Weakly \((m, n)\)-semiprime submodules

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Abstract

This article gives several properties of a new type of submodules, namely weakly \((m, n)\)-semiprime submodules where \(m\) and \(n\) are positive integers satisfying \(m > n\). The primary objectives of the present article are to characterize weakly \((m, n)\)-semiprime submodules and to provide a new characterization of the von Neumann regular modules in terms of weakly \((m, n)\)-semiprime submodules.

Keywords: von Neumann regular module; weakly \((m, n)\)-semiprime submodule; duplication modules.

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1. Introduction

In this paper, all the considered rings are commutative and unitary. Also, all the modules studied in this paper are unitary. Let \(A\) be a commutative ring and consider a proper ideal \(I\) of \(A\). According to [10], \(I\) is said to be weakly semiprime ideal if the following property holds: whenever \(0 \neq r^2 \in I\) for some \(r \in R\) then \(r \in I\). Also, according to [5], \(I\) is the \((m, n)\)-closed ideal if \(x^m \in I\) implies \(x^n \in I\) for each \(x \in A\). A weak version of the \((m, n)\)-closed ideal, namely the weakly \((m, n)\)-closed ideal, was introduced and studied in [6] by Fahid et al., who generalized the concept of weakly semiprime ideal. The ideal \(I\) is said to be weakly \((m, n)\)-closed ideal for some positive integers \(m\) and \(n\), if the following property holds: for each \(x \in A\) with \(0 \neq x^m \in I\), it holds that \(x^n \in I\). Mostafanasab and Darani [21] studied the concept of quasi \(n\)-absorbing ideal; the ideal \(I\) is said to be quasi \(n\)-absorbing ideal if \(r^n x \in I\) for \(r, x \in R\) implies \(r^n \in I\) or \(r^{n-1} x \in I\), where \(n\) is a positive integer. Notice that a semiprime ideal is exactly a quasi 2-absorbing ideal.

In [24], Sarac studied the properties of semiprime submodules. According to [24], a proper submodule \(N\) of an \(A\)-module \(M\) is said to be semiprime submodule if whenever \(a^2 x \in N\) for some \(a \in A\) and \(x \in M\), then \(ax \in N\). The concept of 2-absorbing (also, weakly 2-absorbing) submodules was introduced and investigated in [13] by Darani and Soheilnia. According to [13], a submodule \(N\) of an \(A\)-module \(M\) is said to be a 2-absorbing submodule (respectively, weakly 2-absorbing submodule) of \(M\) if whenever \(a, b \in A\) and \(m \in M\) with \(abm \in N\) (respectively, \(0 \neq abm \in N\)), then \(ab \in (N : M)\) or \(am \in N\) or \(bm \in N\). Darani and Soheilnia [14] introduced the concept of \(n\)-absorbing submodule where \(n\) is a positive integer; a proper submodule \(N\) of \(M\) is an \(n\)-absorbing submodule if whenever \(a_1 \cdots a_n m \in N\) for \(a_1, \ldots, a_n \in A\) and \(m \in M\), then either \(a_1 \cdots a_n \in (N : M)\) or they are \(n - 1\) of \(a_i\)'s whose product with \(m\) is in \(N\). Recently, Issoual et al. [16] studied the concept of weakly quasi \(n\)-absorbing submodule; a proper submodule \(N\) of an \(A\)-module \(M\) is quasi \(n\)-absorbing submodule if whenever \(a \in A\) and \(x \in M\) such that \(a^n x \in N\), then \(a^n \in (N : M)\) or \(a^{n-1} x \in N\). In order to generalize the notion of semiprime submodules, Pekin et al. [23] introduced the concept of \((m, n)\)-semiprime submodule, where \(m\) and \(n\), with \(m > n\), are positive integers; a proper submodule \(N\) of an \(A\)-module \(M\) is said to be \((m, n)\)-semiprime if \(a^m x \in N\) then \(a^n x \in N\) for some \(a \in A\) and \(x \in M\). In the present paper, the concept of weakly \((m, n)\)-semiprime submodules is introduced, which is a proper generalization of \((m, n)\)-semiprime submodules. A proper submodule \(N\) of \(M\) is said to be a weakly \((m, n)\)-semiprime submodule if whenever \(a \in R, x \in M\) with \(0 \neq a^m x \in N\), then \(a^n x \in N\).

The remaining part of this paper is organized as follows. The next section gives several properties of weakly \((m, n)\)-semiprime submodules, including a characterization of weakly \((m, n)\)-semiprime submodules. In Section 3, the modules in which every proper submodule is weakly \((m, n)\)-semiprime are studied. A new characterization of the von Neumann regular module in terms of weakly \((m, n)\)-semiprime submodules is also given in Section 3.

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2. Properties of weakly $(m, n)$–semiprime submodules

Recall the notions of $(m, n)$-semiprime submodules and weakly $(m, n)$-semiprime submodules defined in the introduction. According to [5], a proper ideal $P$ of a commutative ring $A$ is said to be an $(m, n)$-closed ideal of $A$ if $x^m \in P$ implies that $x^n \in P$ for each $x \in A$, where $m$ and $n$, with $m > n$, are positive integers. Also, according to [6], a proper ideal $P$ of a commutative ring $A$ is said to be a weakly $(m, n)$-closed ideal of $A$ if $0 \neq x^m \in P$ implies that $x^n \in P$ for each $x \in A$, where $m$ and $n$, with $m > n$, are positive integers.

