [13] by Darani and Soheilnia. According to [13], a submodule $N$ of an $A$-module $M$ is said to be a 2-absorbing submodule (respectively, weakly 2-absorbing submodule) of $M$ if whenever $a, b \in A$ and $m \in M$ with $abm \in N$ (respectively, $0 \neq abm \in N$), then $ab \in (N : M)$ or $am \in N$ or $bm \in N$. Darani and Soheilnia [14] introduced the concept of $n$-absorbing submodule where $n$ is a positive integer; a proper submodule $N$ of $M$ is an $n$-absorbing submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in A$ and $m \in M$, then either $a_1 \cdots a_n \in (N : M)$ or there are $n-1$ of $a_i's$ whose product with $m$ is in $N$. Recently, Issoual et al. [16] studied the concept of weakly quasi $n$-absorbing submodule; a proper submodule $N$ of an $A$-module $M$ is quasi $n$-absorbing submodule if whenever $a \in A$ and $x \in M$ such that $a^n x \in N$, then either $a^n \in (N : M)$ or $a^{n-1}x \in N$. In order to generalize the notion of semiprime submodules, Pekin et al. [23] introduced the concept of $(m, n)$-semiprime submodule, where $m$ and $n$, with $m > n$, are positive integers; a proper submodule $N$ of an $A$-module $M$ is said to be $(m, n)$-semiprime if $a^m x \in N$ then $a^n x \in N$ for some $a \in A$ and $x \in M$. In the present paper, the concept of *weakly* $(m, n)$*-semiprime submodules* is introduced, which is a proper generalization of $(m, n)$-semirpime submodules. A proper submodule $N$ of $M$ is said to be a *weakly* $(m, n)$*-semiprime submodule* if whenever $a \in R, x \in M$ with $0 \neq a^m x \in N$, then $a^n x \in N$.

The remaining part of this paper is organized as follows. The next section gives several properties of weakly $(m, n)$-semiprime submodules, including a characterization of weakly $(m, n)$-semiprime submodules. In Section 3, the modules in which every proper submodule is weakly $(m, n)$-semiprime are studied. A new characterization of the von Neumann regular module in terms of weakly $(m, n)$-semiprime submodules is also given in Section 3.

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## 2. Properties of weakly $(m, n)$–semiprime submodules

Recall the notions of $(m, n)$-semiprime submodules and weakly $(m, n)$-semiprime submodules defined in the introduction. According to [5], a proper ideal $P$ of a commutative ring $A$ is said to be an $(m, n)$-closed ideal of $A$ if $x^m \in P$ implies that $x^n \in P$ for each $x \in A$, where $m$ and $n$, with $m > n$, are positive integers. Also, according to [6], a proper ideal $P$ of a commutative ring $A$ is said to be a weakly $(m, n)$-closed ideal of $A$ if $0 \neq x^m \in P$ implies that $\neq x^n \in P$ for each $x \in A$, where $m$ and $n$, with $m > n$, are positive integers.

**Remark 2.1.** *If $A$ is an $A$-module then every $(m, n)$-semiprime submodule of $A$ is an $(m, n)$-closed ideal of $A$.*

**Example 2.1.** Let $A$ be a commutative ring and $N$ be a weakly semiprime submodule of an $A$-module $M$; that is, whenever $0 \neq a^2x \in N$, then $ax \in N$ for each $a \in A$ and $x \in M$. Certainly, $N$ is weakly $(m, n)$-semiprime submodule of $M$. Indeed, assume that $0 \neq a^m x \in N$ for some $a \in A, x \in M$. Then note that $0 \neq a^2(a^{m-2}x) \in M$. Since $N$ is weakly semiprime submodule of $M$, we conclude that $0 \neq a^{m-2}x \in N$. Continuing with the same reasoning we get $a^n x \in N$. The converse is false in general as shown in the following example. Let $K$ be a field and let $R$ denote the ring $K[S, T]$ of polynomials over $K$ in the determinates $S, T$. Let $M = RS + RT$ be a maximal ideal of $R$. Let $P = (S, T^2)$. Then $P$ is not weakly semiprime ideal since $0 \neq T^2.1 \in P$, but $T \notin P$. Also, note that $\sqrt{P} = M$, thus $P$ is $M$-primary ideal of $R$. We will show that $P$ is weakly $(3, 1)$-semiprime ideal of $R$. Let $0 \neq h^3.k \in (P) \subseteq (S)$ for some $h, k \in R$. Then $h.h^2k \in P$. If $h^2.k \in P$ we are done. If not, as $P$ is $M$-primary ideal, we get $h \in M$. So $h^2 \in M^2 = (S^2, ST, T^2) \subseteq P$. Finally $h^2.k \in P$. Thus, $P$ is weakly $(3, 1)$-semiprime ideal of $R$.

