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Review Article

Galois and Pataki connections for ordinary functions and super relations

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Abstract

A subset R of a product set $X \times Y$ is called a relation on X to Y. A relation U on the power set $\mathcal{P}(X)$ to Y is called a super relation on X to Y. The relation R can be identified, to some extent, with the set-valued function φ_R defined by $\varphi_R(x) = R(x) = \{ y \in Y : (x, y) \in R \}$ for all $x \in X$, and the union-preserving super relation $R^{\triangleright}$ defined by $R^{\triangleright}(A) = R[A] = \bigcup_{a \in A} R(a)$ for all $A \subseteq X$. By using the relation R, we also define two super relations lb_R and cl_R on Y to X such that $\mathrm{lb}_R(B) = \{ x \in X : \{x\} \times B \subseteq R \}$ and $\mathrm{cl}_R(B) = \{ x \in X : R(x) \cap B \neq \emptyset \}$ for all $B \subseteq X$. By using complement and inverse relations, we prove that $\mathrm{lb}_R = \mathrm{cl}^c_{R^c}$ and $\mathrm{cl}_R(B) = R^{-1}[B]$. We also consider the dual super relations $\mathrm{ub}_R = \mathrm{lb}_{R^{-1}}$ and $\mathrm{int}_R = \mathrm{cl}^c_R \circ \mathcal{C}_Y$. If U is a super relation on X to Y and V is a super relation on Y to X, then having in mind Galois connections and residuated mappings, we say that U is V–normal if, for all $A \subseteq X$ and $B \subseteq Y$, we have $U(A) \subseteq B$ if and only if $A \subseteq V(B)$. Thus, if U is V–normal, then by defining $\Phi = V \circ U$ and following Pataki's ideas, we see that U is Φ–regular in the sense that, for all $A_1, A_2 \subseteq X$, we have $U(A_1) \subseteq U(A_2)$ if and only if $A_1 \subseteq \Phi(A_2)$. In this paper, by considering a relator (family of relations) $\mathcal{R}$ on X to Y, we investigate normality properties of the more general super relations $\mathrm{lb}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathrm{lb}_R$ and $\mathrm{cl}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathrm{cl}_R$, and their duals $\mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^{-1}}$ and $\mathrm{int}_{\mathcal{R}} = \mathrm{cl}^c_{\mathcal{R}} \circ \mathcal{C}_Y$. However, as some applicable results of the paper, we only prove that if R is a relation on X to Y, then the following assertions hold: (1) $\mathrm{cl}_{R^{-1}}$ is int_R–normal, or equivalently cl_R is $\mathrm{int}_{R^{-1}}$–normal; (2) ub^c_R is $\mathrm{lb}_R \circ \mathcal{C}_Y$–normal, or equivalently lb^c_R is $\mathrm{ub}_R \circ \mathcal{C}_X$–normal; (3) R is a function of X to Y if and only if $\mathrm{cl}_{R^{-1}}$ is cl_R–normal, or equivalently int_R is $\mathrm{int}_{R^{-1}}$–normal. The closure-interior and the upper-lower-bound Galois connections, established in assertions (1) and (2), are applied in the calculus of relations and the completion of posets, respectively. Some of the implications in assertion (3) require that $Y \neq \emptyset$.

Keywords: relations; relators; closures; Galois connections.

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1. Introduction

An important particular case of "Galois connections" was already considered by Birkhoff in the first edition of his famous book "Lattice Theory", under the name "polarities" [3, p. 122]. More concretely, Birkhoff offered the following construction: "Let ρ be any binary relation between the members of two classes I and J. For any subsets $X \subset I$ and $Y \subset J$, define $X^* \subset J$ (the "polar" of X) as the set of all $y \in J$ such that $x \rho y$ for all $x \in X$, and we define $Y^\dagger \subset I$ (the "polar" of Y) as the set of all $x \in I$ such that $x \rho y$ for all $y \in Y$." Thus, he established the following basic properties of the operations $*$ and $\dagger$:

(a) $X \subset X_1$ implies $X^* \supset X_1^*$, (b) $Y \subset Y_1$ implies $Y^\dagger \supset Y_1^\dagger$, (c) $X \subset (X^*)^\dagger$ and $Y \subset (Y^\dagger)^*$.

Moreover, he derived some consequences of properties (a)–(c). In particular, he studied the particular case when $I = J$ and ρ is symmetric, and he listed several remarkable illustrating examples for polarities. These already well indicated that polarities can be applied in a great variety of mathematical theories.

The above observations of Birkhoff were extended to posets (partially ordered sets) by Ore [39], who having in mind the classical Galois theory of algebraic equations, introduced the terms "Galois correspondences" and "Galois connexions". More concretely, Ore offered the following definition: "Let P and Q denote two partially ordered sets. We shall assume that there exists a correspondence from P to Q, $p \to \mathfrak{D}(p)$, and also a correspondence from Q to P, $q \to \mathfrak{P}(q)$. These two correspondences $\mathfrak{D}$ and $\mathfrak{P}$ together shall be called a Galois correspondence between P and Q provided that the two conditions given at the start of the the next page are fulfilled.

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(α) When $p_1 \supset p_2$ are two elements in P or $q_1 \supset q_2$ are two elements in Q then

$$\mathfrak{D}(p_1) \subseteq \mathfrak{D}(p_2), \qquad \mathfrak{P}(q_1) \subseteq \mathfrak{P}(q_2).$$

(β) For any element p in P or q in Q

$$\mathfrak{P}\mathfrak{D}(p) \supseteq p, \qquad \mathfrak{D}\mathfrak{P}(q) \supseteq q.$$

We shall also say that there exists a Galois connexion between P and Q when a pair of Galois correspondences $\mathfrak{D}$ and $\mathfrak{P}$ has been defined."

The next important step in the theory of Galois connections was made by Schmidt [54]. Despite his awareness of some former papers of Everett [21], Riguet [50], and Pickert [46] too, he was mainly interested in the original setting of Birkhoff. By considering a relation R between two sets E_1 and E_2, he defined and investigated three important set-functions defined as follows, for any $M_1 \subseteq E_1$,

$$\begin{aligned} R\,(M_1) &= \{x_2: \quad \exists\ x_1 \in M_1: \quad x_1 R x_2\}, \\ R\,[M_1] &= \{x_2: \quad \forall\ x_1 \in M_1: \quad x_1 R x_2\}, \\ R\,)M_1(&= \{x_2: \quad \forall\ x_1 R x_2: \quad x_1 \in M_1\}. \end{aligned}$$

Moreover, having in mind these set-functions generated by the relation R, he assumed that ω_1 is an arbitrary function of $\mathfrak{P}_1 = \mathcal{P}(E_1)$ to $\mathfrak{P}_2 = \mathcal{P}(E_2)$, and defined an associated set-function ω_2 such that

$$\omega_2 M_2 = \max\{M_1: \quad M_2 \subseteq \omega_1 M_1\}$$

for all $M_2 \in \mathfrak{P}_2$ provided that the above maximum exists. The investigations of the relationships between these two functions, led him to the ingenious observation that properties (a)–(c) established by Birkhoff can be replaced by the single requirement that

$$M_2 \subseteq \omega_1 M_1 \quad \Longleftrightarrow \quad M_1 \subseteq \omega_2 M_2$$

for all $M_1 \in \mathfrak{P}_1$ and $M_2 \in \mathfrak{P}_2$. In addition, he also considered modifications and specializations of the above equivalence. For instance, he also investigated the more natural requirement that

$$\omega_1 M_1 \subseteq M_2 \quad \Longleftrightarrow \quad M_1 \subseteq \omega_2 M_2$$

for all $M_1 \in \mathfrak{P}_1$ and $M_2 \in \mathfrak{P}_2$.

This shows that if f is a function of one goset X to another Y and g is a function of Y to X such that

$$f(x) \leq y \quad \Longleftrightarrow \quad x \leq g(y)$$

for all $x \in X$ and $y \in Y$, then one may naturally say that the the pair (f, g) is a Galois connection between X and Y [13, p. 155]. Curiously enough, in [24, p. 18] and [31], the increasingness of the corresponding functions was also postulated. Namely, this is usually a consequence of the above equivalence.

Now, if f and g are as above, then by defining $\varphi = g \circ f$ and having in mind Pataki's ideas [42], we at once see that

$$f(u) \leq f(v) \iff u \leq g\big(f(v)\big) \iff u \leq (g \circ f)(v) \iff u \leq \varphi(v)$$

for all $u, v \in X$. This shows that before Galois connections, it is more convenient to investigate first another, more simple connection which usually lies between closure operations and Galois connections. Thus, if φ is function of the goset X to itself such that

$$f(u) \leq f(v) \iff u \leq \varphi(v)$$

for all $u, v \in X$, then the pair (f, φ) was called a Pataki connection by the third author [73]. Namely, if $\mathfrak{F}$ is a structure (set-valued function) and $\square$ is a unary operation for relators (families of relations) on X, then Pataki [42] called the function $\mathfrak{F}$ to be $\square$–increasing if, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X, we have

$$\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathcal{S} \subseteq \mathcal{R}^{\square}.$$

Several particular cases of the latter connection were formerly also considered by the third author [62]. Moreover, he also determined the Galois adjoint of some particular structures for relators [72].

For an easy illustration of the above situation, we note that if $\mathcal{R}$ is a relator on X, then for any $A \subseteq X$, we may naturally define

$$\operatorname{Int}_{\mathcal{R}}(A) = \{B \subseteq X: \ \exists\ R \in \mathcal{R}: \quad R[B] \subseteq A\}$$

and $\operatorname{int}_{\mathcal{R}}(A)=\{x\in X: \{x\}\in \operatorname{Int}_{\mathcal{R}}(A)\}$.

We may also naturally define $\tau_{\mathcal{R}}=\{A\subseteq X: A\in \operatorname{Int}_{\mathcal{R}}(A)\}$,

$$\mathcal{T}_{\mathcal{R}}=\{A\subseteq X: A\subseteq \operatorname{int}_{\mathcal{R}}(A)\} \qquad \text{and} \qquad \mathcal{E}_{\mathcal{R}}=\{A\subseteq X: \operatorname{int}_{\mathcal{R}}(A)\neq\emptyset\}.$$

Only the most widely used increasing structure $\mathcal{T}$ fails to be $\square$–increasing for some operation $\square$. However, Mala [34, 36] could still find a projection operation $\lozenge$ for relators such that for any two nonvoid relators $\mathcal{R}$ and $\mathcal{S}$ on X, we could have

$$\mathcal{T}_{\mathcal{R}}\subseteq \mathcal{T}_{\mathcal{S}} \iff \mathcal{R}^{\lozenge}\subseteq \mathcal{S}^{\lozenge}.$$

In the sequel, if (f, g) is a Galois connection, then following a more convenient terminology of the third author [75], we say that f is g–normal. While, if (f,φ) is a Pataki connection, then we say that f is φ–regular. Thus, if X and Y are preordered sets, then we prove that φ is a closure operation if and only if φ is increasingly φ–regular, or equivalently there is a function h of X to Y which is increasingly φ–regular.

Beside the normality of a function f of one goset X to another Y, we also investigate the normality of a super relation U on one set X to another Y. That is, an ordinary relation U on $\mathcal{P}(X)$ to Y which can be identified with a function φ_U of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ defined by $\varphi_U(A)=U(A)$ for all $A\subseteq X$. In particular, we investigate the normality of the basic structures $\operatorname{cl}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\operatorname{lb}_{\mathcal{R}}$, $\operatorname{ub}_{\mathcal{R}}$ derived from a relator $\mathcal{R}$ on one set X to another Y. However, these structures are not independent of each other since, by [69], we have $cl_{\mathcal{R}}=\operatorname{int}^c_{\mathcal{R}}\circ\mathcal{C}_Y$ and $\operatorname{lb}_{\mathcal{R}}=\operatorname{ub}_{\mathcal{R}^{-1}}=\operatorname{cl}^c_{\mathcal{R}^c}$.

Now, as some applicable particular cases of our results, we only prove that if R is a relation on X to Y, then the following assertions hold:

(1) $\operatorname{cl}_{R^{-1}}$ is int_R–normal, or equivalently cl_R is $\operatorname{int}_{R^{-1}}$–normal;

(2) ub^c_R is $\operatorname{lb}_R\circ\mathcal{C}_Y$–normal, or equivalently lb^c_R is $\operatorname{ub}_R\circ\mathcal{C}_X$–normal;

(3) R is a function of X to Y if and only if $\operatorname{cl}_{R^{-1}}$ is cl_R–normal, or equivalently int_R is $\operatorname{int}_{R^{-1}}$–normal.

Some of the implications in assertion (3) require the reasonable assumption that $Y\neq\emptyset$.

By the corresponding definitions, the first part of assertion (1) means only that

$$\operatorname{cl}_{R^{-1}}(A)\subseteq B \iff A\subseteq \operatorname{int}_R(B)$$

for all $A\subseteq X$ and $B\subseteq Y$. While, the first part of assertion (2) means only that $\operatorname{ub}^c_R(A)\subseteq B \iff A\subseteq (\operatorname{lb}_R\circ\mathcal{C}_Y)(B)$ for all $A\subseteq X$ and $B\subseteq Y$, or equivalently

$$B\subseteq \operatorname{ub}_R(A) \iff A\subseteq \operatorname{lb}_R(B)$$

for all $A\subseteq X$ and $B\subseteq Y$. The above closure-interior, and the upper-lower-bound Galois connections, which can be reformulated in relational forms by using the equalities

$$\operatorname{cl}_{R^{-1}}(A)=R[A], \qquad \operatorname{ub}_R(A)=R^c[A]^c \qquad \text{and} \qquad \operatorname{int}_R(B)=\operatorname{cl}_R(B^c)^c, \qquad \operatorname{lb}_R(B)=\operatorname{ub}_{R^{-1}}(B),$$

have important applications in the calculus of relations [84] and the completion of posets [13], respectively.

2. A few basic facts on relations

A subset F of a product set $X\times Y$ is called a *relation on* X *to* Y. In particular, a relation R on X to itself is called a *relation on* X. And, $\Delta_X=\{(x,x): x\in X\}$ is called the *identity relation of* X. If F is a relation on X to Y, then for any $x\in X$ and $A\subseteq X$ the sets $F(x)=\{y\in Y: (x,y)\in F\}$ and $F[A]=\bigcup_{a\in A}F(a)$ are called the *images* or *neighbourhoods* of x and A under F, respectively. If $(x,y)\in F$, then instead of $y\in F(x)$, we may also write $x\,F\,y$. However, instead of $F[A]$ we cannot write $F(A)$. Namely, it may occur that, in addition to $A\subseteq X$, we also have $A\in X$. Now, the sets $D_F=\{x\in X: F(x)\neq\emptyset\}$ and $F[X]$ may be called the *domain* and *range* of F, respectively. And, if $D_F=X$, then we may say that F is a *relation of* X *to* Y, or that F is a *non-partial relation on* X *to* Y.

If F is a relation on X to Y and $E\subseteq D_F$, then the relation $F|E=F\cap(E\times Y)$ is called the *restriction* of F to E. While, if F and G are relations on X to Y such that $D_F\subseteq D_G$ and $F=G|D_F$, then G is called an *extension* of F.

In particular, a relation f on X to Y is called a *function* if for each $x\in D_f$ there exists $y\in Y$ such that $f(x)=\{y\}$. In this case, by identifying singletons with their elements, we may simply write $f(x)=y$ instead of $f(x)=\{y\}$. Moreover, a function $\star$ of X to itself is called a *unary operation on* X. While, a function $*$ of X^2 to X is called a *binary operation on* X. Also, for any $x, y\in X$, we usually write $x^\star$ and $x*y$ instead of $\star(x)$ and $*(x,y)$, respectively.

If F is a relation on X to Y, then a function f of D_F to Y is called a *selection function of* F if $f(x) \in F(x)$ for all $x \in D_F$. Thus, by the Axiom of Choice [16], we see that every relation is the union of its selection functions.

For a relation F on X to Y, we may naturally define two *set-valued functions* φ_F of X to $\mathcal{P}(Y)$ and Φ_F of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ such that $\varphi_F(x) = F(x)$ for all $x \in X$ and $\Phi_F(A) = F[A]$ for all $A \subseteq X$.

Functions of X to $\mathcal{P}(Y)$ can be naturally identified with relations on X to Y. While, functions of $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ are more powerful objects than relations on X to Y. In [78,85,86], they were briefly called *corelations on* X *to* Y.

However, if U is a relation on $\mathcal{P}(X)$ to Y and V is a relation on $\mathcal{P}(X)$ to $\mathcal{P}(Y)$, then it is better to say that U is a *super relation* and V is a *hyper relation on* X *to* Y [49,90]. Thus, closures (proximities) [93] are super (hyper) relations.

For a relation F on X to Y, the relation, $F^c = (X \times Y) \setminus F$ is called the *complement* of F. Thus, it can be shown that $F^c(x) = F(x)^c = Y \setminus F(x)$ for all $x \in X$, and $F^c[A]^c = \bigcap_{a \in A} F(a)$ for all $A \subseteq X$.

Moreover, the relation $F^{-1} = \{(y, x) \in Y \times X : \ (x, y) \in F\}$ is called the *inverse* of F. Thus, it can be shown that $F^{-1}(y) = \{x \in X : \ y \in F(x)\}$ for all $y \in Y$, and $F^{-1}[B] = \{x \in X : \ F(x) \cap B \neq \emptyset\}$ for all $B \subseteq Y$.

If F is a relation on X to Y, then we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the values $F(x)$, where $x \in X$, uniquely determine F. Thus, a relation F on X to Y can also be naturally defined by specifying $F(x)$ for all $x \in X$. For instance, if G is a relation on Y to Z, then the *composition relation* $G \circ F$ can be defined such that $(G \circ F)(x) = G[F(x)]$ for all $x \in X$. Thus, it can be shown that $(G \circ F)[A] = G\big[F[A]\big]$ for all $A \subseteq X$. While, if G is a relation on Z to W, then the *box product* $F \boxtimes G$ can be defined such that $(F \boxtimes G)(x, z) = F(x) \times G(z)$ for all $x \in X$ and $z \in Z$. Thus, it can be shown that $(F \boxtimes G)[A] = G \circ A \circ F^{-1}$ for all $A \subseteq X \times Z$ [77]. Hence, by taking $A = \{(x, z)\}$, and $A = \Delta_Y$ if $Y = Z$, one can at once see that the box and composition products are actually equivalent tools. However, the box product can be immediately defined for any family of relations.

The above unary operations c and -1 are inversion compatible in the sense that $\left(F^c\right)^{-1} = \left(F^{-1}\right)^c$. Moreover, concerning the above binary operations $\circ$ and $\boxtimes$, we prove that $(G \circ F)^{-1} = F^{-1} \circ G^{-1}$ and $(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}$.

3. Some important relational properties

Now, a relation R on X may be briefly defined to be *reflexive* if $\Delta_X \subseteq R$, and *transitive* if $R \circ R \subseteq R$. Moreover, R may be briefly defined to be *symmetric* if $R \subseteq R^{-1}$, *antisymmetric* if $R \cap R^{-1} \subseteq \Delta_X$, and *total* if $X^2 \subseteq R \cup R^{-1}$.

In addition to the above well-known, basic properties, several further remarkable relational properties were also studied in [64] with the help of the self closure and interior relations $R^- = R^{-1} \circ R$ and $R^\circ = R^{c-c} = \left(R^{-1} \circ R^c\right)^c$.

In the sequel, as it is usual, a reflexive and transitive (symmetric) relation will be called a *preorder (tolerance) relation*. And, a symmetric (antisymmetric) preorder relation will be called an *equivalence (partial order) relation*.

For a relation R on X, we may now also naturally define $R^0 = \Delta_X$, and $R^n = R \circ R^{n-1}$ if $n \in \mathbb{N}$. Moreover, we may also define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder relation on X containing R [25].

Now, in contrast to $(R^c)^c = R$ and $(R^{-1})^{-1} = R$, we have $(R^\infty)^\infty = R^\infty$. Moreover, analogously to $(R^c)^{-1} = (R^{-1})^c$, we also have $(R^\infty)^{-1} = (R^{-1})^\infty$. Thus, in particular R^{-1} is also a preorder on X if R is a preorder on X.

For $A \subseteq X$, the *Pervin relation* $R_A = A^2 \cup (A^c \times X)$ is an important preorder on X [45]. While, for a *pseudometric* d on X, the *Weil surrounding* $B_r^d = \{(x, y) \in X^2 : \ d(x, y) < r\}$, with $r > 0$, is an important tolerance on X [96].

Note that $S_A = R_A \cap R_A^{-1} = R_A \cap R_{A^c} = A^2 \cap \left(A^c\right)^2$ is already an equivalence relation on X. And, more generally if $\mathcal{A}$ is a *cover (partition)* of X, then $S_{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A^2$ is a tolerance (equivalence) relation on X.

Now, as a straightforward generalization of the Pervin relation R_A, for any $A \subseteq X$ and $B \subseteq Y$, we may also naturally consider the *Hunsaker-Lindgren relation* $R_{(A,B)} = (A \times B) \cup (A^c \times Y)$ [26]. (See also [12, pp. 42 and 351].)

However, it is now more important to note that if $\mathcal{A} = (A_n)_{n=1}^\infty$ is an increasing sequence in $\mathcal{P}(X)$, then the *Cantor relation* $R_{\mathcal{A}} = \Delta_X \cup \bigcup_{n=1}^\infty \left(A_n \times A_n^c\right)$ is also an important preorder on X [40].

Note that if R is only reflexive relation on X and $x \in X$, then $\mathcal{A}_R(x) = \left(R^n(x)\right)_{n=1}^\infty$ is already an increasing sequence in $\mathcal{P}(X)$. Thus, the preorder relation $R_{\mathcal{A}_R(x)}$ may also be naturally investigated.

For a real function φ of X and a quasi-pseudo-metric d on X [22], the *Brøndsted relation*

$$R_{(\varphi,d)} = \{(x, y) \in X^2 : \ d(x, y) \le \varphi(y) - \varphi(x)\}$$

is also an important preorder on X [8] . From this relation, by letting φ and d to be the zero functions, we obtain the *specialization* and *preference relations* $R_d = \{(x, y) \in X^2 : \ d(x, y) = 0\}$ and $R_\varphi = \{(x, y) \in X^2 : \ \varphi(x) \le \varphi(y)\}$, respectively. (See [11,95].)

If R is a relation on X to Y, then the ordered pair $(X, Y)(R) = \big((X, Y), R\big)$ is usually called a *formal context* or *context space* [23]. However, it is better to call it a *relational space* or a *simple relator space* [41].

If in particular R is a relation on X, then having in mind a widely used terminology of Birkhoff [3, p. 1] the ordered pair $X(R) = (X, R)$ may be naturally called a *goset* (generalized ordered set) [81], instead of a *relational system* [51].

If P is a relational property, then the goset $X(R)$ will be said to have property P if the relation R has this property. For instance, the goset $X(R)$ will be called *reflexive* if R is a reflexive relation on X.

In particular, the goset $X(R)$ will be called a *proset* (preordered set) if R is a preorder on X. Moreover, $X(R)$ will be called a *poset* (partially ordered set) if R is a partial order on X. (In [52], the terms "toset" and "woset" were also used.)

Thus, every set X is a poset with the identity relation Δ_X. Moreover, X is a proset with the universal relation X^2. And, the power set $\mathcal{P}(X) = \{A: \ A \subseteq X\}$ of X is a poset with the ordinary set inclusion $\subseteq$.

Several definitions on posets can as well be applied to gosets. For instance, if $X(R)$ is a goset, then for any $Y \subseteq X$ the goset $Y(R \cap Y^2)$ is called a *subgoset* of $X(R)$. While, the goset $X'(R') = X(R^{-1})$ is called the *dual* of $X(R)$.

4. Lower and upper bounds in gosets

Notation 4.1. In this and the next section, we assume that R is a relation on X.

Definition 4.1. For any $A, B \subseteq X$ and $x, y \in X$, we define

(1) $A \in \mathrm{Lb}_R(B)$ *and* $B \in \mathrm{Ub}_R(A)$ *if* $A \times B \subseteq R$;

(2) $x \in \mathrm{lb}_R(B)$ *if* $\{x\} \in \mathrm{Lb}_R(B)$; *(3)* $y \in \mathrm{ub}_R(A)$ *if* $\{y\} \in \mathrm{Ub}_R(A)$.

(4) $B \in \mathfrak{L}_R$ *if* $\mathrm{lb}_R(B) \neq \emptyset$; *(5)* $A \in \mathfrak{U}_R$ *if* $\mathrm{ub}_R(A) \neq \emptyset$.

Thus, for instance, we easily prove the following theorems.

Theorem 4.1. *We have*

(1) $\mathrm{Ub}_R = \mathrm{Lb}_{R^{-1}} = \mathrm{Lb}_{\mathcal{R}}^{-1}$; (2) $\mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{R^{-1}}$; (3) $\mathfrak{U}_R = \mathfrak{L}_{R^{-1}}$.

Theorem 4.2. *For any $A, B \subseteq X$, we have*

(1) $A \in \mathrm{Lb}_R(B) \iff A \subseteq \mathrm{lb}_R(B)$; (2) $B \in \mathrm{Ub}_R(A) \iff B \subseteq \mathrm{ub}_R(A)$.

Proof. By Definition 4.1, we have

$$A \in \mathrm{Lb}_R(B) \iff A \times B \subseteq R \iff \forall\ x \in A: \ \{x\} \times B \subseteq R \iff$$
$$\forall\ x \in A: \ \{x\} \in \mathrm{Lb}_R(B) \iff \forall\ x \in A: \ x \in \mathrm{lb}_R(B) \iff A \subseteq \mathrm{lb}_R(B).$$

Thus, assertion (1) is true.

From assertion (1), by using Theorem 4.1, we infer that

$$B \in \mathrm{Ub}_R(A) \iff B \in \mathrm{Lb}_{R^{-1}}(A) \iff B \subseteq \mathrm{lb}_{R^{-1}}(A) \iff B \subseteq \mathrm{ub}_R(A).$$

Thus, assertion (2) is also true. ■

Remark 4.1. The above two theorems show that the above lower and upper bound relations are actually equivalent tools in the goset $X(R)$.

The following result is an immediate consequence of Theorems 4.1 and 4.2.

Corollary 4.1. *For any $A, B \subseteq X$, we have*

$$A \subseteq \mathrm{lb}_R(B) \iff B \subseteq \mathrm{ub}_R(A).$$

Proof. By Theorems 4.1 and 4.2, it is clear that

$$A \subseteq \mathrm{lb}_R(B) \iff A \in \mathrm{Lb}_R(B) \iff B \in \mathrm{Lb}_R^{-1}(A) \iff B \in \mathrm{Ub}_R(A) \iff B \subseteq \mathrm{ub}_R(A).$$ ■

Hence, by identifying singletons with their elements, we immediately derive the following result.

Corollary 4.2. *For any* $A, B \subseteq X$, *we have*

(1) $\operatorname{lb}_R(B) = \{x \in X : \ B \subseteq \operatorname{ub}_R(x)\}$; (2) $\operatorname{ub}_R(A) = \{y \in X : \ A \subseteq \operatorname{lb}_R(y)\}$.

Proof. To prove assertion (1), note that, by Corollary 4.1, for any $x \in X$ we have

$$x \in \operatorname{lb}_R(B) \iff \{x\} \subseteq \operatorname{lb}_R(B) \iff B \subseteq \operatorname{ub}_R(\{x\}) \iff B \subseteq \operatorname{ub}_R(x). \qquad \blacksquare$$

Remark 4.2. It is important to note that by defining

$$F(A) = \operatorname{ub}_{\mathcal{R}}(A) \qquad \text{and} \qquad G(B) = \operatorname{lb}_{\mathcal{R}}(B),$$

for all $A \subseteq X$ and $B \subseteq Y$, we at once see that

$$F(A) \subseteq' B \iff B \subseteq F(A) \iff B \subseteq \operatorname{ub}_{\mathcal{R}}(A) \iff A \subseteq \operatorname{lb}_{\mathcal{R}}(B) \iff A \subseteq G(B)$$

for all $A \subseteq X$ and $B \subseteq Y$. Thus, the functions F and G establish a *Galois connection* [13, p. 155] between the poset $\mathcal{P}(X)$ and the dual of the poset $\mathcal{P}(Y)$. Therefore, several properties of the super relations ub_R and lb_R can be derived from the extensive theory of Galois connections [5, 13, 14, 20, 23, 24]. (Examples for Galois connections can also be found in [10, 67, 72, 78, 80, 82].)