**Remark 2.1.** If $A$ is an $A$-module then every $(m, n)$-semiprime submodule of $A$ is an $(m, n)$-closed ideal of $A$.

**Example 2.1.** Let $A$ be a commutative ring and $N$ be a weakly semiprime submodule of an $A$-module $M$; that is, whenever $0 \neq a^m x \in N$, then $ax \in N$ for each $a \in A$ and $x \in M$. Certainly, $N$ is weakly $(m, n)$-semiprime submodule of $M$. Indeed, assume that $0 \neq a^m x \in N$ for some $a \in A$, $x \in M$. Then note that $0 \neq a^2(a^m x) \in M$. Since $N$ is weakly semiprime submodule of $M$, we conclude that $0 \neq a^{m+2} x \in N$. Continuing with the same reasoning we get $a^n x \in N$. The converse is false in general as shown in the following example. Let $K$ be a field and let $R$ denote the ring $K[S, T]$ of polynomials over $K$ in the determinates $S, T$. Let $M = RS + RT$ be a maximal ideal of $R$. Let $P = (S, T^2)$. Then $P$ is not weakly semiprime ideal since $0 \neq T^2.1 \in P$, but $T \notin P$. Also, note that $\sqrt{P} = M$, thus $P$ is $M$-primary ideal of $R$. We will show that $P$ is weakly $(3, 1)$-primary ideal of $R$. Let $0 \neq h^3.k \in (P) \subseteq (S)$ for some $h, k \in R$. Then $h^3.k \in P$. If $h^2.k \in P$ we are done. If not, as $P$ is $M$-primary ideal, we get $h \in M$. So $h^2 \in M^2 = (S^2, ST, T^2) \subseteq P$. Finally $h^2.k \in P$. Thus, $P$ is weakly $(3, 1)$-primary ideal of $R$.

**Example 2.2.** Every $(m, n)$-semiprime submodule is weakly $(m, n)$-semiprime submodule. But the converse is not true. For example, let $A = \mathbb{Z}_{p^n}$ where $p$ is a prime number and $n > 2$. Let $N = \{0, p^{n-1}\}$. Since $a^n x = 0$ for every $a, x \in A$, we conclude that $N$ is a weakly $(n, 1)$-semiprime ideal. However, $N$ is not an $(n, 1)$-semiprime ideal since $p^n.T = 0 \in N$ and $p \notin N$. By a similar argument, $N$ is not an $(n, k)$-semiprime ideal for every $n > k > 0$.

**Example 2.3.** We consider in this example the $\mathbb{Z}$-module $\mathbb{Z}_{p^n.q}$, where $p$ and $q$ are prime numbers and $n > 2$. It is easy to see that $N = \{(0)\}$ is a weakly $(n, n - 1)$-semiprime submodule of $M$. Indeed, let $a^n x = 0$ for some $a, x \in \mathbb{Z}$. Then, we have $p^n.q | a^n x$, which yields $a^{n-1} x = 0$. Therefore, $N = \{(0)\}$ is weakly $(n, n - 1)$-semiprime submodule. On the other hand $(N : M) = p^n.q\mathbb{Z}$ which is not an $(n, n - 1)$-closed ideal of $\mathbb{Z}$, since $(pq)^n \in p^n.q\mathbb{Z}$ but $(pq)^{n-1} \notin p^n.q\mathbb{Z}$. We conclude by using Corollary 1 of [23] that $N$ is not an $(n, n - 1)$-semiprime submodule of $M$.

The following propositions show that the notions of $(m, n)$-semiprime and weakly $(m, n)$-semiprime submodule coincide on certain modules.

**Proposition 2.1.** Let $A$ be an integral domain and $M$ be a torsion-free $A$-module. Then every weakly $(m, n)$-semiprime submodule is $(m, n)$-semi prime submodule of $M$.

**Proof.** Let $N$ be a weakly $(m, n)$-semiprime submodule of $M$. Suppose that $a^n x \in N$ for some $a \in A, x \in M$. If $0 \neq a^n x$, then the fact that $N$ is a weakly $(m, n)$-semiprime submodule of $M$, gives $a^n x \in N$. Next, we assume $a^n x = 0$ with $0 \neq x$. Then $a^n = 0$ as $T(M) = \{0\}$. We get $a^m = 0$ and consequently $a = 0$ since $A$ is an integral domain. Hence, $a^n x = 0 \in N$. Thus, $N$ is $(m, n)$-semiprime submodule of $M$.

**Proposition 2.2.** Let $(A, M)$ be a local ring with the maximal ideal $\mathcal{M}$ and let $M$ be an $A$-module such that $\mathcal{M}M = 0$. Then, every weakly $(m, n)$-semiprime submodule is an $(m, n)$-semiprime submodule of $M$.

