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Research Article

Hermite–Hadamard–type fractional–integral inequalities for (p,h)–convex fuzzy–interval–valued mappings

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Abstract

In this paper, fuzzy-interval-valued functions of the (p,h)-convex type, defined recently by Khan et al. [*AIMS Math.* **8** (2023) 7437–7470], are studied. Several Hermite-Hadamard-type inequalities in the said setting are obtained. A Hermite-Fejer-type inequality is also obtained, which generalizes several recently published results. Moreover, in order to supplement the obtained results, suitable numerical examples are given.

Keywords: Hermite-Hadamard-type inequality; (p,h)-convex fuzzy-interval-valued mapping; fuzzy generalized fractional integral operator; Hermite-Hadamard-Fejer-type inequality.

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1. Introduction

Convex inequalities have been an active topic of mathematical research since the introduction of the first convex inequality, known as Jensen inequality. Many inequalities derived using convexity exist in the literature; for example, see the books [25, 31]. Inequalities have various applications to different branches of mathematics, including numerical analysis, probability density functions, and optimization; see the papers [2, 6, 8, 13, 26, 27, 34].

The Hermite-Hadamard inequality [12], proved independently by Charles Hermite and Jacques Hadamard, is among those inequalities that have attracted the most attention in the mathematical community. This inequality has been generalized in various ways by many mathematicians. If $\mathcal{F}:\mathbb{I}\to\mathbb{R}$ is a convex function on $\mathbb{I}$ and $n,m\in\mathbb{I}$ with $n<m$, then the said inequality is stated as

$$\mathcal{F}\left(\frac{m+n}{2}\right)\leqslant\frac{1}{m-n}\int_n^m\mathcal{F}(\xi)d\xi\leqslant\frac{\mathcal{F}(n)+\mathcal{F}(m)}{2}.$$

Many researchers obtained this inequality as a consequence of the generalization using different kinds of convexity with fractional operators [3, 7, 21, 23, 28, 32, 35–39]. Additional detail about the Hermite-Hadamard and convex inequalities can be found in the papers [4, 5, 15]. In the present paper, the fuzzy-interval-valued setting together with newly defined fuzzy convexity is utilized to derive various convex inequalities together with fractional integral operators. The recent results obtained by Khan et al. [22] are generalized in this paper.

It is remarked here that the initial idea of fractional calculus was given by L'Hospital and Leibniz in 1695. This concept was extended by many mathematicians, including Riemann, Grünwald, Letnikov, Hadamard, and Weyl. These mathematicians made valuable contributions not only to fractional calculus but also to its various applications. Nowadays, fractional calculus is being used widely in describing various phenomena, such as the fractional conservation of mass and fractional Schrödinger equation in quantum theory; more detail about fractional calculus can be found in [14, 29, 41].

2. Definitions and preliminaries

Let $\mathbb{K}_c$ and $\mathbb{F}_c(\mathbb{R})$ be the collections of all closed and bounded intervals, and fuzzy intervals of $\mathbb{R}$, respectively. Denote by $\mathbb{K}_c^+$ the set of all positive intervals. The collections of all Riemann-integrable real-valued functions, Riemann-integrable interval-valued functions (IVFs), and all Riemann-integrable fuzzy-interval-valued functions (FIVFs) over $[u,v]$ are denoted by $\mathfrak{R}_{[\mathfrak{u},\mathfrak{v}]}$, $\mathfrak{IR}_{[\mathfrak{u},\mathfrak{v}]}$, and $\mathfrak{FR}_{[\mathfrak{u},\mathfrak{v}]}$, respectively. A brief overview of the interval-valued analysis and notions is given in this section; for additional detail, see [9, 24, 42].

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Let $\mathbb{K}_c = \{[\Xi_*, \Xi^*] : \Xi_*, \Xi^* \in \mathbb{R}\ and\ \Xi_* \leqslant \Xi^*\}$. An interval $[\Xi_*, \Xi^*]$ is said to be a positive interval if $\Xi_* \geqslant 0$. Take $\mathbb{K}_c^+ = \{[\Xi_*, \Xi^*] : \Xi_*, \Xi^* \in \mathbb{R}\ and\ \Xi_* \geqslant 0\}$. The algebraic addition, the algebraic multiplication, and the scalar multiplication for $[\Lambda_*, \Lambda^*], [\Xi_*, \Xi^*] \in \mathbb{K}_c$ and $\zeta \in \mathbb{R}$ are defined as

$$[\Lambda_*, \Lambda^*] + [\Xi_*, \Xi^*] = [\Lambda_* + \Xi_*, \Lambda^* + \Xi^*],$$

$$[\Lambda_*, \Lambda^*] \cdot [\Xi_*, \Xi^*] = \left[\min\{\Lambda_*\Xi_*, \Lambda^*\Xi_*, \Lambda_*\Xi^*, \Lambda^*\Xi^*\}, \max\{\Lambda_*, \Xi_*, \Lambda^*\Xi_*, \Lambda_*\Xi^*, \Lambda^*\Xi^*\}\right],$$

and

$$\zeta \cdot [\Lambda_*, \Lambda^*] = \begin{cases} [\zeta\Lambda_*, \zeta\Lambda^*] & \text{if } \zeta > 0 \\ \{0\} & \text{if } \zeta = 0 \\ [\zeta\Lambda^*, \zeta\Lambda_*] & \text{if } \zeta < 0, \end{cases}$$

respectively. Also, the difference is defined as $[\Lambda_*, \Lambda^*] - [\Xi_*, \Xi^*] = [\Lambda_* - \Xi^*, \Lambda^* - \Xi_*]$. The inclusion relation $[\Xi_*, \Xi^*] \supseteq [\Lambda_*, \Lambda^*]$ holds if and only if $\Lambda_* \geqslant \Xi_*$ with $\Xi^* \geqslant \Lambda^*$.