**Example 2.2.** Every $(m, n)$-semiprime submodule is weakly $(m, n)$-semiprime submodule. But the converse is not true. For example, let $A = \mathbb{Z}_{p^n}$ where $p$ is a prime number and $n > 2$. Let $N = \{0, p^{n-1}\}$. Since $a^n x = 0$ for every $a, x \in A$, we conclude that $N$ is a weakly $(n, 1)$-semiprime ideal. However, $N$ is not an $(n, 1)$-semiprime ideal since $p^n.\bar{1} = 0 \in N$ and $p \notin N$. By a similar argument, $N$ is not an $(n, k)$-semiprime ideal for every $n > k > 0$.

**Example 2.3.** We consider in this example the $\mathbb{Z}$-module $\mathbb{Z}_{p^n q}$, where $p$ and $q$ are prime numbers and $n > 2$. It is easy to see that $N = (\bar{0})$ is a weakly $(n, n-1)$-semiprime submodule of $M$. Indeed, let $a^n \bar{x} = \bar{0}$ for some $a, x \in \mathbb{Z}$. Then, we have $p^n q | a^n x$, which yields that $p^{n-1}q | a^{n-1}x$. Thus, we have $a^{n-1}\bar{x} = \bar{0}$. Therefore, $N = (\bar{0})$ is weakly $(n, n-1)$-semiprime submodule. On the other hand $(N : M) = p^n q\mathbb{Z}$ which is not an $(n, n-1)$-closed ideal of $\mathbb{Z}$, since $(pq)^n \in p^n q\mathbb{Z}$ but $(pq)^{n-1} \notin p^n q\mathbb{Z}$. We conclude by using Corollary 1 of [23] that $(\bar{0})$ is not an $(n, n-1)$-semiprime submodule of $M$.

The following propositions show that the notions of $(m, n)$-semiprime and weakly $(m, n)$-semiprime submodule coincide on certain modules.

**Proposition 2.1.** *Let $A$ be an integral domain and $M$ be a torsion-free $A$-module. Then every weakly $(m, n)$-semiprime submodule is $(m, n)$-semi prime submodule of $M$.*

*Proof.* Let $N$ be a weakly $(m, n)$-semiprime submodule of $M$. Suppose that $a^m x \in N$ for some $a \in A, x \in M$. If $0 \neq a^m x$, then the fact that $N$ is a weakly $(m, n)$-semiprime submodule of $M$, gives $a^n x \in N$. Next, we assume $a^m x = 0$ with $0 \neq x$. Then $a^m = 0$ as $T(M) = \{0\}$. We get $a^m = 0$ and consequently $a = 0$ since $A$ is an integral domain. Hence, $a^n x = 0 \in N$. Thus, $N$ is $(m, n)$-semiprime submodule of $M$. □

**Proposition 2.2.** *Let $(A, \mathcal{M})$ be a local ring with the maximal ideal $\mathcal{M}$ and let $M$ be an $A$-module such that $\mathcal{M}M = 0$. Then, every weakly $(m, n)$-semiprime submodule is an $(m, n)$-semiprime submodule of $M$.*

*Proof.* Let $N$ be a submodule of $M$ which is a weakly $(m, n)$-semiprime $M$. Choose $a \in A$ and $x \in M$ such that $a^m x \in N$. If $a$ is unit, then $x \in N$ and so $a^n x \in N$. Next, we assume that $a$ is not unit. As $(A, \mathcal{M})$ is a local ring we get $a \in \mathcal{M}$. On the other hand, $\mathcal{M}M = 0$, which implies $aM = 0$. Thus, $a^n x = 0 \in N$. consequently, $N$ is an $(m, n)$-semiprime submodule of $M$. □

The following theorem gives a characterization of weakly $(m, n)$-semiprime submodules.

**Theorem 2.1.** *Let $M$ be an $A$-module and $N$ be a proper submodule of $M$. Then the following statements are equivalent.*

**(1).** *$N$ is weakly $(m, n)$-semiprime submodule of $M$.*

**(2).** *For every $a \in A$ such that $0 \neq a^m N$, it holds that $(N :_M a^m) = (N :_M a^n)$.*

**(3).** *For every $a \in A$ and $L$ submodule of $M$ with $0 \neq a^m L \subset N$, then $a^n L \subset N$.*

*Proof.* (1) $\Rightarrow$ (2). As $m > n$, it is clear that $(N : a^n) \subseteq (N : a^m)$. Now, let $x \in (N : a^m)$. Then $a^m x \in N$. If $0 \neq a^m x \in N$, we get $a^n x \in N$ and so $x \in (N : a^n)$ (as $N$ is a weakly $(m, n)$-semiprime submodule). Now, suppose that $a^m x = 0$. The fact that $0 \neq a^m N$ gives $0 \neq a^m y$ for some $y \in N$, it follows $a^n y \in N$ as $N$ is weakly $(m, n)$-semiprime submodule of $M$. Set $z = x + y$, so $0 \neq a^m z \in N$, and by the same argument as above, we get $a^n z \in N$. Therefore, $a^n x \in N$. Hence, $x \in (N : a^n)$. Finally, we conclude that $(N : a^m) \subseteq (N : a^n)$ and $(N : a^m) = (N : a^n)$.