**Proof.** Let $N$ be a submodule of $M$ which is a weakly $(m, n)$-semiprime $M$. Choose $a \in A$ and $x \in M$ such that $a^n x \in N$. If $a$ is unit, then $x \in N$ and so $a^n x \in N$. Next, we assume that $a$ is not unit. As $(A, M)$ is a local ring we get $a \in \mathcal{M}$. On the other hand, $\mathcal{M}M = 0$, which implies $a(M) = 0$. Thus, $a^n x = 0 \in N$. Consequently, $N$ is an $(m, n)$-semiprime submodule of $M$.

The following theorem gives a characterization of weakly $(m, n)$-semiprime submodules.

**Theorem 2.1.** Let $M$ be an $A$-module and $N$ be a proper submodule of $M$. Then the following statements are equivalent.

1. $N$ is weakly $(m, n)$-semiprime submodule of $M$.

2. For every $a \in A$ such that $0 \neq a^m N$, it holds that $(N ;_M a^m) = (N ;_M a^n)$.

3. For every $a \in A$ and $L$ submodule of $M$ with $0 \neq a^n L \subset N$, then $a^n L \subset N$. 

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Proof: (1) $\Rightarrow$ (2). As $m > n$, it is clear that $(N : a^n) \subseteq (N : a^m)$. Now, let $x \in (N : a^m)$. Then $a^m x \in N$. If $0 \neq a^m x \in N$, we get $a^n x \in N$ and so $x \in (N : a^n)$ (as $N$ is a weakly $(m, n)$-semiprime submodule). Now, suppose that $a^m x = 0$. The fact that $0 \neq a^m N$ gives $0 \neq a^m y$ for some $y \in N$, it follows $a^m y \in N$ as $N$ is weakly $(m, n)$-semiprime submodule of $M$. Set $z = x + y$, so $0 \neq a^m z \in N$, and by the same argument as above, we get $a^m z \in N$. Therefore, $a^m x \in N$. Hence, $x \in (N : a^n)$. Finally, we conclude that $(N : a^n) \subseteq (N : a^m)$ and $(N : a^m) = (N : a^n)$.

(2) $\Rightarrow$ (3). Suppose that $0 \neq a^n L \subseteq N$ for some $a \in A$ and that $L$ is a submodule of $M$. We have $0 \neq a^m N$ and $L \subseteq (N : a^m)$. Then by (2), we conclude that $L \subseteq (N : a^n)$, and this implies that $a^n L \subseteq N$.

(3) $\Rightarrow$ (1). Assume that $a^n L \subseteq N$ for every $a \in A$, where $L$ is a submodule of $M$ with $0 \neq a^n L \subseteq N$. Let $0 \neq a^m x \in N$ for some $a \in A$ and $x \in M$. We take $L = Ax$. Then $0 \neq a^n L \subseteq N$. By our assumption, we get $a^n L \subseteq N$ and so $a^n x \in N$. Thus, $N$ is a weakly $(m, n)$-semiprime submodule of $M$.

Remark 2.2. In Corollary 1 of [23] if $N$ is $(m, n)$-semiprime submodule of $M$, then $(N : M)$ is an $(m, n)$-closed ideal. If $N$ is a weakly $(m, n)$-semiprime submodule of $M$, then the residual $(N : M)$ need not to be a weakly $(m, n)$-closed ideal of $A$, as shown in the next example.

Example 2.4. Consider the $\mathbb{Z}$-module $M := \mathbb{Z}_{p^m}$ where $p$ is a prime number and $m > n > 2$. Set $N = \{0\}$. We will show that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Let $a^m x = 0$ for some $a, x \in \mathbb{Z}$. Then $p^m | a^m x$. Necessarily, $p^m | a^x$ otherwise $p$ and $a^x$ are coprime and this is a contradiction. Thus, we have $a^x = 0$. Hence, $N$ is $(m, n)$-semiprime and thus it is weakly $(m, n)$-semiprime. We remark that $(N : M) = p^m \mathbb{Z}$ is not weakly $(m, n)$-closed ideal of $\mathbb{Z}$ since $p^m \in (N : M)$ but $p^n \notin (N : M)$.

The following corollary gives a characterization of weakly $(m, n)$-semiprime submodules by using the concept of weakly $(m, n)$-closed ideals when $M$ is an $A$-faithful module. Recall that an $A$-module $M$ is said to be faithful module if

$$\text{ann}_A(M) := \{a \in A; a.m = 0\} = \{0\}.$$

Corollary 2.1. Let $M$ be a faithful $A$-module and let $N$ be a proper submodule of $M$. Then the following statements are equivalent:

(1). $N$ is a weakly $(m, n)$-semiprime submodule of $M$.

(2). For every $L$ submodule of $M$, the residual $(L : M)$ is a weakly $(m, n)$-closed ideal of $A$.

Proof: (1) $\Rightarrow$ (2). Suppose that $N$ is a weakly $(m, n)$-semiprime submodule of $M$. Let $0 \neq a^m \in (L : M)$. Since $M$ is a faithful $A$-module, we have $0 \neq a^m L \subseteq N$. By Theorem 2.1, we conclude that $a^n L \subseteq N$ and hence $a^n \in (L : N)$. Therefore, $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$.

(2) $\Rightarrow$ (1). Assume that $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$ for each submodule $L$. Let $0 \neq a^m x \in N$ for $a \in A$ and $x \in M$. Set $L = Ax$, then $0 \neq a^n L \subseteq N$, which implies $0 \neq a^m \in (L : N)$. By our assumption, we get $a^n \in (L : N)$, and hence $a^n x \in N$. Thus, $N$ is a weakly $(m, n)$-semiprime submodule of $M$.