From Corollary 4.1 we can already derive the following theorem. However, it is frequently more convenient to apply some direct proofs.

Theorem 4.3. *If* $A \subseteq X$, *then*

(1) $\operatorname{lb}_R(A) \subseteq \operatorname{lb}_R(B)$ *for all* $B \subseteq A$; (2) $A \subseteq \operatorname{ub}_R(\operatorname{lb}_R(A))$; (3) $\operatorname{lb}_R(A) = \operatorname{lb}_R(\operatorname{ub}_R(\operatorname{lb}_R(A)))$.

In addition to Corollary 4.2, it is also worth proving the following result.

Theorem 4.4. *For any* $A, B \subseteq X$, *we have*

(1) $\operatorname{ub}_R(A) = \bigcap_{x \in A} \operatorname{ub}_R(x)$; (2) $\operatorname{lb}_R(B) = \bigcap_{y \in B} \operatorname{lb}_R(y)$.

Now, by this theorem and Corollary 4.2, we also state the following result.

Corollary 4.3. *For any* $A, B \subseteq X$, *we have*

(1) $\operatorname{ub}_R(A) = \bigcap_{x \in A} R(x)$; (2) $\operatorname{lb}_R(B) = \{x \in X : \ B \subseteq R(x)\}$.

Proof. Note that, for any $x, y \in X$, we have

$$y \in \operatorname{ub}_R(x) \iff y \in \operatorname{ub}_R(\{x\}) \iff \{y\} \in \operatorname{Ub}_R(\{x\}) \iff$$
$$\{x\} \times \{y\} \subseteq R \iff \{(x, y)\} \subseteq R \iff (x, y) \in R \iff y \in R(x).$$

Therefore, $\operatorname{ub}_R(x) = R(x)$. $\blacksquare$

Remark 4.3. Assertion (1) of Theorem 4.4 can be generalized by showing that the relation $F = \operatorname{ub}_R$ is *union-reversing* in the sense that, for any $\mathcal{A} \subseteq \mathcal{P}(X)$, we have $F(\bigcup \mathcal{A}) = \bigcap_{A \in \mathcal{A}} F(A)$.

5. Some further important basic tools in gosets

Now, by using Definition 4.1, we also naturally introduce the following definition.

Definition 5.1. For any $A \subseteq X$, we define

(1) $\min_R(A) = A \cap \operatorname{lb}_R(A)$; *(2)* $\max_R(A) = A \cap \operatorname{ub}_R(A)$;

(3) $\operatorname{Min}_R(A) = \mathcal{P}(A) \cap \operatorname{Lb}_R(A)$; *(4)* $\operatorname{Max}_R(A) = \mathcal{P}(A) \cap \operatorname{Ub}_R(A)$;

(5) $\inf_R(A) = \max_R(\operatorname{lb}_R(A))$; *(6)* $\sup_R(A) = \min_R(\operatorname{ub}_R(A))$;

(7) $\operatorname{Inf}_R(A) = \operatorname{Max}_R[\operatorname{Lb}_R(A)]$; *(8)* $\operatorname{Sup}_R(A) = \operatorname{Min}_R[\operatorname{Ub}_R(A)]$;

(9) $A \in \ell_R$ *if* $A \in \operatorname{Lb}_R(A)$; *(10)* $A \in \mathcal{L}_R$ *if* $A \subseteq \operatorname{lb}_R(A)$.

By using this definition, for instance, we prove the following theorems.

Theorem 5.1. *We have*

(1) $\operatorname{Max}_R = \operatorname{Min}_{R^{-1}}$; (2) $\operatorname{Sup}_R = \operatorname{Inf}_{R^{-1}}$; (3) $\ell_R = \ell_{R^{-1}}$

(4) $\max_R = \min_{R^{-1}}$; (5) $\sup_R = \inf_{R^{-1}}$; (6) $\ell_R = \mathcal{L}_R$.

Theorem 5.2. *For any* $A \subseteq X$, *we have*

(1) $\max_R(A) = \bigcap_{x \in A} A \cap \operatorname{ub}_R(x)$; (2) $\max_R(A) = \{x \in A : A \subseteq \operatorname{lb}_R(x)\}$.

Theorem 5.3. *For any* $A \subseteq X$, *we have*

(1) $\sup_R(A) = \operatorname{ub}_R(A) \cap \operatorname{lb}_R(\operatorname{ub}_R(A))$;

(2) $\max_R(A) = A \cap \sup_R(A)$; (3) $\sup_R(A) = \inf_R(\operatorname{ub}_R(A))$.

Proof. To prove (3), note that by assertion (1) and Theorem 4.3, and their duals, we have

$$\sup_R(A) = \operatorname{lb}_R(\operatorname{ub}_R(A)) \cap \operatorname{ub}_R(A) = \operatorname{lb}_R(\operatorname{ub}_R(A)) \cap \operatorname{ub}_R\left(\operatorname{lb}_R(\operatorname{ub}_R(A))\right) = \inf_R(\operatorname{ub}_R(A)).$$ ■

Theorem 5.4. *For any* $A \subseteq X$, *we have*

$$\sup_R(A) = \{x \in X : \operatorname{ub}_R(x) = \operatorname{ub}_R(A)\} = \{x \in \operatorname{ub}_R(A) : \operatorname{ub}_R(A) \subseteq \operatorname{ub}_R(x)\}.$$

Theorem 5.5. *For any* $A \subseteq X$ *the following assertions are equivalent:*

(1) $A \in \mathcal{L}_R$; (2) $A \in \operatorname{Ub}_R(A)$; (3) $A \in \operatorname{Min}_R(A)$; (4) $A \in \operatorname{Max}_R(A)$.

Corollary 5.1. *For any* $A \subseteq X$ *the following assertions are equivalent:*

(1) $\operatorname{ub}_R(A) \in \mathcal{L}_R$; (2) $\operatorname{ub}_R(A) = \sup_R(A)$; (3) $\operatorname{ub}_R(A) \subseteq \operatorname{lb}_R(\operatorname{ub}_R(A))$.

Theorem 5.6. *We have*

$$\mathcal{L}_R = \{\min_R(A) : \quad A \subseteq X\} = \{\max_R(A) : \quad A \subseteq X\}.$$

Theorem 5.7. *If* R *is reflexive on* X, *then following assertions are equivalent:*

(1) R *is antisymmetric;*

(2) $\operatorname{card}(A) \leq 1$ *for all* $A \in \mathcal{L}_R$;

(3) $\max_R$ *is a function;* (4) $\sup_R$ *is a function.*

Remark 5.1. The implications $(1) \Longrightarrow (3) \Longleftrightarrow (4)$ do not require the relation R to be reflexive.

Definition 5.2. The relation R on X, or the goset $X(R)$ will be called

(1) *inf–complete* if $\inf_R(A) \neq \emptyset$ for all $A \subseteq X$;

(2) *min–complete* if $\min_R(A) \neq \emptyset$ for all $\emptyset \neq A \subseteq X$.

Remark 5.2. Thus, for instance, the set $\mathbb{Z}$ of all integers is min-complete, but not inf-complete. While, the set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ of all extended real numbers is inf-complete, but not min–complete.

Now, by letting A to be a singleton, and then a doubleton, we obtain

Theorem 5.8. *If* R *is min-complete, then* R *is reflexive and total.*

Moreover, by using Theorem 5.3, we also easily prove the following result.

Theorem 5.9. *The following assertions are equivalent:*

(1) R *is inf–complete;* (2) R *is sup–complete.*

Proof. By Theorem 5.3, for any $A \subseteq X$, we have $\sup_R(A) = \inf_R(\operatorname{ub}_R(A))$. Hence, the implication $(1) \Longrightarrow (2)$ immediately follows. ■

Remark 5.3. For several other reasonable order-theoretic completeness properties, and their relationships, see [7] and [6].

6. A few basic facts on increasing functions

Notation 6.1. In this section, we assume that f is a function of one goset $X(R)$ to another $Y(S)$.

Definition 6.1. The function f will be called *increasing* if

$$u\,R\,v \quad \Longrightarrow \quad f(u)\,S\,f(v)$$

for all $u, v \in X$.

Remark 6.1. Now, the function f may be briefly defined to be *decreasing* if it is increasing as a function $X(R)$ to $Y\big(S^{-1}\big)$. Moreover, the function may, for instance, be briefly defined to be *strictly increasing* if it is increasing as a function of $X(R \setminus \Delta_X)$ to $Y(S \setminus \Delta_Y)$. However, to define a strict form of the relation R, the relation $R \setminus R^{-1}$ can also be well-used instead of $R \setminus \Delta_X$. (See, for instance, Patrone [44].)

The following theorem shows that the strictly increasing functions are closely related to the injective, increasing ones.

Theorem 6.1. *If R is total on X and S is reflexive on Y, then the following assertions are equivalent:*

(1) *f is strictly increasing;* (2) *f is injective and increasing.*

Remark 6.2. To prove the implication (2) $\Longrightarrow$ (1) we do not need any extra conditions on the relations R and S. While, if assertion (1) holds with $f[X] = Y$, then to prove that f^{-1} is also strictly increasing, we have to assume that R is total and S is antisymmetric.

Concerning increasing functions, we also prove the following theorems.

Theorem 6.2. *The following assertions are equivalent:*

(1) *f is increasing;*

(2) $f[\operatorname{ub}_R(x)] \subseteq \operatorname{ub}_S\big(f(x)\big)$ *for all* $x \in X$; (3) $f[\operatorname{ub}_R(A)] \subseteq \operatorname{ub}_S\big(f[A]\big)$ *for all* $A \subseteq X$.

Theorem 6.3. *If R is reflexive on X, then the following assertions are equivalent:*

(1) *f is increasing;*

(2) $f[\max_R(A)] \subseteq \operatorname{ub}_S\big(f[A]\big)$ *for all* $A \subseteq X$; (3) $f[\max_R(A)] \subseteq \max_S\big(f[A]\big)$ *for all* $A \subseteq X$;

From Theorem 6.2, by using Theorem 4.3, we immediately derive the following result.

Theorem 6.4. *If f is increasing, then for any $A \subseteq X$, we have*

$$\operatorname{lb}_S\big(\operatorname{ub}_S(f[A])\big) \subseteq \operatorname{lb}_S\big(f[\operatorname{ub}_R(A)]\big).$$

Moreover, by using Theorems 6.2 and 5.7, we also prove the next theorem.

Theorem 6.5. *If f is increasing and R and S are antisymmetric and sup–complete, then for any $A \subseteq X$ we have*

$$\sup_S\big(f[A]\big)\,S\,f\big(\sup_R(A)\big).$$

Finally, we note that, by the results of [87], the following theorems are also true. Therefore, instead of "increasing", we may also naturally say "continuous".

Theorem 6.6. *The following assertions are equivalent:*

(1) *f is increasing;*

(2) $(u, v) \in R$ *implies* $\big(f(u), f(v)\big) \in S$; (3) $v \in R(u)$ *implies* $f(v) \in S\big(f(u)\big)$ *for all* $u \in X$.

Theorem 6.7. *The following assertions are equivalent:*

(1) *f is increasing;*

(2) $f \circ R \subseteq S \circ f$, (3) $R \subseteq f^{-1} \circ S \circ f$,

(4) $f \circ R \circ f^{-1} \subseteq S$, (5) $R \circ f^{-1} \subseteq f^{-1} \circ S$.

Theorem 6.8. *The following assertions are equivalent:*

(1) f *is increasing;*

(2) $(f \boxtimes f)[R] \subseteq S$;

(3) $\big(f \boxtimes R\big)[\Delta_X] \subseteq (S \boxtimes f)^{-1}[\Delta_Y]$;

(4) $R \subseteq (f \boxtimes f)^{-1}[S]$;

(5) $\big(R^{-1} \boxtimes f\big)[\Delta_X] \subseteq \big(f^{-1} \boxtimes S\big)[\Delta_Y]$.

Remark 6.3. Now, a relation F on the goset $X(R)$ to a set Y may be naturally called *increasing* if the associated set-valued function φ_F is increasing. That is, $u\,R\,v$ implies $F(u) \subseteq F(v)$ for all $u, v \in X$. However, if F is a relation on $X(R)$ to $Y(S)$, then in addition to the above inclusion-increasingness of F, we may also define an *order-increasingness* of F by requiring the implication $u \in \mathrm{lb}_R(v) \implies F(u) \in \mathrm{Lb}_S\big(F(v)\big)$ for all $u, v \in X$. Thus, it can be shown that F is inclusion-increasing if and only if $R \circ F^{-1} \subseteq F^{-1}$, or equivalently F^{-1} is ascending-valued. And, F is order-increasing if and only if $F \circ R \circ F^{-1} \subseteq S$, or equivalently $F[R(u)] \subseteq \mathrm{ub}_S\big(F(u)\big)$ for all $u \in X$ [87].

7. The induced order and interior relations

Notation 7.1. In this section, we assume that f is a function of a set X to a goset $Y(S)$.

Definition 7.1. For each $u \in X$ and $y \in Y$, we define

$$\mathrm{Ord}_f(u) = \big\{ v \in X: \quad f(u)\,S\,f(v) \big\} \qquad \text{and} \qquad \mathrm{Int}_f(y) = \big\{ x \in X: \quad f(x)\,S\,y \big\}.$$

The relations Ord_f and Int_f will be called the *natural order* and the *proximal interior* induced by f, respectively.

Remark 7.1. If F is a relation on X to Y, then by using the associated set-valued function φ_F, we may naturally define $\mathrm{Ord}_F = \mathrm{Ord}_{\varphi_F}$ and $\mathrm{Int}_F = \mathrm{Int}_{\varphi_F}$. Moreover, if U is super relation on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, we may also define $\mathrm{ord}_U(A) = X \cap \mathrm{Ord}_U(A)$ and $\mathrm{int}_U(B) = X \cap \mathrm{Int}_U(B)$.

Concerning the relations Ord_f and Int_f, we easily prove the following four theorems.

Theorem 7.1. Ord_f *is the largest relation on* X *making the function* f *to be increasing.*

Proof. If R is a relation on X making f to be increasing, then

$$v \in R(u) \implies u\,R\,v \implies f(u) S f(v) \implies v \in \mathrm{Ord}_f(u),$$

and thus $R(u) \subseteq \mathrm{Ord}_f(u)$ for all $u \in X$. Therefore, $R \subseteq \mathrm{Ord}_f$ also holds. ■

Theorem 7.2. *The following assertions hold:*

(1) Ord_f *is a preorder on* X *if* S *is a preorder on* Y;

(2) Ord_f *is a partial order on* X *if* f *is injective and* S *is a partial order on* Y.

Theorem 7.3. *If* f *is a function of a goset* $X(R)$ *to a proset* $Y(S)$, *then the following assertions are equivalent:*

(1) f *is increasing;*

(2) Ord_f *is decreasing;*

(3) Ord_f *is ascending valued.*

Proof. If $u\,R\,v$ and (1) holds, then $f(u)\,S\,f(v)$. Moreover, if $w \in \mathrm{Ord}_f(v)$, then $f(v)\,S\,f(w)$. Hence, by the transitivity of S, we infer that $f(u)\,S\,f(w)$, and thus $w \in \mathrm{Ord}_f(u)$. Therefore, $\mathrm{Ord}_f(v) \subseteq \mathrm{Ord}_f(u)$, and thus (2) also holds.

Conversely, if $u\,R\,v$ and (2) holds, then $\mathrm{Ord}_f(v) \subseteq \mathrm{Ord}_f(u)$. Moreover, by the reflexivity of S, we also have $f(v)\,S\,f(v)$, and thus $v \in \mathrm{Ord}_f(v)$. Therefore, $v \in \mathrm{Ord}_f(u)$, and thus $f(u)\,S\,f(v)$ is also true. Consequently, (1) also holds. ■

Theorem 7.4. *If* f *is a function of a goset* $X(R)$ *to a transitive goset* $Y(S)$, *then*

(1) Int_f *is increasing;*

(2) Int_f *is descending valued if* f *is increasing.*

Proof. To prove (2), note that if $y \in Y$ and $x \in \mathrm{Int}_f(y)$, then $f(x)\,S\,y$. Moreover, if $u \in X$ such that $u\,R\,x$ and f increasing, then $f(u)\,S\,f(x)$. Thus, by the transitivity of S, we also have $f(u)\,S\,y$, and thus $u \in \mathrm{Int}_f(y)$. Therefore, $\mathrm{Int}_f(y)$ is a descending subset of X. ■

The next two theorems show that the relations Ord_f and Int_f are not independent of each other, and they are also closely related to the relations lb_S and ub_S.

Theorem 7.5. *We have*

$$\mathrm{Ord}_f = \big(\mathrm{Int}_f \circ f\big)^{-1} = f^{-1} \circ \mathrm{Int}_f^{-1}.$$

Proof. By the corresponding definitions, it is clear that

$$v \in \mathrm{Ord}_f(u) \iff f(u) \le f(v) \iff u \in \mathrm{Int}_f\big(f(v)\big) \iff u \in \big(\mathrm{Int}_f \circ f\big)(v) \iff v \in \big(\mathrm{Int}_f \circ f\big)^{-1}(v)$$

for all $u, v \in X$. Therefore, $\mathrm{Ord}_f(u) = \big(\mathrm{Int}_f \circ f\big)^{-1}(u)$ for all $u \in X$, and thus the equality $\mathrm{Ord}_f = \big(\mathrm{Int}_f \circ f\big)^{-1}$ is also true. ∎

Theorem 7.6. *For any $y \in Y$ and $x \in X$, we have*

$$\mathrm{Int}_f(y) = f^{-1}\big[\,\mathrm{lb}(y)\,\big] \qquad \textit{and} \qquad \mathrm{Int}_f^{-1}(x) = \mathrm{ub}_S\big(f(x)\big).$$

Remark 7.2. In this respect, it is also worth noticing that

$$y \in \mathrm{ub}_S\big(f\,[\mathrm{Int}_f(y)\,]\big)$$

for all $y \in Y$. Namely, for every $x \in \mathrm{Int}_f(y)$, we have $f\,(x)\,S\,y$.

Now, we also easily prove the following result.

Theorem 7.7. *If*

$$f\,[\sup_R(A)\,] \subseteq \mathrm{lb}_S\,\big(\mathrm{ub}_S\big(f\,[A\,]\big)\big)$$

for all $A \subseteq X$, then

$$\max_R\big(\mathrm{Int}_f(y)\big) = \sup_R\big(\mathrm{Int}_f(y)\big)$$

for all $y \in Y$.

Proof. If $y \in Y$, then by Theorem 5.3 we have

$$\max\big(\mathrm{Int}_f(y)\big) \subseteq \sup\big(\mathrm{Int}_f(y)\big).$$

Therefore, we need actually prove only the converse inclusion.

For this, note that if $x \in \sup_R\big(\mathrm{Int}_f(y)\big)$, then by the assumed property of f we have

$$f(x) \in f\,\big[\sup_R\big(\mathrm{Int}_f(y)\big)\big] \subseteq \mathrm{lb}_S\,\big(\mathrm{ub}_S\big(f\,[\mathrm{Int}_f(y)\,]\big)\big).$$

Moreover, by Remark 7.2, we also have $y \in \mathrm{ub}_S\big(f\,[\mathrm{Int}_f(y)\,]\big)$. Therefore, we necessarily have $f(x) \le y$, and thus $x \in \mathrm{Int}_f(y)$. Hence, by Theorem 5.3, we see that

$$x \in \mathrm{Int}_f(y) \cap \sup_R\big(\mathrm{Int}_f(y)\big) = \max_R\big(\mathrm{Int}_f(y)\big).$$

Therefore, $\sup_R\big(\mathrm{Int}_f(y)\big) \subseteq \max_R\big(\mathrm{Int}_f(y)\big)$, and thus the required equality is also true. ∎

Remark 7.3. Note that, by Theorem 5.3, for a subset A of the goset $X(R)$ we have $\max_R(A) = \sup_R(A)$ if and only if $\sup_R(A) \subseteq A$.

8. Extensive, involutive, and idempotent functions

Notation 8.1. In this and the next section, we assume that φ is a function of a goset $X\,(R)$ to itself.

Definition 8.1. The function φ will be called

(1) extensive if $\Delta_X\,R\,\varphi$; *(2) intensive* if $\varphi\,R\,\Delta_X$;

(3) right–semi–involutive if $\Delta_X\,R\,\varphi^2$; *(4) left–semi–involutive* if $\varphi^2\,R\,\Delta_X$;

(5) right–semi–idempotent if $\varphi\,R\,\varphi^2$; *(6) left–semi–idempotent* if $\varphi^2\,R\,\varphi$.

Remark 8.1. Property (3), in detailed form, means only that $\Delta_X\,(x)\,R\,\varphi^2\,(x)$, i.e., $x\,R\,\varphi\big(\varphi\,(x)\big)$ for all $x \in X$.

By using Definition 8.1, we easily establish the following result.

Theorem 8.1. *The following assertions hold ;*

(1) *φ is right-semi-idempotent if φ is extensive ;*

(2) *φ is right-semi-involutive if and only if φ^2 is extensive ;*

(3) *φ is right-semi-idempotent if and only if $\varphi \,|\, \varphi[X]$ is extensive.*

Proof. If φ is extensive, then $x \,R\, \varphi(x)$ for all $x \in X$. Hence, taking $u \in X$ and writing $\varphi(u)$ in place of x, we infer that $\varphi(u) \,R\, \varphi^2(u)$. Thus, φ is right-semi-idempotent.

Moreover, if $y \in \varphi[X]$, then there exists $x \in X$ such that $y = \varphi(x)$, and thus $\varphi(y) = \varphi^2(x)$. Moreover, if φ is right-semi-idempotent, then $\varphi(x) \,R\, \varphi^2(x)$, and thus $y \,R\, \varphi(y)$. Therefore, the restriction $\varphi \,|\, \varphi[X]$ is extensive. ■

Remark 8.2. In addition to the above observations, it is also worth noticing that φ is extensive with respect R if and only if $\varphi(x) \in R(x)$ for all $x \in X$. That is, φ is a selection function of R.

Thus, analogously to a relational reformulation of the Axiom of Choice, the following generalization of a theorem of Bourbaki [4, p. 4] may also be considered as a selection theorem.

Theorem 8.2. *If φ is strictly increasing and R is antisymmetric and min-complete, then φ is extensive.*

Proof. Assume on the contrary that φ is not extensive. Then, by Remark 8.2, φ is not a selection function of R. Thus,

$$A = \left\{ x \in X : \ \varphi(x) \notin R(x) \right\} \neq \emptyset .$$

Therefore, by the min-completeness of R, there exists $a \in X$ such that $a \in \min_R(A)$. Hence, by the definition of $\min_R$, we infer that

$$a \in A \qquad \text{and} \qquad a \in \mathrm{lb}_R(A),$$

and thus $a \,R\, x$ for all $x \in A$.

Now, since $a \in A$, we also note that $a \,R\, a$, and thus $a \in R(a)$. Moreover, by the definition of A, we also note that $\varphi(a) \notin R(a)$. Therefore, $\varphi(a) \neq a$. Moreover, from Theorem 5.8, we know that R is total. Thus, since $a \,R\, \varphi(a)$ does not hold, we necessarily have $\varphi(a) \,R\, a$.

Hence, by using that $\varphi(a) \neq a$ and φ is strictly increasing, we infer that $\varphi\big(\varphi(a)\big) \,R\, \varphi(a)$ and $\varphi\big(\varphi(a)\big) \neq \varphi(a)$. Thus, by the antisymmetry of R, $\varphi(a) \,R\, \varphi\big(\varphi(a)\big)$ cannot hold. This, shows that $\varphi\big(\varphi(a)\big) \notin R\big(\varphi(a)\big)$, and thus $\varphi(a) \in A$. Hence, by using that $a \,R\, x$ for all $x \in A$, we infer that $a \,R\, \varphi(a)$, and thus $\varphi(a) \in R(a)$. This contradiction shows that φ is extensive. ■

Remark 8.3. Note that if φ is extensive, R is antisymmetric and x is a maximal element of $X(R)$ in the sense that $x \,R\, y$ implies $y \,R\, x$ for all $y \in X$, then x is already a fixed point of φ in the sense that $\varphi(x) = x$. This simple, but important fact was first explicitly stated by Brøndsted [9]. Also, fixed point theorems for extensive maps (which were sometimes also called expansive, progressive, increasing, or inflationary) were proved by several authors.

9. Involution, projection, and closure operations

Definition 9.1. The function φ is called

(1) *involution operation* if it is increasing and both left and right semi-involutive ;

(2) *projection operation* if it is increasing and both left and right semi-idempotent ;

(3) *closure (interior) operation* if it is an extensive (intensive) projection operation .

Remark 9.1. Moreover, φ may, for instance be called a

(1) *preclosure operation* if it is increasing and extensive ;

(2) *semi-closure operation* if it is extensive and left-semi-idempotent .

Note that, by Theorem 8.1, an extensive operation is right-semi-idempotent. Moreover, the corresponding interior operations can be briefly defined by using the dual of $X(R)$. In connection with Definition 8.1, it is also worth mentioning if, for instance, φ is both left and right semi-idempotent and R is antisymmetric, then φ is idempotent in the sense that $\varphi^2 = \varphi$. However, if φ is idempotent and R is not reflexive, then φ need not be either left or right semi-idempotent.

Concerning closure operations, for instance, we prove the following result.

Theorem 9.1. *If φ is a closure operation, and R is antisymmetric and inf-complete, then for any $A \subseteq X$ we have*

$$\inf_R(\varphi[A]) = \varphi\left(\inf_R(\varphi[A])\right).$$

Proof. By the dual of Theorem 6.5, we have

$$\inf_R(\varphi[A]) \in R\left(\varphi(\inf_R(A))\right).$$

Hence, by writing $\varphi[A]$ in place of A, we see that

$$\inf_R(\varphi[\varphi[A]]) \in R\left(\varphi\left(\inf_R(\varphi[A])\right)\right).$$

Moreover, because of the antisymmetry of R, we note that φ is now idempotent. Therefore, $\varphi[\varphi[A]] = (\varphi \circ \varphi)[A] = \varphi^2[A] = \varphi[A]$. Thus, we actually have

$$\inf_R(\varphi[A]) \in R\left(\varphi\left(\inf_R(\varphi[A])\right)\right).$$

Moreover, by extensivity of φ, the converse inclusion is also true. Hence, by using the antisymmetry of R, we see that the required equality is also true. ■

Remark 9.2. It can be easily seen that an operation φ on a set X is idempotent if and only if $\varphi[X]$ is the family of all fixed points of φ.

Therefore, by using Theorem 9.1, we also prove the following result.

Corollary 9.1. *Under the conditions of Theorem 9.1, for any $A \subseteq \varphi[X]$, we have*

$$\inf_R(A) = \varphi(\inf_R(A)).$$

Proof. Now, because of the antisymmetry of R, the operation φ is idempotent. Thus, by Remark 9.2, we have $\varphi(y) = y$ for all $y \in \varphi[X]$. Hence, by using the assumption $A \subseteq \varphi[X]$, we see that $\varphi[A] = A$. Thus, Theorem 9.1 gives the required equality. ■

Remark 9.3. Note that if φ is an extensive and left-semi-idempotent, and R reflexive and antisymmetric, then $\varphi[X]$ is also the family of all elements x of X which are φ–closed in the sense that $\varphi(x) R x$. Therefore, if in addition to the conditions of Theorem 9.1, R is reflexive, then the assertion of Corollary 9.1 can also be expressed by stating that the infimum of any family of φ–closed elements of $X(R)$ is also φ–closed.

Now, instead of a counterpart of Theorem 9.1, we only prove the following result.