Remark 2.1 (see [42]). The relation "$\leqslant_l$" is defined on $\mathbb{K}_c$ as follows

$$[r_*, r^*] \leqslant_l [m_*, m^*] \text{ if and only if } r_* \leqslant m_*,\ r^* \leqslant m^*,$$

for all $[r_*, r^*], [m_*, m^*] \in \mathbb{K}_c$; it is an order relation.

Remark 2.2 (see [16]). Let $F_c(\mathbb{R})$ be a set of fuzzy numbers. If $\zeta, w \in F_c(\mathbb{R})$, then the relation "$\preccurlyeq$" is defined on $F_c(\mathbb{R})$ as follows

$$\zeta \preccurlyeq w \ \text{ if and only if } \ [\zeta]^\phi \leqslant_l [w]^\phi, \ \ for\ all\ \phi \in [0, 1];$$

this relation is known as partial order relation.

Theorem 2.1 (see [16]). *Let $\mathcal{F} : [u, v] \subset \mathbb{R} \to \mathbb{F}_c(\mathbb{R})$ be a FIVF, while ϕ levels define the family of IVFs $\mathcal{F}_\phi : [u, v] \subset \mathbb{R} \to K_c$ given by $\mathcal{F}_\phi(x) = [\mathcal{F}_*(x, \phi), \mathcal{F}^*(x, \phi)]$ for all $x \in [u, v]$ and for all $\phi \in [0, 1]$. Then $\mathcal{F}$ is a fuzzy Riemann integrable over $[u, v]$ if and only if $\mathcal{F}_*(x, \phi)$ and $\mathcal{F}^*(x, \phi)$ both are Riemann integrable over $[u, v]$. Moreover, if $\mathcal{F}$ is fuzzy Riemann integrable over $[u, v]$, then*

$$[(FR)\int_u^v \mathcal{F}(x)dx]^\phi = [(R)\int_u^v \mathcal{F}_*(x, \phi)dx, (R)\int_u^v \mathcal{F}^*(x, \phi)dx] = (IR)\int_u^v \mathcal{F}_\phi(x)dx)$$

for all $\phi \in (0, 1]$.

Next, the notion of convexity and generalized convexity is defined, which is used throughout the rest of the paper.

Definition 2.1. *For an interval $\mathcal{I}$ in $\mathbb{R}$, a function $f : \mathcal{I} \to \mathbb{R}$ is said to be convex on $\mathcal{I}$ if the inequality*

$$f(\zeta x + (1 - \zeta)y) \leqslant \zeta f(x) + (1 - \zeta)f(y)$$

holds for all $x, y \in \mathcal{I}$ and $\zeta \in [0, 1]$; and is said to be a concave function if the above inequality is reversed.

The (s, m)-convexity generalized the s convexity; J. Park asserted a new definition given below and gave some properties about this class of functions in [30].

Definition 2.2. *For some fixed $s \in (0, 1]$ and $m \in [0, 1]$, a mapping $f : [0, +\infty) \to \mathbb{R}$ is said to be (s, m)-convex in the second sense on $\mathcal{I}$ if the inequality*

$$f(tx + m(1 - t)y) \leqslant t^s f(x) + m(1 - t)^s f(y)$$

holds for all $x, y \in \mathcal{I}$ and $t \in [0, 1]$.

The following definition, introduced in [10], generalizes the p-convexity.

Definition 2.3. *Let $h : J \to \mathbb{R}$ be a non-negative and non-zero function. We say that $f : \mathcal{I} \to \mathbb{R}$ is a (p, h)-convex function or that f belongs to the class $ghx(h, p, \mathcal{I})$, if f is non-negative and*

$$f([\alpha x^p + (1 - \alpha)y^p]^{\frac{1}{p}}) \leqslant h(\alpha)f(x) + h(1 - \alpha)f(y)$$

for all $x, y \in \mathcal{I}$ and $\alpha \in (0, 1)$. Similarly, if the inequality is reversed, then f is said to be a (p, h)-concave function or belong to the class $ghv(h, p, \mathcal{I})$.

Definition 2.4. *Let $p \in \mathbb{R} - 0$. A mapping $\mathcal{C} : [a, b] \subset (0, +\infty) \to \mathbb{R}$ is said to be p-symmetric with respect to $\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}}$ if $\mathcal{C}(x) = \mathcal{C}\left([a^p + b^p - x^p]^{\frac{1}{p}}\right)$ holds for every $x \in [a, b]$.*

The following definition given by Khan et al. [20] generalizes the previously defined convex types of functions.

Definition 2.5. *Let K_p be a p-convex set, let $J \subset \mathbb{R}$ be an interval containing $(0, 1)$, and let $h : J \to \mathbb{R}$ be a non-negative function. Then FIVF $\mathcal{F} : K_p \to F_c(\mathbb{R})$ is named as (p, h)-convex FIVF on K_p such that*

$$\mathcal{F}\left([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}\right) \preccurlyeq h(\alpha)\mathcal{F}(x)\tilde{+}h(1-\alpha)\mathcal{F}(y),$$

for all $x, y \in K_p, \alpha \in [0, 1]$, where $\mathcal{F}(x) \succcurlyeq 0$. If the inequality is reversed, then $\mathcal{F}$ is named as (p, h)-concave FIVF on $[u, v]$. Also, $\mathcal{F}$ is affine if and only if it is both (p, h)-convex and (p, h)-concave FIVF.