It remarked here that for some submodule $L$ of $M$ if $0 \neq a^m L \subseteq N$, then $a^n L \subseteq N$ is equivalent to say that $(L : N)$ is a weakly $(m, n)$-closed ideal of $A$ since $A$ is faithful.

Remark 2.3. In Corollary 2.1, the condition "$M$ is a faithful module" is necessary, as shown in the following example.

Example 2.5. Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z}$, which is not a faithful $\mathbb{Z}$-module and consider a submodule $N = \{0\} \times 16\mathbb{Z}$. Then, $N$ is not a weakly $(3, 2)$-semiprime submodule since $2^3(0, 2) = (0, 16) \in N$ and $2^2(0, 2) = (0, 8) \notin N$. On the other hand, $(N : M) = \{0\}$ is a weakly $(3, 2)$-closed ideal of $\mathbb{Z}$ since $(0)$ is a prime ideal of $\mathbb{Z}$.

In the next theorem, we show the relationship between a weakly $(m, n)$-semiprime submodule and the fact that $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$ where $x \in M \setminus N$. Recall that a ring $A$ is said to be reduced if $\text{Nil}(A) = \{0\}$.

Theorem 2.2. Let $M$ be an $A$-module and $N$ be a proper submodule of $M$.

(1). If $(N : x)$ is a weakly $(m, n)$-closed ideal of $A$ for every $x \in M \setminus N$, then $N$ is a weakly $(m, n)$-semiprime submodule of $M$.

(2). If $A$ is a reduced ring, then every submodule of $M$ is a weakly $(m, n)$-semiprime.

(3). If $N$ is a weakly $(m, n)$-semiprime submodule of $M$ and $\text{ann}(x)$ is a weakly $(m, n)$-closed ideal of $A$ for every each $x \in M \setminus N$, then $(N : x)$ is a weakly $(m, n)$-closed ideal of $R$. 

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Proof. (1). Suppose that \( 0 \neq a^my \in N \) for some \( a \in A \) and \( y \in M \). If \( y \in N \), we are done. Next, we assume that \( y \notin M \backslash N \). Since \( (N : y) \) is a weakly \((m, n)\)-closed ideal of \( A \), we have \( a^n \in (N : y) \), and so \( a^my \in N \). Hence, \( N \) is a weakly \((m, n)\)-semiprime submodule of \( M \).

(2). It follows from Corollary 3.6 of [6]. Indeed, if \( A \) is reduced, then every proper ideal of \( A \) is weakly \((m, n)\)-closed ideal, by Corollary 3.6(1) of [6].

(3). Let \( x \in M \backslash N \). Suppose that \( 0 \neq a^n \in (N : x) \) for some \( a \in A \). First, we consider the case \( 0 \neq a^nx \in N \). Since \( N \) is a weakly \((m, n)\)-semiprime submodule of \( M \), we have \( a^nx \in N \), and hence \( a^n \in (N : x) \). Now, consider the case \( a^nx = 0 \). Then, \( 0 \neq a^n \in \text{ann}(x) \), and so \( a^n \in \text{ann}(x) \). Consequently, \( (N : x) \) is a weakly \((m, n)\)-closed ideal of \( A \). \( \square \)

Example 2.6. Let \( N \) be a weakly \((m, n)\)-semiprime submodule of \( M \) and \( x \in M \backslash N \). We will show that \((N : x)\) need not to be a weakly \((m, n)\)-closed ideal of \( A \). Indeed, let \( m \) and \( n \), with \( m > n \), be positive integers and \( M = \mathbb{Z}p^m \) be a \( \mathbb{Z} \)-module, where \( p \) is a prime number. Let \( N = (0) \). It is clear that \( N \) is a weakly \((m, n)\)-semiprime submodule of \( M \). On the other hand, \((N : p) = p^{m-1}\mathbb{Z} \) is not a weakly \((m, n)\)-closed ideal of \( \mathbb{Z} \) since \( p^m \in (N : p) \) but \( p^m \notin (N : p) \).

Let \( A \) be an integral domain. An \( A \)-module \( M \) is said to be torsion-free if \( ma = 0 \) for some \( a \in A \) and \( m \in M \), implies \( a = 0 \) or \( m = 0 \).

Theorem 2.3. Let \( A \) be an integral domain, \( M \) be a torsion-free \( A \)-module and \( N \) be a proper submodule of \( M \). The following statements are equivalent.

(1). \( N \) is weakly \((m, n)\)-semiprime submodule of \( M \).

(2). \((N : x)\) is weakly \((m, n)\)-closed ideal of \( A \) for each \( x \in M \backslash N \).

Proof. (2) \( \Rightarrow \) (1). It follows from the Theorem 2.2.