Theorem 9.2. *If φ is a closure operation, and R is transitive, antisymmetric and sup-complete, then for any $A \subseteq X$ we have*

$$\varphi(\sup_R(A)) = \varphi\left(\sup_R(\varphi[A])\right).$$

Proof. Define $\alpha = \sup_R(A)$ and $\beta = \sup_R(\varphi[A])$. Then, by Theorem 6.5, we have $\beta\, R\, \varphi(\alpha)$. Hence, since φ is increasing, we infer that $\varphi(\beta)\, R\, \varphi(\varphi(\alpha))$. Moreover, since φ is now idempotent, we also have $\varphi(\varphi(\alpha)) = \varphi(\alpha)$. Therefore, $\varphi(\beta) R \varphi(\alpha)$.

On the other hand, since φ is extensive, for any $x \in A$ we have $x\, R\, \varphi(x)$. Moreover, since $\beta \in \mathrm{ub}_R(\varphi[A])$, we also have $\varphi(x) R \beta$. Hence, by using the transitivity of R, we infer that $x\, R\, \beta$. Therefore, $\beta \in \mathrm{ub}_R(A)$. Now, by using that $\alpha \in \mathrm{lb}_{RX}(\mathrm{ub}_X(A))$, we see that $\alpha\, R\, \beta$. Hence, by using the increasingness of φ, we infer that $\varphi(\alpha) R \varphi(\beta)$. Therefore, by the antisymmetry of R, we actually have $\varphi(\alpha) = \varphi(\beta)$, and thus the required equality is also true. ■

By using this theorem, we prove the following result.

Corollary 9.2. *Under the conditions of Theorem 9.2, for any $A \subseteq X$, the following assertions are equivalent:*

(1) $\sup_R(\varphi[A]) = \varphi(\sup_R(A))$, (2) $\sup_R(\varphi[A]) = \varphi\left(\sup_R(\varphi[A])\right)$.

10. Normal and regular functions of one goset to another

Notation 10.1. In this and the next five sections, we assume that:

(1) $X(\leq)$ and $Y(\leq)$ are gosets;

(2) φ is a function of X to itself;

(3) f is a function of X to Y and g is a function of Y to X.

In [88], slightly extending the ideas of Ore [39], Schmidt [54, p. 209], Blyth and Janowitz [5, p. 11] and [75] on Galois connections, residuated mappings, and normal and regular functions, the third author has introduced the following definition.

Definition 10.1. We say that the function f is

(1) *right g–seminormal* if for any $x \in X$ and $y \in Y$

$$f(x) \leq y \quad \Longrightarrow \quad x \leq g(y);$$

(2) *left g–seminormal* if for any $x \in X$ and $y \in Y$

$$x \leq g(y) \quad \Longrightarrow \quad f(x) \leq y.$$

Remark 10.1. Thus, if the function f is both left and right g–seminormal, then we may naturally say that f *is g–normal*. Moreover, if for instance the function f is left h–seminormal for some function h of Y to X, then we may naturally say that f *is left seminormal*.

In [88], having in mind the properties of the function $g \circ f$, and slightly extending the ideas of Pataki [42] and [75], the third author has also introduced the following definition.

Definition 10.2. We say that the function f is

(1) *right φ–semiregular* if for any $u, v \in X$

$$f(u) \leq f(v) \quad \Longrightarrow \quad u \leq \varphi(v);$$

(2) *left φ–semiregular* if for any $u, v \in X$

$$u \leq \varphi(v) \quad \Longrightarrow \quad f(u) \leq f(v).$$

Remark 10.2. Thus, if the function f is both left and right φ–semiregular, then we may naturally say that f *is φ–regular*. Moreover, if for instance the function f is left ψ–semiregular for some function ψ of X to itself, then we may naturally say that f *is left semiregular*.

In [88], to clarify the relationship between normal and regular functions, the third author has proved the following two simple theorems and their corollaries.

Theorem 10.1. *If f is left (right) g–seminormal and $\varphi = g \circ f$, then f is left (right) φ–semiregular.*

Corollary 10.1. *If f is g–normal and $\varphi = g \circ f$, then f is φ–regular.*

Theorem 10.2. *If f is a left (right) φ–semiregular, $\varphi = g \circ f$ and $Y = f[X]$, then f is left (right) g–seminormal.*

Proof. Suppose that $x \in X$ and $y \in Y$. Then, since $Y = f[X]$, there exists $v \in X$ such that $y = f(v)$.

Now, if f is right φ–semiregular, then we easily see that

$$f(x) \leq y \implies f(x) \leq f(v) \implies x \leq \varphi(v) \implies x \leq \big(g \circ f\big)(v)\big) \implies x \leq g\big(f(v)\big) \implies x \leq g(y).$$

Therefore, f is right g–seminormal. The corresponding statement for the left φ–semiregularity of f can be proved by reversing the above implications. ■

Corollary 10.2. *If f is φ–regular, $\varphi = g \circ f$ and $Y = f[X]$, then f is g–normal.*

Remark 10.3. Note that if in particular X and Y are transitive, then by using the inequalities $\varphi \leq g \circ f$ and $g \circ f \leq \varphi$, instead of the equality $\varphi = g \circ f$, some similar statements can also be proved.

By Corollary 10.1, it is clear that several properties of normal functions can be immediately derived from those of the regular ones. Therefore, the latter ones have to be studied before the former ones. Moreover, from Corollary 10.2, we see that regular functions are still less general objects than the normal ones. Actually, we shall see that they are strictly between closure operations and normal functions.

In [88], the third author has also proved the following dualization principle.

Theorem 10.3. *If f is a left (right) g–seminormal as a function of X to Y, then g is a right (left) f–seminormal as a function of Y' to X'.*

Proof. If for instance f is right g–seminormal as a function of X to Y, then by the corresponding definitions we see that

$$y \leq' f(x) \implies f(x) \leq y \implies x \leq g(y) \implies g(y) \leq' x$$

for all $y \in Y$ and $x \in X$ Therefore, g is left f–seminormal as a function of Y' to X'. ■

Corollary 10.3. *If f is g–normal as a function of X to Y, then g is f–normal as a function of Y' to X'.*

Remark 10.4. By using Theorem 10.3 and its corollary, the relevant properties of the functions g and $f \circ g$ can be frequently derived from those of f and $g \circ f$. However, it is sometimes more convenient to apply a direct proof.

11. Some basic properties of regular functions

The following two theorems were also proved in [88]. Their proofs are included here only for the readers convenience.

Theorem 11.1. *If f is φ–regular, X is preordered and Y is reflexive, then*

(1) *φ is extensive;* (2) *f is increasing;* (3) $f \circ \varphi \leq f \leq f \circ \varphi$.

Proof. If $x \in X$, then by the reflexivity of X and Y we have

$$\varphi(x) \leq \varphi(x) \qquad \text{and} \qquad f(x) \leq f(x).$$

Hence, by using the right and left φ-semiregularity of f, we infer that

$$x \leq \varphi(x) \qquad \text{and} \qquad f\big(\varphi(x)\big) \leq f(x).$$

Therefore, assertion (1) holds and $f \circ \varphi \leq f$.

Moreover, if $u, v \in X$ such that $u \leq v$, then by assertion (1) and the transitivity of X we have

$$v \leq \varphi(v), \qquad \text{and thus} \qquad u \leq \varphi(v).$$

Hence, by using the left φ–semiregularity of f, we infer that $f(u) \leq f(v)$. Therefore, assertion (2) also holds.

Now, if $x \in X$, then by using assertions (1) and (2), we can also see that

$$x \leq \varphi(x), \qquad \text{and thus} \qquad f(x) \leq f\big(\varphi(x)\big).$$

Therefore, $f \leq f \circ \varphi$, and thus assertion (3) also holds. ■

Remark 11.1. Note that if in addition Y antisymmetric, then instead of assertion (3) we can also state that $f = f \circ \varphi$.

Theorem 11.2. *If X is a proset, then the following assertions are equivalent:*

(1) *φ is φ–regular;*

(2) *φ is a closure operation on X;*

(3) *there exists a φ–regular function h of X to a proset Z.*

Proof. Suppose first that (2) holds and $u, v \in X$. If $u \leq \varphi(v)$, then by the increasingness of φ we have $\varphi(u) \leq \varphi\big(\varphi(v)\big)$. Moreover, by the left semiidempotency of φ, we have $\varphi^2 \leq \varphi$, and thus $\varphi\big(\varphi(v)\big) \leq \varphi(v)$. Therefore, by the transitivity of X, we also have $\varphi(u) \leq \varphi(v)$. Thus, φ is left φ–semiregular.

On the other hand, by the extensivity of φ we have $u \leq \varphi(u)$. Therefore, if $\varphi(u) \leq \varphi(v)$, then by the transitivity of X we also have $u \leq \varphi(v)$. This shows that φ is right φ–semiregular too, and thus (1) also holds. Moreover, if (1) holds, then by taking $h = \varphi$ and $Z = X$, we at once see that (3) also holds.

Therefore, to complete the proof, we need only show that (3) also implies (2). For this, assume that (3) holds. Then, by Theorem 11.1, φ is extensive and $h\circ\varphi \leq h$. Hence, we infer that $h\circ\varphi^2 = h\circ\varphi\circ\varphi \leq h\circ\varphi$. Therefore, by the transitivity of Z, we also have $h\circ\varphi^2 \leq h$. Thus, for any $u\in X$, we have $h\left(\varphi^2(u)\right) \leq h(u)$. Hence, by using the right φ–regularity of h, we can infer that $\varphi^2(u) \leq \varphi(u)$. Therefore, $\varphi^2 \leq \varphi$, and thus φ is left semiidempotent.

Moreover, if $u, v\in X$ such that $u\leq v$, then Theorem 11.1 we also have $h(u)\leq h(v)$ and $h\left(\varphi(u)\right) \leq h(u)$. Therefore, by the transitivity of Z, we also have $h\left(\varphi(u)\right) \leq h(v)$. Hence, by using the right φ–regularity of h, we already infer that $\varphi(u)\leq\varphi(v)$. Therefore, φ is increasing, and thus (2) also holds. ■

Remark 11.2. By Erné [14, p. 50], the origins of the equivalence of assertions (2) and (1) goes back to R. Dedekind. While, by the third author [73], the equivalence of assertions (2) and (3) are mainly due to Pataki [42, Theorem 1.9].

From Theorem 11.2, by using the corresponding definitions, we easily derive the following result.

Corollary 11.1. *If X and Y are prosets, then the following assertions are equivalent :*

(1) *f is φ–regular ;* (2) *φ is a closure and* $\mathrm{Ord}_\varphi = \mathrm{Ord}_f$.

Hint. If $\mathrm{Ord}_\varphi = \mathrm{Ord}_f$ holds, then by Definition 7.1, for any $u, v\in X$, we have $\varphi(u) R\, \varphi(v) \iff f(u) S\, f(v)$.

Moreover, if φ is a closure operation on X, then by Theorem 11.2, for any $u, v\in X$, we have

$$\varphi(u)\, R\, \varphi(v) \iff u\, R\, \varphi(v).$$

Therefore, in contrast to the implication (1) $\Longrightarrow$ (2), the converse implication (2) $\Longrightarrow$ (1) does not need any particular property of S. ■

In addition to the above two theorems, we also prove the following result.

Theorem 11.3. *If f is φ–regular, and X and Y are posets, then the following assertions are equivalent :*

(1) *f is injective ;* (2) *φ is the identity function.*

Proof. By Theorem 11.1 and the antisymmetry of Y, for any $x\in X$, we have $f\left(\varphi(x)\right) = f(x)$. Hence, if assertion (1) holds, we infer that $\varphi(x) = x$. Thus, assertion (2) also holds.

To prove the converse implication, suppose now that $u, v\in X$ such that $f(u) = f(v)$. Then, by the reflexivity of Y, we also have $f(u)\leq f(v)$ and $f(v)\leq f(u)$. Hence, by using the right φ–semiregularity of f, we infer that $u\leq\varphi(v)$ and $v\leq\varphi(u)$. Hence, if (2) holds, we infer that $u\leq v$ and $v\leq u$. Thus, by the antisymmetry of X, we also have $u=v$. Therefore, assertion (1) also holds. ■

Remark 11.3. Note that, by Definition 10.2, f is Δ_X–regular if and only if, for all $u, v\in X$, we have

$$u\leq v \iff f(u)\leq f(v).$$

12. Some basic properties of normal functions

From Theorem 11.1, by using Corollary 10.1, we immediately derive the following result.

Theorem 12.1. *If f is g–normal, X is preordered and Y is reflexive, then*

(1) *$g\circ f$ is extensive ;* (2) *f is increasing ;* (3) $f\circ g\circ f\leq f\leq f\circ g\circ f$.

From this theorem, by using Corollary 10.3, we easily derive the next result.

Corollary 12.1. *If f is g–normal, X is reflexive and Y is preordered, then*

(1) *$f\circ g$ is intensive ;* (2) *g is increasing ;* (3) $g\circ f\circ g\leq g\leq g\circ f\circ g$.

Proof. By Corollary 10.3, g is f–normal as a function of Y' to X'. Hence since Y' is preordered and X' is reflexive, by using Theorem 12.1, we infer that

(a) $f\circ g$ is extensive as a function of Y' to itself;

(b) g is increasing as a function of Y' to X'; (c) $g\circ f\circ g\leq' g\leq' g\circ f\circ g$.

Thus, (3) is true. Moreover, form (a) and (b), by the corresponding definitions, we see that (1) and (2) are also true. ■

From Theorem 11.2, we quite similarly derive the following result.

Theorem 12.2. *If f is g–normal and X and Y are prosets, then*

(1) *f and g are increasing;*

(2) *$g \circ f$ is a closure operation;* (3) *$g \circ f$ is an interior operation.*

Proof. By Corollary 10.3, g is an f–normal function of Y' to X'. Thus, by Corollary 10.1, g is a $f \circ g$–regular function of Y' to X'. Hence, by using Theorems 11.1 and 11.2, we infer that g is an increasing function of Y' to X', and $f \circ g$ is a closure operation on Y'. Hence, by the corresponding definitions, we see that g is increasing, and assertion (3) is also true. ■

Now, as a useful characterization of normal functions, we also prove the next result.

Theorem 12.3. *If X and Y are prosets, then the following assertions are equivalent:*

(1) *f is g–normal;*

(2) *f and g are increasing, $g \circ f$ is extensive and $f \circ g$ is intensive.*

Proof. From Theorem 12.2, we at once see that (1) implies (2). Therefore, we need actually prove the converse implication.

For this, assume that assertion (2) holds, $x \in X$ and $y \in Y$. Now, if $f(x) \leq y$, then by using the increasingness of g we see that $g\big(f(x)\big) \leq g(y)$. Hence, by using that $x \leq (g \circ f)(x) = g\big(f(x)\big)$, we already infer that $x \leq g(y)$. Therefore, f is right g–seminormal.

Conversely, if $x \leq g(y)$, then by using the increasingness of f we see that $f(x) \leq f\big(g(y)\big)$. Hence, by using that $f\big(g(y)\big) = (f \circ g)(y) \leq y$, we already infer that $f(x) \leq y$. Therefore, f is also left g–seminormal. Thus, assertion (1) also holds. ■

Remark 12.1. This theorem shows that the recent definition of Galois connections [13, p. 155], suggested by Schmidt [54, p. 209], is equivalent to the old one given by Ore [39].

Finally, we note that the following counterpart of Theorem 11.2 is also true.

Theorem 12.4. *If X is proset, then the following assertions are equivalent:*

(1) *φ is φ–normal;* (2) *φ is an involution operation on X.*

Proof. If assertion (1) holds, then by Theorem 12.2, φ is increasing and φ^2 is both extensive and intensive. Thus, by Theorem 8.1 and its dual, φ is both right and left semi-involutive. Therefore, by Definition 9.1, assertion (2) also holds.

Conversely, if assertion (2) holds, then by Definition 9.1, φ is increasing and φ^2 is both extensive and intensive. Thus,

$$u \leq \varphi\big(\varphi(u)\big) \qquad \text{and} \qquad \varphi\big(\varphi(v)\big) \leq v$$

for all $u, v \in X$. Hence, by using the increasingness of φ and the transitivity of X, we see that

$$\varphi(u) \leq v \implies \varphi\big(\varphi(u)\big) \leq \varphi(v) \implies u \leq \varphi(v)$$

and

$$u \leq \varphi(v) \implies \varphi(u) \leq \varphi\big(\varphi(v)\big) \implies \varphi(u) \leq v\,.$$

Thus, by Definition 10.1 and Remark 10.1, assertion (1) also holds. ■

Moreover, in addition to Theorem 11.3, we also prove the following result.

Theorem 12.5. *If f is g–normal, and X and Y are posets, then the following assertions are equivalent:*

(1) *f is onto Y;* (2) *g is injective;* (3) *$f \circ g$ is the identity function.*

Proof. Now, by Corollary 10.1, f is $g \circ f$–regular. Thus, by Theorem 11.1 and the antisymmetry of Y, we have

$$f(x) = \big(f \circ (g \circ f)\big)(x) = f\,\big(g\big(f(x)\big)\big)$$

for all $x \in X$. Hence, if assertion (1) holds, i. e., $Y \subseteq f[X]$, we infer that

$$y = f\big(g(y)\big) = (f \circ g)(y)$$

for all $y \in Y$. Therefore, assertions (2) and (3) also hold. On the other hand, by using Theorem 11.3, we see that assertions (2) and (3) are equivalent. Moreover, if assertion (3) holds, then we have $f\big(g(x)\big)=y$ for all $y \in Y$. Therefore,

$$f\big[g[Y]\big]=Y,$$

and thus assertion (1) also holds. ■

Now, by using this theorem and Theorem 11.3, we also prove the following result.

Corollary 12.2. *If X and Y are posets, f is g–normal, injective and onto Y, then $g=f^{-1}$.*

Proof. Namely, by Theorem 12.5, we have $f\circ g=\Delta_Y$. Moreover, by Corollary 10.1 and Theorem 11.3, we have $g\circ f=\Delta_X$. Therefore, $g=f^{-1}$. ■

Remark 12.2. Thus, if f is g–normal, then g may be considered as a certain generalized inverse function of f.

13. Characterizations of normal functions

Simple reformulations of properties (1) and (2) in Definition 10.1 yield the next result.

Theorem 13.1. *The following assertions are equivalent:*

(1) *f is g–normal;* (2) $\operatorname{Int}_f(y)=\operatorname{lb}\big(g(y)\big)$ *for all* $y \in Y$.

Proof. If assertion (1) holds, then by the corresponding definitions, for all $x \in X$ and $y \in Y$, we have

$$x\in \operatorname{Int}_f(y) \iff f(x)\le y \iff x\le g(y) \iff x\in \operatorname{lb}\big(g(y)\big).$$

Therefore, assertion (2) also holds. The converse implication can be proved quite similarly. ■

From this theorem, by using Theorem 12.4, we immediately derive the following result.

Corollary 13.1. *If X is a proset, then the following assertions are equivalent:*

(1) *φ is an invoulution operation;* (2) $\operatorname{Int}_\varphi(v)=\operatorname{lb}\big(\varphi(v)\big)$ *for all* $v\in X$.

Now, by using our former results, we also prove the next theorem.

Theorem 13.2. *If X and Y are prosets, then the following assertions are equivalent:*

(1) *f is g–normal;* (2) *f is increasing and* $g(y)\in \max\big(\operatorname{Int}_f(y)\big)$ *for all* $y\in Y$.

Proof. If assertion (1) holds, then by Theorem 12.1, f is increasing. Moreover, by Corollary 12.1, $f\circ g$ is intensive. Therefore, for any $y\in Y$, we have $f\big(g(y)\big)\le y$, and thus $g(y)\in \operatorname{Int}_f(y)$. On the other hand, from Theorem 13.1 we see that $\operatorname{Int}_f(y)=\operatorname{lb}\big(g(y)\big)$, and thus $\operatorname{Int}_f(y)\subseteq \operatorname{lb}\big(g(y)\big)$. Hence, by using Theorem 5.2, we already infer that $g(y)\in\max\big(\operatorname{Int}_f(y)\big)$, and thus assertion (2) also holds.

To prove the converse implication, suppose now that assertion (2) holds, and $x\in X$ and $y\in Y$. Then, by using Theorem 13.1, we can see that

$$f(x)\le y \implies x\in\operatorname{Int}_f(y) \implies x\in \operatorname{lb}\big(g(y)\big) \implies x\le g(y).$$

Therefore, f is right g–seminormal. Moreover, if $x\le g(y)$, then by using the increasingness of f we infer that

$$f(x)\le f\big(g(y)\big).$$

On the other hand, from the assumption that $g(y)\in\max\big(\operatorname{Int}_f(y)\big)$, we infer that $g(y)\in\operatorname{Int}_f(y)$, and thus $f\big(g(y)\big)\le y$. Hence, by using the transitivity of Y, we already see that $f(x)\le y$, and thus f is also left g–seminormal. Therefore, assertion (1) also holds. ■

From this theorem, by using Theorem 12.4, we immediately derive the following corollary.

Corollary 13.2. *If X is a proset, then the following assertions are equivalent:*

(1) *φ is an invoulution operation on X;* (2) *φ is increasing and* $\varphi(v)\in\max\big(\operatorname{Int}_\varphi(v)\big)$ *for all* $v\in Y$.

Moreover, as an immediate consequence of Theorem 13.2, we also state the next result.

Theorem 13.3. *If X and Y are prosets, then the following assertions are equivalent:*

(1) *f is normal;* (2) *f is increasing and $\max\big(\mathrm{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$.*

Proof. To prove the implication (2) $\Longrightarrow$ (1), note that if the second part of assertion (2) holds, then by the Axiom of Choice there exists a function h of Y to X such that $h(y) \in \max\big(\mathrm{Int}_f(y)\big)$ for all $y \in Y$. Thus, if in addition f is increasing, then by Theorem 13.2, f is h–normal. Thus, in particular assertion (1) also holds. ∎

Corollary 13.3. *If X and Y are prosets and X is $\max$-complete, then the following assertions are equivalent:*

(1) *f is normal;* (2) *f is increasing and $f[X]$ is cofinal in the dual of Y.*

Proof. Since X is $\max$–complete, for any $y \in Y$, we have

$$\max\big(\mathrm{Int}_f(y)\big) \neq \emptyset \iff \mathrm{Int}_f(y) \neq \emptyset \iff \exists\ x \in X: \quad x \in \mathrm{Int}_f(y) \iff \exists\ x \in X: \quad f(x) \leq y\,.$$

Therefore,

$$\forall\ y \in Y: \quad \max\big(\mathrm{Int}_f(y)\big) \neq \emptyset \iff \forall\ y \in Y: \quad \exists\ x \in X: \quad y \leq' f(x)\,.$$

That is, $F[X]$ is cofinal (dense) in Y'. Thus, Theorem 13.3 can be applied to obtain the required equivalence. ∎

Corollary 13.4. *If X and Y are prosets, X is $\max$-complete and f is onto Y, then the following assertions are equivalent:*

(1) *f is normal;* (2) *f is increasing.*

Remark 13.1. Because of the above results, the normality of f may be naturally considered as a strong increasingness of f. Moreover, by Theorem 13.2, we may naturally define a relation G_f on Y to X such that, for all $y \in Y$,

$$G_f(y) = \max\big(\mathrm{Int}_f(y)\big)\,.$$

Thus, the second part of assertion (2) of Theorem 13.2 can be reformulated in form that g is a selection of G_f. Moreover, G_f may be studied separately.

14. Supremum properties of normal functions

The following theorem will allow us to easily prove a useful supremum property of normal functions which fails to hold for increasing functions.

Theorem 14.1. *If f is normal, then for any $A \subseteq X$ we have*

$$f\big[\mathrm{lb}\big(\mathrm{ub}(A)\big)\big] \subseteq \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big)\,.$$

Proof. If $y \in f\big[\mathrm{lb}\big(\mathrm{ub}(A)\big)\big]$, then there exists $x \in \mathrm{lb}\big(\mathrm{ub}(A)\big)$ such that $y = f(x)$. Moreover, if $b \in \mathrm{ub}\big(f[A]\big)$, then for any $a \in A$ we have $f(a) \leq b$. Hence, by using that f is h–normal, for some function h of Y to X, we infer that $a \leq h(b)$. Therefore, $h(b) \in \mathrm{ub}(A)$, and thus because of $x \in \mathrm{lb}\big(\mathrm{ub}(A)\big)$ we have $x \leq h(b)$. Hence, by using that f is h–normal, we infer that $f(x) \leq b$, and thus $y \leq b$. Therefore, $y \in \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big)$ also holds. ∎

From this theorem, by using Theorem 7.7, we derive the following corollary.

Corollary 14.1. *If f is normal, then for any $y \in Y$ we have*

$$\max\big(\mathrm{Int}_f(y)\big) = \sup\big(\mathrm{Int}_f(y)\big)\,.$$

Proof. By Theorems 5.3 and 14.1, we have

$$f\big[\sup(A)\big] = f\big[\mathrm{ub}(A) \cap \mathrm{lb}\big(\mathrm{ub}(A)\big)\big] \subseteq f\big[\mathrm{lb}\big(\mathrm{ub}(A)\big)\big] \subseteq \mathrm{lb}\big(\mathrm{ub}\big(f[A]\big)\big)\,.$$

for all $A \subseteq X$. Hence, by using Theorem 7.7, we already see that the required equality is also true. ∎

Moreover, from Theorem 14.1, by using Theorem 6.2, we also derive the following result.

Theorem 14.2. *If f is increasing and normal, then for any $A \subseteq X$ we have*

$$f\big[\sup(A)\big] \subseteq \sup\big(f[A]\big)\,.$$

Proof. By Theorems 5.3, 6.2 and 14.1, we have

$$f\left[\sup(A)\right]=f\left[\mathrm{ub}(A)\cap\mathrm{lb}\left(\mathrm{ub}(A)\right)\right]\subseteq f\left[\mathrm{ub}(A)\right]\cap f\left[\mathrm{lb}\left(\mathrm{ub}(A)\right)\right]\subseteq\mathrm{ub}\left(f\left[A\right]\right)\cap\mathrm{lb}\left(\mathrm{ub}\left(f\left[A\right]\right)\right)=\sup\left(f\left[A\right]\right).$$

■

From Theorems 12.1 and 14.2, the next corollary follows.

Corollary 14.2. *If f is normal, X is transitive and Y is reflexive, then for any $A\subseteq X$ we have*

$$f\left[\sup(A)\right]\subseteq\sup\left(f\left[A\right]\right).$$

Now, to obtain some partial converses of the above theorems, we also prove the following result.

Theorem 14.3. *If X is a sup–complete proset and Y is an arbitrary proset, then the following assertions are equivalent :*

(1) *f is normal ;*

(2) *$f\left[\sup(A)\right]\subseteq\sup\left(f\left[A\right]\right)$ for all $A\subseteq X$;*

(3) *f is increasing and $\sup\left(\mathrm{Int}_f(y)\right)\subseteq\mathrm{Int}_f(y)\big)$ for all $y\in Y$;*

(4) *f is increasing and $\max\left(\mathrm{Int}_f(y)\right)=\sup\left(\mathrm{Int}_f(y)\right)$ for all $y\in Y$;*

(5) *f is increasing and $f\left[\sup(A)\right]\subseteq\mathrm{lb}\left(\mathrm{ub}\left(f\left[A\right]\right)\right)$ for all $A\subseteq X$;*

(6) *f is increasing and $f\left[\mathrm{lb}\left(\mathrm{ub}(A)\right)\right]\subseteq\mathrm{lb}\left(\mathrm{ub}\left(f\left[A\right]\right)\right)$ for all $A\subseteq X$.*

Proof. From Corollary 14.2, we see that (1) implies (2). Moreover, from Theorems 12.1 and 14.1, we see that (1) also implies (6). On the other hand, from Theorem 5.3, we see that $\sup(A)\subseteq\mathrm{lb}\big(\mathrm{ub}(A)$, and thus

$$f\left[\sup(A)\right]\subseteq f\left[\mathrm{lb}\left(\mathrm{ub}(A)\right)\right].$$

Therefore, (6) implies (5). Moreover, if (5) holds, then by Theorem 7.7 we see that (4) also holds. While, if (4) holds, then by sup–completeness of X, we have

$$\max\left(\mathrm{Int}_f(y)\right)=\sup\left(\mathrm{Int}_f(y)\right)\neq\emptyset$$

for all $y\in Y$. Thus, from Theorem 13.3, we see that (1) also holds. On the other hand, if (2) holds, then by using Theorem 5.3 we see that

$$f\left[\max(A)\right]\subseteq f\left[\sup(A)\right]\subseteq\sup\left(f\left[A\right]\right)\subseteq\mathrm{ub}\left(f\left[A\right]\right)$$

for all $A\subseteq X$. Thus, by Theorem 6.3, f is increasing. Moreover, we also note that

$$f\left[\sup(A)\right]\subseteq\sup\left(f\left[A\right]\right)\subseteq\mathrm{lb}\left(\mathrm{ub}\left(f\left[A\right]\right)\right)$$

for all $A\subseteq X$. Therefore, (5), and thus (1) also holds. Now, to complete the proof, it remains to note only that, because of Theorem 5.3, (3) and (4) are also equivalent. ■

Remark 14.1. Note that if in addition X is antisymmetric, then instead of (2) we may write that $f\left(\sup(A)\right)\in\sup\left(f\left[A\right]\right)$ for all $A\subseteq X$. While, if in addition both X and Y are antisymmetric, then instead of (2) we may write that

$$f\left(\sup(A)\right)=\sup\left(f\left[A\right]\right)$$

for all $A\subseteq X$.