Remark 2.3. The following properties hold for the (p, h)-convex FIVF $\mathcal{F}$:

- If we take $h(\alpha) = \alpha^s$ we obtain the (p, s)-convex [22] FIVF, that is

$$\mathcal{F}\left([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}\right) \preccurlyeq \alpha^s\mathcal{F}(x)\tilde{+}(1-\alpha)^s\mathcal{F}(y).$$

- If we take $h \equiv \mathbb{I}$ we get p-convex FIVF, that is

$$\mathcal{F}\left([\alpha x^p + (1-\alpha)y^p]^{\frac{1}{p}}\right) \preccurlyeq \alpha\mathcal{F}(x)\tilde{+}(1-\alpha)\mathcal{F}(y).$$

- If we take $p = 1$ and $h(\alpha) = \alpha^s$ then from (p, h)-convex FIVF we archieve s-convex FIVF [19]; that is

$$\mathcal{F}(\alpha x + (1-\alpha)y) \preccurlyeq \alpha^s\mathcal{F}(x)\tilde{+}(1-\alpha)^s\mathcal{F}(y).$$

- If we take $p \equiv 1$ and $h \equiv \mathbb{I}$ then from (p, h)-convex FIVF we archieve convex FIVF (see [11, 19]); that is

$$\mathcal{F}(\alpha x + (1-\alpha)y) \preccurlyeq \alpha\mathcal{F}(x)\tilde{+}(1-\alpha)\mathcal{F}(y).$$

Example 2.1. *By setting $h(\alpha) = \alpha^s$, one gets the (p, s)-convex FIVF. Consider the FIVF $\mathcal{F} : [0, 1] \to \mathbb{F}_c(\mathbb{R})$ defined by*

$$\mathcal{F}(x)(\sigma) = \begin{cases} \dfrac{\sigma}{2x^p} & \text{if } \sigma \in [0, 2x^p], \\ \dfrac{4x^p - \sigma}{2x^p} & \text{if } \sigma \in (2x^p, 4x^p], \\ 0 & \text{otherwise.} \end{cases}$$

Then, for each $\phi \in [0, 1]$, one has $\mathcal{F}_\phi(x) = [2\phi x^p, (4 - 2\phi)x^p]$. Since the end-point functions $\mathcal{F}_*(x, \phi)$ and $\mathcal{F}^*(x, \phi)$ are (p, s)-convex functions in the second sense for each $\phi \in [0, 1]$ and $s \in [0, 1]$. Hence, $\mathcal{F}(x)$ is a (p, s)-convex FIVF in the second sense.

Next, some fractional-type integrals are defined, which are used in the rest of the paper. The following definition defines Katugampola-fractional integrals, due to Udita Katugampola [17], which generalizes the Riemann-Liouville fractional integrals.

Definition 2.6. *Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left-and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in [a, b]$ are defined by*

$${}^pI^\alpha_{a+}f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{t^{p-1}}{(x^p - t^p)^{1-\alpha}}f(t)dt$$

and

$${}^pI^\alpha_{b-}f(x) := \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{t^{p-1}}{(t^p - x^p)^{1-\alpha}}f(t)dt$$

with $a < x < b$ and $p > 0$, provided that the integrals exist, where

$$\Gamma(t) = \int_0^{+\infty} e^{-z}z^{t-1}dz$$

is the gamma function [1].

Definition 2.7. *Let $p,\alpha > 0$ and $L([\rho,\zeta],\mathbb{E})$ be the collection of all Lebesgue measurable fuzzy-interval-valued mappings (FIVMs) on $[\rho,\zeta]$. Then the fuzzy interval left and right generalized fractional integrals of $\mathcal{F} \in L([\rho,\zeta],\mathbb{E})$ with order $\alpha > 0$ are defined by*

$$I^{p,\alpha}_{\rho^+}\mathcal{F}(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_\rho^x (x^p - v^p)^{\alpha-1}v^{p-1}\mathcal{F}(v)dv, (x > \rho),$$

and

$$I^{p,\alpha}_{\zeta^-}\mathcal{F}(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^\zeta (v^p - x^p)^{\alpha-1}v^{p-1}\mathcal{F}(v)dv, (x < \zeta),$$

respectively. The fuzzy interval left and right generalized fractional integral based on end-point mappings can be defined as

$$[I^{p,\alpha}_{\rho^+}\mathcal{F}(x)]^\phi = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_\rho^x (x^p - v^p)^{\alpha-1}v^{p-1}[\mathcal{F}_*(v,\phi),\mathcal{F}^*(v,\phi)]dv, (x > \rho),$$

where

$$I^{p,\alpha}_{\rho^+}\mathcal{F}_*(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_\rho^x (x^p - v^p)^{\alpha-1}v^{p-1}\mathcal{F}_*(v)dv, (x > \rho),$$

and

$$I^{p,\alpha}_{\zeta^-}\mathcal{F}^*(x) = \frac{p^{1-\alpha}}{\Gamma(\alpha)}\int_x^\zeta (v^p - x^p)^{\alpha-1}v^{p-1}\mathcal{F}^*(v)dv, (x < \zeta).$$

Similarly, we can define right-generalized fractional integral $\mathcal{F}$ of x based on end-point mappings.

3. Main results

The first result presented is a variation of the Hermite-Hadamard type inequality in the fractional convex FIVF sense.