(1) \( \Rightarrow \) (2). Let \( 0 \neq a^m \in (N : x) \) for some \( a \in A \) and \( x \in M \). If \( x = 0 \), then \((N : x) = A \), and we are done. Next, we assume that \( 0 \neq x \). Since \( M \) is a torsion-free \( A \)-module, we get \( 0 \neq a^nx \in N \). The fact that \( N \) is a weakly \((m, n)\)-semiprime submodule of \( M \), gives \( a^nx \in N \), and thus \( a^n \in (N : x) \). Hence, \((N : x)\) is a weakly \((m, n)\)-closed ideal of \( A \). \( \square \)

Our next objective is the study of the stability of the tensor product of weakly \((m, n)\)-semiprime submodules.

Theorem 2.4. Let \((A, M)\) be a local ring and \( M \) be an \( A \)-module.

(1). If \( F \) is a non-zero finitely generated flat \( A \)-module and \( N \) is a finitely generated weakly \((m, n)\)-semiprime submodule of \( M \) such that \( F \otimes N \neq F \otimes M \), then \( F \otimes N \) is a weakly \((m, n)\)-semiprime submodule of \( F \otimes M \).

(2). If \( F \) is a finitely generated faithful flat \( A \)-module and \( N \) is a finitely generated submodule of \( M \), then the following statements are equivalent:

(a). \( N \) is a weakly \((m, n)\)-semiprime submodule of \( M \).

(b). \( F \otimes N \) is a weakly \((m, n)\)-semiprime submodule of \( F \otimes M \).

Proof. (1). Let \( F \) be a finitely flat module, \( N \) be a finitely \((m, n)\)-semiprime submodule of \( M \) and take \( a \in R \) such that \( 0 \neq a^m(F \otimes N) \). Since \( 0 \neq a^m(F \otimes N) = F \otimes a^nN \) and \((A, M)\) is a local ring, we deduce that \( 0 \neq a^nN \) (see Exercise 3 of Chapter 2 in [8]). Also, by Theorem 6 of [9], we have \((F \otimes N : F\otimes M a^m) = F \otimes (N : M a^m)\), and by Theorem 2.1 we get \((F \otimes N : F\otimes M a^m) = (F \otimes N : F\otimes M a^n)\). By Theorem 2.1 we deduce that \( F \otimes N \) is a weakly \((m, n)\)-semiprime submodule of \( F \otimes M \).

(2). (a) \( \Rightarrow \) (b). Since \( F \) is a faithful flat module and \( N \) is a proper submodule of \( M \), we have \( F \otimes N \neq F \otimes M \). Now, the result follows from Part (1).

(b) \( \Rightarrow \) (a). Suppose that \( F \otimes N \) is a weakly \((m, n)\)-semiprime submodule of \( F \otimes M \). Take \( a \in R \) with \( 0 \neq a^nN \). Since \((A, M)\) is a local ring and \( F, N \), are finitely generated modules, we get \( 0 \neq a^n(F \otimes N) \). By Theorem 2.1 and Lemma 3.2 of [9], we have

\[
F \otimes (N : M a^m) = (F \otimes N : F\otimes M a^n) = F \otimes (N : M a^n).
\]

Thus, \( F \otimes (N : M a^m) = F \otimes (N : M a^n) \). Since the sequence

\[
0 \rightarrow F \otimes (N : M a^n) \hookrightarrow F \otimes (N : M a^m) \rightarrow 0
\]

is exact and \( F \) is a faithful module, we get the exact sequence

\[
0 \rightarrow (N : M a^n) \hookrightarrow (N : M a^m) \rightarrow 0,
\]

which implies \((N : M a^n) = (N : M a^m)\). Now, the desired result follows from Theorem 2.1. \( \square \)
Theorem 2.5. Let $f : M \rightarrow M'$ be a homomorphism of $A$-modules.

(1). If $N$ is a weakly $(m,n)$-semiprime submodule of $M$ containing $\ker(f)$ and if $f$ is surjective, then $f(N)$ is a weakly $(m,n)$-semiprime submodule of $M'$.

(2). If $N'$ is a weakly $(m,n)$-semiprime submodule of $M'$ and if $f$ is injective, then $f^{-1}(N')$ is a weakly $(m,n)$-semiprime submodule of $M$.

Proof. (1). Suppose that $f$ is surjective and $\ker(f) \subseteq N$, where $N$ is a weakly $(m,n)$-semiprime submodule of $M$. Take $0 \neq a^m x' \in N'$ for some $a \in A$ and $x' \in M'$. Then there exists $x \in M$ such that $x' = f(x)$. Since $0 \neq f(a^m x) \in f(N)$ and $\ker(f) \subseteq N$, we have $0 \neq a^m x \in N$. Since $N$ is a weakly $(m,n)$-semiprime submodule of $M$, we get $a^n x \in N$. It follows that $a^n f(x) = a^n x' \in f(N)$. Hence, $f(N)$ is a weakly $(m,n)$-semiprime submodule of $M'$, as desired.

(2). Assume that $f$ is a monomorphism of $A$-modules and $N'$ is a weakly $(m,n)$-semiprime submodule of $M'$. Take $0 \neq a^m x \in f^{-1}(N')$ for some $a \in A, x \in M$. So, $0 \neq a^m f(x) \in N'$. The fact that $N$ is a weakly $(m,n)$-semiprime submodule of $M'$, gives $a^n f(x) \in N'$. Therefore, $a^n x \in f^{-1}(N')$. Hence, $f^{-1}(N')$ is a weakly $(m,n)$-semiprime submodule of $M$. □

Corollary 2.2. Let $N$ be a proper submodule of $M$.