(2) $\Rightarrow$ (3). Suppose that $0 \neq a^m L \subseteq N$ for some $a \in A$ and that $L$ is a submodule of $M$. We have $0 \neq a^m N$ and $L \subseteq (N : a^m)$. Then by (2), we conclude that $L \subseteq (N : a^n)$, and this implies that $a^n L \subseteq N$.

(3) $\Rightarrow$ (1). Assume that $a^n L \subseteq N$ for every $a \in A$, where $L$ is a submodule of $M$ with $0 \neq a^m L \subseteq N$. Let $0 \neq a^m x \in N$ for some $a \in A$ and $x \in M$. We take $L = Ax$. Then $0 \neq a^m L \subseteq N$. By our assumption, we get $a^n L \subseteq N$ and so $a^n x \in N$. Thus, $N$ is a weakly $(m, n)$-semiprime submodule of $M$. ☐

**Remark 2.2.** *In Corollary 1 of* [23] *if $N$ is $(m, n)$-semiprime submodule of $M$, then $(N : M)$ is an $(m, n)$-closed ideal. If $N$ is a weakly $(m, n)$-semiprime submodule of $M$, then the residual $(N : M)$ need not to be a weakly $(m, n)$-closed ideal of $A$, as shown in the next example*

**Example 2.4.** Consider the $\mathbb{Z}$-module $M := \mathbb{Z}_{p^m}$ where $p$ is a prime number and $m > n > 2$. Set $N = \{\overline{0}\}$. We will show that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Let $a^m \overline{x} = \overline{0}$ for some $a, x \in \mathbb{Z}$. Then $p^m | a^m x$. Necessarily, $p^n | a^n x$ otherwise $p$ and $a^n x$ are coprime and this is a contradiction. Thus, we have $a^n \overline{x} = \overline{0}$. Hence, $N$ is $(m, n)$-semiprime and thus it is weakly $(m, n)$-semiprime. We remark that $(N : M) = p^m \mathbb{Z}$ is not weakly $(m, n)$-closed ideal of $\mathbb{Z}$ since $p^m \in (N : M)$ but $p^n \notin (N : M)$.

The following corollary gives a characterization of weakly $(m, n)$-semiprime submodules by using the concept of weakly $(m, n)$-closed ideals when $M$ is an $A$-faithful module. Recall that an $A$-module $M$ is said to be faithful module if

$$ann_A(M) := \{a \in A; a.m = 0\} = \{0\}.$$

**Corollary 2.1.** *Let $M$ be a faithful $A$-module and let $N$ be a proper submodule of $M$. Then the following statements are equivalent:*

**(1).** *$N$ is a weakly $(m, n)$-semiprime submodule of $M$.*

**(2).** *For every $L$ submodule of $M$, the residual $(L : M)$ is a weakly $(m, n)$-closed ideal of $A$.*

*Proof.* (1) $\Longrightarrow$ (2). Suppose that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Let $0 \neq a^m \in (L : M)$. Since $M$ is a faithful $A$-module, we have $0 \neq a^m L \subseteq N$. By Theorem 2.1, we conclude that $a^n L \subseteq N$ and hence $a^n \in (L : N)$. Therefore, $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$.

(2) $\Longrightarrow$ (1). Assume that $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$ for each submodule $L$. Let $0 \neq a^m x \in N$ for $a \in A$ and $x \in M$. Set $L = Ax$, then $0 \neq a^m L \in N$, which implies $0 \neq a^m \in (L : N)$. By our assumption, we get $a^n \in (L : N)$, and hence $a^n x \in N$. Thus, $N$ is a weakly $(m, n)$-semiprime submodule of $M$. ☐

It remarked here that for some submodule $L$ of $M$ if $0 \neq a^m L \subset N$, then $a^n L \subset N$ is equivalent to say that $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$ since $A$ is faithful.

**Remark 2.3.** *In Corollary 2.1, the condition "$M$ is a faithful module" is necessary, as shown in the following example.*

**Example 2.5.** Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$, which is not a faithful $\mathbb{Z}$- module and consider a submodule $N = (0) \times 16\mathbb{Z}$. Then, $N$ is not a weakly $(3, 2)$-semiprime submodule since $2^3(0, 2) = (0, 16) \in N$ and $2^2(0, 2) = (0, 8) \notin N$. On the other hand, $(N : M) = (0)$ is a weakly $(3, 2)$-closed ideal of $\mathbb{Z}$ since $(0)$ is a prime ideal of $\mathbb{Z}$.