Theorem 3.1. *Let $\mathcal{F} : [s,t] \to \mathbb{F}_c(\mathbb{R})$ be a (p,h)-convex fuzzy-interval-valued mapping (FIVM) on $[s,t]$, as well as ϕ-levels define the family of interval-valued mappings (IVMs) $\mathcal{F}_\phi : [s,t] \subset \mathbb{R} \to K_c^+$, satisfying that $\mathcal{F}_\phi(x,\phi) = [\mathcal{F}_*(x,\phi),\mathcal{F}^*(x,\phi)]$ for every $x \in [s,t]$, and for every $\phi \in [0,1]$. If $\mathcal{F} \in L([s,t],F_c(\mathbb{R}))$, then*

$$\frac{\mathcal{F}\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)}) \preccurlyeq \frac{p^\alpha\Gamma(\alpha+1)}{(t^p-s^p)^\alpha}(I^{p,\alpha}_{s^+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t^-}\mathcal{F}(s))) \preccurlyeq \alpha(\mathcal{F}(s)\tilde{+}\mathcal{F}(t))\int_0^1(h(\zeta)\tilde{+}h(1-\zeta))\zeta^{\alpha-1}d\zeta.$$

Proof. Let $\mathcal{F} : [s,t] \to \mathbb{F}_c(\mathbb{R})$ be a (p,h)-convex FIVM. Then, for $a,b \in [s,t]$, one has

$$\mathcal{F}\left([\alpha a^p + (1-\alpha)b^p]^{\frac{1}{p}}\right) \preccurlyeq h(\alpha)\mathcal{F}(a)\tilde{+}h(1-\alpha)\mathcal{F}(b).$$

If $\alpha = \frac{1}{2}$, then one has

$$\frac{\mathcal{F}\left([\frac{a^p+b^p}{2}]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \preccurlyeq \mathcal{F}(a)\tilde{+}\mathcal{F}(b).$$

Let $a^p = \zeta t^p + (1-\zeta)s^p$ and $y^p = (1-\zeta)t^p + \zeta s^p$. Then, by the above inequality, one has

$$\frac{\mathcal{F}\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \preccurlyeq \mathcal{F}\left([\zeta t^p + (1-\zeta)s^p]^{\frac{1}{p}}\right)\tilde{+}\mathcal{F}\left([\zeta s^p + (1-\zeta)t^p]^{\frac{1}{p}}\right).$$

Therefore, for every $\phi \in [0,1]$, one has

$$\frac{\mathcal{F}_*\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)} \leqslant \mathcal{F}_*\left([\zeta t^p + (1-\zeta)s^p]^{\frac{1}{p}},\phi\right) + \mathcal{F}_*\left([\zeta s^p + (1-\zeta)t^p]^{\frac{1}{p}},\phi\right),$$

$$\frac{\mathcal{F}^*\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)} \leqslant \mathcal{F}^*\left([\zeta t^p + (1-\zeta)s^p]^{\frac{1}{p}},\phi\right) + \mathcal{F}^*\left([\zeta s^p + (1-\zeta)t^p]^{\frac{1}{p}},\phi\right).$$

Multiplying both sides by $\zeta^{\alpha-1}$ and integrating the obtained result with respect to ζ over $(0,1)$, one has

$$\int_0^1 \zeta^{\alpha-1}\frac{\mathcal{F}_*\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)}d\zeta \leqslant \int_0^1 \zeta^{\alpha-1}\mathcal{F}_*\left([\zeta t^p + (1-\zeta)s^p]^{\frac{1}{p}},\phi\right)d\zeta + \int_0^1 \zeta^{\alpha-1}\mathcal{F}_*\left([\zeta s^p + (1-\zeta)t^p]^{\frac{1}{p}},\phi\right)d\zeta,$$

$$\int_0^1 \zeta^{\alpha-1}\frac{\mathcal{F}^*\left([\frac{t^p+s^p}{2}]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)}d\zeta \leqslant \int_0^1 \zeta^{\alpha-1}\mathcal{F}^*\left([\zeta t^p + (1-\zeta)s^p]^{\frac{1}{p}},\phi\right)d\zeta + \int_0^1 \zeta^{\alpha-1}\mathcal{F}^*\left([\zeta s^p + (1-\zeta)t^p]^{\frac{1}{p}},\phi\right)d\zeta.$$

Let $\zeta t^p + s^p - \zeta s^p = k^p$ and $\zeta s^p + t^p - \zeta t^p = k^p$. Then one has

$$\frac{\mathcal{F}_*\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}},\phi\right)}{\alpha h\left(\frac{1}{2}\right)} \leqslant \frac{p}{(t^p-s^p)^\alpha}\int_s^t \mathcal{F}_*(k,\phi)(k^p-s^p)^{\alpha-1}k^{p-1}dk + \frac{p}{(t^p-s^p)^\alpha}\int_s^t \mathcal{F}_*(k,\phi)(t^p-k^p)^{\alpha-1}k^{p-1}dk$$
$$= \frac{p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}_*(t,\phi) + I^{p,\alpha}_{t-}\mathcal{F}_*(s,\phi)\right).$$

Analogously, for $\mathcal{F}^*(x,\phi)$ one has

$$\frac{\mathcal{F}^*\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}},\phi\right)}{\alpha h\left(\frac{1}{2}\right)} \leqslant \frac{p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}^*(t,\phi) + I^{p,\alpha}_{t-}\mathcal{F}^*(s,\phi)\right).$$

That is,

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)}\left(\mathcal{F}_*\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}},\phi\right),\mathcal{F}^*\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}},\phi\right)\right) \leqslant_I \frac{p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}_*(t,\phi) + I^{p,\alpha}_{t-}\mathcal{F}_*(s,\phi), I^{p,\alpha}_{s+}\mathcal{F}^*(t,\phi) + I^{p,\alpha}_{t-}\mathcal{F}^*(s,\phi)\right).$$