(1). If $L$ is a submodule of $M$ with $L \subseteq N$ and if $N$ is a weakly $(m,n)$-semiprime submodule of $M$, then $N/L$ is a weakly $(m,n)$-semiprime submodule of $M/L$.

(2). If $L$ is a submodule of $M$ with $L \subseteq N$ and if $N/L$ is a weakly $(m,n)$-semiprime submodule of $M/L$, and if $L$ is a weakly $(m,n)$-semiprime submodule of $M$, then $N$ is a weakly $(m,n)$-semiprime submodule of $M$.

Proof. (1). It is a direct consequence of Theorem 2.5(1).

(2). Assume that $N/L$ is a weakly $(m,n)$-semiprime submodule of $M/L$ and $L$ is a weakly $(m,n)$-semiprime submodule of $M$. Take $0 \neq a^m x \in N$ for some $a \in A, x \in M$. Then $a^m(x + L) \in N/L$. If $a^m(x + L) = 0_{M/L}$, then $0 \neq a^m x \in L$, which is a weakly $(m,n)$-semiprime submodule of $M$. Thus, $a^m x \in L$ and so $a^n x \in N$. Next, we assume that $0 \neq a^m(x + L)$. The fact that $N/L$ is a weakly $(m,n)$-semiprime submodule of $M/L$ gives that $a^m(x + L) \in N/L$. Hence, $a^n x \in N$ and $N$ is a weakly $(m,n)$-semiprime submodule of $M$. Therefore, $N$ is weakly $(m,n)$-semiprime submodule of $M$. □

Pekin et al. [23] studied the concept of $(m,n)$-semiprime submodules over the trivial extension ring $A(+1)M$, where $A$ is a commutative ring and $M$ is an $A$-module. For more detail on trivial extensions of rings, see [7]. We end this section by giving another way to construct weakly $(m,n)$-semiprime submodules that are not $(m,n)$-semiprime. Let $A$ be a ring, $I$ be an ideal of $A$, and $M$ be an $A$-module, and set

$$M \bowtie I := \{(x,x') \in M \times M | x - x' \in IM\},$$

which is a $A \bowtie I$-module with the multiplication given by

$$(r,r+i)(x,x') = (rx, (r+i)x') \quad \text{where} \quad r \in A, i \in I, \quad \text{and} \quad (x,x') \in M \bowtie I.$$  

According to [12], $M \bowtie I$ is known as the duplication of the $A$-module $M$ along the ideal $I$. If $N$ is a submodule of $M$, then it is clear that

$$N \bowtie I := \{(x,x') \in N \times M | x - x' \in IM\} \quad \text{and} \quad \overline{N} := \{(x,x') \in M \times N | x - x' \in IM\}$$

are submodules of $M \bowtie I$.

Lemma 2.1. Let $A$ be a ring, $I$ be an ideal of $A$, and $M$ be an $A$-module. Let $N$ be a submodule of $M$. Let $m$ and $n$ be positive integers satisfying $m > n$.

(1). $N \bowtie I$ is an $(m,n)$-semiprime submodule of $M \bowtie I$ if and only if $N$ is an $(m,n)$-semiprime submodule of $M$.

(2). $N \bowtie I$ is an $(m,n)$-semiprime submodule of $M \bowtie I$ if and only if $N$ is an $(m,n)$-semiprime submodule of $M$.

Proof. (1). Assume that $N \bowtie I$ is an $(m,n)$-semiprime submodule of $M \bowtie I$. Take $a^m x \in N$ for some $a \in A, x \in M$. Then $(a, a)^m (x,x) \in N \bowtie I$. The fact that $N \bowtie I$ is an $(m,n)$-semiprime submodule of $M \bowtie I$, gives that $(a,a)^m (x,x) \in N \bowtie I$. So, $a^m x \in N$. Hence, $N$ is an $(m,n)$-semiprime submodule of $M$. Conversely, assume that $N$ is an $(m,n)$-semiprime submodule of $M$. Take $(a, a+i)^m(x,x')$ for some $(a, a+i) \in A \bowtie I, (x,x') \in M \bowtie I$. Then $(a^m x, (a+i)^m x') \in N$ and so $a^m x \in N$. As $N$ is an $(m,n)$-semiprime submodule of $M$, we conclude that $a^n x \in N$, which implies $(a, a+i)^m(x,x') \in N \bowtie I$ and this shows that $N \bowtie I$ is an $(m,n)$-semiprime submodule of $M \bowtie I$.

(2). The proof is similar to the proof of (1) and so is omitted. □
The following definition is useful for studying weakly \((m, n)\)-semiprime submodules that are not \((m, n)\)-semiprime submodules.

**Definition 2.1.** Let \(M\) be an \(A\)-module where \(A\) is a commutative ring, \(m \geq n > 0\) are positive integers, and \(N\) a weakly \((m, n)\)-semiprime submodule of \(M\). Then \(a \in A\) is an \((m, n)\)-unbreakable-zero element of \(N\) if there exists \(x \in M\) such that \(a^m x = 0\) and \(a^n x \notin N\). (Thus, \(N\) has an \((m, n)\)-unbreakable-zero element if and only if \(N\) is a weakly \((m, n)\)-semiprime submodule of \(M\) that is not \((m, n)\)-semiprime.)