In the next theorem, we show the relationship between a weakly $(m, n)$-semiprime submodule and the fact that $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$ where $x \in M \setminus N$. Recall that a ring $A$ is said to be reduced if $Nil(A) = \{0\}$.

**Theorem 2.2.** *Let $M$ be an $A$-module and $N$ be a proper submodule of $M$.*

**(1).** *If $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$ for every $x \in M \backslash N$, then $N$ is a weakly $(m, n)$-semiprime submodule of $M$.*

**(2).** *If $A$ is a reduced ring, then every submodule of $M$ is a weakly $(m, n)$-semiprime.*

**(3).** *If $N$ is a weakly $(m, n)$-semiprime submodule of $M$ and $ann(x)$ is a weakly $(m, n)$-closed ideal of $A$ for every each $x \in M \backslash N$, then $(N : x)$ is a weakly $(m, n)$-closed ideal of $R$.*

*Proof.* (1). Suppose that $0 \neq a^m y \in N$ for some $a \in A$ and $y \in M$. If $y \in N$, we are done. Next, we assume that $y \in M \backslash N$. Since $(N : y)$ is a weakly $(m, n)$-closed ideal of $A$, we have $a^n \in (N : y)$, and so $a^n y \in N$. Hence, $N$ is a weakly $(m, n)$-semiprime submodule of $M$.

(2). It follows from Corollary 3.6 of [6]. Indeed, if $A$ is reduced, then every proper ideal of $A$ is weakly $(m, n)$-closed ideal, by Corollary 3.6(1) of [6].

(3). Let $x \in M \backslash N$. Suppose that $0 \neq a^m \in (N : x)$ for some $a \in A$. First, we consider the case $0 \neq a^m x \in N$. Since $N$ is a weakly $(m, n)$-semiprime submodule of $M$, we have $a^n x \in N$, and hence $a^n \in (N : x)$. Now, consider the case $a^m x = 0$. Then, $0 \neq a^m \in \text{ann}(x)$, and so $a^n \in \text{ann}(x)$. Consequently, $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$. □

**Example 2.6.** Let $N$ be a weakly $(m, n)$-semiprime submodule of $M$ and take $x \in M \backslash N$. We will show that $(N : x)$ need not to be a weakly $(m, n)$-closed ideal of $A$. Indeed, let $m$ and $n$, with $m > n$, be positive integers and $M = \mathbb{Z}^{p^m}$ be a $\mathbb{Z}$-module, where $p$ is a prime number. Let $N = (\bar{0})$. It is clear that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. On the other hand, $(N : p) = p^{m-1}\mathbb{Z}$ is not a weakly $(m, n)$-closed ideal of $\mathbb{Z}$ since $p^m \in (N : p)$ but $p^n \notin (N : p)$.

Let $A$ be an integral domain. An $A$-module $M$ is said to be torsion-free if $ma = 0$ for some $a \in A$ and $m \in M$, implies $a = 0$ or $m = 0$.

**Theorem 2.3.** *Let $A$ be an integral domain, $M$ be a torsion-free $A$-module and $N$ be a proper submodule of $M$. The following statements are equivalent.*

**(1).** *$N$ is weakly $(m, n)$-semiprime submodule of $M$.*

**(2).** *$(N : x)$ is weakly $(m, n)$-closed ideal of $R$ for each $x \in M \backslash N$.*

*Proof.* $(2) \Rightarrow (1)$. It follows from the Theorem 2.2.

$(1) \Rightarrow (2)$. Let $0 \neq a^m \in (N : x)$ for some $a \in A$ and $x \in M$. If $x = 0$, then $(N : x) = A$, and we are done. Next, we assume that $0 \neq x$. Since $M$ is a torsion-free $A$-module, we get $0 \neq a^m x \in N$. The fact that $N$ is a weakly $(m, n)$-semiprime submodule of $M$, gives $a^n x \in N$, and thus $a^n \in (N : x)$. Hence, $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$. □

Our next objective is the study of the stability of the tensor product of weakly $(m, n)$-semiprime submodules.

**Theorem 2.4.** *Let $(A, \mathcal{M})$ be a local ring and $M$ be an $A$-module.*

**(1).** *If $F$ is a non-zero finitely generated flat $A$-module and $N$ is a finitely generated weakly $(m, n)$-semiprime submodule of $M$ such that $F \otimes N \neq F \otimes M$, then $F \otimes N$ is a weakly $(m, n)$-semiprime submodule of $F \otimes M$.*

**(2).** *If $F$ is a finitely generated faithful flat $A$-module and $N$ is a finitely generated submodule of $M$, then the following statements are equivalent:*