Thus, one has

$$\frac{\mathcal{F}\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}}\right)}{\alpha h\left(\frac{1}{2}\right)} \preccurlyeq \frac{p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t-}\mathcal{F}(s)\right).$$

In a similar way as above, one gets

$$\frac{p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t-}\mathcal{F}(s)\right) \preccurlyeq \left(\mathcal{F}(s)\tilde{+}\mathcal{F}(t)\right)\int_0^1 (h(\zeta)\tilde{+}h(1-\zeta))\zeta^{\alpha-1}d\zeta.$$

Combining the left-and right-hand sides, one arrives at

$$\frac{\mathcal{F}\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \preccurlyeq \frac{p^\alpha\Gamma(\alpha+1)}{(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t-}\mathcal{F}(s)\right) \preccurlyeq \alpha\left(\mathcal{F}(s)\tilde{+}\mathcal{F}(t)\right)\int_0^1 (h(\zeta)\tilde{+}h(1-\zeta))\zeta^{\alpha-1}d\zeta.$$

□

By setting $h = \mathbb{I}$ in Theorem 3.1, one gets Theorem 5 of Khan et al. [18].

Corollary 3.1.

$$\mathcal{F}\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}}\right) \preccurlyeq \frac{p^\alpha\Gamma(\alpha+1)}{2(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t-}\mathcal{F}(s)\right) \preccurlyeq \frac{\mathcal{F}(s)\tilde{+}\mathcal{F}(t)}{2}.$$

Remark 3.1. By setting $h = \mathbb{I}$, $\mathcal{F}_*(x,\phi) = \mathcal{F}^*(x,\phi)$, and $\phi = 1$ in Theorem 3.1, one gets Theorem 2.1 of [40]. Also, the setting $h = \mathbb{I}$, $p = \phi = 1$, and $\mathcal{F}_*(x,\phi) = \mathcal{F}^*(x,\phi)$ yields a result reported in [33]. Moreover, the setting $\alpha = p = \phi = 1$ and $\mathcal{F}_*(x,\phi) = \mathcal{F}^*(x,\phi)$ gives the classical Hermite-Hadamard inequality [12].

Corollary 3.2. *The setting* $h(\alpha) = \alpha^2 - 1/2$ *provides the following new* (p,h)*-convex-FIVF inequality*

$$\mathcal{F}\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}}\right) \preccurlyeq \frac{3p^\alpha\Gamma(\alpha+1)}{4(t^p-s^p)^\alpha}\left(I^{p,\alpha}_{s+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t-}\mathcal{F}(s)\right) \preccurlyeq \frac{3\alpha\left(\mathcal{F}(s)+\mathcal{F}(t)\right)}{4}\left(\frac{7}{4\alpha} + \frac{\alpha}{2+3\alpha+\alpha^2}\right).$$

Example 3.1. *Let* p *be an odd number and* $s \in [0,1]$. *Let* $\mathcal{F} : [t,s] = [0,2] \to \mathbb{F}_c(\mathbb{R})$ *be a FIVF defined by*

$$\mathcal{F}(x)(\sigma) = \begin{cases} \dfrac{\sigma}{(2-x^{\frac{p}{2}})} & \text{if } \sigma \in [0, 2-x^{\frac{p}{2}}], \\ \dfrac{2(2-x^{\frac{p}{2}})-\sigma}{(2-x^{\frac{p}{2}})} & \text{if } \sigma \in (2-x^{\frac{p}{2}},\ 2(2-x^{\frac{p}{2}})], \\ 0 & \text{otherwise}. \end{cases}$$

Then, for each $\phi \in [0,1]$, *one has*

$$\mathcal{F}_\phi(x) = \left[\phi(2-x^{\frac{p}{2}}),\ (2-\phi)(2-x^{\frac{p}{2}})\right].$$

Since the end-point functions $\mathcal{F}_*(x,\phi) = \phi(2-x^{\frac{p}{2}})$ *and* $\mathcal{F}^*(x,\phi) = (2-\phi)(2-x^{\frac{p}{2}})$ *are* (p,s)*-convex functions for each* $\phi \in [0,1]$. *Thus, the function* $\mathcal{F}(x)$ *is a* (p,s)*-convex FIVF.*

Now, by setting $\alpha = 2,\ p = 1$, and $h(t) = t$, one gets

$$\frac{\mathcal{F}\left([\frac{s^p+t^p}{2}]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \preccurlyeq \frac{p^{\alpha}\Gamma(\alpha+1)}{(t^p-s^p)^{\alpha}}\left(I^{p,\alpha}_{s^+}\mathcal{F}(t)\tilde{+}I^{p,\alpha}_{t^-}\mathcal{F}(s)\right) \preccurlyeq \alpha\left(\mathcal{F}(s)\tilde{+}\mathcal{F}(t)\right)\int_0^1 (h(\zeta)\tilde{+}h(1-\zeta))\zeta^{\alpha-1}d\zeta,$$

$$\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) = 2\phi,$$

$$\frac{p^{\alpha}\Gamma(\alpha+1)}{(t^p-s^p)^{\alpha}}\left(I^{p,\alpha}_{s^+}\mathcal{F}_*(t)+I^{p,\alpha}_{t^-}\mathcal{F}_*(s)\right) = \frac{4}{3}(3-\sqrt{2})\phi,$$

and

$$\alpha\left(\mathcal{F}_*(s)+\mathcal{F}_*(t)\right)\int_0^1 (h(\zeta)+h(1-\zeta))\zeta^{\alpha-1}d\zeta = 4\phi.$$