15. Characterizations of regular functions

Simple reformulations of properties (1) and (2) in Definition 10.2 yield the next theorem.

Theorem 15.1. *The following assertions are equivalent :*

(1) *f is φ–regular ;*

(2) *$\mathrm{Ord}_f^{-1}(x)=\mathrm{lb}\left(\varphi(x)\right)$ for all $x\in X$;* (3) *$\mathrm{Int}_f\left(f(x)\right)=\mathrm{lb}\left(\varphi(x)\right)$ for all $x\in X$.*

Proof. If assertion (1) holds, then by the corresponding definition we have

$$u \in \operatorname{Int}_f\big(f(x)\big) \iff f(u) \leq f(x) \iff u \leq \varphi(x) \iff u \in \operatorname{lb}\big(\varphi(x)\big)$$

for all $x, u \in X$. Therefore, assertion (3) also holds. The converse implication can be proved quite similarly. Moreover, by Theorem 7.5, we see that assertions (2) and (3) are also equivalent. ■

From this theorem, by using Theorem 11.2, we immediately derive the following result.

Corollary 15.1. *If X is a proset, then the following assertions are equivalent:*

(1) *φ is a closure operation;*

(2) $\operatorname{Ord}_\varphi^{-1}(x) = \operatorname{lb}\big(\varphi(x)\big)$ *for all* $x \in X$; (3) $\operatorname{Int}_\varphi\big(\varphi(x)\big) = \operatorname{lb}\big(\varphi(x)\big)$ *for all* $x \in X$.

Now, analogously to Theorem 13.2, we also prove the following result.

Theorem 15.2. *If X and Y are prosets, then the following assertions are equivalent:*

(1) *f is φ–regular;*

(2) *f is increasing and* $\varphi(x) \in \max\big(\operatorname{Ord}_f^{-1}(x)\big)$ *for all* $x \in X$;

(3) *f is increasing and* $\varphi(x) \in \max\big(\operatorname{Int}_f\big(f(x)\big)\big)$ *for all* $x \in X$.

Proof. If assertion (1) holds, then by Theorem 11.1, f is increasing. Moreover, $f\big(\varphi(x)\big) \leq f(x)$, and thus $\varphi(x) \in \operatorname{Int}_f\big(f(x)\big)$. On the other hand, by Theorem 15.1, we have $\operatorname{Int}_f\big(f(x)\big) = \operatorname{lb}\big(\varphi(x)\big)$, and thus $\operatorname{Int}_f\big(f(x)\big) \subseteq \operatorname{lb}\big(\varphi(x)\big)$. Hence, by using Theorem 5.2, we already infer that $\varphi(x) \in \max\big(\operatorname{Int}_f\big(f(x)\big)\big)$, and thus assertion (3) also holds.

To prove the converse implication, suppose now that assertion (3) holds, and $u, v \in X$. Then, by using Theorem 15.1, we see that

$$f(u) \leq f(v) \implies u \in \operatorname{Int}_f\big(f(v)\big) \implies u \in \operatorname{lb}\big(\varphi(v)\big) \implies u \leq \varphi.$$

Therefore, f is right φ–semiregular. Moreover, if $u \leq \varphi(v)$, then by the increasingness of f we also have $f(u) \leq f\big(\varphi(v)\big)$. On the other hand, from the assumption $\varphi(v) \in \max\big(\operatorname{Int}_f\big(f(v)\big)\big)$, we infer that $\varphi(v) \in \operatorname{Int}_f\big(f(v)\big)$, and thus $f\big(\varphi(v)\big) \leq f(v)$. Hence, by using the transitivity of Y, we already see that $f(u) \leq f(v)$, and thus f is also left seminormal. Therefore, assertion (1) also holds. ■

From this theorem, by using Theorem 11.2, we immediately derive the following result.

Corollary 15.2. *If X is a proset, then the following assertions are equivalent:*

(1) *φ is a closure operation;*

(2) *φ is increasing and* $\varphi(x) \in \max\big(\operatorname{Ord}_\varphi^{-1}(x)\big)$ *for all* $x \in X$;

(3) *f is increasing and* $\varphi(x) \in \max\big(\operatorname{Int}_\varphi\big(\varphi(x)\big)\big)$ *for all* $x \in X$.

Moreover, as an immediate consequence of Theorem 15.2, we also state the following result.

Theorem 15.3. *If X and Y are prosets, then the following assertions are equivalent:*

(1) *f is regular;*

(2) *f is increasing and* $\max\big(\operatorname{Ord}_f^{-1}(x)\big) \neq \emptyset$ *for all* $x \in X$;

(3) *f is increasing and* $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) \neq \emptyset$ *for all* $x \in X$.

Now, by using this theorem and Theorem 13.3, we also prove the following result.

Corollary 15.3. *If X and Y are prosets and f is onto Y, then the following assertions are equivalent:*

(1) *f is regular;* (2) *f is normal.*

Proof. If assertion (1) holds, then by Theorem 15.2 $\max\big(\operatorname{Int}_f\big(f(x)\big)\big) \neq \emptyset$ for all $x \in X$. Hence, since $Y = f[X]$, we infer that $\max\big(\operatorname{Int}_f(y)\big) \neq \emptyset$ for all $y \in Y$. Therefore, by Theorem 13.3, assertion (2) also holds. Moreover, by Corollary 10.1, the converse implication is always true. ■

Now, analogously to Theorem 14.3, we also prove the following result.

Theorem 15.4. *If X is a* sup*–complete proset, Y is an arbitrary proset and f is onto Y, then the following assertions are equivalent:*

(1) *f is regular;*

(2) $f\,[\sup(A)] \subseteq \sup\big(f\,[A]\big)$ *for all* $A \subseteq X$;

(3) *f is increasing and* $\max\big(\operatorname{Ord}_f^{-1}(x)\big) = \sup\big(\operatorname{Ord}_f^{-1}(x)\big)$ *for all* $x \in X$.

From this theorem, by using Theorem 11.2, we immediately derive the following result.

Corollary 15.4. *If X is a* sup*–complete proset and φ is onto X, then the following assertions are equivalent:*

(1) *φ is a closure operation;*

(2) $\varphi\,[\sup(A)] \subseteq \sup\big(\varphi\,[A]\big)$ *for all* $A \subseteq X$;

(3) *f is increasing and* $\max\big(\operatorname{Ord}_\varphi^{-1}(x)\big) = \sup\big(\operatorname{Ord}_\varphi^{-1}(x)\big)$ *for all* $x \in X$.

Remark 15.1. Note that, in Theorem 15.4 and Corollary 15.4, we may write $\operatorname{Int}_f\big(f(x)\big)$ and $\operatorname{Int}_\varphi\big(\varphi(x)\big)$ in place of $\operatorname{Ord}_f^{-1}(x)$ and $\operatorname{Ord}_\varphi^{-1}(x)$, respectively. Moreover, by Theorem 15.2, for the function f we may naturally define a relation Φ_f on X such that, for all $x \in X$,

$$\Phi_f(x) = \max\big(\operatorname{Int}_f\big(f(x)\big)\big)\,.$$

Thus, we have $\Phi_f = G_f \circ f$. Moreover, the second part of assertion (3) of Theorem 15.2 can be reformulated in the more instructive form that φ is a selection function of the relation Φ_f.

16. Some basic properties of super relations

Notation 16.1. In this section, we assume that U is a super relation on X to Y.

Remark 16.1. Thus, by our former definition, U is actually an ordinary relation on $\mathcal{P}(X)$ to Y, i.e., it is an arbitrary subset of $\mathcal{P}(X)\times Y$. Moreover, U can be identified with the set-valued function φ_U, defined by $\varphi_U(A) = U(A)$ for all $A \subseteq X$, which is a particular subset of $\mathcal{P}(X)\times\mathcal{P}(Y)$.

Several properties of the super relation U can be easily defined with the help of the set-valued function φ_U. For instance, we naturally introduce the following definition.

Definition 16.1. The super relation U will be called

(1) *increasing* if $U(A) \subseteq U(B)$ for all $A \subseteq B \subseteq X$;

(2) *quasi-increasing* if $U(\{x\}) \subseteq U(A)$ for all $x \in A \subseteq X$;

(3) *union-preserving* if $U\big(\bigcup \mathcal{A}\big) = \bigcup_{A\in\mathcal{A}} U(A)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$.

We easily establish the following two theorems.

Theorem 16.1. *The following assertions are equivalent:*

(1) *U is quasi-increasing;* (2) $\bigcup_{x\in A} U\big(\{x\}\big) \subseteq U(A)$ *for all* $A \subseteq X$.

Theorem 16.2. *The following assertions are equivalent:*

(1) *U is increasing;*

(2) $U\big(\bigcap \mathcal{A}\big) \subseteq \bigcap_{A\in\mathcal{A}} U(A)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$; (3) $\bigcup_{A\in\mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ *for all* $\mathcal{A} \subseteq \mathcal{P}(X)$.

Proof. If $A_1 \subseteq A_2 \subseteq X$, then by using a particular case of (3) we see that

$$U(A_1) \subseteq U(A_1) \cup U(A_2) \subseteq U(A_1 \cup A_2) = U(A_2)\,,$$

and thus (1) also holds. ■

Moreover, by using Definition 16.1 and Theorem 16.2, we also prove the next theorem.

Theorem 16.3. *The following assertions are equivalent:*

(1) *U is union-preserving;* (2) $U(A) = \bigcup_{x\in A} U\big(\{x\}\big)$ *for all* $A \subseteq X$.

Proof. Since $A = \bigcup_{x\in A}\{x\}$ for all $A \subseteq X$, it is clear that (1) implies (2). While, if (2) holds, then we at once see that U is increasing. Thus, by Theorem 16.2, we have $\bigcup_{A\in\mathcal{A}} U(A) \subseteq U\big(\bigcup \mathcal{A}\big)$ for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Therefore, to obtain (1), we need only prove the converse inclusion. For this, note that if $\mathcal{A} \subseteq \mathcal{P}(X)$, then by (2) we have

$$U\big(\bigcup \mathcal{A}\big) = \bigcup_{x\in\bigcup\mathcal{A}} U(\{x\}).$$

Therefore, if $y \in U\big(\bigcup \mathcal{A}\big)$, then there exists $x \in \bigcup \mathcal{A}$ such that $y \in U\big(\{x\}\big)$. Thus, in particular, there exists $A_0 \in \mathcal{A}$ such that $x \in A_0$, and so, $\{x\} \subseteq A_0$. Hence, by using the increasingness of U, we already see that

$$y \in U\big(\{x\}\big) \subseteq U(A_0) \subseteq \bigcup_{A\in\mathcal{A}} U(A).$$

Therefore, $U\big(\bigcup \mathcal{A}\big) \subseteq \bigcup_{A\in\mathcal{A}} U(A)$ also holds. ∎

Remark 16.2. In particular, a super relation U on X to itself may be simply called a super relation on X. Thus, a super relation U on X may be called *extensive, intensive, involutive* and *idempotent* if $A \subseteq U(A)$, $U(A) \subseteq A$, $U\big(U(A)\big) = A$ and $U\big(U(A)\big) = U(A)$ for all $A \subseteq X$, respectively. Moreover, an increasing involutive (idempotent) super relation may be called an *involution (projection) relation.* While, an extensive (intensive) projection relation may be called a *closure (interior) relation.*

17. Relationships between ordinary and super relations

Notation 17.1. In this and the next two sections, we assume that

(1) U and V are super relations on X to Y;

(2) R and S are ordinary relations on X to Y.

In [85], having in mind Galois connections [13, 75], we have introduced the following definition.

Definition 17.1. For the ordinary relation R, we define a super relation $R^{\triangleright}$ on X to Y such that

$$R^{\triangleright}(A) = R[A] \quad \textit{for all} \quad A \subseteq X.$$

While, for the super relation U, we define an ordinary relation $U^{\triangleleft}$ on X to Y such that

$$U^{\triangleleft}(x) = U\big(\{x\}\big) \quad \textit{for all} \quad x \in X.$$

The appropriateness of these definitions is apparent from the following two theorems whose proofs are included here only for the reader's convenience.

Theorem 17.1. $R^{\triangleright} \subseteq U$ *implies* $R \subseteq U^{\triangleleft}$.

Proof. If $R^{\triangleright} \subseteq U$, then, in particular, we have

$$R(x) = R\big[\{x\}\big] = R^{\triangleright}\big(\{x\}\big) \subseteq U\big(\{x\}\big) = U^{\triangleleft}(x)$$

for all $x \in X$. Therefore, $R \subseteq U^{\triangleleft}$ also holds. ∎

This theorem shows that the operation $\triangleright$ is right $\triangleleft$–seminormal. While, from the next theorem we can see that the operation $\triangleright$ need not be left $\triangleleft$–seminormal.

Theorem 17.2. *The following assertions are equivalent:*

(1) $U^{\triangleleft\triangleright} \subseteq U$; (2) U *is quasi-increasing;* (3) $R \subseteq U^{\triangleleft}$ *implies* $R^{\triangleright} \subseteq U$.

Proof. If $R \subseteq U^{\triangleleft}$ and assertion (2) holds, then

$$R^{\triangleright}(A) = R[A] = \bigcup_{x\in A} R(x) \subseteq \bigcup_{x\in A} U^{\triangleleft}(x) = \bigcup_{x\in A} U\big(\{x\}\big) \subseteq U(A)$$

for all $A \subseteq X$. Therefore, $R^{\triangleright} \subseteq U$, and thus assertion (3) also holds.

Now, since by taking $R = U^{\triangleleft}$, assertion (3) trivially implies (1), we need only show that (1) also implies (2). For this, note that if assertion (1) holds, then for any $A \subseteq X$ we have $U^{\triangleleft\triangleright}(A) \subseteq U(A)$. Moreover, by using the corresponding definitions, we see that

$$U^{\triangleleft\triangleright}(A) = \big(U^{\triangleleft}\big)^{\triangleright}(A) = U^{\triangleleft}[A] = \bigcup_{x\in A} U^{\triangleleft}(x) = \bigcup_{x\in A} U\big(\{x\}\big).$$

Therefore, $\bigcup_{x\in A} U\big(\{x\}\big) \subseteq U(A)$. Thus, assertion (2) also holds. ∎

Now, as an immediate consequence of the above two theorems, we also state the following result.

Corollary 17.1. *If U is quasi-increasing, then*

$$R^{\triangleright} \subseteq U \iff R \subseteq U^{\triangleleft}.$$

Remark 17.1. This shows that the operations $\triangleright$ and $\triangleleft$ establish a "partial Galois connection between" the power sets $\mathcal{P}(X \times Y)$ and $\mathcal{P}\big(\mathcal{P}(X) \times Y\big)$.

Therefore, by Theorem 12.2, we also naturally introduce the following definition.

Definition 17.2. The super relation

$$U^{\circ} = U^{\triangleleft \triangleright}$$

will be called the *Galois interior* of U.

Thus, by the proof of Theorem 17.2, we at once state the following result.

Theorem 17.3. *For any $A \subseteq X$, we have*

$$U^{\circ}\big(A\big) = U^{\triangleleft}[A] = \bigcup_{x \in A} U^{\triangleleft}(x) = \bigcup_{x \in A} U\big(\{x\}\big).$$

Moreover, by letting A to be a singleton, we also state the following result.

Corollary 17.2. *For any $x \in X$, we have*

$$U^{\circ \triangleleft}(x) = U^{\circ}\big(\{x\}\big) = U^{\triangleleft}(x) = U\big(\{x\}\big).$$

Theorem 17.3 allows us to easily determine the Galois interior of some very particular super relations.

Example 17.1. If $U\big(\{x\}\big) = \emptyset$ for all $x \in X$, then $U^{\circ}(A) = \emptyset$ for all $A \subseteq X$.

Example 17.2. If $U\big(\{x\}\big) = \{x\}$ for all $x \in X$, then $U^{\circ}(A) = A$ for all $A \subseteq X$.

Example 17.3. If $U(A) = A^c$ for all $A \subseteq X$, then for any $A \subseteq X$ we have

$$U^{\circ}(A) = \begin{cases} \emptyset & \text{if} \quad \operatorname{card}(A) = 0, \\ A^c & \text{if} \quad \operatorname{card}(A) = 1, \\ X & \text{if} \quad \operatorname{card}(A) > 1. \end{cases}$$

Namely, by Theorem 17.3 and De Morgan's law, we have

$$U^{\circ}(A) = \bigcup_{x \in A} U\big(\{x\}\big) = \bigcup_{x \in A} \{x\}^c = \Big(\bigcap_{x \in A} \{x\}\Big)^c,$$

whence the required equalities immediately follow.

Remark 17.2. Note that if U is as in Example 17.1, then $U^{\triangleleft} = \emptyset$ and $U^{\circ} = \emptyset$. While, if U is as in Example 17.2, then $U^{\triangleleft} = \Delta_X$ and $U^{\circ} = \Delta_{\mathcal{P}(X)}$. However, if U is as in Example 17.3, then for any $x, y \in X$ we have

$$y \in U^{\triangleleft}(x) \iff y \in U\big(\{x\}\big) \iff y \in \{x\}^c \iff y \notin \{x\} \iff y \neq x.$$

Therefore, $U^{\triangleleft}$ is just the diversity relation on X.

18. Some further theorems on the operations $\triangleright$, $\triangleleft$, and $\circ$

Several properties of the operations $\triangleright$, $\triangleleft$, and $\circ$ can be immediately derived from the general theory of Galois and Pataki connections [13, 88]. However, because of the simplicity of Definition 17.1, it is now more convenient to apply some direct proofs to establish the following four theorems.

Theorem 18.1. *The operations $\triangleright$, $\triangleleft$, and $\circ$ are increasing.*

Proof. For instance, if $U \subseteq V$, then $U(A) \subseteq V(A)$ for all $A \subseteq X$. Thus, in particular, we also have

$$U^{\triangleleft}(x) = U\big(\{x\}\big) \subseteq V\big(\{x\}\big) = V^{\triangleleft}(x)$$

for all $X \in X$. Therefore, $U^{\triangleleft} \subseteq V^{\triangleleft}$ also holds. ■

Theorem 18.2. *$R^{\triangleright}$ is a union-preserving super relation on X to Y such that*

(1) $R^{\triangleright\triangleleft} = R$; (2) $R^{\triangleright\circ} = R^{\triangleright}$.

Proof. By the corresponding definitions, we have

$$R^{\triangleright}(A) = R[A] = \bigcup_{x\in A} R(x) = \bigcup_{x\in A} R[\{x\}] = \bigcup_{x\in A} R^{\triangleright}(\{x\})$$

for all $A \subseteq X$. Thus, by Theorem 16.3, the super relation $R^{\triangleright}$ is union-preserving. Moreover, we easily see that

$$R(x) = R[\{x\}] = R^{\triangleright}(\{x\}) = R^{\triangleright\triangleleft}(x)$$

for all $x \in X$. Therefore, assertion (1) is true. Now, by using Definition 17.2 and assertion (1), we also easily see that

$$R^{\triangleright\circ} = \left(R^{\triangleright}\right)^{\circ} = \left(R^{\triangleright}\right)^{\triangleleft\triangleright} = \left(R^{\triangleright\triangleleft}\right)^{\triangleright} = R^{\triangleright}.$$

Therefore, assertion (2) is also true. ∎

By using the above two theorems, we easily establish the following result.

Corollary 18.1. *We have*

$$R \subseteq S \iff R^{\triangleright} \subseteq S^{\triangleright}.$$

Moreover, in addition to Theorem 18.2, we also prove the following result.

Theorem 18.3. *U° is a union-preserving super relation on X to Y such that*

(1) $U^{\circ\triangleleft} = U^{\triangleleft}$; (2) $U^{\circ\circ} = U^{\circ}$.

Proof. From Definition 17.2, by Theorem 18.2, it is clear that U° is union-preserving. Moreover, from Corollary 17.2, we see that $U^{\circ\triangleleft}(x) = U^{\triangleleft}(x)$ for all $x \in X$, and thus $U^{\circ\triangleleft} = U^{\triangleleft}$ is also true. Furthermore, by using Theorem 17.3, we easily see that

$$U^{\circ\circ}(A) = \left(U^{\circ}\right)^{\circ}(A) = \bigcup_{x\in A} U^{\circ}(\{x\}) = \bigcup_{x\in A} U(\{x\}) = U^{\circ}(A)$$

for all $A \subseteq X$, and thus $U^{\circ\circ} = U^{\circ}$ is also true. ∎

Now, to characterize union-preserving super relations, we also prove the following result.

Theorem 18.4. *The following assertions are equivalent:*

(1) $U^{\circ} = U$; (2) *U is union-preserving;* (3) *$U = R^{\triangleright}$ for some relation R on X to Y.*

Proof. If (2) holds, then by Theorem 17.3 we see that $U^{\circ}(A) = \bigcup_{x\in A} U(\{x\}) = U(A)$ for all $A \subseteq X$. Therefore, (1) also holds. Moreover, if (1) holds, then by Definition 17.2 it is clear that (3) also holds. Therefore, to complete the proof, we need only note that if (3) holds, then by Theorem 18.2 assertion (2) also holds. ∎

Next, by using Theorems 18.1 and 18.4, we also easily establish the following result.

Corollary 18.2. *If U and V are union-preserving, then*

$$U \subseteq V \iff U^{\triangleleft} \subseteq V^{\triangleleft}.$$

By using our former results, the following four theorems can also be proved.

Theorem 18.5. *We have*

(1) $U \subseteq U^{\circ} \iff U(A) \subseteq U^{\triangleleft}[A]$ *for all* $A \subseteq X$;

(2) $U^{\circ} \subseteq U \iff$ *U is quasi-increasing* $\iff U^{\triangleleft}[A] \subseteq U(A)$ *for all* $A \subseteq X$;

(3) $U^{\circ} = U \iff$ *U is union-preserving* $\iff U(A) = U^{\triangleleft}[A]$ *for all* $A \subseteq X$.

Theorem 18.6. *We have*

(1) $U^{\circ} \subseteq V \implies U^{\circ} \subseteq V^{\circ} \iff U^{\triangleleft} \subseteq V^{\triangleleft}$;

(2) $U^{\circ} \subseteq V^{\circ} \implies U^{\circ} \subseteq V$ *if V is quasi-increasing;*

(3) $U \subseteq V \iff U^{\circ} \subseteq V^{\circ}$ *if U and V are union-preserving.*

Theorem 18.7. *If* $U = R^{\triangleright}$*, then*

(1) U *is an union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} = R$*;*

(2) U *is the smallest quasi-increasing super relation on* X *to* Y *such that* $R \subseteq U^{\triangleleft}$*;*

(3) U *is the largest union-preserving super relation on* X *to* Y *such that* $U^{\triangleleft} \subseteq R$.

Theorem 18.8. *If* $R = U^{\triangleleft}$*, then*

(1) $R^{\triangleright} \subseteq U$ *if and only if* U *is quasi-increasing;*

(2) $R^{\triangleright} = U$ *if and only if* U *is union-preserving;*

(3) *if* U *is quasi-increasing, then* R *is the largest relation on* X *to* Y *such that* $R^{\triangleright} \subseteq U$*;*

(4) *if* U *is union-preserving, then* R *is the smallest relation on* X *to* Y *such that* $U \subseteq R^{\triangleright}$.

19. Relationally defined inverses of super relations

Because of Remark 17.1, we also naturally introduce the following definition.

Definition 19.1. The super relation

$$U^{\mathbf{-1}} = U^{\triangleleft\, -1\, \triangleright}$$

will be called the *relationally defined inverse* of U.

Remark 19.1. To feel the necessity of this bold inverse $U^{\mathbf{-1}}$, note that the ordinary inverse U^{-1} of U is not a super relation. While, the ordinary inverse φ_U^{-1} of the associated set-valued function φ_U, which can be identified with U, is usually a hyper relation.

Now, by using the corresponding definitions and Theorem 18.2, we easily prove the following three theorems.

Theorem 19.1. *We have*

(1) $R^{\triangleright\, \mathbf{-1}} = R^{-1\, \triangleright}$*;* (2) $R^{\triangleright\, \mathbf{-1}\, \triangleleft} = R^{-1}$.

Proof. By Definition 19.1 and Theorem 18.2, we have

$$R^{\triangleright\, \mathbf{-1}} = R^{\triangleright \triangleleft\, -1\, \triangleright} = R^{-1\, \triangleright}, \qquad \text{and thus also} \qquad R^{\triangleright\, \mathbf{-1}\, \triangleleft} = R^{-1\, \triangleright \triangleleft} = R^{-1}. \qquad \blacksquare$$

Theorem 19.2. $U^{\mathbf{-1}}$ *is a union-preserving super relation on* Y *to* X *such that*

(1) $U^{\mathbf{-1}\, \triangleleft} = U^{\triangleleft\, -1}$*;* (2) $U^{\circ\, \mathbf{-1}} = U^{\mathbf{-1}}$.

Proof. By Definitions 19.1 and 17.2 and Theorem 18.2, we have

$$U^{\mathbf{-1}\, \triangleleft} = U^{\triangleleft\, -1\, \triangleright \triangleleft} = U^{\triangleleft\, -1} \qquad \text{and} \qquad U^{\circ\, \mathbf{-1}} = U^{\triangleleft \triangleright \triangleleft\, -1\, \triangleright} = U^{\triangleleft\, -1\, \triangleright} = U^{\mathbf{-1}}. \qquad \blacksquare$$

Remark 19.2. Note that if $U^{\triangleleft}$ is symmetric, then $U^{\mathbf{-1}} = U^{\triangleleft\, -1\, \triangleright} = U^{\triangleleft \triangleright} = U^{\circ}$. Thus, if in addition U is union-preserving, then $U^{\mathbf{-1}} = U$. Moreover, if in particular U is as in Example 17.3, then by Remark 17.2 the relation $U^{\triangleleft}$ is symmetric. Thus, by the above observation, $U^{\mathbf{-1}} = U^{\circ}$.

Theorem 19.3. *We have*

(1) $U^{\mathbf{-1}\, \circ} = U^{\mathbf{-1}}$*;* (2) $\left(U^{\mathbf{-1}}\right)^{\mathbf{-1}} = U^{\circ}$.

Proof. Assertion (1) can be derived from Theorem 19.2 by using Theorem 18.3. Moreover, by the corresponding definitions and Theorem 18.2, we see that

$$\left(U^{\mathbf{-1}}\right)^{\mathbf{-1}} = U^{\triangleleft\, -1\, \triangleright \triangleleft\, -1\, \triangleright} = U^{\triangleleft\, -1\, -1\, \triangleright} = U^{\triangleleft \triangleright} = U^{\circ}. \qquad \blacksquare$$

Hence, by using Theorem 18.4, we immediately derive the following result.

Corollary 19.1. *The following assertions are equivalent:*

(1) $U = \left(U^{\mathbf{-1}}\right)^{\mathbf{-1}}$*;* (2) U *is union-preserving.*

Moreover, as a counterpart of Theorem 18.6, we also prove the following result.