**(a).** *$N$ is a weakly $(m, n)$-semiprime submodule of $M$.*

**(b).** *$F \otimes N$ is a weakly $(m, n)$-semiprime submodule of $F \otimes M$.*

*Proof.* (1). Let $F$ be a finitely flat module, $N$ be a finitely weakly $(m, n)$-semiprime submodule of $M$ and take $a \in R$ such that $0 \neq a^m(F \otimes N)$. Since $0 \neq a^m(F \otimes N) = F \otimes a^m N$ and $(A, \mathcal{M})$ is a local ring, we deduce that $0 \neq a^m N$ (see Exercise 3 of Chapter 2 in [8]). Also, by Theorem 6 of [9], we have $(F \otimes N :_{F \otimes M} a^m) = F \otimes (N :_M a^m)$, and by Theorem 2.1 we get $(F \otimes N :_{F \otimes M} a^m) = (F \otimes N :_{F \otimes M} a^n)$. By Theorem 2.1 we deduce that $F \otimes N$ is a weakly $(m, n)$-semiprime submodule of $F \otimes M$.

(2). $(a) \Rightarrow (b)$. Since $F$ is a faithful flat module and $N$ is a proper submodule of $M$, we have $F \otimes N \neq F \otimes M$. Now, the result follows from Part (1).

$(b) \Rightarrow (a)$. Suppose that $F \otimes N$ is a weakly $(m, n)$-semiprime submodule of $F \otimes M$. Take $a \in R$ with $0 \neq a^m N$. Since $(A, \mathcal{M})$ is a local ring and $F, N$, are finitely generated modules, we get $0 \neq a^m(F \otimes N)$. By Theorem 2.1 and Lemma 3.2 of [9], we have

$$F \otimes (N :_M a^m) = (F \otimes N :_{F \otimes M} a^n) = F \otimes (N ;_M a^n).$$

Thus, $F \otimes (N :_M a^m) = F \otimes (N :_M a^n)$. Since the sequence

$$0 \to F \otimes (N :_M a^n) \hookrightarrow F \otimes (N :_M a^m) \to 0$$

is exact and $F$ is a faithful module, we get the exact sequence

$$0 \to (N :_M a^n) \hookrightarrow (N :_M a^m) \to 0,$$

which implies $(N :_M a^n) = (N :_M a^m)$. Now, the desired result follows from Theorem 2.1. □

**Theorem 2.5.** *Let $f : M \longrightarrow M'$ be a homomorphism of $A$-modules.*

**(1).** *If $N$ is a weakly $(m, n)$-semiprime submodule of $M$ containing $\ker(f)$ and if $f$ is surjective, then $f(N)$ is a weakly $(m, n)$-semiprime submodule of $M'$.*

**(2).** *If $N'$ is a weakly $(m, n)$-semiprime submodule of $M'$ and if $f$ is injective, then $f^{-1}(N')$ is a weakly $(m, n)$-semiprime submodule of $M$.*

*Proof.* (1). Suppose that $f$ is surjective and $\ker(f) \subseteq N$, where $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Take $0 \neq a^m x' \in N'$ for some $a \in A$ and $x' \in M'$. Then there exists $x \in M$ such that $x' = f(x)$. Since $0 \neq f(a^m x) \in f(N)$ and $\ker(f) \subseteq N$, we have $0 \neq a^m x \in N$. Since $N$ is a weakly $(m, n)$-semiprime submodule of $M$, we get $a^n x \in N$. It follows that $a^n f(x) = a^n x' \in f(N)$. Hence, $f(N)$ is a weakly $(m, n)$-semiprime submodule of $M'$, as desired.

(2). Assume that $f$ is a monomorphism of $A$-modules and $N'$ is a weakly $(m, n)$-semiprime submodule of $M'$. Take $0 \neq a^m x \in f^{-1}(N')$ for some $a \in A, x \in M$. So, $0 \neq a^m f(x) \in N.'$ The fact that $N$ is a weakly $(m, n)$-semiprime submodule of $M'$, gives $a^n f(x) \in N'$. Therefore, $a^n x \in f^{-1}(N')$. Hence, $f^{-1}(N')$ is a weakly $(m, n)$-semiprime submodule of $M$. □

**Corollary 2.2.** *Let $N$ be a proper submodule of $M$.*

**(1).** *If $L$ is a submodule of $M$ with $L \subseteq N$ and if $N$ is a weakly $(m, n)$-semiprime submodule of $M$, then $N/L$ is a weakly $(m, n)$-semiprime submodule of $M/L$.*

**(2).** *If $L$ is a submodule of $M$ with $L \subseteq N$ and if $N/L$ is a weakly $(m, n)$-semiprime submodule of $M/L$, and if $L$ is a weakly $(m, n)$-semiprime submodule of $M$, then $N$ is a weakly $(m, n)$-semiprime submodule of $M$.*

*Proof.* (1). It is a direct consequence of Theorem 2.5(1).