Therefore,

$$2\phi \leqslant \frac{4}{3}(3-\sqrt{2})\phi \leqslant 4\phi.$$

Now, we compute the upper end-point function as follows

$$\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) = 2(2-\phi),$$

$$\frac{p^{\alpha}\Gamma(\alpha+1)}{(t^p-s^p)^{\alpha}}\left(I^{p,\alpha}_{s^+}\mathcal{F}^*(t)+I^{p,\alpha}_{t^-}\mathcal{F}^*(s)\right) = \frac{4}{3}(\sqrt{2}-3)(\phi-2)\phi,$$

$$\alpha\left(\mathcal{F}^*(s)+\mathcal{F}^*(t)\right)\int_0^1 (h(\zeta)+h(1-\zeta))\zeta^{\alpha-1}d\zeta = 4(2-\phi).$$

From this we get

$$2(2-\phi) \leqslant \frac{4}{3}(\sqrt{2}-3)(\phi-2) \leqslant 4(2-\phi).$$

Thus,

$$\left[2\phi, 2(2-\phi)\right] \leqslant_p \left[\frac{4}{3}\phi(3-\sqrt{2}), \frac{4}{3}(\sqrt{2}-3)(\phi-2)\right] \leqslant_p \left[4\phi, 4(2-\phi)\right].$$

Theorem 3.2. *Let $\mathcal{F} : [s^p, t^p] \to \mathbb{F}_c(\mathbb{R})$ be a (p,h)-convex FIVM on $[s,t]$, as well as ϕ-levels define the family of IVMs $\mathcal{F}_\phi : [s,t] \subset \mathbb{R} \to K_c^+$, satisfying that $\mathcal{F}_\phi(x,\phi) = [\mathcal{F}_*(x,\phi), \mathcal{F}^*(x,\phi)]$ for every $x \in [s,t]$ and for every $\phi \in [0,1]$. If $\mathcal{F} \in L([s,t], F_c(\mathbb{R}))$, then*

$$\frac{\mathcal{F}\left([t^p+s^p]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \preccurlyeq \frac{2^{\alpha}p^{\alpha}\Gamma(\alpha+1)}{(t^p-s^p)^{\alpha}}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\tilde{+}\mathcal{I}_{s^+}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\right)$$

$$\preccurlyeq \alpha\left(\mathcal{F}(s)\tilde{+}\mathcal{F}(t)\right)\int_0^1 h\left(\frac{1-\zeta}{2}\right)\zeta^{\alpha-1}d\zeta\ \tilde{+}\alpha\left(\mathcal{F}(t)\tilde{+}\mathcal{F}(s)\right)\int_0^1 h\left(\frac{1+\zeta}{2}\right)\zeta^{\alpha-1}d\zeta.$$

Proof. Since $\mathcal{F}$ is a(p,h)-convex FIVF, one has

$$\mathcal{F}\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{\frac{1}{p}}\right) \preccurlyeq h(\alpha)\mathcal{F}(x)\tilde{+}h(1-\alpha)\mathcal{F}(y).$$

By setting $\alpha = \frac{1}{2},\ x^p = \frac{1-\zeta}{2}s^p + \frac{1+\zeta}{2}t^p$, and $y^p = \frac{1+\zeta}{2}s^p + \frac{1-\zeta}{2}t^p$, one gets

$$\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right) \preccurlyeq h\left(\frac{1}{2}\right)\left(\mathcal{F}\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}}\right)\tilde{+}\mathcal{F}\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}}\right)\right).$$

Therefore, for every $\phi \in [0,1]$, one has

$$\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)} \leqslant \mathcal{F}_*\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right)+\mathcal{F}_*\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right),$$

and

$$\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{h\left(\frac{1}{2}\right)} \leqslant \mathcal{F}^*\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}}\right)+\mathcal{F}^*\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}}\right).$$

Multiplying both sides by $\zeta^{\alpha-1}$ and integrating the obtained result with respect to ζ, one gets

$$\int_0^1 \zeta^{\alpha-1}\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)}d\zeta \leqslant \int_0^1 \zeta^{\alpha-1}\mathcal{F}_*\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right)d\zeta+\int_0^1 \zeta^{\alpha-1}\mathcal{F}_*\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right)d\zeta,$$

and

$$\int_0^1 \zeta^{\alpha-1}\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{h\left(\frac{1}{2}\right)}d\zeta \leqslant \int_0^1 \zeta^{\alpha-1}\mathcal{F}^*\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right)d\zeta+\int_0^1 \zeta^{\alpha-1}\mathcal{F}^*\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}},\phi\right)d\zeta.$$

Considering the following substitutions $x^p=\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p$ and $y^p=\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p$, one has

$$\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{\alpha h\left(\frac{1}{2}\right)} \leqslant \frac{p2^\alpha}{(t^p-s^p)^\alpha}\int_{\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}}^{t}\left(k^p-\frac{t^p+s^p}{2}\right)^{\alpha-1}\mathcal{F}_*(k,\phi)k^{p-1}dk$$
$$+\frac{p2^\alpha}{(t^p-s^p)^\alpha}\int_s^{\left(\frac{t^p+s^p}{2}\right)^{\frac{1}{p}}}\left(\frac{t^p+s^p}{2}-k^p\right)^{\alpha-1}k^{p-1}\mathcal{F}_*(k,\phi)dk.$$

Identifying in terms of Katugampola integrals, one gets

$$\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{\alpha h\left(\frac{1}{2}\right)} \leqslant \frac{2^\alpha p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}_*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)+\mathcal{I}_{s^+}\mathcal{F}_*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)\right).$$