Theorem 19.4. *For any $B \subseteq Y$, we have*

$$U^{-1}(B) = U^{\triangleleft -1}[B] = \{ x \in X : \quad U^{\triangleleft}(x) \cap B \neq \emptyset \} = \{ x \in X : \quad U(\{x\}) \cap B \neq \emptyset \}.$$

Proof. By the corresponding definitions, we have

$$U^{-1}(B) = U^{\triangleleft -1 \triangleright}(B) = U^{\triangleleft -1}[B].$$

Moreover, it is clear that, for any $x \in X$, we have

$$x \in U^{\triangleleft -1}[B] \iff U^{\triangleleft}(x) \cap B \neq \emptyset \iff U(\{x\}) \cap B \neq \emptyset.$$

Therefore, the required equalities are true. ■

Remark 19.3. From the above theorem, by Theorem 19.2, we also see that

$$U^{-1}(B) = U^{\triangleleft -1}[B] = U^{-1 \triangleleft}[B].$$

20. Functionally defined compositions of super relations

Notation 20.1. In this and the next section, we assume that:

(1) R is an ordinary relation and U is super relation on X to Y;

(2) S is an ordinary relation and V is super relation on Y to Z.

By the usual identification of the relation U with the function φ_U, we naturally introduce the following definition.

Definition 20.1. The super relation $V \circ U$, defined such that

$$(V \circ U)(A) = V(U(A))$$

for all $A \subseteq X$, will be called the *functionally defined composition* of V and U.

Remark 20.1. Namely, thus we easily check that $\varphi_{V \circ U}(A) = (\varphi_V \circ \varphi_U)(A)$ for all $A \subseteq X$, and thus $\varphi_{V \circ U} = \varphi_V \circ \varphi_U$ also holds.

The appropriateness of Definition 20.1 is also quite obvious from the following three theorems and their corollaries.

Theorem 20.1. *We have*

$$(S \circ R)^{\triangleright} = S^{\triangleright} \circ R^{\triangleright}.$$

Proof. By the Definitions 17.1 and 20.1, we have

$$(S \circ R)^{\triangleright}(A) = (S \circ R)[A] = S[R[A]] = S^{\triangleright}(R^{\triangleright}(A)) = (S^{\triangleright} \circ R^{\triangleright})(A)$$

for all $A \subseteq X$. Thus, the required equality is also true. ■

From this theorem, by using Definition 17.2 and Theorem 18.4, we derive the following result.

Corollary 20.1. *We have*

(1) $(S \circ U^{\triangleleft})^{\triangleright} = S^{\triangleright} \circ U$ *if U is union-preserving;* (2) $(V^{\triangleleft} \circ R)^{\triangleright} = V \circ R^{\triangleright}$ *if V is union-preserving.*

Theorem 20.2. *If V is union-preserving, then*

$$(V \circ U)^{\triangleleft} = V^{\triangleleft} \circ U^{\triangleleft}.$$

Proof. By the corresponding definitions and Theorem 18.4, we have

$$(V \circ U)^{\triangleleft}(x) = (V \circ U)(\{x\}) = V(U(\{x\})) = V(U^{\triangleleft}(x)) = V^{\circ}(U^{\triangleleft}(x)) = V^{\triangleleft \triangleright}(U^{\triangleleft}(x)) = V^{\triangleleft}[U^{\triangleleft}(x)] = (V^{\triangleleft} \circ U^{\triangleleft})(x)$$

for all $x \in X$. Therefore, the required equality is also true. ■

Corollary 20.2. *We have*

(1) $(S^{\triangleright} \circ U)^{\triangleleft} = S \circ U^{\triangleleft}$; (2) $(V \circ R^{\triangleright})^{\triangleleft} = V^{\triangleleft} \circ R$ *if* V *is union-preserving*.

Theorem 20.3. *If* V *is union-preserving, then*

$$(V \circ U)^{-1} = U^{-1} \circ V^{-1}.$$

Proof. By Definition 19.1 and Theorems 20.2, and 20.1, we have

$$(V \circ U)^{-1} = (V \circ U)^{\triangleleft -1 \triangleright} = (V^{\triangleleft} \circ U^{\triangleleft})^{-1 \triangleright} = \big(U^{\triangleleft -1} \circ V^{\triangleleft -1}\big)^{\triangleright} = U^{\triangleleft -1 \triangleright} \circ V^{\triangleleft -1 \triangleright} = U^{-1} \circ V^{-1}. \qquad \blacksquare$$

From this theorem, by using Theorem 19.1, we immediately derive the following result.

Corollary 20.3. *We have*

$$(S^{\triangleright} \circ U)^{-1} = U^{-1} \circ S^{-1 \triangleright}.$$

Remark 20.2. By using Definition 20.1, we also easily see that the functionally defined composition of super relations is associative.

21. Relationally defined compositions of super relations

Now, analogously to Definition 19.1, we may also naturally introduce the following definition.

Definition 21.1. The super relation

$$V \bullet U = \big(V^{\triangleleft} \circ U^{\triangleleft}\big)^{\triangleright}$$

will be called the *relationally defined composition* of V and U.

The appropriateness of this definition is apparent from the following theorems.

Theorem 21.1. *We have*

(1) $S^{\triangleright} \bullet R^{\triangleright} = \big(S \circ R\big)^{\triangleright}$; (2) $(S^{\triangleright} \bullet R^{\triangleright})^{\triangleleft} = S \circ R$.

Proof. By Definition 21.1 and Theorem 18.2, we have

$$S^{\triangleright} \bullet R^{\triangleright} = \big(S^{\triangleright\triangleleft} \circ R^{\triangleright\triangleleft}\big)^{\triangleright} = \big(S \circ R\big)^{\triangleright},$$

and thus also $(S^{\triangleright} \bullet R^{\triangleright})^{\triangleleft} = \big(S \circ R\big)^{\triangleright\triangleleft} = S \circ R$. $\blacksquare$

Theorem 21.2. $V \bullet U$ *is a union-preserving super relation such that*

(1) $\big(V \bullet U\big)^{\triangleleft} = V^{\triangleleft} \circ U^{\triangleleft}$; (2) $V \bullet U = V^{\circ} \circ U^{\circ}$.

Proof. From Definition 21.1, by Theorem 18.2, it is clear that $V \bullet U$ is union-preserving and

$$\big(V \bullet U\big)^{\triangleleft} = \big(V^{\triangleleft} \circ U^{\triangleleft}\big)^{\triangleright\triangleleft} = V^{\triangleleft} \circ U^{\triangleleft}.$$

Moreover, by using Theorem 20.1 and Definition 17.2, we see that

$$V \bullet U = (V^{\triangleleft} \circ U^{\triangleleft})^{\triangleright} = V^{\triangleleft\triangleright} \circ U^{\triangleleft\triangleright} = V^{\circ} \circ U^{\circ}. \qquad \blacksquare$$

From assertion (2), by using Theorem 18.4, we immediately derive

Corollary 21.1. *If both* U *and* V *are union-preserving, then*

$$V \bullet U = V \circ U.$$

Moreover, we note that the relationally defined composition of super relations may greatly differ from the functionally defined one.

Example 21.1. If $U(A) = A^{c}$ for all $A \subseteq X$, then by Definition 20.1 we have

$$(U \circ U)(A) = U\big(U(A)\big) = U\big(A^{c}\big) = A^{cc} = A$$

for all $A \subseteq X$. However, for instance, if $\operatorname{card}(X) > 2$ and $A \subseteq X$ such that $\operatorname{card}(A) = 1$, then by Theorem 21.2 and Eaxample 17.3, we have

$$(U \bullet U)(A) = \big(U^{\circ} \circ U^{\circ}\big)(A) = U^{\circ}\big(U^{\circ}(A)\big) = U^{\circ}\big(A^{c}\big) = X.$$

By using Theorem 21.2, we also prove the following two theorems.

Theorem 21.3. *For any* $A \subseteq X$, *we have*

$$\big(V \bullet U\big)(A) = \bigcup_{x \in A} \ \bigcup_{y \in U^{\triangleleft}(x)} V^{\triangleleft}(y)\,.$$

Proof. By using Theorem 21.2, we see that

$$\big(V \bullet U\big)(A) = \bigcup_{x \in A} \big(V \bullet U\big)\big(\{x\}\big) = \bigcup_{x \in A} \big(V \bullet U\big)^{\triangleleft}(x) = \bigcup_{x \in A} \big(V^{\triangleleft} \circ U^{\triangleleft}\big)(x) = \bigcup_{x \in A} V^{\triangleleft}\big[\,U^{\triangleleft}(x)\,\big] = \bigcup_{x \in A} \ \bigcup_{y \in U^{\triangleleft}(x)} V^{\triangleleft}(y)\,. \qquad \blacksquare$$

Theorem 21.4. *We have*

$$\big(V \bullet U\big)^{-1} = U^{-1} \circ V^{-1}.$$

Proof. By using Theorems 21.2, 20.3 and 19.2, we see that

$$\big(V \bullet U\big)^{-1} = \big(\,V^{\circ} \circ U^{\circ}\,\big)^{-1} = U^{\circ\,-1} \circ V^{\circ\,-1} = U^{-1} \circ V^{-1}. \qquad \blacksquare$$

Remark 21.1. Moreover, by using Theorem 21.2 and Remark 20.2, it can be easily seen that the relationally defined composition of super relations is also associative.

22. The duals of super relations

Having in mind the usual closure and interior operations, we naturally introduce the following definition.

Definition 22.1. For a super relation U on X to Y, we define a *dual super relation* $U^{\star}$ on X to Y such that

$$U^{\star}(A) = U\,(\,A^{c}\,)^{\,c}$$

for all $A \subseteq X$.

Remark 22.1. Thus, if R is an ordinary relation R on X to Y, then by defining $R^{\star} = R^{\triangleright\star}$, we note that

$$R^{\star}(A) = R^{\triangleright\star}(A) = R^{\triangleright}(A^{c})^{c} = R\,[\,A^{c}\,]^{\,c}\,.$$

for all $A \subseteq X$.

Moreover, we also easily prove the following four theorems.

Theorem 22.1. *If* U *and* V *are super relations on* X *to* Y, *then*

(1) $U = U^{\star\star}$ *;* (2) $U \subseteq V$ *implies* $V^{\star} \subseteq U^{\star}$.

Proof. To prove (2), note that if $U \subseteq V$, then $U\,(A^{c}) \subseteq V\,(A^{c})$, and thus

$$V^{\star}(A) = V\,(A^{c})^{c} \subseteq U\,(A^{c})^{c} = U^{\star}(A)$$

for all $A \subseteq X$. Therefore, $V^{\star} \subseteq U^{\star}$ also holds. $\blacksquare$

Theorem 22.2. *If* U *is a super relation on* X *to* Y, *then*

(1) $U^{\star}$ *is increasing if and only if* U *is increasing ;*

(2) $U^{\star}$ *is union-preserving if and only if* U *is intersection-preserving ;*

(3) $U^{\star}$ *is intersection-preserving if and only if* U *is union-preserving .*

Proof. For instance, if U is union-preserving, then by the corresponding definitions and De Morgan's law we have

$$U^{\star}\big(\bigcap_{A \in \mathcal{A}} A\big) = U\,\Big(\big(\bigcap_{A \in \mathcal{A}} A\big)^{c}\Big)^{c} = U\,\big(\bigcup_{A \in \mathcal{A}} A^{c}\big)^{c} = \big(\bigcup_{A \in \mathcal{A}} U\,(A^{c})\big)^{c} = \bigcap_{A \in \mathcal{A}} U\,(A^{c})^{c} = \bigcap_{A \in \mathcal{A}} U^{\star}(A)$$

for all $\mathcal{A} \subseteq \mathcal{P}(X)$. Therefore, $U^{\star}$ is intersection-preserving.

Thus, the "if part" of assertion (3) is true. Hence, since $U^{\star\star} = U$, it is clear that the "only if part" of assertion (2) is also true. $\blacksquare$

Theorem 22.3. *If U is a super relation on X to Y, then*

(1) *$U^\star$ is intensive if and only if U is extensive ;*

(2) *$U^\star$ is extensive if and only if U is intensive ;*

(3) *$U^\star$ is involutive if and only if U is involutive ;*

(4) *$U^\star$ is idempotent if and only if U is idempotent .*

Proof. For instance if U is idempotent, then by the corresponding definitions

$$U^\star\big(U^\star(A)\big) = U^\star\big(U(A^c)^c\big) = U\big(U(A^c)\big)^c = U(A^c)^c = U^\star(A)$$

for all $A \subseteq X$. Therefore, $U^\star$ is also idempotent.

Thus, the "if part" of assertion (4) is true. Hence, since $U^{\star\star} = U$, it is clear that the "only if part" of assertion (4) is also true. ■

Theorem 22.4. *If U is a super relation on X to Y and V is a super relation on Y to Z, then*

(1) *$(V \circ U)^\star = V^\star \circ U^\star$;* (2) *$(V \bullet U)^\star = V^{\circ\star} \circ U^{\circ\star}$.*

Proof. By Definitions 22.1 and 20.1, we have

$$(V \circ U)^\star(A) = (V \circ U)(A^c)^c = V\big(U(A^c)\big)^c = V\big(U^\star(A)^c\big)^c = V^\star\big(U^\star(A)\big) = \big(V^\star \circ U^\star\big)(A)$$

for all $A \subseteq X$. Thus, assertion (1) is true.

Now, by using Theorem 21.2 and assertion (1), we also see that

$$(V \bullet U)^\star = \big(V^\circ \circ U^\circ\big)^\star = V^{\circ\star} \circ U^{\circ\star}. \qquad ■$$

To determine the super relation $U^{\circ\star}$, we only prove the following result.

Theorem 22.5. *If R is an ordinary and U is super relation on X to Y, then for any $A \subseteq X$ we have*

(1) $R^{\triangleright\star}(A) = R[A^c]^c = \bigcap_{x \in A^c} R(x)^c$; (2) $U^{\circ\star}(A) = U^{\triangleleft}[A^c]^c = \bigcap_{x \in A^c} U^{\triangleleft}(x)^c$.

Proof. By the corresponding definitions and De Morgan's law, we have

$$R^{\triangleright\star}(A) = R^{\triangleright}\big(A^c\big)^c = R[A^c]^c = \textstyle\bigcap_{x\in A^c} R(x)^c,$$

and thus also

$$U^{\circ\star}(A) = U^{\triangleleft\triangleright\star}(A) = U^{\triangleleft}[A^c]^c = \textstyle\bigcap_{x\in A^c} U^{\triangleleft}(x)^c. \qquad ■$$

23. Some further theorems on dual super relations

Theorem 23.1. *We have*

(1) $U^\star = U^c \circ \mathcal{C}_X$; *(2) $U^\star = \mathcal{C}_Y \circ U \circ \mathcal{C}_X$.*

Proof. By the Definitions 22.1 and 20.1, for any $A \subseteq X$, we have

$$U^\star(A) = U(A^c)^c = U^c(A^c) = U^c\big(\mathcal{C}_X(A)\big) = \big(U^c \circ \mathcal{C}_X\big)(A).$$

Therefore, assertion (1) is true.

From assertion (1), by using that

$$U^c(A) = U(A)^c = \mathcal{C}_Y\big(U(A)\big) = \big(\mathcal{C}_Y \circ U\big)(A)$$

for all $A \subseteq X$, and thus $U^c = \mathcal{C}_Y \circ U$, we see that assertion (2) also holds. ■

Corollary 23.1. *We have $\mathcal{C}_X^\star = \mathcal{C}_X$.*

Proof. By Theorem 23.1, it is clear that $\mathcal{C}_X^\star = \mathcal{C}_X \circ \mathcal{C}_X \circ \mathcal{C}_X = \mathcal{C}_X$. ■

Theorem 23.2. *We have*

$$(V\circ U)^{\star}=V^{c}\circ U\circ\mathcal{C}_X\,.$$

Proof. From Theorem 23.1, by using that

$$(V\circ U)^{c}(A)=(V\circ U)(A)^{c}=V\big(U(A)\big)^{c}=V^{c}\big(U(A)\big)=(V^{c}\circ U)(A)$$

for all $A\subseteq X$, and thus $(V\circ U)^{c}=V^{c}\circ U$, we see that

$$(V\circ U)^{\star}=(V\circ U)^{c}\circ\mathcal{C}_X=\big(V^{c}\circ U\big)\circ\mathcal{C}_X\,. \qquad \blacksquare$$

Corollary 23.2. *We have*

(1) $\big(U\circ\mathcal{C}_X\big)^{\star}=U^{c}$; (2) $\big(U^{c}\big)^{\star}=U\circ\mathcal{C}_X$.

Proof. By Theorem 23.2, it is clear that

$$\big(U\circ\mathcal{C}_X\big)^{\star}=U^{c}\circ\mathcal{C}_X\circ\mathcal{C}_X=U^{c}\,,$$

and thus $\big(U^{c}\big)^{\star}=\big(U\circ\mathcal{C}_X\big)^{\star\star}=U\circ\mathcal{C}_X$ is also true. $\blacksquare$

Analogously to Definition 22.1, we also naturally introduce the following definition.

Definition 23.1. For a hyper relation V on X to Y, we define two *dual hyper relations* $V^{\star}$ and $V^{\bigstar}$ on X to Y such that

$$V^{\star}(A)=V(A^{c})^{c}=\mathcal{P}(Y)\setminus V(A^{c}) \qquad \text{and} \qquad V^{\bigstar}(A)=\big[V(A^{c})\big]^{c}=\big\{B^{c}:\ B\in V(A^{c})\big\}$$

for all $A\subseteq X$.

Remark 23.1. Thus, some properties of the hyper relations $V^{\star}$ and $V^{\bigstar}$ can also be easily derived from those of the hyper relation V.

Keeping in mind the derivations of small closures and interiors from the big ones [57,61], we also naturally introduce the following definition.

Definition 23.2. For a hyper relation V on X to Y, we define a super relation $V^{\blacktriangleleft}$ on X to Y such that

$$V^{\blacktriangleleft}(A)=\big\{\,y\in Y:\ \ \{y\}\in V(A)\,\big\}$$

for all $A\subseteq X$.

Remark 23.2. Thus, we may also naturally consider the ordinary relation $V^{\blacktriangleleft\triangleleft}$. In particular, we may define $\operatorname{int}_{\mathcal{R}}=\operatorname{Int}_{\mathcal{R}}^{\blacktriangleleft}$ and $\sigma_{\mathcal{R}}=\operatorname{int}_{\mathcal{R}}^{\triangleleft}$ for any relator $\mathcal{R}$. Moreover, from [85,91], we see that hyper relations can be derived from super, and thus also from ordinary relations and relators in several natural ways.

24. Normal and regular super relations

Notation 24.1. In this and the next three sections, we assume that:

(1) X and Y are sets; (2) Φ is a super relation on X;

(3) U is a super relation on X to Y and V is a super relation on Y to X.

Remark 24.1. Thus, Φ is an ordinary relation on $\mathcal{P}(X)$ to X, U is an ordinary relation on $\mathcal{P}(X)$ to Y and V is an ordinary relation on $\mathcal{P}(Y)$ to X which can be identified with the set-valued functions φ_{Φ}, φ_U and φ_V, defined by $\varphi_{\Phi}(A)=\Phi(A)$, $\varphi_U(A)=U(A)$ and $\varphi_V(B)=V(B)$ for all $A\subseteq X$ and $B\subseteq Y$.

By specializing Definitions 10.1 and 10.2, we naturally introduce the following two definitions.

Definition 24.1. We say that the super relation U is

(1) *right V–seminormal* if for any $A\subseteq X$ and $B\subseteq Y$

$$U(A)\subseteq B \quad\Longrightarrow\quad A\subseteq V(B)\,;$$

(2) *left V–seminormal* if for any $A\subseteq X$ and $B\subseteq Y$

$$A\subseteq V(B) \quad\Longrightarrow\quad U(A)\subseteq B\,.$$

Remark 24.2. If the super relation U is both left and right V–seminormal, then we may naturally say that U *is V–normal*. Moreover, if for instance the super relation U is left W–seminormal for some super relation W on Y to X, then we may naturally say that U *is left seminormal*.

Definition 24.2. We say that the super relation U is

(1) *is right Φ–semiregular* if for any $A_1, A_2 \subseteq X$

$$U(A_1) \subseteq U(A_2) \quad \Longrightarrow \quad A_1 \subseteq \Phi(A_2);$$

(2) *is left Φ–semiregular* if for any $A_1, A_2 \subseteq X$

$$A_1 \subseteq \Phi(A_2) \quad \Longrightarrow \quad U(A_1) \subseteq U(A_2).$$

Remark 24.3. If the super relation U is both left and right Φ–semiregular, then we may naturally say that U *is Φ–regular*. Moreover, if for instance the super relation U is left Ψ–semiregular for some super relation Ψ on X, then we may naturally say that U *is left semiregular*.

Now, by specializing Theorems 10.1 and 10.2 and their corollaries, we easily establish the following two theorems and their corollaries.

Theorem 24.1. *If U is left (right) V–seminormal and $\Phi = V \circ U$, then U is left (right) Φ–semiregular.*

Corollary 24.1. *If $\mathcal{U}$ is V–normal and $\Phi = V \circ U$, then U is Φ–regular.*

Theorem 24.2. *If U is left (right) Φ–semiregular, $\Phi = V \circ U$ and $\mathcal{P}(Y) = \{ U(A) : \ A \subseteq X \}$, then U is left (right) V–seminormal.*

Corollary 24.2. *If U is Φ–regular, $\Phi = V \circ U$ and $\mathcal{P}(Y) = \{ U(A) : \ A \subseteq X \}$, then U is V–normal.*

Remark 24.4. By Theorem 24.1, it is clear that several properties of the normal super relations can be immediately derived from those of the regular ones. Therefore, the latter ones have to studied before the former ones. Also, from Theorem 24.2, we see that the regular super relations are still less general objects than the normal ones. Actually, we see that they are strictly between closure relations and normal super relations.

25. Some further properties of normal and regular super relations

By specializing some of the results of Sections 12–14, we also easily establish the following three theorems.

Theorem 25.1. *If U is V–normal, then*

(1) *U and V are increasing;*

(2) *$V \circ U$ is a closure operation;* (3) *$U \circ V$ is an interior operation;*

(4) *$U = U \circ V \circ U$;* (5) *$V = V \circ U \circ V$.*

Theorem 25.2. *The following assertions are equivalent:*

(1) *U is V–normal;*

(2) *U is union-preserving and for all $B \subseteq Y$*

$$V(B) = \bigcup \{ A \subseteq X : \ U(A) \subseteq B \}.$$

Proof. If assertion (1) holds, then by Theorems 5.7 and 14.3, for any $\mathcal{A} \subseteq \mathcal{P}(X)$, we have

$$U(\sup(\mathcal{A})) = \sup(U[\mathcal{A}]), \qquad \text{and thus} \qquad U(\sup(\mathcal{A})) = \textstyle\bigcup_{A \in \mathcal{A}} U(A).$$

Therefore, U is union-preserving. Moreover, by using Theorem 13.2, Corollary 14.1 and Definition 7.1, we see that

$$V(B) = \max\big(\mathrm{Int}_U(B)\big) = \sup\big(\mathrm{Int}_U(B)\big) = \sup\big(\{ A \subseteq X : \ U(A) \subseteq B \}\big) = \bigcup \{ A \subseteq X : \ U(A) \subseteq B \}$$

for all $B \subseteq Y$. Therefore, assertion (2) also holds.

Conversely if assertion (2) holds and $A \subseteq X$ and $B \subseteq Y$, then it is clear that

$$U(A) \subseteq B \implies A \subseteq \bigcup \{ C \subseteq X : \quad U(C) \subseteq B \} \implies A \subseteq V(B).$$

Therefore, U is right V–seminormal. Moreover, we also see that

$$A \subseteq V(B) \implies A \subseteq \bigcup \{ C \subseteq X : \quad U(C) \subseteq B \} \implies \forall\, x \in A : \ \exists\, C_x \subseteq X : \quad x \in C_x\,, \quad U(C_x) \subseteq B.$$

Hence, by using that U is union-preserving, we already infer that

$$U(A) = \bigcup_{x \in A} U(\{x\}) \subseteq \bigcup_{x \in A} U(C_x) \subseteq B.$$

Therefore, U is also left V–seminormal, and thus assertion (1) also holds. ■

Remark 25.1. In addition to this theorem, it is also worth noticing that if U is V–normal, then for any $x \in X$ and $B \subseteq Y$ we also have

$$V(B) = \{ x \in X : \quad U^{\triangleleft}(x) \subseteq B \}.$$

Namely, for any $x \in X$, we have

$$x \in V(B) \iff \{x\} \subseteq V(B) \iff U(\{x\}) \subseteq B \iff U^{\triangleleft}(x) \subseteq B.$$

Theorem 25.3. *The following assertions are equivalent :*

(1) *U is normal ;* (2) *U is union-preserving .*

Now, by specializing some of the results of Sections 11 and 15, we also easily establish the following three theorems.

Theorem 25.4. *If U is Φ–regular, then*

(1) *U is increasing ;* (2) *Φ is a closure operation ;* (3) $U = U \circ \Phi$.

Theorem 25.5. *The following assertions are equivalent :*

(1) *U is Φ–regular ;*

(2) *U is union-preserving and for all $B \subseteq X$*

$$\Phi(B) = \bigcup \{ A \subseteq X : \quad U(A) \subseteq U(B) \}.$$

Remark 25.2. In addition to this theorem, it is also worth noticing that if U is Φ–regular, then for any $B \subseteq Y$ we also have

$$\Phi(B) = \{ x \in X : \quad U^{\triangleleft}(x) \subseteq U(B) \}.$$

Theorem 25.6. *If $\mathcal{P}(Y) = \{ U(A) : \ A \subseteq X \}$, then the following assertions are equivalent :*

(1) *U is regular ;* (2) *U is union-preserving .*

26. Normalities of complement and dual super relations

Theorem 26.1. *The following assertions are equivalent :*

(1) *U^{c} is right V^{c}–seminormal ;* (2) *$V \circ \mathcal{C}_Y$ is left $U \circ \mathcal{C}_X$–seminormal .*

Proof. By the definitions of complement relations and sets, it is clear that the following assertions are equivalent :

(a) $U^{c}(A) \subseteq B \implies A \subseteq V^{c}(B)$ for all $A \subseteq X$ and $B \subseteq Y$;

(b) $U(A)^{c} \subseteq B \implies A \subseteq V(B)^{c})$ for all $A \subseteq X$ and $B \subseteq Y$;

(c) $B^{c} \subseteq U(A) \implies V(B) \subseteq A^{c}$ for all $A \subseteq X$ and $B \subseteq Y$;

(d) $B \subseteq U(A^{c}) \implies V(B^{c}) \subseteq A$ for all $A \subseteq X$ and $B \subseteq Y$;

(e) $B \subseteq (U \circ \mathcal{C}_X)(A) \implies (V \circ \mathcal{C}_Y)(B) \subseteq A$ for all $A \subseteq X$ and $B \subseteq Y$.

Moreover, by Definition 24.1, we see that assertion (1) is equivalent to assertion (a), and assertion (2) is equivalent to assertion (e), and thus assertions (1) and (2) are also equivalent. ■

Quite similarly, we also prove the following result.

Theorem 26.2. *The following assertions are equivalent:*

(1) U^c *is left* V^c–*seminormal;* (2) $V \circ \mathcal{C}_Y$ *is right* $U \circ \mathcal{C}_X$–*seminormal.*

Remark 26.1. Moreover, Theorem 26.2 can also be derived from Theorem 26.1 by using Theorem 10.3.

The next corollary is an an immediate consequence of Theorems 26.1 and 26.2.

Corollary 26.1. *The following assertions are equivalent:*

(1) U^c *is* V^c–*normal;* (2) $V \circ \mathcal{C}_Y$ *is* $U \circ \mathcal{C}_X$–*normal.*

Remark 26.2. Hence, by writing V^c in place of V, we at once see that the following assertions are also equivalent:

(1) U^c is V–normal; (2) $V^\star$ is $U \circ \mathcal{C}_X$–normal.

From Theorems 26.1 and 26.2, by writing U^c and V^c in place of U and V, respectively, we also derive the following two theorems.