(2). Assume that $N/L$ is a weakly $(m, n)$-semiprime submodule of $M/L$ and $L$ is a weakly $(m, n)$-semiprime submodule of $M$. Take $0 \neq a^m x \in N$ for some $a \in A, x \in M$. Then $a^m(x + L) \in N/L$. If $a^m(x + L) = 0_{M/L}$, then $0 \neq a^m x \in L$, which is a weakly $(m, n)$-semiprime submodule of $M$. Thus, $a^n x \in L$ and so $a^n x \in N$. Next, we assume that $0 \neq a^m(x + L)$. The fact that $N/L$ is a weakly $(m, n)$-semiprime submodule of $M/L$ gives that $a^n(x + L) \in N/L$. Hence, $a^n x \in N$ and $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Therefore, $N$ is weakly $(m, n)$-semiprime submodule of $M$. □

Pekin et al. [23] studied the concept of $(m, n)$-semiprime submodules over the trivial extension ring $A(+)M$, where $A$ is a commutative ring and $M$ is an $A$-module. For more detail on trivial extensions of rings, see [7]. We end this section by giving another way to construct weakly $(m, n)$-semiprime submodules that are not $(m, n)$-semiprime. Let $A$ be a ring, $I$ be an ideal of $A$, and $M$ be an $A$-module, and set

$$M \bowtie I := \{(x, x') \in M \times M | x - x' \in IM\},$$

which is a $A \bowtie I$-module with the multiplication given by

$$(r, r + i)(x, x') = (rx, (r + i)x'), \quad \text{where} \quad r \in A, i \in I, \quad \text{and} \quad (x, x') \in M \bowtie I.$$

According to [12], $M \bowtie I$ is known as the *duplication of the $A$-module $M$ along the ideal $I$*. If $N$ is a submodule of $M$, then it is clear that

$$N \bowtie I := \{(x, x') \in N \times M | x - x' \in IM\} \quad \text{and} \quad \overline{N} := \{(x, x') \in M \times N | x - x' \in IM\}$$

are submodules of $M \bowtie I$.

**Lemma 2.1.** *Let $A$ be a ring, $I$ be an ideal of $A$, and $M$ be an $A$-module. Let $N$ be a submodule of $M$. Let $m$ and $n$ be positive integers satisfying $m > n$.*

**(1).** *$N \bowtie I$ is an $(m, n)$-semiprime submodule of $M \bowtie I$ if and only if $N$ is an $(m, n)$-semiprime submodule of $M$.*

**(2).** *$\overline{N}$ is an $(m, n)$-semiprime submodule of $M \bowtie I$ if and only if $N$ is an $(m, n)$-semiprime submodule of $M$.*

*Proof.* (1). Assume that $N \bowtie I$ is an $(m, n)$-semiprime submodule of $M \bowtie I$. Take $a^m x \in N$ for some $a \in A, x \in M$. Then $(a, a)^m(x, x) \in N \bowtie I$. The fact that $N \bowtie I$ is an $(m, n)$-semiprime submodule of $M \bowtie I$, gives that $(a, a)^n(x, x) \in N \bowtie I$. So, $a^n x \in N$. Hence, $N$ is an $(m, n))$-semiprime submodule of $M$. Conversely, assume that $N$ is an $(m, n)$-semiprime submodule of $M$. Take $(a, a + i)^m(x, x')$ for some $(a, a + i) \in A \bowtie I, (x, x') \in M \bowtie I$. Then $(a^m x, (a + i)^m x') \in N$ and so $a^m x \in N$. As $N$ is an $(m, n)$-semiprime submodule of $M$, we conclude that $a^n x \in N$, which implies $(a, a + i)^n(x, x') \in N \bowtie I$ and this shows that $N \bowtie$ is an $(m, n)$-semiprime submodule of $M \bowtie I$.

(2). The proof is similar to the proof of (1) and so is omitted.

□

The following definition is useful for studying weakly $(m, n)$-semirpime submodules that are not $(m, n)$-semirpime submodules.

**Definition 2.1.** *Let $M$ be an $A$-module where $A$ is a commutative ring, $m \geq n > 0$ are a positive integers, and $N$ a weakly $(m, n)$-semiprime submodule of $M$. Then $a \in A$ is an $(m, n)$-unbreakable-zero element of $N$ if there exists $x \in M$ such that $a^m x = 0$ and $a^n x \notin N$. (Thus, $N$ has an $(m, n)$-unbreakable-zero element if and only if $N$ is a weakly $(m, n)$-semiprime submodule of $M$ that is not $(m, n)$-semiprime.)*

**Theorem 2.6.** *The following statements are equivalents:*

**(1).** *$N \bowtie I$ is a weakly $(m, n)$-semiprime submodule which is not an $(m, n)$-semiprime submodule of $M \bowtie I$.*