Similarly, for the upper end-point function, one has

$$\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{\alpha h\left(\frac{1}{2}\right)} \leqslant \frac{2^\alpha p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}^*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)+\mathcal{I}_{s^+}\mathcal{F}^*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)\right).$$

That is, one has the following

$$\frac{1}{\alpha h\left(\frac{1}{2}\right)}\left(\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right),\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)\right)$$
$$\leqslant_l \frac{2^\alpha\Gamma(\alpha)p^\alpha}{(t^p-s^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}_*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)+\mathcal{I}_{s^+}\mathcal{F}_*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)\right.$$
$$\left.,\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}^*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)+\mathcal{I}_{s^+}\mathcal{F}^*\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}},\phi\right)\right).$$

Theefore,

$$\frac{\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)}{\alpha h\left(\frac{1}{2}\right)} \preccurlyeq \frac{2^\alpha p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\tilde{+}\mathcal{I}_{s^+}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\right).$$

Thus, we obtain the left-hand side inequality. Focusing on to the right-hand side, we proceed by using the (p,h)-FIVF convexity on the right-hand side

$$\mathcal{F}\left(\left[\frac{1-\zeta}{2}s^p+\frac{1+\zeta}{2}t^p\right]^{\frac{1}{p}}\right)\tilde{+}\mathcal{F}\left(\left[\frac{1+\zeta}{2}s^p+\frac{1-\zeta}{2}t^p\right]^{\frac{1}{p}}\right).$$

Proceeding using the technique similar to the one shown above, we obtain the second inequality, namely

$$\frac{2^\alpha p^\alpha\Gamma(\alpha)}{(t^p-s^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\tilde{+}\mathcal{I}_{s^+}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\right)$$
$$\preccurlyeq (\mathcal{F}(s)\tilde{+}\mathcal{F}(t))\int_0^1 h\left(\frac{1-\zeta}{2}\right)\zeta^{\alpha-1}d\zeta\ \tilde{+}\ (\mathcal{F}(t)\tilde{+}\mathcal{F}(s))\int_0^1 h\left(\frac{1+\zeta}{2}\right)\zeta^{\alpha-1}d\zeta.$$

Combining the left-and right-hand sides, we obtain the desired inequality. □

Corollary 3.3. *By setting h to be equal to the identity mapping, one obtains the following inequality*

$$\mathcal{F}\left([t^p+s^p]^{\frac{1}{p}}\right) \preccurlyeq \frac{2^{\alpha-1}p^{\alpha}\Gamma(\alpha+1)}{(t^p-s^p)^{\alpha}}\left(\mathcal{I}^{p,\alpha}_{t^-}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\tilde{+}\mathcal{I}_{s^+}\mathcal{F}\left(\left(\frac{s^p+t^p}{2}\right)^{\frac{1}{p}}\right)\right)$$
$$\preccurlyeq \frac{\left(\mathcal{F}(s)\tilde{+}\mathcal{F}(t)\right)}{4\alpha(\alpha+1)}\tilde{+}\frac{\left(\mathcal{F}(t)\tilde{+}\mathcal{F}(s)\right)(2\alpha+1)}{2\alpha(\alpha+1)}.$$

Next, we present a generalized Hermite-Hadamard-Fejer inequality for the convex FIVF.

Theorem 3.3. *Let $\mathcal{F}:[a^p,b^p]\to\mathbb{F}_c(\mathbb{R})$ be a (p,h)-convex FIVM on $[s,t]$, as well as ϕ -levels define the family of IVMs $\mathcal{F}_\phi:[a,b]\subset\mathbb{R}\to K_c^+$, satisfying that $\mathcal{F}_\phi(x,\phi)=[\mathcal{F}_*(x,\phi),\mathcal{F}^*(x,\phi)]$ for every $x\in[s,t]$ and for every $\phi\in[0,1]$. If $\mathcal{F}\in L([a,b],F_c(\mathbb{R}))$ and $\mathcal{C}:[a,b]\to\mathbb{R},\mathcal{C}\geqslant 0$, be a p-symmetric function with respect to*

$$\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},$$

then

$$\frac{\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right)}{2h(\frac{1}{2})} \preccurlyeq \mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}(b)\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{F}(a)\mathcal{C}(a).$$

Proof. Since $\mathcal{F}$ is a (p,h)-convex FIVF, one has

$$\mathcal{F}\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{\frac{1}{p}}\right)\preccurlyeq h(\alpha)\mathcal{F}(x)\tilde{+}h(1-\alpha)\mathcal{F}(y).$$

By setting $\alpha=\frac{1}{2},\ x^p=ta^p+(1-t)b^p$, and $y^p=tb^p+(1-t)a^p$, one gets

$$\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\preccurlyeq h\left(\frac{1}{2}\right)\left(\mathcal{F}\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}}\right)\tilde{+}\mathcal{F}\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}}\right)\right).$$

Therefore, for every $\phi\in[0,1]$ one has

$$\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{h(\frac{1}{2})}\leqslant \mathcal{F}_*\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\phi\right)+\mathcal{F}_*\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}},\phi\right)$$

and

$$\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{h(\frac{1}{2})}\leqslant \mathcal{F}^*\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\phi\right)+\mathcal{F}^*\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}},\phi\right).$$

Multiplying both sides of the inequality with

$$t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)$$

and integrating with respect to t from 0 to 1, one arrives at

$$\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}},\phi\right)}{h(\frac{1}{2})}dt$$
$$\leqslant \int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}_*\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\phi\right)dt$$
$$+\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}_*\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}},\phi\right)dt,$$

and

$$\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{h(\frac{1}{2})}dt$$

$$\leqslant \int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}^*\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\phi\right)dt$$

$$+\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}^*\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}},\phi\right)dt.$$