Theorem 26.3. *The following assertions are equivalent:*

(1) U *is right* V–*seminormal;* (2) $V^\star$ *is left* $U^\star$–*seminormal.*

Theorem 26.4. *The following assertions are equivalent:*

(1) U *is left* V–*seminormal;* (2) $V^\star$ *is right* $U^\star$–*seminormal.*

Proof. By using Theorems 26.2 and 23.1, we see that

$$(1) \iff (U^c)^c \text{ is left } (V^c)^c\text{–seminormal} \iff V^c \circ \mathcal{C}_Y \text{ is right } U^c \circ \mathcal{C}_X\text{–seminormal} \iff (2). \qquad \blacksquare$$

Now, as an immediate consequence of Theorems 26.3 and 26.4, we also state the following result.

Corollary 26.2. *The following assertions are equivalent:*

(1) U *is* V–*normal;* (2) $V^\star$ *is* $U^\star$–*normal.*

Remark 26.3. Hence, by writing $U^\star$ in place of U, we at once see that the following assertions are also equivalent:

(1) $U^\star$ is V–normal; (2) $V^\star$ is U–normal.

27. Some further theorems on the normalities of dual and complement relations

From Theorems 26.4 and 26.3, by writing U^* and V^* in place of U and V, respectively, we also derive the following two theorems.

Theorem 27.1. *The following assertions are equivalent:*

(1) V *is right* U–*seminormal;* (2) U^* *is left* V^*–*seminormal.*

Theorem 27.2. *The following assertions are equivalent:*

(1) V *is left* U–*seminormal;* (2) U^* *is right* V^*–*seminormal.*

Remark 27.1. The latter two theorems can also be derived from Theorems 26.2 and 26.1, by writing $U \circ \mathcal{C}_X$ and $V \circ \mathcal{C}_Y$ in place of U and V, respectively.

Now, as an immediate consequence of Theorems 27.1 and 27.2, we also state the following result.

Corollary 27.1. *The following assertions are equivalent:*

(1) V *is* U–*normal;* (2) U^* *is* V^*–*normal.*

Remark 27.2. Hence, by writing $U^\star$ in place of U, we at once see that the following assertions are also equivalent:

(1) V is $U^\star$–normal; (2) U is V^*–normal.

From Theorems 26.1 and 26.2, by writing $V^\star$ in place of V, we also derive the following two theorems.

Theorem 27.3. *The following assertions are equivalent:*

(1) $B \subseteq U(A) \implies A \subseteq V(B)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$;

(2) U^c *is right* $V \circ \mathcal{C}_Y$*–seminormal;* (3) V^c *is left* $U \circ \mathcal{C}_X$*–seminormal.*

Theorem 27.4. *The following assertions are equivalent:*

(1) $A \subseteq V(B) \implies B \subseteq U(A)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$;

(2) U^c *is left* $V \circ \mathcal{C}_Y$*–seminormal;* (3) V^c *is right* $U \circ \mathcal{C}_X$*–seminormal.*

Proof. By the corresponding definitions, it is clear that following assertions are equivalent:

(a) $A^c \subseteq V(B) \implies B \subseteq U(A^c)$ for all $A \subseteq X$ and $B \subseteq Y$;

(b) $V(B)^c \subseteq A \implies B \subseteq U(A^c)$ for all $A \subseteq X$ and $B \subseteq Y$;

(c) $V^c(B) \subseteq A \implies B \subseteq (U \circ \mathcal{C}_X)(A)$ for all $A \subseteq X$ and $B \subseteq Y$.

Since assertion (1) is equivalent to assertion (a) and and assertion (3) is equivalent to assertion (c), we see that assertions (1) and (3) are also equivalent.

Moreover, by using Theorems 23.1 and 26.2, we see that

$$(2) \iff U^c \text{ is left } (V^\star)^c\text{–seminormal} \iff V^\star \circ \mathcal{C}_Y \text{ is right } U \circ \mathcal{C}_X\text{–seminormal} \iff (3). \qquad \blacksquare$$

Now, as an immediate consequence of Theorems 27.3 and 27.4, we also state the following result.

Corollary 27.2. *The following assertions are equivalent:*

(1) $A \subseteq V(B) \iff B \subseteq U(A)$ *for all* $A \subseteq X$ *and* $B \subseteq Y$;

(2) U^c *is* $V \circ \mathcal{C}_Y$*–normal;* (3) V^c *is* $U \circ \mathcal{C}_X$*–normal.*

Remark 27.3. To obtain some more instructive reformulations of the corresponding results of Sections 26 and 27, note that $U^c = \mathcal{C}_Y \circ U$ *and* $V^c = \mathcal{C}_X \circ V$.

28. A few basic facts on relators

A family $\mathcal{R}$ of relations on one set X to another Y is called a *relator on* X *to* Y, and the ordered pair $(X, Y)(\mathcal{R}) = \big((X, Y), \mathcal{R}\big)$ is called a *relator space*. For the origins of this notion, see [56, 57, 68, 69], and the references in [57]. If in particular $\mathcal{R}$ is a relator on X to itself, then $\mathcal{R}$ is simply called a *relator on* X. Thus, by identifying singletons with their elements, we may naturally write $X(\mathcal{R})$ instead of $(X, X)(\mathcal{R})$. Namely, $(X, X) = \{\{X\}, \{X, X\}\} = \{\{X\}\}$. Relator spaces of this simpler type are already substantial generalizations of the various *ordered sets* [13, 81] and *uniform spaces* [22, 93]. However, they are insufficient for some important purposes. (See [23, 66, 68, 76, 79, 83, 92].)

A relator $\mathcal{R}$ on X to Y, or the relator space $(X, Y)(\mathcal{R})$, is called *simple* if $\mathcal{R} = \{R\}$ for some relation R on X to Y. Simple relator spaces $(X, Y)(R)$ and $X(R)$ were called *formal contexts* and *gosets* in [23] and [81], respectively. Moreover, a relator $\mathcal{R}$ on X, or the relator space $X(\mathcal{R})$, may, for instance, be naturally called *reflexive* if each member of $\mathcal{R}$ is reflexive on X. Thus, we may also naturally speak of *preorder, tolerance and equivalence relators*. For instance, for a family $\mathcal{A}$ of subsets of X, the family $\mathcal{R}_\mathcal{A} = \{R_A : A \in \mathcal{A}\}$, where $R_A = A^2 \cup (A^c \times X)$, is an important preorder relator on X. Such relators were first used by Pervin [45] and Levine [33]. While, for a family $\mathcal{D}$ of *pseudo-metrics* on X, the family $\mathcal{R}_\mathcal{D} = \{B^d_r : r > 0, \ d \in \mathcal{D}\}$, where $B^d_r = \{(x, y) : d(x, y) < r\}$, is an important tolerance relator on X. Such relators were first considered by Weil [96]. Moreover, if $\mathfrak{S}$ is a family of *covers (partitions)* of X, then the family $\mathcal{R}_\mathfrak{S} = \{S_\mathcal{A} : \mathcal{A} \in \mathfrak{S}\}$, where $S_\mathcal{A} = \bigcup_{A \in \mathcal{A}} A^2$, is an important tolerance (equivalence) relator on X. Equivalence relators were first studied by Levine [32]. (Preorder relators were also considered in [2, 74].)

If $\star$ is a unary operation for relations on X to Y, then for any relator $\mathcal{R}$ on X to Y we define $\mathcal{R}^\star = \{R^\star : R \in \mathcal{R}\}$. However, this plausible notation may cause confusions if $\star$ is a set-theoretic operation. For instance, for any relator $\mathcal{R}$ on X to Y, we may naturally define the *elementwise complement* $\mathcal{R}^c = \{R^c : R \in \mathcal{R}\}$, which may easily be confused with the *global complement* $\mathcal{R}^c = \mathcal{P}(X \times Y) \setminus \mathcal{R}$ of $\mathcal{R}$. However, for instance, the practical notations $\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$, and $\mathcal{R}^\infty = \{R^\infty : R \in \mathcal{R}\}$ whenever $\mathcal{R}$ is only a relator on X, will certainly not cause confusions in the sequel. In particular, for a relator $\mathcal{R}$ on X, we may also naturally define $\mathcal{R}^\partial = \{S \subseteq X^2 : S^\infty \in \mathcal{R}\}$. Namely, for any two relators $\mathcal{R}$ and

$\mathcal{S}$ on X, we evidently have $\mathcal{R}^{\infty} \subseteq \mathcal{S} \iff \mathcal{R} \subseteq \mathcal{S}^{\partial}$. That is, the operation ∞ is ∂–normal. The operations ∞ and ∂ were introduced by Mala [34, 36] and Pataki [42, 43], respectively. These two former PhD students of the third author, together with Kurdics [28–30], have made substantial developments in the theory of relators. Moreover, if $*$ is a binary operation for relations, then for any two relators $\mathcal{R}$ and $\mathcal{S}$ we may naturally define $\mathcal{R} * \mathcal{S} = \{ R * S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \}$. However, this notation may again cause confusions if $*$ is a set-theoretic operation Therefore, in the former papers, we wrote $\mathcal{R} \wedge \mathcal{S} = \{ R \cap S : \ R \in \mathcal{R}, \ S \in \mathcal{S} \}$. Moreover, for instance, we also wrote $\mathcal{R} \triangle \mathcal{R}^{-1} = \{ R \cap R^{-1} : \ R \in \mathcal{R} \}$. Thus, $\mathcal{R} \triangle \mathcal{R}^{-1}$ is a symmetric relator such that $\mathcal{R} \triangle \mathcal{R}^{-1} \subseteq \mathcal{R} \wedge \mathcal{R}^{-1}$.

A function $\square$ of the family of all relators on X to Y is called a *direct (indirect) unary operation for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathcal{R}^{\square} = \square(\mathcal{R})$ is a relator on X to Y (on Y to X). More generally, a function $\mathfrak{F}$ of the family of all relators on X to Y is called a *structure for relators* if, for every relator $\mathcal{R}$ on X to Y, the value $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}(\mathcal{R})$ is in a power set depending only on X and Y.

Concerning structures and operation for relators, we freely use some basic terminology on set-to-set functions. However, for closures and projections, we now also use the terms *refinements and modifications*, respectively. For instance, c and -1 are *involution operations* for relators. While, ∞ and ∂ are *projection operations* for relators. Moreover, the operation $\square = c$, ∞ or ∂ is *inversion compatible* in the sense that $\big(\mathcal{R}^{\square}\big)^{-1} = \big(\mathcal{R}^{-1}\big)^{\square}$. While, if for instance

$$\operatorname{Int}_{\mathcal{R}}(B) = \{ A \subseteq X : \ \exists \ R \in \mathcal{R} : \ \ R[A] \subseteq B \}$$

for every relator $\mathcal{R}$ on X to Y and $B \subseteq Y$, then the function $\mathfrak{F}$, defined by $\mathfrak{F}(\mathcal{R}) = \operatorname{Int}_{\mathcal{R}}$, is a union-preserving structure for relators.

The first basic problem in the theory of relators is that, for any increasing structure $\mathfrak{F}$, we have to find an operation $\square$ for relators such that, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y we could have $\mathfrak{F}_{\mathcal{S}} \subseteq \mathfrak{F}_{\mathcal{R}} \iff \mathcal{S} \subseteq \mathcal{R}^{\square}$. By using regular functions [42, 88], several closure operations can be derived from union-preserving structures. However, more generally, one can find first the Galois adjoint $\mathfrak{G}$ of such a structure $\mathfrak{F}$, and then take $\square_{\mathfrak{F}} = \mathfrak{G} \circ \mathfrak{F}$ [72]. By finding the Galois adjoint of the structure $\mathfrak{F}$, the second basic problem for relators, that which structures can be derived from relators, can also be solved. However, for this, some direct methods can also be well used [61, 74].

Now, for an operation $\square$ for relators, a relator $\mathcal{R}$ on X to Y may be naturally called $\square$*–fine* if $\mathcal{R}^{\square} = \mathcal{R}$. Also, for some structure $\mathfrak{F}$ for relators, two relators $\mathcal{R}$ and $\mathcal{S}$ on X to Y may be naturally called $\mathfrak{F}$*–equivalent* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{S}}$. Moreover, for a structure $\mathfrak{F}$ for relators, a relator $\mathcal{R}$ on X to Y may, for instance, be naturally called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{R}$ for some relation R on X to Y. Thus, singleton relators have to be actually called *properly simple*.

29. Closures and interiors derived from relators

Notation 29.1. In this section, we assume that $\mathcal{R}$ is a relator on X to Y.

Remark 29.1. Though, somewhat more generally, we should assume that $\mathcal{U}$ is a super relator on X to Y. Namely, the following definition can be trivially extended to to super relators. Moreover, if $\mathfrak{F}$ is a structure for super relators, then we may naturally define $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\triangleright}}$ with $\mathcal{R}^{\triangleright} = \{ R^{\triangleright} : \ \ R \in \mathcal{R} \}$. However, in this paper, we are mainly interested in those structures $\mathfrak{F}$ for which $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{U}^{\triangleleft}}$ holds with $\mathcal{U}^{\triangleleft} = \{ U^{\triangleleft} : \ U \in \mathcal{U} \}$.

Definition 29.1. For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$, we define:

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *if* $R[A] \subseteq B$ *for some* $R \in \mathcal{R}$*;*

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *if* $R[A] \cap B \neq \emptyset$ *for all* $R \in \mathcal{R}$*;*

(3) $x \in \operatorname{int}_{\mathcal{R}}(B)$ *if* $\{x\} \in \operatorname{Int}_{\mathcal{R}}(B)$*;*

(4) $x \in \sigma_{\mathcal{R}}(y)$ *if* $x \in \operatorname{int}_{\mathcal{R}}(\{y\})$*;*

(5) $x \in \operatorname{cl}_{\mathcal{R}}(B)$ *if* $\{x\} \in \operatorname{Cl}_{\mathcal{R}}(B)$*;*

(6) $x \in \rho_{\mathcal{R}}(y)$ *if* $x \in \operatorname{cl}_{\mathcal{R}}(\{y\})$*;*

(7) $B \in \mathcal{E}_{\mathcal{R}}$ *if* $\operatorname{int}_{\mathcal{R}}(B) \neq \emptyset$*;*

(8) $B \in \mathcal{D}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(B) = X$.

Remark 29.2. The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called *the proximal, topological and infinitesimal interiors* generated by $\mathcal{R}$, respectively. While, the members of the families, $\mathcal{E}_{\mathcal{R}}$ and $\mathcal{D}_{\mathcal{R}}$ are called the *fat and dense subsets* of the relator space $(X, Y)(\mathcal{R})$, respectively. The origins of the relations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$ go back to Efremović's proximity δ [17] and Smirnov's strong inclusion $\Subset$ [55], respectively. While, the convenient notations $\operatorname{Cl}_{\mathcal{R}}$ and $\operatorname{Int}_{\mathcal{R}}$, and the family $\mathcal{E}_{\mathcal{R}}$, together with its dual $\mathcal{D}_{\mathcal{R}}$, were first explicitly used by the third author in [57, 60, 61, 71]. (See also [89, 91].)

The following theorem shows that, in a relator space, the closure of a set can be more directly described than in a topological one. Moreover, the corresponding closure and interior relations are equivalent tools.

Theorem 29.1. *For any* $B \subseteq X$, *we have*

(1) $\operatorname{cl}_{\mathcal{R}}(B) = \bigcap_{R \in \mathcal{R}} R^{-1}[B]$;

(2) $\operatorname{cl}_{\mathcal{R}}(B) = \big(\operatorname{int}_{\mathcal{R}} \circ \mathcal{C}_Y\big)^c = X \setminus \operatorname{int}_{\mathcal{R}}\big(Y \setminus B\big)$;

(3) $\operatorname{Cl}_{\mathcal{R}}(B) = \big(\operatorname{Int}_{\mathcal{R}} \circ \mathcal{C}_Y\big)^c = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}\big(Y \setminus B\big)$.

Remark 29.3. From assertion (2), we at once see that

(1) $\operatorname{cl}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}}^{\star}$;

(2) $\operatorname{cl}_{\mathcal{R}} = \big(\operatorname{int}_{\mathcal{R}}\big)^c \circ \mathcal{C}_Y$.

While, from assertion (1), we easily derive the following result.

Corollary 29.1. *We have*

(1) $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \big(\bigcap \mathcal{R}\big)^{-1}$;

(2) $\rho_{\mathcal{R}^{-1}} = \rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R}$.

The small closure and interior relations are usually much weaker tools than the big ones. Namely, in general, we only prove the following result.

Theorem 29.2. *For any* $A \subseteq X$ *and* $B \subseteq Y$

(1) $A \in \operatorname{Int}_{\mathcal{R}}(B)$ *implies* $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$;

(2) $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ *implies* $A \in \operatorname{Cl}_{\mathcal{R}}(B)$.

The following theorem shows that, in contrast to their equivalence, the big closure relation is usually a more convenient tool than the big interior one.

Theorem 29.3. *We have*

(1) $\operatorname{Cl}_{\mathcal{R}^{-1}} = \operatorname{Cl}_{\mathcal{R}}^{-1}$;

(2) $\operatorname{Int}_{\mathcal{R}^{-1}} = \mathcal{C}_Y \circ \operatorname{Int}_{\mathcal{R}}^{-1} \circ \mathcal{C}_X$.

By using Theorem 29.1 and Definition 29.1, we easily establish the following result.

Theorem 29.4. *We have*

(1) $\mathcal{D}_{\mathcal{R}} = \big\{ B \subseteq Y : \ \forall\ R \in \mathcal{R} : \ X = R^{-1}[B] \big\}$;

(2) $\mathcal{E}_{\mathcal{R}} = \bigcup_{x \in X} \mathcal{U}_{\mathcal{R}}(x)$, *where* $\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x)$.

Remark 29.4. Note that thus

$$\mathcal{U}_{\mathcal{R}}(x) = \operatorname{int}_{\mathcal{R}}^{-1}(x) = \big\{ B \subseteq Y : \ x \in \operatorname{int}_{\mathcal{R}}(B) \big\}$$

is just the family of all *neighbourhoods* of the point x of X in Y.

The following theorem shows that the families of fat and dense sets are also equivalent tools.

Theorem 29.5. *We have*

(1) $\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \ D^c \notin \mathcal{E}_{\mathcal{R}} \big\}$;

(2) $\mathcal{D}_{\mathcal{R}} = \big\{ D \subseteq Y : \ \forall\ E \in \mathcal{E}_{\mathcal{R}} : \ E \cap D \neq \emptyset \big\}$.

By Definition 29.1 and Theorem 16.3, we also state the next theorem.

Theorem 29.6. *The structures* Int, int, *and* $\mathcal{E}$ *are union-preserving.*

Remark 29.5. Instead of the union-preservingness of the hyper relation Int, we usually need only that $\operatorname{Int}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \operatorname{Int}_R$. Therefore, Theorem 29.6 is, in general, only of some terminological importance for us.

30. Open and closed sets derived from relators

Notation 30.1. In this section, we assume that $\mathcal{R}$ is a relator on X.

By using Definition 29.1, we introduce the following definition.

Definition 30.1. For any $A \subseteq X$, we define:

(1) $A \in \tau_{\mathcal{R}}$ *if* $A \in \operatorname{Int}_{\mathcal{R}}(A)$;

(2) $A \in \tau\!\!\!-_{\mathcal{R}}$ *if* $A^c \notin \operatorname{Cl}_{\mathcal{R}}(A)$;

(3) $A \in \mathcal{T}_{\mathcal{R}}$ *if* $A \subseteq \operatorname{int}_{\mathcal{R}}(A)$;

(4) $A \in \mathcal{F}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(A) \subseteq A$;

(5) $A \in \mathcal{N}_{\mathcal{R}}$ *if* $\operatorname{cl}_{\mathcal{R}}(A) \notin \mathcal{E}_{\mathcal{R}}$;

(6) $A \in \mathcal{M}_{\mathcal{R}}$ *if* $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{D}_{\mathcal{R}}$.

Remark 30.1. The members of the families, $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{N}_{\mathcal{R}}$ are called the *proximally open, topologically open and rare (or nowhere dense) subsets* of the relator space $X(\mathcal{R})$, respectively. The families $\tau_{\mathcal{R}}$ and $\tau\!\!\!-_{\mathcal{R}}$ were first explicitly used by the third author in [60, 61]. While, the practical notation $\tau\!\!\!-_{\mathcal{R}}$ has been suggested by J. Kurdics who first noticed that connectedness is a particular case of well-chainedness [28, 30, 43].

By using Definition 30.1 and the corresponding results of Section 29, we easily establish the following two theorems.

Theorem 30.1. *We have*

(1) $\tau\!\!\!-_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$; *(2)* $\tau\!\!\!-_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \tau_{\mathcal{R}} \}$;

(3) $\mathcal{F}_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \mathcal{T}_{\mathcal{R}} \}$; *(4)* $\mathcal{M}_{\mathcal{R}} = \{ A \subseteq X : \ A^c \in \mathcal{N}_{\mathcal{R}} \}$.

Theorem 30.2. *We have*

(1) $\tau_{\mathcal{R}} \subseteq \mathcal{T}_{\mathcal{R}}$; *(2)* $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subseteq \mathcal{E}_{\mathcal{R}}$; *(3)* $\mathcal{D}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} \subseteq \{X\}$.

Remark 30.2. In addition to assertion (1), it also worth noticing that $\tau_R = \mathcal{T}_R$ for any $R \in \mathcal{R}$. Moreover, from assertion (3), by using global complementations, we easily infer that $\mathcal{F}_{\mathcal{R}} \subseteq \left(\mathcal{D}_{\mathcal{R}}\right)^c \cup \{X\}$ and $\mathcal{D}_{\mathcal{R}} \subseteq \left(\mathcal{F}_{\mathcal{R}}\right)^c \cup \{X\}$.

We also have the following result.

Theorem 30.3. *For any* $A \subseteq X$ *we have*

(1) $\mathcal{P}(A) \cap \left(\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}\right) \neq \emptyset$ *implies* $A \in \mathcal{E}_{\mathcal{R}}$;

(2) $\bigcup \mathcal{T}_{\mathcal{R}} \cap \mathcal{P}(A) \subseteq \operatorname{int}_{\mathcal{R}}(A)$; *(3)* $\mathcal{P}\left[\tau_{\mathcal{R}} \cap \mathcal{P}(A)\right] \subseteq \operatorname{Int}_{\mathcal{R}}(A)$.

Remark 30.3. The fat sets are frequently more important tools than the open ones. For instance, if R is a relation on X, then $\mathcal{T}_R$ and $\mathcal{E}_R$ are just the families of all *ascending and residual subsets* of the goset $X(R)$, respectively.

This fact, stressed first by the third author in [59], can also be well seen from the next example.

Example 30.1. If in particular $X = \mathbb{R}$ and

$$R(x) = \{x - 1\} \cup [x, +\infty[$$

for all $x \in X$, then R is a reflexive relation on X such that $\mathcal{T}_R = \{\emptyset, X\}$, but $\mathcal{E}_R$ is quite a large family.

Remark 30.4. If the relator $\mathcal{R}$ is *topological or proximal* in the sense that:

(1) for each $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subseteq R(x)$;

(2) for each $A \subseteq X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subseteq V \subseteq R[A]$;

respectively, then the converses of the assertions (1)–(3) of Theorem 30.3 can also be proved. Therefore, in these cases, the families $\mathcal{T}_{\mathcal{R}}$ and $\tau_{\mathcal{R}}$ are also quite powerful tools.

By Definition 30.1 and Theorem 16.3, we state the next theorem.

Theorem 30.4. *The structure* τ *is also union-preserving.*

The following example shows that the increasing structure $\mathcal{T}$ need not be union-preserving. This is a serious disadvantage of the topologically open sets.

Example 30.2. If $\operatorname{card}(X) > 2$ and $x_1, x_2 \in X$ such that $x_1 \neq x_2$, and

$$R_i = \{x_i\}^2 \cup \left(\{x_i\}^c\right)^2$$

for all $i = 1, 2$, then $\mathcal{R} = \{R_1, R_2\}$ is an equivalence relator on X such that

$$\{x_1, x_2\} \in \mathcal{T}_{\mathcal{R}} \setminus \left(\mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}\right),$$

and thus $\mathcal{T}_{\mathcal{R}} \not\subseteq \mathcal{T}_{R_1} \cup \mathcal{T}_{R_2}$.

Remark 30.5. Later, by using the topological refinement $\mathcal{R}^{\wedge}$ of $\mathcal{R}$, we see that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^{\wedge}} \mathcal{T}_R$, whenever $\mathcal{R}$ is nonvoid.

31. Convergences and adherences derived from relators

Notation 31.1. In this and the subsequent sections, we assume that $\mathcal{R}$ is a relator on X to Y.

The importance of fat and dense lies mainly in the following definition of the convergence and adherence of nets to nets and points suggested by Efremović and Švarc [18]. This definition has also been used in [1, 47, 48].

Definition 31.1. If φ and ψ are functions of a relator space $\Gamma(\mathcal{U})$ to X and Y, respectively, and

$$(\varphi, \psi)(\gamma) = \big(\varphi(\gamma), \psi(\gamma)\big)$$

for all $\gamma \in \Gamma$, then we define:

(1) $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{E}_{\mathcal{U}}$ for all $R \in \mathcal{R}$;

(2) $\varphi \in \operatorname{Adh}_{\mathcal{R}}(\psi)$ if $(\varphi, \psi)^{-1}[R] \in \mathcal{D}_{\mathcal{U}}$ for all $R \in \mathcal{R}$.

Moreover, if $x \in X$ and $x_\Gamma(\gamma) = x$ for all $\gamma \in \Gamma$, then we also define

(3) $x \in \lim_{\mathcal{R}}(\psi)$ if $x_\Gamma \in \operatorname{Lim}_{\mathcal{R}}(\psi)$; (4) $x \in \operatorname{adh}_{\mathcal{R}}(\psi)$ if $x_\Gamma \in \operatorname{Adh}_{\mathcal{R}}(\psi)$.

Remark 31.1. This definition can be immediately generalized to the case when Φ and Ψ are relations on Γ to X and Y, respectively, and $(\Phi \otimes \Psi)(\gamma) = \Phi(\gamma) \times \Psi(\gamma)$ for all $\gamma \in \Gamma$. Moreover, $A \subseteq X$ and $A_\Gamma(\gamma) = A$ for all $\gamma \in \Gamma$.

However, to make the above big limit and adherence relations to be stronger tools than the big closure and interior ones, it is sufficient to consider only a poset $\Gamma(\leq)$ instead of the relator space $\Gamma(\mathcal{U})$.

Theorem 31.1. *If $\mathcal{R}$ is a relator on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, the following assertions are equivalent:*

(1) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$;

(2) *there exist a poset $\Gamma(\leq)$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$;*

(3) *there exist a non-partial relator space $\Gamma(\mathcal{U})$ and functions φ and ψ of Γ to A and B, respectively, such that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$.*

Proof. For instance, if assertion (1) holds, then for each $R \in \mathcal{R}$ we have $R[A] \cap B \neq \emptyset$. Therefore, there exist $\varphi(R) \in A$ and $\psi(R) \in B$ such that $\psi(R) \in R\big(\varphi(R)\big)$. Hence, we already infer that $(\varphi, \psi)(R) = \big(\varphi(R), \psi(R)\big) \in R$, and thus

$$R \in (\varphi, \psi)^{-1}[R].$$

Now, by taking $\Gamma = \mathcal{R}$ and $\leq = \supseteq$, we see that $\Gamma(\leq)$ is poset such that $\Gamma \neq \emptyset$ if $\mathcal{R} \neq \emptyset$. Moreover, if $R \in \mathcal{R}$, then $R \in \Gamma$ such that for any $S \in \Gamma$, with $S \geq R$, we have $S \subseteq R$, and thus

$$S \in (\varphi, \psi)^{-1}[S] \subseteq (\varphi, \psi)^{-1}[R].$$

This shows that $(\varphi, \psi)^{-1}[R]$ is a residual, and thus fat subset of the poset $\Gamma(\leq)$. Therefore, $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$, and thus assertion (2) also holds. ■

Remark 31.2. To prove an analogous theorem for the relation $\operatorname{Adh}_{\mathcal{R}}$, on the set $\Gamma = \mathcal{R}$ we have to consider the preorder $\leq = \Gamma^2$. Therefore, posets are usually not sufficient. If $\mathcal{R}$ is *uniformly filtered* in the sense that for every $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T \subseteq R \cap S$, then in the proof of the corresponding theorem we also take $\leq = \supseteq$.