**(2).** *$N$ is a weakly $(m, n)$-semiprime submodule that is not an $(m, n)$-semiprime submodule of $M$, and for every $(m, n)$-unbreakable-zero element $a \in A$ of $N$, it holds that $(a + i)^m M = 0$ for every $i \in I$.*

*Proof.* $(1) \Rightarrow (2)$. Suppose that $N \bowtie I$ is a weakly $(m, n)$-semiprime submodule of $M \bowtie I$. Take $0 \neq a^m x \in N$ for some $a \in A, x \in M$. Then $0 \neq (a, a)^m (x, x) \in N \bowtie I$. As $N \bowtie I$ is a weakly $(m, n)$-semiprime submodule of $M \bowtie I$, we obtain that $(a, a)^n (x, x) \in N \bowtie I$, which implies $a^n x \in N$ and shows that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. By Lemma 2.1, $N$ is not an $(m, n)$-semiprime submodule of $M$. Now, let $a \in A$ be an $(m, n)$-unbreakable-zero element of $N$; that is, there exists $x \in M$ such that $a^m x = 0$ and $a^n x \notin N$. We will show that $(a + i)^m M = 0$ for every $i \in I$. Since $N$ is a weakly $(m, n)$-semiprime submodule of $M$ that is not $(m, n)$-semiprime, $N$ has an $(m, n)$-unbreakable-zero $a \in A$. By the way of contradiction, suppose that there exists $i \in I$ such that $(a + i)^m y = 0$ for some $y \in M$. Then, $0 \neq (a, a + i)^m = (0, (a + i)^m y) \in N \bowtie I$. As $N \bowtie I$ is a weakly $(m, n)$-semiprime submodule of $M \bowtie I$, we conclude that $(a, a + i)^n (x, y) \in N \bowtie I$ and so $a^n x \in I$, which is a contradiction. Hence, $(a + i)^m M = 0$ for every $i \in I$.

$(2) \Rightarrow (1)$. Suppose that $N$ is a weakly $(m, n)$-semiprime submodule which is not $(m, n)$-semiprime and $(a + i)^m M = 0$ if $a \in A$ is a $(m, n)$-unbreakable-zero element of $N$. Let $0 \neq (a, a+i)^m (x, x') \in N \bowtie I$. Then $a^m x \in N$ and $(a+i)^m x' - a^m x \in IM$. Assume that $0 \neq a^m x \in N$. As $N$ is a weakly $(m, n)$-semiprime submodule of $M$ we get $a^n x \in N$. Now, assume that $a^m x = 0$, then necessarily $0 \neq (a + i)^m x$. If $a^n x \notin N$, then $a$ is an $(m, n)$-unbreakable-zero element of $N$. By our assumption, we have $(a + i)^m M = 0$. This is a contradiction. Hence, $a^n x \in N$. □

**Theorem 2.7.** *Let $M$ be an $A$-module, $N$ be a submodule of $M$, and $m, n,$ are positive integers satisfying $m > n$. Let*

$$\overline{N} := \{(x, x') \in M \times N; x - x' \in IM\}.$$

*The following statements are equivalent:*

**(1).** *$\overline{N}$ is a weakly $(m, n)$-submodule of $M \bowtie I$.*

**(2).** *$N$ is a weakly $(m, n)$-submodule of $M$ and the equation $(a - i)^m M = 0$ holds for every $i \in I$ and for an $(m, n)$-unbreakable-zero element $a \in A$ of $N$.*

*Proof.* It is the same as the proof of Theorem 2.6. □

## 3. Modules over which every submodule is weakly $(m, n)$–semiprime

The following result gives the constraints under which every given proper submodule is a weakly $(m, n)$-semiprime submodule.

**Theorem 3.1.** *Let $M$ be an $A$-module and $m, n,$ be two positive integers such that $m > n$. The following statements are equivalents:*

**(1).** *Every proper submodule is a weakly $(m, n)$-semiprime submodule of $M$.*

**(2).** *For every submodule $N$ of $M$ and for every $a \in A$ such that $0 \neq a^m N$, the descending chain*

$$aN \supseteq a^2 N \supseteq \cdots \supseteq a^m N \supseteq \cdots$$

*of submodules of $M$ terminates at the $n^{th}$ step.*

**(3).** *For every submodule $N$ of $M$ and for every $a \in A$ with $0 \neq a^m N$, it holds that $a^n N = a^m N$.*

*Proof.* $(1) \Rightarrow (2)$. Take $a \in A$ and let $N$ be a submodule of $M$ such that $0 \neq a^m N$. If $a^m N = M$ then we are done. Next, we assume that $a^m N$ is a proper submodule of $M$. Since $0 \neq a^m N \subseteq a^m N$ and $N$ is a weakly $(m, n)$-semiprime submodule of $M$, by Theorem 2.1 we have $a^n N \subseteq a^m N$, which implies that $a^n N = a^m N$. Hence, the descending chain $aN \supseteq a^2 N \supseteq \cdots \supseteq a^m N \supseteq \cdots$, terminates at the $n^{th}$ step.