By using the fact that $\mathcal{C}$ is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}$, one concludes that the left-hand side is equal to

$$\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{h(\frac{1}{2})}dt,$$

which is equal to

$$\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{2h(\frac{1}{2})}\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left(\mathcal{I}_{a^+}^{p,\alpha}\mathcal{C}(b)+\mathcal{I}_{b^-}^{p,\alpha}\mathcal{C}(a)\right).$$

Now, we focus on the right-hand side. By utilizing the substitution $(1-t)a^p+b^pt=k^p$ in the first integral on the right-hand side, we obtain

$$\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}^*\left(\left[ta^p+(1-t)b^p\right]^{\frac{1}{p}},\phi\right)dt$$

$$=\frac{p}{(b^p-a^p)^\alpha}\int_a^b (k^p-a^p)^{\alpha-1}\mathcal{C}(k)\mathcal{F}^*\left(\left[a^p+b^p-k^p\right]^{\frac{1}{p}}\right)k^{p-1}dk.$$

By using the substitution $a^p+b^p-k^p=u^p$ and the fact that $\mathcal{C}$ is p-symmetric, we arrive at

$$\frac{p}{(b^p-a^p)^\alpha}\int_a^b (k^p-a^p)^{\alpha-1}\mathcal{C}(k)\mathcal{F}^*\left(\left[a^p+b^p-k^p\right]^{\frac{1}{p}}\right)k^{p-1}dk.$$

$$=\frac{p}{b^p-a^p}\int_a^b (b^p-u^p)^{\alpha-1}\mathcal{C}(u)\mathcal{F}^*(u)du=\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\mathcal{I}_{a^+}^{p,\alpha}\mathcal{F}^*(b)\mathcal{C}(b).$$

The second integral from the right-hand side follows similarly by introducing the same substitution. Namely, we obtain

$$\int_0^1 t^{\alpha-1}\mathcal{C}\left(\left[(1-t)a^p+tb^p\right]^{\frac{1}{p}}\right)\mathcal{F}^*\left(\left[tb^p+(1-t)a^p\right]^{\frac{1}{p}},\phi\right)dt=\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}I_{b^-}^{\alpha,p}\mathcal{F}^*(a)\mathcal{C}(a).$$

Combining the left-and right-hand sides, we get

$$\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{2h(\frac{1}{2})}\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left(\mathcal{I}_{a^+}^{p,\alpha}\mathcal{C}(b)+\mathcal{I}_{b^-}^{p,\alpha}\mathcal{C}(a)\right)$$

$$\leqslant \frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}I_{b^-}^{\alpha,p}\mathcal{F}^*(a)\mathcal{C}(a)+\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\mathcal{I}_{a^+}^{p,\alpha}\mathcal{F}^*(b)\mathcal{C}(b).$$

In a similar manner, we obtain the inequality with the lower-end function

$$\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{2h(\frac{1}{2})}\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left(\mathcal{I}_{a^+}^{p,\alpha}\mathcal{C}(b)+\mathcal{I}_{b^-}^{p,\alpha}\mathcal{C}(a)\right)$$

$$\leqslant \frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}I_{b^-}^{\alpha,p}\mathcal{F}_*(a)\mathcal{C}(a)+\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\mathcal{I}_{a^+}^{p,\alpha}\mathcal{F}_*(b)\mathcal{C}(b).$$

From the last inequality, one gets

$$\left(\frac{\mathcal{F}^*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{2h(\frac{1}{2})}\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)+\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right),\frac{\mathcal{F}_*\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\phi\right)}{2h(\frac{1}{2})}\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)+\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right)\right)$$

$$\leqslant_l\left(\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}I^{\alpha,p}_{b^-}\mathcal{F}^*(a)\mathcal{C}(a)+\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}^*(b)\mathcal{C}(b),\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}I^{\alpha,p}_{b^-}\mathcal{F}_*(a)\mathcal{C}(a)+\frac{p^\alpha\Gamma(\alpha)}{(b^p-a^p)^\alpha}\mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}_*(b)\mathcal{C}(b)\right).$$

Finally, we obtain the required inequality

$$\frac{\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right)}{2h(\frac{1}{2})}\preccurlyeq\mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}(b)\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{F}(a)\mathcal{C}(a).$$

Corollary 3.4. *By setting $h\equiv\mathbb{I}$ in Theorem 3.3, one gets Theorem 6 of Khan et al. [18]:*

$$\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right)\preccurlyeq\mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}(b)\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{F}(a)\mathcal{C}(a).$$

Corollary 3.5. *The setting $h(t)=t^2-\frac{1}{6}$ in Theorem 3.3 yields the following new fractional (p,h)-FIVF inequality:*

$$\mathcal{F}\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}\right)\left(\mathcal{I}^{p,\alpha}_{a^+}\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{C}(a)\right)\preccurlyeq\frac{\mathcal{I}^{p,\alpha}_{a^+}\mathcal{F}(b)\mathcal{C}(b)\tilde{+}\mathcal{I}^{p,\alpha}_{b^-}\mathcal{F}(a)\mathcal{C}(a)}{6}.$$

□

4. Conclusion

The recently defined (p,h)-convex FIVFs are investigated in this paper. The topic of FIVFs is interesting because of its application to numerical integration and probability density functions. The results obtained by Khan et al. [18] are generalized in the setting of (p,h)-fuzzy-convex functions. A new Hermite-Hadamard-type inequality as well as a Hermite-Hadamard-Fejer-type inequality are obtained. Questions arise about whether generalizations of the convex-fractional inequalities obtained in this paper are obtainable.

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