From Theorem 31.1, by letting φ to be a constant function, we derive the next corollary.

Corollary 31.1. *For any $x \in X$ and $B \subseteq Y$, the following assertions are equivalent:*

(1) $x \in \operatorname{cl}_{\mathcal{R}}(B)$;

(2) *there exist a poset $\Gamma(\leq)$ and a function ψ of Γ to B such that $x \in \lim_{\mathcal{R}}(\psi)$;*

(3) *there exist a non-partial relator space $\Gamma(\mathcal{U})$ and a function ψ of Γ to B such that $x \in \lim_{\mathcal{R}}(\psi)$.*

In addition this corollary, it is also worth proving the following result.

Theorem 31.2. *For any function ψ of a relator space $\Gamma(\mathcal{U})$ to Y, we have*

(1) $\lim_{\mathcal{R}}(\psi) = \bigcap_{D \in \mathcal{D}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[D]\big)$ (2) $\operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{E \in \mathcal{E}_{\mathcal{U}}} \operatorname{cl}_{\mathcal{R}}\big(\psi[E]\big)$.

Remark 31.3. The above results show that the relations $\operatorname{cl}_{\mathcal{R}}$ and $\lim_{\mathcal{R}}$ are usually also equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

By using the corresponding definitions, we more easily prove the following result.

Theorem 31.3. *For any function* ψ *of a goset* $\Gamma(\leq)$ *to* Y, *we have*

(1) $\lim_{\mathcal{R}}(\psi) = \bigcap_{R\in\mathcal{R}} \underline{\lim}_{\gamma\in\Gamma} R^{-1}(\psi(\gamma)) = \bigcap_{R\in\mathcal{R}} \bigcup_{\alpha\in\Gamma} \bigcap_{\beta\geq\alpha} R^{-1}(\psi(\beta))$;

(2) $\operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{R\in\mathcal{R}} \overline{\lim}_{\gamma\in\Gamma} R^{-1}(\psi(\gamma)) = \bigcap_{R\in\mathcal{R}} \bigcap_{\alpha\in\Gamma} \bigcup_{\beta\geq\alpha} R^{-1}(\psi(\beta))$.

Remark 31.4. Beside assertion (1) of Theorem 29.1, this theorem also shows a remarkable advantage of relator spaces over the topological ones.

By Definition 31.1 and a dual of Theorem 16.3, the following theorem is also true.

Theorem 31.4. *The structures* Lim, $\lim$, Adh *and* adh *are intersection-preserving.*

Remark 31.5. Thus, in particular, for a function ψ of a relator space $\Gamma(\mathcal{U})$ to Y, we have

$$\lim_{\mathcal{R}}(\psi) = \bigcap_{R\in\mathcal{R}} \lim_R(\psi) \qquad \textit{and} \qquad \operatorname{adh}_{\mathcal{R}}(\psi) = \bigcap_{R\in\mathcal{R}} \operatorname{adh}_R(\psi).$$

Therefore, the function ψ may be naturally called *convergence (adherence) Cauchy* if $\lim_R(\psi) \neq \emptyset$ $\left(\operatorname{adh}_R(\psi) \neq \emptyset\right)$ for all $R \in \mathcal{R}$. Thus, "convergent (adherent)" trivially implies "convergence (adherence) Cauchy". Moreover, if $\mathcal{R}$ is topologically fine, then it can be shown that the converse implication is also true. (See [63, 65].) Analogously to *completeness and compactness*, the *Lebesgue* and *Baire properties* can also be most nicely treated in relator spaces [58, 70]. By [43, 53], the same is true for the *well-chainedness* and *connectedness properties*.

32. Lower and upper bounds derived from relators

According to [69], we introduce the following definition.

Definition 32.1. For any $A \subseteq X$, $B \subseteq Y$ and $x \in X$, $y \in Y$, we define

(1) $A \in \operatorname{Lb}_{\mathcal{R}}(B)$ *and* $B \in \operatorname{Ub}_{\mathcal{R}}(A)$ *if* $A \times B \subseteq R$ *for some* $R \in \mathcal{R}$;

(2) $x \in \operatorname{lb}_{\mathcal{R}}(B)$ *if* $\{x\} \in \operatorname{Lb}_{\mathcal{R}}(B)$; *(3)* $y \in \operatorname{ub}_{\mathcal{R}}(A)$ *if* $\{y\} \in \operatorname{Ub}_{\mathcal{R}}(A)$;

(4) $B \in \mathfrak{L}_{\mathcal{R}}$ *if* $\operatorname{lb}_{\mathcal{R}}(B) \neq \emptyset$; *(5)* $A \in \mathfrak{U}_{\mathcal{R}}$ *if* $\operatorname{ub}_{\mathcal{R}}(A) \neq \emptyset$.

Thus, for instance, we easily prove the following two theorems.

Theorem 32.1. *We have*

(1) $\operatorname{Ub}_{\mathcal{R}} = \operatorname{Lb}_{\mathcal{R}^{-1}} = \operatorname{Lb}_{\mathcal{R}}^{-1}$; (2) $\operatorname{ub}_{\mathcal{R}} = \operatorname{lb}_{\mathcal{R}^{-1}}$; (3) $\mathfrak{U}_{\mathcal{R}} = \mathfrak{L}_{\mathcal{R}^{-1}}$.

Theorem 32.2. *We have*

(1) $\operatorname{Lb}_{\mathcal{R}} = \operatorname{Cl}^c_{\mathcal{R}^c} = \operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$; (2) $\operatorname{lb}_{\mathcal{R}} = \operatorname{cl}^c_{\mathcal{R}^c} = \operatorname{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y$; (3) $\mathfrak{L}_{\mathcal{R}} = \mathcal{P}(Y) \setminus \mathcal{D}_{\mathcal{R}^c}$.

Proof. By Definitions 32.1 and 29.1, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\begin{aligned} A \in \operatorname{Lb}_{\mathcal{R}}(B) &\iff \exists\ R \in \mathcal{R}: \quad A \times B \subseteq R \\ &\iff \exists\ R \in \mathcal{R}: \quad \forall\ (a, b) \in A \times B: \quad (a, b) \notin R^c \\ &\iff \exists\ R \in \mathcal{R}: \quad \forall\ a \in A, \quad b \in B: \quad b \notin R^c(a) \\ &\iff \exists\ R \in \mathcal{R}: \quad R^c[A] \cap B = \emptyset \\ &\iff A \notin \operatorname{Cl}_{\mathcal{R}^c}(B) \\ &\iff A \in \operatorname{Cl}_{\mathcal{R}^c}(B)^c \\ &\iff A \in \operatorname{Cl}^c_{\mathcal{R}^c}(B). \end{aligned}$$

Therefore, $\operatorname{Lb}_{\mathcal{R}}(B) = \operatorname{Cl}^c_{\mathcal{R}^c}(B)$ for all $B \subseteq Y$, and thus the first part of (1) is true. The second part of (1) follows from Theorem 29.1. ∎

Remark 32.1. The above two theorems show that, for instance, the relations $\mathrm{Lb}_{\mathcal{R}}$, $\mathrm{Ub}_{\mathcal{R}}$, $\mathrm{Cl}_{\mathcal{R}}$ and $\mathrm{Int}_{\mathcal{R}}$ are also equivalent tools in the relator space $(X, Y)(\mathcal{R})$.

Notation 32.1. In the remaining part of this section, we assume that $\mathcal{R}$ is a relator on X.

In addition to Definition 32.1, we also introduce the following definition.

Definition 32.2. For any $A \subseteq X$, we define

(1) $\min_{\mathcal{R}}(A) = A \cap \mathrm{lb}_{\mathcal{R}}(A)$; *(2)* $\max_{\mathcal{R}}(A) = A \cap \mathrm{ub}_{\mathcal{R}}(A)$;

(3) $\mathrm{Min}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Lb}_{\mathcal{R}}(A)$; *(4)* $\mathrm{Max}_{\mathcal{R}}(A) = \mathcal{P}(A) \cap \mathrm{Ub}_{\mathcal{R}}(A)$;

(5) $\inf_{\mathcal{R}}(A) = \max_{\mathcal{R}}\big(\mathrm{lb}_{\mathcal{R}}(A)\big)$; *(6)* $\sup_{\mathcal{R}}(A) = \min_{\mathcal{R}}\big(\mathrm{ub}_{\mathcal{R}}(A)\big)$;

(7) $\mathrm{Inf}_{\mathcal{R}}(A) = \mathrm{Max}_{\mathcal{R}}\big[\mathrm{Lb}_{\mathcal{R}}(A)\big]$; *(8)* $\mathrm{Sup}_{\mathcal{R}}(A) = \mathrm{Min}_{\mathcal{R}}\big[\mathrm{Ub}_{\mathcal{R}}(A)\big]$;

(9) $A \in \ell_{\mathcal{R}}$ *if* $A \in \mathrm{Lb}_{\mathcal{R}}(A)$; *(10)* $A \in \mathcal{L}_{\mathcal{R}}$ *if* $A \subseteq \mathrm{lb}_{\mathcal{R}}(A)$.

Moreover, for instance, we easily prove the following three theorems.

Theorem 32.3. *We have*

(1) $\mathrm{Max}_{\mathcal{R}} = \mathrm{Min}_{\mathcal{R}^{-1}}$; (2) $\mathrm{Sup}_{\mathcal{R}} = \mathrm{Inf}_{\mathcal{R}^{-1}}$;

(3) $\max_{\mathcal{R}} = \min_{\mathcal{R}^{-1}}$; (4) $\sup_{\mathcal{R}} = \inf_{\mathcal{R}^{-1}}$; (5) $\ell_{\mathcal{R}} = \ell_{\mathcal{R}^{-1}}$.

Theorem 32.4. *For any $A \subseteq X$ we have*

(1) $\mathrm{Max}_{\mathcal{R}}(A) \subseteq \mathcal{P}\big(\max_{\mathcal{R}}(A)\big)$; (2) $\mathrm{Max}_{\mathcal{R}}(A) = \big\{B \subseteq X : \ \mathcal{P}(A) \subseteq \mathrm{Lb}_{\mathcal{R}}(B)\big\}$.

Theorem 32.5. *For any $A \subseteq X$ the following assertions are equivalent:*

(1) $A \in \ell_{\mathcal{R}}$; (2) $A \in \mathrm{Ub}_{\mathcal{R}}(A)$; (3) $A \in \mathrm{Min}_{\mathcal{R}}(A)$; (4) $A \in \mathrm{Max}_{\mathcal{R}}(A)$.

Remark 32.2. Concerning the family $\mathcal{L}_{\mathcal{R}}$, we only prove that $A \in \mathcal{L}_{\mathcal{R}}$ if and only if $A = \min_{\mathcal{R}}(A)$.

Moreover, it is also worth mentioning that the following theorem is true.

Theorem 32.6. *We have*

(1) $\ell_{\mathcal{R}} \subseteq \mathcal{L}_{\mathcal{R}} \cap \mathcal{L}_{\mathcal{R}^{-1}}$; (2) $\mathcal{L}_{\mathcal{R}} = \big\{ \min_{\mathcal{R}}(A) : \ A \subseteq X \big\}$.

33. The most important closure operations for relators

Some of the following operations were already considered by Kenyon [27] and Nakano and Nakano [38].

Definition 33.1. The relators

$$\begin{aligned}
\mathcal{R}^{*} &= \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad R \subseteq S \big\}, \\
\mathcal{R}^{\#} &= \big\{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad \exists\ R \in \mathcal{R} : \quad R[A] \subseteq S[A] \big\}, \\
\mathcal{R}^{\wedge} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ R \in \mathcal{R} : \quad R(x) \subseteq S(x) \big\}, \\
\mathcal{R}^{\triangle} &= \big\{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad \exists\ u \in X : \quad \exists\ R \in \mathcal{R} : \quad R(u) \subseteq S(x) \big\}
\end{aligned}$$

are called the *uniform, proximal, topological and paratopological closures (or refinements)* of the relator $\mathcal{R}$, respectively.

Remark 33.1. Thus, we evidently have

$$\mathcal{R} \subseteq \mathcal{R}^{*} \subseteq \mathcal{R}^{\#} \subseteq \mathcal{R}^{\wedge} \subseteq \mathcal{R}^{\triangle}.$$

Moreover, if in particular $\mathcal{R}$ is a relator on X, then by using the notation $\mathcal{R}^{\infty} = \{ R^{\infty} : \ R \in \mathcal{R} \}$ we easily prove that $\mathcal{R}^{\infty} \subseteq \mathcal{R}^{*\infty} \subseteq \mathcal{R}^{\infty *} \subseteq \mathcal{R}^{*}$.

However, it is now more important to note that, by using Definition 29.1, the definitions of the operations $\#$, $\wedge$, and $\triangle$ can be quite briefly reformulated.

Theorem 33.1. *We have*

(1) $\mathcal{R}^{\#} = \{ S \subseteq X \times Y : \quad \forall\ A \subseteq X : \quad A \in \operatorname{Int}_{\mathcal{R}}(S[A]) \}$;

(2) $\mathcal{R}^{\wedge} = \{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}(S(x)) \}$;

(3) $\mathcal{R}^{\vartriangle} = \{ S \subseteq X \times Y : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \}$.

The importance of the operation $*$ lies mainly in the following result.

Theorem 33.2. $*$ *is a closure operation for relators on* X *to* Y *such that, if* $\mathcal{R}$ *is nonvoid, then for any relator* $\mathcal{S}$ *on* X *to* Y *we have*

$$\mathcal{S} \subseteq \mathcal{R}^* \quad \Longleftrightarrow \quad \operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}} \quad \Longleftrightarrow \quad \operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}} .$$

Proof. For instance, if $\mathcal{S} \not\subseteq \mathcal{R}^*$, then there exists $S \in \mathcal{S}$ such that $S \notin \mathcal{R}^*$. Thus, by the definition of $\mathcal{R}^*$, for any $R \in \mathcal{R}$ we have $R \not\subseteq S$. Therefore, there exists $\big(\varphi(R),\ \psi(R)\big) \in R$ such that $\big(\varphi(R),\ \psi(R)\big) \notin S$. Hence, since $(\varphi,\ \psi)(R) = \big(\varphi(R),\ \psi(R)\big)$, we infer that

$$R \in (\varphi,\ \psi)^{-1}[R] \qquad \text{and} \qquad (\varphi,\ \psi)^{-1}[S] = \emptyset .$$

Now, by taking $\Gamma = \mathcal{R}$ and $\leq = \supseteq$, we see that $\Gamma(\leq)$ is a poset. Moreover, as in the proof of Theorem 31.1, we show that $\varphi \in \operatorname{Lim}_{\mathcal{R}}(\psi)$.

Furthermore, since $\leq$ is reflexive, and thus in particular it is non-partial, we note that $\emptyset \notin \mathcal{E}_{\leq}$, and thus $(\varphi,\ \psi)^{-1}[S] \notin \mathcal{E}_{\leq}$. Therefore, by Definition 31.1, we have $\varphi \notin \operatorname{Lim}_{\mathcal{S}}(\psi)$, and thus $\operatorname{Lim}_{\mathcal{S}} \not\subseteq \operatorname{Lim}_{\mathcal{R}}$. This proves that $\operatorname{Lim}_{\mathcal{S}} \subseteq \operatorname{Lim}_{\mathcal{R}}$ implies $\mathcal{S} \subseteq \mathcal{R}^*$ even if $\mathcal{R} = \emptyset$. ■

Remark 33.2. Actually, the equivalence $\mathcal{S} \subseteq \mathcal{R}^* \iff \operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$ does not require the relator $\mathcal{R}$ to be nonvoid. However, to prove the implication $\operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}} \implies \mathcal{S} \subseteq \mathcal{R}^*$, on the set $\Gamma = \mathcal{R}$ we to consider the preorder $\leq = \Gamma^2$. Moreover, we have to note that $\emptyset \notin \mathcal{D}_{\leq}$ if $\mathcal{R} \neq \emptyset$.

Corollary 33.1. *If* $\mathcal{R}$ *is nonvoid, then* $\mathcal{S} = \mathcal{R}^*$ *is the largest relator on* X *to* Y *such that*

$$\operatorname{Lim}_{\mathcal{S}} = \operatorname{Lim}_{\mathcal{R}} \qquad (\operatorname{Adh}_{\mathcal{S}} = \operatorname{Adh}_{\mathcal{R}}) .$$

Proof. To prove (1), note that, by Theorem 33.2, the inclusion $\mathcal{R}^* \subseteq \mathcal{R}^*$ implies $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{R}^*}$, and $\mathcal{R} \subseteq \mathcal{R}^{**}$ implies $\operatorname{Lim}_{\mathcal{R}^*} \subseteq \operatorname{Lim}_{\mathcal{R}}$. Therefore, we actually have $\operatorname{Lim}_{\mathcal{R}^*} = \operatorname{Lim}_{\mathcal{R}}$.

Moreover, if $\mathcal{S}$ is a relator on X to Y such that $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$ then by Theorem 33.2 we have $\mathcal{S} \subseteq \mathcal{R}^*$. Therefore, $\mathcal{S} = \mathcal{R}^*$ is actually the largest relator on X to Y such that $\operatorname{Lim}_{\mathcal{R}} \subseteq \operatorname{Lim}_{\mathcal{S}}$. ■

Remark 33.3. Note that the implication $\mathcal{S} \subseteq \mathcal{R}^* \implies \operatorname{Adh}_{\mathcal{R}} \subseteq \operatorname{Adh}_{\mathcal{S}}$ is always true. Therefore, by the first part of the above proof, the equality $\operatorname{Adh}_{\mathcal{R}^*} = \operatorname{Adh}_{\mathcal{R}}$ is also always true.

Now, analogously to Theorem 33.2 and it corollary, we also prove the following theorem and its corollary.

Theorem 33.3. *If* $\mathcal{R}$ *is nonvoid, then for any relator* $\mathcal{S}$ *on* X *to* Y*, we have*

$$\mathcal{S} \subseteq \mathcal{R}^{\wedge} \quad \Longleftrightarrow \quad \operatorname{lim}_{\mathcal{R}} \subseteq \operatorname{lim}_{\mathcal{S}} \quad \Longleftrightarrow \quad \operatorname{adh}_{\mathcal{R}} \subseteq \operatorname{adh}_{\mathcal{S}} .$$

Corollary 33.2. *If* $\mathcal{R}$ *is nonvoid, then* $\mathcal{S} = \mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that*

$$\operatorname{lim}_{\mathcal{S}} = \operatorname{lim}_{\mathcal{R}} \qquad (\operatorname{adh}_{\mathcal{S}} = \operatorname{adh}_{\mathcal{R}}) .$$

34. Some basic theorems on the operations $\#$, $\wedge$, and $\vartriangle$

The following theorem and its corollary can be proved in a unified way by using some basic theorems on regular functions [42, 53].

Theorem 34.1. *The operations* $\#$, $\wedge$, *and* $\vartriangle$ *are closure operations for relators on* X *to* Y *such that, for any relator* $\mathcal{S}$ *on* X *to* Y*,*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\#} \iff \operatorname{Int}_{\mathcal{S}} \subseteq \operatorname{Int}_{\mathcal{R}} \iff \operatorname{Cl}_{\mathcal{R}} \subseteq \operatorname{Cl}_{\mathcal{S}}$;

(2) $\mathcal{S} \subseteq \mathcal{R}^{\wedge} \iff \operatorname{int}_{\mathcal{S}} \subseteq \operatorname{int}_{\mathcal{R}} \iff \operatorname{cl}_{\mathcal{R}} \subseteq \operatorname{cl}_{\mathcal{S}}$;

(3) $\mathcal{S} \subseteq \mathcal{R}^{\vartriangle} \iff \mathcal{E}_{\mathcal{S}} \subseteq \mathcal{E}_{\mathcal{R}} \iff \mathcal{D}_{\mathcal{R}} \subseteq \mathcal{D}_{\mathcal{S}}$.

Remark 34.1. Here, the statement that $\#$, $\wedge$, and $\vartriangle$ are closure operations need not be proved directly since this is a consequence of the regularity properties (1), (2), and (3).

Corollary 34.1. *The following assertions are true :*

(1) $\mathcal{S}=\mathcal{R}^{\#}$ *is the largest relator on* X *to* Y *such that* $\mathrm{Int}_{\mathcal{S}}=\mathrm{Int}_{\mathcal{R}}$ $\big(\mathrm{Cl}_{\mathcal{S}}=\mathrm{Cl}_{\mathcal{R}}\big)$*;*

(2) $\mathcal{S}=\mathcal{R}^{\wedge}$ *is the largest relator on* X *to* Y *such that* $\mathrm{int}_{\mathcal{S}}=\mathrm{int}_{\mathcal{R}}$ $\big(\mathrm{cl}_{\mathcal{S}}=\mathrm{cl}_{\mathcal{R}}\big)$*;*

(3) $\mathcal{S}=\mathcal{R}^{\vartriangle}$ *is the largest relator on* X *to* Y *such that* $\mathcal{E}_{\mathcal{S}}=\mathcal{E}_{\mathcal{R}}$ $\big(\mathcal{D}_{\mathcal{S}}=\mathcal{D}_{\mathcal{R}}\big)$*.*

Remark 34.2. Actually, for instance, it can also be proved that $\mathcal{S}=\mathcal{R}^{\#}$ is the largest relator on X to Y such that $\mathrm{Int}_{\mathcal{S}}\subseteq\mathrm{Int}_{\mathcal{R}}$.

Concerning the above basic closure operations, we also easily prove the following result.

Theorem 34.2. *We have*

(1) $\mathcal{R}^{*}=\mathcal{R}^{**}$*;*

(2) $\mathcal{R}^{\#}=\mathcal{R}^{\Diamond\#}=\mathcal{R}^{\#\Diamond}$ *with* $\Diamond=*$ *and* $\#$*;*

(3) $\mathcal{R}^{\wedge}=\mathcal{R}^{\Diamond\wedge}=\mathcal{R}^{\wedge\Diamond}$ *with* $\Diamond=*$, $\#$ *and* $\wedge$*;*

(4) $\mathcal{R}^{\vartriangle}=\mathcal{R}^{\Diamond\vartriangle}=\mathcal{R}^{\vartriangle\Diamond}$ *with* $\Diamond=*$, $\#$, $\wedge$ *and* $\vartriangle$*.*

Proof. To prove (2), note that, by Remark 33.1 and the closure properties, we have

$$\mathcal{R}^{\#}\subseteq\mathcal{R}^{\#*}\subseteq\mathcal{R}^{\#\#}=\mathcal{R}^{\#}\qquad\text{and}\qquad\mathcal{R}^{\#}\subseteq\mathcal{R}^{*\#}\subseteq\mathcal{R}^{\#\#}=\mathcal{R}^{\#}.$$ ∎

Remark 34.3. If in particular $\mathcal{R}$ is a relator on X, then by using Remark 33.1 we also prove that

$$\mathcal{R}^{*\infty}=\mathcal{R}^{\infty*\infty}\quad\text{and}\quad\mathcal{R}^{\infty*}=\mathcal{R}^{*\infty*}.$$

Theorem 34.3. *We have*

(1) $\mathcal{R}^{*-1}=\mathcal{R}^{-1*}$*;* (2) $\mathcal{R}^{\#-1}=\mathcal{R}^{-1\#}$*.*

Proof. To prove (2), note that by Theorems 29.3 and Corollary 34.1 we have

$$\mathrm{Cl}_{\mathcal{R}^{\#-1}}=\mathrm{Cl}_{\mathcal{R}^{\#}}^{-1}=\mathrm{Cl}_{\mathcal{R}}^{-1}=\mathrm{Cl}_{\mathcal{R}^{-1}}\,,$$

and thus in particular $\mathrm{Cl}_{\mathcal{R}^{-1}}\subseteq\mathrm{Cl}_{\mathcal{R}^{\#-1}}$.

Hence, by using Theorem 34.1, we infer that $\mathcal{R}^{\#-1}\subseteq\mathcal{R}^{-1\#}$. Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$, we see that assertion (2) is also true. ∎

Remark 34.4. It can be more easily shown that the elementwise operations c, ∞ and ∂ are also inversion compatible. However, the operations $\wedge$ and $\vartriangle$ are not inversion compatible. Therefore, in addition to Definition 33.1, we must also introduce the following definition.

Definition 34.1. We define

$$\mathcal{R}^{\vee}=\mathcal{R}^{\wedge-1}\qquad\text{and}\qquad\mathcal{R}^{\triangledown}=\mathcal{R}^{\vartriangle-1}.$$

Remark 34.5. The latter operations have very curious properties. For instance, if $\mathcal{R}\neq\emptyset$, then $\mathcal{R}^{\vee\wedge}=\{\rho_{\mathcal{R}}\}^{\wedge}$, and thus in particular the relator $\mathcal{R}^{\vee}$ is topologically simple. (For some generalizations, see [35].) The operations $\vee\vee$ and $\triangledown\triangledown$, already coincide with the extremal closure operations $\bullet$ and $\blacklozenge$, defined for any relator $\mathcal{R}$ on X to Y such that $\mathcal{R}^{\bullet}=\{\rho_{\mathcal{R}^{-1}}\}^{*}$, and moreover

$$\mathcal{R}^{\blacklozenge}=\mathcal{R}\quad\text{if}\quad\mathcal{R}=\{X\times Y\}\qquad\text{and}\qquad\mathcal{R}^{\blacklozenge}=\mathcal{P}(X\times Y)\quad\text{if}\quad\mathcal{R}\neq\{X\times Y\}.$$

The importance of the operation $\blacklozenge$ lies in the fact that it is the ultimate stable unary operation for relators on X to Y in the sense that $\{X\times Y\}^{\blacklozenge}=\{X\times Y\}$.

35. Some further theorems on the operations $\wedge$ and $\vartriangle$

A preliminary version of the following basic theorem was proved in [57].

Theorem 35.1. *If $\mathcal{R}$ is nonvoid relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $\operatorname{Int}_{\mathcal{R}^\wedge}(B) = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$; (2) $\operatorname{Cl}_{\mathcal{R}^\wedge}(B) = \mathcal{P}^\star\big(\operatorname{cl}_{\mathcal{R}}(B)\big)$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^\wedge}(B)$, then by Theorem 29.2 and Corollary 34.1 we have $A \subseteq \operatorname{int}_{\mathcal{R}^\wedge}(B) = \operatorname{int}_{\mathcal{R}}(B)$, and thus also $A \in \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$. Therefore, $\operatorname{Int}_{\mathcal{R}^\wedge}(B) \subseteq \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$.

While, if $A \in \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big)$, then $A \subseteq \operatorname{int}_{\mathcal{R}}(B)$. Therefore, by Definition 29.1, for each $x \in A$ there exists $R_x \in \mathcal{R}$ such that $R_x(x) \subseteq B$. Now, by defining

$$S(x) = R_x(x) \quad \text{for all} \quad x \in A \qquad \text{and} \qquad S(x) = Y \quad \text{for all} \quad x \in A^c,$$

we at once state that $S[A] \subseteq B$. Moreover, by using that $\mathcal{R} \neq \emptyset$, we also easily note that $S \in \mathcal{R}^\wedge$. Therefore, by Definition 29.1, we also have $A \in \operatorname{Int}_{\mathcal{R}^\wedge}(B)$. Consequently, $\mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(B)\big) \subseteq \operatorname{Int}_{\mathcal{R}^\wedge}(B)$, and thus assertion (1) also holds. ■

Remark 35.1. By assertion (2) and Definition 23.1, for any $A \subseteq X$, we have

$$A \in \operatorname{Cl}_{\mathcal{R}^\wedge}(B) \iff A \in \mathcal{P}^\star\big(\operatorname{cl}_{\mathcal{R}}(B)\big) \iff A \in \mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big)^c \iff$$
$$A \notin \mathcal{P}\big(\operatorname{cl}_{\mathcal{R}}(B)^c\big) \iff A \not\subseteq \operatorname{cl}_{\mathcal{R}}(B)^c \iff A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset.$$

From the above theorem, by using Definition 30.1, we immediately derive the following result.

Corollary 35.1. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$; (2) $\tau\!\!\!-_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$.