$(2) \Rightarrow (3)$. It is trivial.

$(3) \Rightarrow (1)$. Let $N$ be a proper submodule of $M$. Take $a \in A$ and let $K$ be a submodule of $M$ such that $0 \neq a^m K \subseteq N$. By our assumption $a^n K = a^m K \subseteq N$. Now, by Theorem 2.1, we conclude that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. □

According to [11, 15], an $A$-module $M$ is said to be *multiplication module* if every submodule $N$ of $M$ has the form $N = IM$, where $I$ is an ideal of $A$. In this case, we have $N = (N : M)M$. For more detail about multiplication modules, see [1–4]

Let $M$ be an $A$-module. According to [20], $M$ is a *reduced module* if for every $a \in A, x \in M$ with $ax = 0, aM \cap Ax = 0$, or equivalently $a^2 x = 0$ implies $ax = 0$.

According to [22], a commutative ring $A$ is a *von Neumann regular ring* if for every $a \in A$, there exist $b \in A$ such that $a = a^2 b$. In [17], Jayarm and Tekir studied the concept of von Neumann regular modules (see also [18, 19]) by introducing the concept *$M$-von Neumann regular* elements of modules as follows. If $M$ is an $A$-module, then an element $a \in A$ is an *$M$-von Neumann regular element* if $aM = a^2 M$. Also, an $A$-module $M$ is said to be a *von Neumann regular module* if for every $x \in M$, $Ax = aM = a^2 M$ for some $a \in A$.

Our next objective is to give a characterization of von Neumann regular modules using the properties of weakly $(m, n)$-semi prime submodules.

**Theorem 3.2.** *Let $M$ be a finitely generated $A$-module. The following statements are equivalents:*

**(1).** *$M$ is an von Neumann regular module.*

**(2).** *$M$ is a multiplication reduced module in which every submodule is weakly $(m, n)$-semiprime submodule*

*Proof.* We follow the same reasoning as $(1) \Leftrightarrow (2)$ in Theorem 8 of [23].

$(2) \Rightarrow (1)$. Let $M$ be a finitely generated reduced multiplication module in which every proper submodule is a weakly $(m, n)$-semiprime. Take $a \in A$. We will show that $aM = a^2 M$. If $a^m M = M$, then clearly we have $aM = a^2 M$. Next, we assume that $a^m M$ is a proper submodule of $M$. First, consider the case $a^m M = 0$. Since $M$ is reduced, we have $\text{ann}(M)$ is a semiprime ideal, which implies that $a \in \text{ann}(M)$. Thus, $aM = a^2 M = 0$. Next, we consider the case $0 \neq a^m M$. As $a^m M$ is a weakly $(m, n)$-semiprime submodule and $0 \neq a^m M \subseteq a^m M$, we conclude by Theorem 2.1 that $a^n M = a^m M$. We deduce that $a^{n+1} M = a^n M = a(a^n M)$. Since $a^n M$ is a finitely generated module, by Corollary 2.5 of [8] we have $x(a^n M) = 0$ for some $x \in A$ such that $x \equiv 1((a^n))$. Thus, there exists $b \in A$ such that $(1 - ab)a^n M = 0$. AS $M$ is reduced, we get $(1 - ab)aM = 0$, Which implies $aM = a^2 bM \subseteq a^2 M$. Consequently, we obtain $aM = a^2 M$, as desired.

$(1) \Rightarrow (2)$. Suppose that $M$ is a von Neumann regular module. By the proof of Theorem 8 of [23], $M$ is a multiplication-reduced module. Now, let $N$ be a proper submodule of $M$. Take $a \in A$ and let $L$ be a submodule of $M$ such that $0 \neq a^m L \subseteq N$. Since $M$ is a multiplication module, we have $L = (L : M)M$, which then gives $0 \neq a^m L = (L : M)a^m M = (L : M)a^n M$, and hence $0 \neq a^m N = a^n N$. Therefore, $N$ is a weakly $(m, n)$-semiprime submodule. □

**Corollary 3.1.** *Let $M$ be a finitely generated $A$-module. The following statements are equivalent:*

**(1).** *$M$ is a von Neumann regular module.*

**(2).** *$M$ is a multiplication-reduced module in which every submodule is an $(m, n)$-semiprime submodule.*

**(3).** *$M$ is a multiplication-reduced module in which every submodule is a weakly $(m, n)$-semiprime submodule.*

*Proof.* $(1) \Leftrightarrow (2)$. It follows from Theorem 8 of [23].

$(2) \Rightarrow (3)$. It is trivial.

$(3) \Rightarrow (1)$. It follows from Theorem 3.2. □

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