**Theorem 2.6.** The following statements are equivalent:

1. \(N \triangleleft I\) is a weakly \((m, n)\)-semiprime submodule which is not an \((m, n)\)-semiprime submodule of \(M \triangleright I\).
2. \(N\) is a weakly \((m, n)\)-semiprime submodule that is not an \((m, n)\)-semiprime submodule of \(M\), and for every \((m, n)\)-unbreakable-zero element \(a \in A\) of \(N\), it holds that \((a + i)^m M = 0\) for every \(i \in I\).

*Proof:* (1) \(\Rightarrow\) (2). Suppose that \(N \triangleleft I\) is a weakly \((m, n)\)-semiprime submodule of \(M \triangleright I\). Take \(0 \neq a^m x \in N\) for some \(a \in A, x \in M\). Then \(0 \neq (a, a)^m (x, x) \in N \triangleleft I\). As \(N \triangleleft I\) is a weakly \((m, n)\)-semiprime submodule of \(M \triangleright I\), we obtain that \((a, a)^m (x, x) \in N \triangleleft I\), which implies \(a^m x \in N\) and shows that \(N\) is a weakly \((m, n)\)-semiprime submodule of \(M\). By Lemma 2.1, \(N\) is not an \((m, n)\)-semiprime submodule of \(M\). Now, let \(a \in A\) be an \((m, n)\)-unbreakable-zero element of \(N\); that is, there exists \(x \in M\) such that \(a^m x = 0\) and \(a^n x \notin N\). We will show that \((a + i)^m M = 0\) for every \(i \in I\).

Since \(N\) is a weakly \((m, n)\)-semiprime submodule of \(M\) that is not \((m, n)\)-semiprime, \(N\) has an \((m, n)\)-unbreakable-zero \(a \in A\). By the way of contradiction, suppose that there exists \(i \in I\) such that \((a + i)^m y = 0\) for some \(y \in M\). Then, \(0 \neq (a, a + i)^m = (0, (a + i)^m y) \in N \triangleleft I\). As \(N \triangleleft I\) is a weakly \((m, n)\)-semiprime submodule of \(M \triangleright I\), we conclude that \((a + i)^m (x, y) \in N \triangleright I\) and so \(a^n x \in I\), which is a contradiction. Hence, \((a + i)^m M = 0\) for every \(i \in I\).

(2) \(\Rightarrow\) (1). Suppose that \(N\) is a weakly \((m, n)\)-semiprime submodule which is not \((m, n)\)-semiprime and \((a + i)^m M = 0\) if \(a \in A\) is an \((m, n)\)-unbreakable-zero element of \(N\). Let \(0 \neq (a, a + i)^m (x, x^\prime) \in N \triangleright I\). Then \(a^n x \in N\) and \((a + i)^m x^\prime - a^m x \in IM\). Assume that \(0 \neq a^m x \in N\). As \(N\) is a weakly \((m, n)\)-semiprime submodule of \(M\) we get \(a^n x \in N\). Now, assume that \(a^m x = 0\), then necessarily \(0 \neq (a + i)^m x^\prime\). If \(a^n x \notin N\), then \(a\) is an \((m, n)\)-unbreakable-zero element of \(N\). By our assumption, we have \((a + i)^m M = 0\). This is a contradiction. Hence, \(a^n x \in N\).

**Theorem 2.7.** Let \(M\) be an \(A\)-module, \(N\) be a submodule of \(M\), and \(m, n\) are positive integers satisfying \(m > n\). Let

\[ N := \{(x, x') \in M \times N; x - x' \in IM\}. \]

The following statements are equivalent:

1. \(N\) is a weakly \((m, n)\)-submodule of \(M \triangleright I\).
2. \(N\) is a weakly \((m, n)\)-submodule of \(M\) and the equation \((a - i)^m M = 0\) holds for every \(i \in I\) and for an \((m, n)\)-unbreakable-zero element \(a \in A\) of \(N\).

*Proof.* It is the same as the proof of Theorem 2.6.

**3. Modules over which every submodule is weakly \((m, n)\)-semiprime**

The following result gives the constraints under which every given proper submodule is a weakly \((m, n)\)-semiprime submodule.

**Theorem 3.1.** Let \(M\) be an \(A\)-module and \(m, n\), be two positive integers such that \(m > n\). The following statements are equivalents:

1. Every proper submodule is a weakly \((m, n)\)-semiprime submodule of \(M\).
2. For every submodule \(N\) of \(M\) and for every \(a \in A\) such that \(0 \neq a^m N\), the descending chain

\[ aN \supseteq a^2 N \supseteq \cdots \supseteq a^n N \supseteq \cdots \]

of submodules of \(M\) terminates at the \(n\)th step.
3. For every submodule \(N\) of \(M\) and for every \(a \in A\) with \(0 \neq a^m N\), it holds that \(a^n N = a^m N\).
**Proof.** (1) ⇒ (2). Take \( a \in A \) and let \( N \) be a submodule of \( M \) such that \( 0 \neq a^nN \). If \( a^nN = M \) then we are done. Next, we assume that \( a^nN \) is a proper submodule of \( M \). Since \( 0 \neq a^nN \subseteq a^nN \) and \( N \) is a weakly \((m,n)\)-semiprime submodule of \( M \), by Theorem 2.1 we have \( a^nN \subseteq a^nN \), which implies that \( a^nN = a^nN \). Hence, the descending chain 
\[ aN \supseteq a^2N \supseteq \cdots \supseteq a^nN \supseteq \cdots, \]
terminates at the \( n^{th} \) step.

(2) ⇒ (3). It is trivial.

(3) ⇒ (1). Let \( N \) be a proper submodule of \( M \). Take \( a \in A \) and let \( K \) be a submodule of \( M \) such that \( 0 \neq a^nK \subseteq N \). By our assumption \( a^nK = a^nK \subseteq N \). Now, by Theorem 2.1, we conclude that \( N \) is a weakly \((m,n)\)-semiprime submodule of \( M \).

According to [11, 15], an \( A \)-module \( M \) is said to be *multiplication module* if every submodule \( N \) of \( M \) has the form \( N = IM \), where \( I \) is an ideal of \( A \). In this case, we have \( N = (N : M)M \). For more detail about multiplication modules, see [1–4].

Let \( M \) be an \( A \)-module. According to [20], \( M \) is a *reduced module* if for every \( a \in A \), \( x \in M \) with \( ax = 0 \), \( aM \cap Ax = 0 \), or equivalently \( a^2x = 0 \) implies \( ax = 0 \).

According to [22], a commutative ring \( A \) is a *von Neumann regular ring* if for every \( a \in A \), there exist \( b \in A \) such that \( a = a^2b \). In [17], Jayarm and Tekir studied the concept of von Neumann regular modules (see also [18, 19]) by introducing the concept \( M \)-von Neumann regular elements of modules as follows. If \( M \) is an \( A \)-module, then an element \( a \in A \) is an \( M \)-von Neumann regular element if \( aM = a^2M \). Also, an \( A \)-module \( M \) is said to be a *von Neumann regular module* if for every \( x \in M \), \( Ax = aM = a^2M \) for some \( a \in A \).

Our next objective is to give a characterization of von Neumann regular modules using the properties of weakly \((m,n)\)-semi prime submodules.

**Theorem 3.2.** Let \( M \) be a finitely generated \( A \)-module. The following statements are equivalents:

1. \( M \) is an von Neumann regular module.
2. \( M \) is a multiplication reduced module in which every submodule is weakly \((m,n)\)-semiprime submodule

**Proof.** We follow the same reasoning as (1) ⇔ (2) in Theorem 8 of [23].

(2) ⇒ (1). Let \( M \) be a finitely generated reduced multiplication module in which every proper submodule is a weakly \((m,n)\)-semiprime. Take \( a \in A \). We will show that \( aM = a^2M \). If \( a^nM = M \), then clearly we have \( aM = a^2M \). Next, we assume that \( a^nM \) is a proper submodule of \( M \). First, consider the case \( a^nM = 0 \). Since \( M \) is reduced, we have \( \text{ann}(M) \) is a semiprime ideal, which implies that \( a \in \text{ann}(M) \). Thus, \( aM = a^2M = 0 \). Next, we consider the case \( 0 \neq a^nM \). As \( a^nM \) is a weakly \((m,n)\)-semiprime submodule and \( 0 \neq a^nM \subseteq a^nM \), we conclude by Theorem 2.1 that \( a^nM = a^2M \). We deduce that \( a^{n+1}M = a^nM = a(a^nM) \). Since \( a^nM \) is a finitely generated module, by Corollary 2.5 of [8] we have \( x(a^nM) = 0 \) for some \( x \in A \) such that \( x \equiv 1 \mod(a^n) \). Thus, there exists \( b \in A \) such that \( (1-ab)a^nM = 0 \). AS \( M \) is reduced, we get \( (1-ab)aM = 0 \), Which implies \( aM = a^2bM \subseteq a^2M \). Consequently, we obtain \( aM = a^2M \), as desired.

(1) ⇒ (2). Suppose that \( M \) is a von Neumann regular module. By the proof of Theorem 8 of [23], \( M \) is a multiplication-reduced module. Now, let \( N \) be a proper submodule of \( M \). Take \( a \in A \) and let \( L \) be a submodule of \( M \) such that \( 0 \neq a^nL \subseteq N \). Since \( M \) is a multiplication module, we have \( L = (L : M)M \), which then gives \( 0 \neq a^nL = (L : M)a^nM = (L : M)a^nM \), and hence \( 0 \neq a^nN = a^nN \). Therefore, \( N \) is a weakly \((m,n)\)-semiprime submodule.

**Corollary 3.1.** Let \( M \) be a finitely generated \( A \)-module. The following statements are equivalent:

1. \( M \) is a von Neumann regular module.
2. \( M \) is a multiplication-reduced module in which every submodule is an \((m,n)\)-semiprime submodule.
3. \( M \) is a multiplication-reduced module in which every submodule is a weakly \((m,n)\)-semiprime submodule.

**Proof.** (1) ⇔ (2). It follows from Theorem 8 of [23].

(2) ⇒ (3). It is trivial.

(3) ⇒ (1). It follows from Theorem 3.2.

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References