Remark 35.2. Note that, by Theorem 30.4 and Remark 30.2, we have

$$\tau_{\mathcal{R}} = \textstyle\bigcup_{R \in \mathcal{R}} \tau_R = \bigcup_{R \in \mathcal{R}} \mathcal{T}_R .$$

Hence, by writing $\mathcal{R}^\wedge$ in place of $\mathcal{R}$ and using Corollary 35.1, we immediately infer that $\mathcal{T}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}^\wedge} \mathcal{T}_R$.

From Corollary 35.1, by using Theorem 34.2, we also immediately derive the following result.

Corollary 35.2. *If $\mathcal{R}$ is a nonvoid relator on X, then*

(1) $\tau_{\mathcal{R}^\vartriangle} = \mathcal{T}_{\mathcal{R}^\vartriangle}$; (2) $\tau\!\!\!-_{\mathcal{R}^\vartriangle} = \mathcal{F}_{\mathcal{R}^\vartriangle}$.

Concerning the operation $\vartriangle$, we also prove the following result.

Theorem 35.2. *If $\mathcal{R}$ is a nonvoid relator on X to Y, then for any $B \subseteq Y$ we have*

(1) $\operatorname{Int}_{\mathcal{R}^\vartriangle}(B) = \{\emptyset\}$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^\vartriangle}(B) = \mathcal{P}(X)$ *if* $B \in \mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(B) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $B \in \mathcal{D}_{\mathcal{R}}$.

Proof. If $A \in \operatorname{Int}_{\mathcal{R}^\vartriangle}(B)$, then there exists $S \in \mathcal{R}^\vartriangle$ such that $S[A] \subseteq B$. Therefore, if $A \neq \emptyset$, then there exists $x \in X$ such that $S(x) \subseteq B$. Hence, by using that $S(x) \in \mathcal{E}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ is ascending, we infer that $B \in \mathcal{E}_{\mathcal{R}}$. Therefore, if $B \notin \mathcal{E}_{\mathcal{R}}$, then we necessarily have $\operatorname{Int}_{\mathcal{R}^\vartriangle}(B) \subseteq \{\emptyset\}$. Moreover, since $\mathcal{R} \neq \emptyset$, we also note that $\mathcal{R}^\vartriangle \neq \emptyset$, and thus $\emptyset \in \operatorname{Int}_{\mathcal{R}^\vartriangle}(B)$. Therefore, the first part of assertion (1) is true.

On the other hand, if $B \in \mathcal{E}_{\mathcal{R}}$, then by defining $R = X \times B$ and using Theorem 33.1, we see that $R \in \mathcal{R}^\vartriangle$. Moreover, we also note that $R[A] \subseteq B$, and thus $A \in \operatorname{Int}_{\mathcal{R}^\vartriangle}(B)$ for all $A \subseteq X$. Therefore, the second part of assertion (1) is also true. ■

From this theorem, by Definition 29.1, it is clear that in particular we also have the following result.

Corollary 35.3. *If $\mathcal{R}$ is a nonvoid relator on X to Y and $B \subseteq Y$, then*

(1) $\operatorname{cl}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{cl}_{\mathcal{R}^\vartriangle}(B) = X$ *if* $B \in \mathcal{D}_{\mathcal{R}}$;

(2) $\operatorname{int}_{\mathcal{R}^\vartriangle}(B) = \emptyset$ *if* $B \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{int}_{\mathcal{R}^\vartriangle}(B) = X$ *if* $B \in \mathcal{E}_{\mathcal{R}}$.

Hence, by using Definition 30.1, we immediately derive the following result.

Corollary 35.4. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{T}_{\mathcal{R}^{\triangle}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$; (2) $\mathcal{F}_{\mathcal{R}^{\triangle}} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

Remark 35.3. Note that if in particular $\mathcal{R} = \emptyset$, then $\mathcal{E}_{\mathcal{R}} = \emptyset$. Moreover, $\mathcal{R}^{\triangle} = \emptyset$ if $X \neq \emptyset$, and $\mathcal{R}^{\triangle} = \{\emptyset\}$ if $X = \emptyset$. Therefore, $\mathcal{T}_{\mathcal{R}^{\triangle}} = \{\emptyset\}$, and thus assertion (1) is still true.

Now, since $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ if $\mathcal{R}$ is non-partial, we also state the following result.

Corollary 35.5. *If $\mathcal{R}$ is a non-partial relator on X, then*

(1) $\mathcal{E}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\triangle}} \setminus \{\emptyset\}$, (2) $\mathcal{D}_{\mathcal{R}} = \big(\mathcal{P}(X) \setminus \mathcal{F}_{\mathcal{R}^{\triangle}}\big) \cup \{X\}$.

36. Some further closure operations for relators

By using the following theorem, from the basic closure operations $*$, $\#$, $\wedge$ and $\triangle$, we easily derive some further important closure operations.

Theorem 36.1. *If $\square$ is a closure (projection) and $\diamond$ is an involution operation for relators, then $\Diamond = \diamond\,\square\,\diamond$ is also a closure (projection) operation for relators.*

Proof. To prove that $\Diamond$ is also idempotent, note that

$$\Diamond\,\Diamond = (\diamond\square\diamond)(\diamond\square\diamond) = (\diamond\square)\big((\diamond\diamond)(\square\diamond)\big) = (\diamond\square)\big(\Delta(\square\diamond)\big) = (\diamond\square)(\square\diamond) = \diamond\big((\square\square)\diamond\big) = \diamond(\square\diamond) = \Diamond,$$

where Δ is the identity operation for relators. ■

Now, by using the elementwise complementation and inversion, we also introduce the following definition.

Definition 36.1. For an operation $\square$ for relators, we define

$$\textcircled{\scriptsize$\square$} = c\,\square\,c \qquad \textit{and} \qquad \boxdot = -1\,\square\,-1\,.$$

Remark 36.1. Thus, by Theorem 36.1, for instance $\circledast$ is also a closure operation for relators. This statement is also quite obvious from the fact that

$$\mathcal{R}^{\circledast} = \bigcup_{R \in \mathcal{R}} \mathcal{P}(R) = \big\{ S \subseteq X \times Y : \quad \exists\ R \in \mathcal{R} : \quad S \subseteq R \big\}.$$

To check this, note that if for instance $S \in \mathcal{R}^{\circledast}$, then $S \in \mathcal{R}^{c*c}$, and thus $S^c \in \mathcal{R}^{c*}$. Therefore, there exists $R \in \mathcal{R}$ such that $R^c \subseteq S^c$. Hence, it follows that $S \subseteq R$, and thus $S \in \mathcal{P}(R)$. Therefore, $S \in \bigcup_{R \in \mathcal{R}} \mathcal{P}(R)$ also holds.

The importance of Definition 36.1 lies mainly in the following counterpart of Theorem 34.1.

Theorem 36.2. *$\textcircled{\scriptsize\#}$ and $\textcircled{\scriptsize$\wedge$}$ are closure operations for relators such that, for any relator $\mathcal{S}$ on X to Y,*

(1) $\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\scriptsize\#}} \iff \operatorname{Lb}_{\mathcal{S}} \subseteq \operatorname{Lb}_{\mathcal{R}}$; (2) $\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\scriptsize$\wedge$}} \iff \operatorname{lb}_{\mathcal{S}} \subseteq \operatorname{lb}_{\mathcal{R}}$.

Proof. By the corresponding definitions and Theorems 34.1 and 32.2, we have

$$\mathcal{S} \subseteq \mathcal{R}^{\textcircled{\scriptsize\#}} \iff \mathcal{S} \subseteq \mathcal{R}^{c\#c} \iff \mathcal{S}^{c} \subseteq \mathcal{R}^{c\#} \iff \operatorname{Int}_{\mathcal{S}^c} \subseteq \operatorname{Int}_{\mathcal{R}^c} \iff \operatorname{Int}_{\mathcal{S}^c} \circ \mathcal{C}_Y \subseteq \operatorname{Int}_{\mathcal{R}^c} \circ \mathcal{C}_Y \iff \operatorname{Lb}_{\mathcal{S}} \subseteq \operatorname{Lb}_{\mathcal{R}}.$$

Therefore, assertion (1) is true. The proof of assertion (2) is quite similar. ■

Now, analogously to Corollary 34.1, we also state the following result.

Corollary 36.1. *The following assertions are true:*

(1) $\mathcal{S} = \mathcal{R}^{\textcircled{\scriptsize\#}}$ *is the largest relator on X to Y such that* $\operatorname{Lb}_{\mathcal{S}} = \operatorname{Lb}_{\mathcal{R}}$;

(2) $\mathcal{S} = \mathcal{R}^{\textcircled{\scriptsize$\wedge$}}$ *is the largest relator on X to Y such that* $\operatorname{lb}_{\mathcal{S}} = \operatorname{lb}_{\mathcal{R}}$.

Remark 36.2. Because of Theorem 32.1, in Theorem 36.2 and its corollary, we may write Ub instead of Lb. Concerning the structure ub, by using Theorems 32.1 and 36.2, we only prove that

$$\operatorname{ub}_{\mathcal{S}} \subseteq \operatorname{ub}_{\mathcal{R}} \iff \operatorname{lb}_{\mathcal{S}^{-1}} \subseteq \operatorname{lb}_{\mathcal{R}^{-1}} \iff \mathcal{S}^{-1} \subseteq \mathcal{R}^{-1\textcircled{\scriptsize$\wedge$}} \iff \mathcal{S} \subseteq \mathcal{R}^{-1\textcircled{\scriptsize$\wedge$}-1} \iff \mathcal{S} \subseteq \mathcal{R}^{\boxed{\textcircled{\scriptsize$\wedge$}}}$$

Remark 36.3. In this respect, it is also worth mentioning that, by using the associativity of composition and the inversion compatibility of c, we also easily see that

$$\boxed{\textcircled{$\wedge$}} = -1 \textcircled{$\wedge$} -1 = -1\, c \wedge c -1 = c -1 \wedge -1\, c = c \boxed{\wedge} c = \textcircled{$\boxed{\wedge}$}\,.$$

Now, by using the corresponding definitions and Theorem 34.2, we prove the next theorem.

Theorem 36.3. *We have*

(1) $\mathcal{R}^{\circledast} = \mathcal{R}^{\circledast\circledast}$;

(2) $\mathcal{R}^{\textcircled{\#}} = \mathcal{R}^{\Diamond\textcircled{\#}} = \mathcal{R}^{\textcircled{\#}\Diamond}$ *with* $\Diamond = \circledast$ *and* $\textcircled{\#}$;

(3) $\mathcal{R}^{\textcircled{$\wedge$}} = \mathcal{R}^{\Diamond\textcircled{$\wedge$}} = \mathcal{R}^{\textcircled{$\wedge$}\Diamond}$ *with* $\Diamond = \circledast$, $\textcircled{\#}$ *and* $\textcircled{$\wedge$}$;

(4) $\mathcal{R}^{\textcircled{$\wedge$}} = \mathcal{R}^{\Diamond\textcircled{$\wedge$}} = \mathcal{R}^{\Diamond\textcircled{$\wedge$}\Diamond}$ *with* $\Diamond = \circledast$, $\textcircled{\#}$, $\textcircled{$\wedge$}$ *and* $\textcircled{$\wedge$}$.

Proof. To prove (2), for instance, note that

$$\circledast\textcircled{\#} = c * c\, c \# c = c * \# c = c \# c = \textcircled{\#}\,. \qquad \blacksquare$$

37. A few basic facts on simple relators

Definition 37.1. For some operation $\Box$ for relators, the relator $\mathcal{R}$ will be called $\Box$*–simple* if $\mathcal{R}^{\Box} = \{R\}^{\Box}$ for some for some relation R on X to Y. More generally, for some structure $\mathfrak{F}$ for relators, the relator $\mathcal{R}$ will be is called $\mathfrak{F}$*–simple* if $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_R$ for some relation R on X to Y.

Remark 37.1. Thus, in particular, a singleton relator has to be called *properly simple*. Moreover, for instance, a $\#$–simple relator has to be called *proximally simple*. Now, by using Theorem 34.1, we see that the relator $\mathcal{R}$ is proximally simple if and only if it is Cl–simple, or equivalently Int–simple. Thus, in particular, $\mathcal{R}$ is also cl–simple and int–simple.

Simple relators have mainly been intensively investigated by the third author, Mala, and Pataki [35, 41, 60]. However, concerning them, we only prove here the following basic characterization theorems.

Theorem 37.1. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly simple;* (2) $\mathcal{R} = \{\rho_{\mathcal{R}}^{-1}\}$.

Proof. If (1) holds, then by Remark 37.1, there exists a relation R on X to Y such that $\mathcal{R} = \{R\}$. Hence, by using Corollary 29.1, we infer that $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = R$. Therefore, (2) also holds. Thus, since the converse implication trivially holds, the proof is complete. $\blacksquare$

Theorem 37.2. *The following assertions are equivalent:*

(1) $\mathcal{R}$ *is uniformly simple;*

(2) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}$; (3) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^*$; (4) $\mathcal{R}^* = \{\rho_{\mathcal{R}}^{-1}\}^*$;

(5) $R \in \mathcal{R}^*$ *and* $\mathcal{R} \subseteq \{R\}^*$ *for some relation* R *on* X *to* Y.

Proof. If (1) holds, then there exists a relation on X to Y such that $\mathcal{R}^* = \{R\}^*$. Hence, by using Theorem 34.2, we infer that $\mathcal{R}^{\wedge} = \mathcal{R}^{*\wedge} = \{R\}^{*\wedge} = \{R\}^{\wedge}$. Therefore, by Corollary 34.1, we also state that $\mathrm{cl}_{\mathcal{R}} = \mathrm{cl}_{\{R\}}$. Thus, in particular, we also see that

$$\rho_{\mathcal{R}}(y) = \mathrm{cl}_{\mathcal{R}}(\{y\}) = \mathrm{cl}_{\{R\}}(\{y\}) = R^{-1}[\{y\}] = R^{-1}(y)$$

for all $y \in Y$. Therefore, $\rho_{\mathcal{R}} = R^{-1}$, and thus $R = \rho_{\mathcal{R}}^{-1}$.

Now, we already see that

$$\rho_{\mathcal{R}}^{-1} \in \{\rho_{\mathcal{R}}^{-1}\}^* = \{R\}^* = \mathcal{R}^*.$$

Therefore, there exists $S \in \mathcal{R}$ such that $S \subseteq \rho_{\mathcal{R}}^{-1}$. Hence, by using Corollary 29.1, we infer that

$$\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} \subseteq S \subseteq \rho_{\mathcal{R}}^{-1}.$$

Therefore, $\rho_{\mathcal{R}}^{-1} = S \in \mathcal{R}$, and thus (2) also holds.

From Remark 33.1, we at once see that (2) $\Longrightarrow$ (3). Moreover, by using the corresponding properties of $*$, we easily see that

$$(3) \;\Longrightarrow\; \{\rho_{\mathcal{R}}^{-1}\} \subseteq \mathcal{R}^* \;\Longrightarrow\; \{\rho_{\mathcal{R}}^{-1}\}^* \subseteq \mathcal{R}^{**} \;\Longrightarrow\; \{\rho_{\mathcal{R}}^{-1}\}^* \subseteq \mathcal{R}^*.$$

Therefore, to prove the implication (3) $\Longrightarrow$ (4), we need only note that

$$\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} \subseteq R \quad \text{for all} \quad R \in \mathcal{R}.$$

Therefore, $\mathcal{R} \subseteq \{\rho_{\mathcal{R}}^{-1}\}^*$, and thus $\mathcal{R}^* \subseteq \{\rho_{\mathcal{R}}^{-1}\}^{**} = \{\rho_{\mathcal{R}}^{-1}\}^*$ also holds.

On the other hand, if (4) holds, then by taking $R = \rho_{\mathcal{R}}^{-1}$, we at once see that R is a relation on X to Y such that $\mathcal{R}^* = \{R\}^*$, and thus

$$R \in \{R\}^* = \mathcal{R}^* \qquad \text{and} \qquad \mathcal{R} \subseteq \mathcal{R}^* = \{R\}^*.$$

Therefore, (5) also holds.

Finally, if (5) holds, then by using the corresponding properties of $*$, we see easily see that

$$\{R\}^* \subseteq \mathcal{R}^{**} = \mathcal{R}^* \qquad \text{and} \qquad \mathcal{R}^* \subseteq \{R\}^{**} = \{R\}^*.$$

Therefore, $\mathcal{R}^* = \{R\}^*$, and thus (1) also holds. ■

By using Theorem 34.2, analogously to the above theorem, we also easily prove the following result.

Theorem 37.3. *For $\square = \#$ and $\wedge$, the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\square$*–simple;*

(2) $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}^{\square}$*;* (3) $\mathcal{R}^{\square} = \{\rho_{\mathcal{R}}^{-1}\}^{\square}$*;*

(4) $R \in \mathcal{R}^{\square}$ *and* $\mathcal{R} \subseteq \{R\}^*$ *for some relation* R *on* X *to* Y.

Remark 37.2. For the operation $\square = \triangle$, we quite similarly prove the implications (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1).

However, the following modification of Pataki (see [41, Example 3.4]) shows that the implication (1) $\Longrightarrow$ (2) need not be true even for a finite equivalence relator.

Example 37.1. If $X = \{1, 2, 3\}$ and

$R_1(1) = \{1\}$, $R_1(2) = \{2, 3\}$, $R_1(3) = \{2, 3\}$;

$R_2(1) = \{1, 3\}$, $R_2(2) = \{2\}$, $R_2(3) = \{1, 3\}$;

then $\mathcal{R} = \{R_1, R_2\}$ is paratopologically simple equivalence relator on X such that $\rho_{\mathcal{R}}^{-1} \notin \mathcal{R}^{\triangle}$. Namely, if $R(1) = \{1\}$, $R(2) = \{2\}$ and $R(3) = X$, then by Definition 29.1 we have $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_R$. Hence, by using Theorem 34.1, we infer that $\mathcal{R}^{\triangle} = \{R\}^{\triangle}$, and thus $\mathcal{R}$ is paratopologically simple. However, by Corollary 29.1, we have

$$\rho_{\mathcal{R}}^{-1} = (R_1 \cap R_2)(3) = R_1(3) \cap R_2(3) = \{3\} \notin \mathcal{E}_{\mathcal{R}}.$$

Therefore, by Remark 33.1, we also state that $\rho_{\mathcal{R}}^{-1} \notin \mathcal{R}^{\triangle}$.

Remark 37.3. At several topological conferences, to justify the appropriateness of a theory of generalized nets, the third author asked the participants to construct a relator which is not paratopologically simple. This problem was finally solved by Pataki [41, Example 5.11] by using some former observations of Jenő Deák. He constructed an equivalence relator $\mathcal{R} = \{R_1, R_2, R_3\}$ on the set $X = \{1, 2, 3, 4\}$ which is not paratopologically simple.

Remark 37.4. To motivate the study of simple relators, we note that a relator $\mathcal{R}$ on X may be called *properly well-chained* if $\mathcal{R}^{\infty} = \{X^2\}^{\infty}$. Also, the relator $\mathcal{R}$ may be called $\square$*–well-chained*, for some unary operation $\square$ for relators, if the relator $\mathcal{R}^{\square}$ is properly well-chained. Furthermore, the relator $\mathcal{R}$ may be called $\square$*–connected* if the relator $\mathcal{R}^{\square} \triangledown \mathcal{R}^{\square\,-1}$ or $\mathcal{R}^{\square} \vee \mathcal{R}^{\square\,-1}$ is properly well-chained [43].

38. Right seminormality properties of closure relations

Theorem 38.1. *The following assertions are equivalent:*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is right* $\mathrm{cl}_{\mathcal{R}}$*–seminormal;* (2) $\mathrm{int}_{\mathcal{R}}$ *is left* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal.*

Proof. By Theorem 26.3, we have

$$\mathrm{cl}_{\mathcal{R}^{-1}} \text{ is right } \mathrm{cl}_{\mathcal{R}}\text{–seminormal} \quad \Longleftrightarrow \quad \mathrm{cl}^{\star}_{\mathcal{R}} \text{ is left } \mathrm{cl}^{\star}_{\mathcal{R}^{-1}}\text{–seminormal}.$$

Moreover, by Theorem 29.1 and Definition 22.1, we have

$$\mathrm{cl}^{\star}_{\mathcal{R}} = \mathrm{int}_{\mathcal{R}}, \qquad \text{and thus} \qquad \mathrm{cl}^{\star}_{\mathcal{R}^{-1}} = \mathrm{int}_{\mathcal{R}^{-1}}.$$

Therefore, assertions (1) and (2) are also equivalent. ∎

Theorem 38.2. *If* $\rho_{\mathcal{R}}^{-1}$ *is non-partial, then*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is right* $\mathrm{cl}_{\mathcal{R}}$*–seminormal;* (2) $\mathrm{int}_{\mathcal{R}}$ *is left* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal.*

Proof. By Theorem 38.1, we need only prove assertion (1). For this, suppose that $A \subseteq X$ and $B \subseteq Y$ such that

$$\mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B, \qquad \text{and moreover} \qquad x \in A.$$

Then, by the assumption of the theorem, there exists $y \in Y$ such that $y \in \rho_{\mathcal{R}}^{-1}(x)$. Hence, by using Corollary 29.1, Definition 29.1 and our former assumptions, we infer that

$$y \in \rho_{\mathcal{R}^{-1}}(x) = \mathrm{cl}_{\mathcal{R}^{-1}}\big(\{x\}\big) \subseteq \mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B.$$

Now, we also easily note that

$$x \in \rho_{\mathcal{R}}(y) = \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big) \subseteq \mathrm{cl}_{\mathcal{R}}(B), \qquad \text{and thus} \qquad A \subseteq \mathrm{cl}_{\mathcal{R}}(B).$$

Therefore,

$$\mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B \quad \Longrightarrow \quad A \subseteq \mathrm{cl}_{\mathcal{R}}(B),$$

and thus by Definition 24.1 assertion (1) also holds. ∎

Remark 38.1. Note that $\rho_{\mathcal{R}}^{-1}$ is a non-partial relation on X to Y if and only if $\rho_{\mathcal{R}}$ is a surjective relation on Y to X.

Now, as a partial converse to Theorem 38.2, we also prove the following result.

Theorem 38.3. *If* $Y \neq \emptyset$, *and any one of assertions*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is right* $\mathrm{cl}_{\mathcal{R}}$*–seminormal,* (2) $\mathrm{int}_{\mathcal{R}}$ *is left* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal*

holds, then $\rho_{\mathcal{R}}^{-1}$ *is non-partial.*

Proof. By Theorem 38.1, we may suppose that assertion (1) holds. Then, by Definition 24.1, for any $A \subseteq X$ and $B \subseteq Y$

$$\mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B \quad \Longrightarrow \quad A \subseteq \mathrm{cl}_{\mathcal{R}}(B).$$

Thus, in particular, for any $x \in X$ and $y \in Y$

$$\mathrm{cl}_{\mathcal{R}^{-1}}\big(\{x\}\big) \subseteq \{y\} \quad \Longrightarrow \quad \{x\} \subseteq \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big) \quad \Longrightarrow \quad x \in \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big).$$

Hence, since

$$\rho_{\mathcal{R}}(y) = \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big) \qquad \text{and} \qquad \rho_{\mathcal{R}}^{-1}(x) = \rho_{\mathcal{R}^{-1}}(x) = \mathrm{cl}_{\mathcal{R}^{-1}}\big(\{x\}\big),$$

we infer that

$$\rho_{\mathcal{R}}^{-1}(x) \subseteq \{y\} \quad \Longrightarrow \quad x \in \rho_{\mathcal{R}}(y) \quad \Longrightarrow \quad y \in \rho_{\mathcal{R}}^{-1}(x).$$

Moreover, if $\rho_{\mathcal{R}}^{-1}$ fails to be non-partial, then there exists $x_0 \in X$ such that

$$\rho_{\mathcal{R}}^{-1}(x_0) = \emptyset \subseteq \{y\}.$$

Therefore, by our former implication, for any $y \in Y$ we have

$$y \in \rho_{\mathcal{R}}^{-1}(x_0), \qquad \text{and thus} \qquad Y = \rho_{\mathcal{R}}^{-1}(x_0) = \emptyset.$$

This, contradiction proves that $\rho_{\mathcal{R}}^{-1}$ is non-partial. ∎

Now, combining our former observations, we also state the following result.

Theorem 38.4. *If* $Y \neq \emptyset$, *then the following assertions are equivalent:*

(1) $\rho_{\mathcal{R}}$ *is surjective;* (2) $\rho_{\mathcal{R}}^{-1}$ *is non-partial;*

(3) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is right* $\mathrm{cl}_{\mathcal{R}}$*–seminormal;* (4) $\mathrm{int}_{\mathcal{R}}$ *is left* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal.*

Remark 38.2. Note that, under the notation $R = \bigcap \mathcal{R}$, we have $\rho_{\mathcal{R}} = R^{-1}$, and thus $\rho_{\mathcal{R}}^{-1} = R$.

39. Left seminormality properties of closure relations

Quite similarly to Theorem 38.1, we also prove the following result.

Theorem 39.1. *The following assertions are equivalent :*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is left* $\mathrm{cl}_{\mathcal{R}}$*–seminormal ;* (2) $\mathrm{int}_{\mathcal{R}}$ *is right* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal .*

Proof. By Theorem 26.4, we have

$$\mathrm{cl}_{\mathcal{R}^{-1}} \text{ is left } \mathrm{cl}_{\mathcal{R}}\text{–seminormal} \quad \Longleftrightarrow \quad \mathrm{cl}_{\mathcal{R}}^{\star} \text{ is right } \mathrm{cl}_{\mathcal{R}^{-1}}^{\star}\text{–seminormal.}$$

Moreover, by Theorem 29.1 and Definition 22.1, we have

$$\mathrm{cl}_{\mathcal{R}}^{\star} = \mathrm{int}_{\mathcal{R}}\,, \qquad \text{and thus} \qquad \mathrm{cl}_{\mathcal{R}^{-1}}^{\star} = \mathrm{int}_{\mathcal{R}^{-1}}\ .$$

Therefore, assertions (1) and (2) are equivalent. ■

However, instead of an analogue of Theorem 38.2, we only prove the following result.

Theorem 39.2. *If there exists* $R \in \mathcal{R}$ *such that* R *is a function, then*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is left* $\mathrm{cl}_{\mathcal{R}}$*–seminormal ;* (2) $\mathrm{int}_{\mathcal{R}}$ *is right* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal .*

Proof. By Theorem 39.1, we need only prove assertion (1). For this, suppose that $A \subseteq X$ and $B \subseteq Y$ such that

$$A \subseteq \mathrm{cl}_{\mathcal{R}}(B)\,, \qquad \text{and moreover} \qquad y \in cl_{\mathcal{R}^{-1}}(A)\,.$$

Then, by Definition 29.1, in particular we have $R^{-1}(y) \cap A \neq \emptyset$. Thus, there exists

$$x \in A \qquad \text{such that} \qquad x \in R^{-1}(y)\,, \quad \text{and thus} \quad y \in R(x)\,.$$

Moreover, since $x \in A \subseteq \mathrm{cl}_{\mathcal{R}}(B)$, we also note that $R(x) \cap B \neq \emptyset$. Thus, there exists

$$z \in B \qquad \text{such that} \qquad z \in R(x)\,.$$

Now, since R is a function, we already see that

$$y = z \in B\,, \qquad \text{and thus} \qquad \mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B\,.$$

Therefore,

$$A \subseteq \mathrm{cl}_{\mathcal{R}}(B) \quad \Longrightarrow \quad \mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B\,.$$

Thus, by Definition 24.1, assertion (1) is true. ■

Remark 39.1. If $R \in \mathcal{R}$ then we have $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} \subseteq R$, and thus $\rho_{\mathcal{R}}^{-1}(x) \subseteq R(x)$ for all $x \in X$. Hence, if in particular R is a function and $\rho_{\mathcal{R}}^{-1}$ is non-partial, then we infer that $\rho_{\mathcal{R}}^{-1}(x) = R(x)$ for all $x \in X$, and thus $R = \rho_{\mathcal{R}}^{-1}$.

The next corollary is a natural consequence of Theorem 39.2.

Corollary 39.1. *If* $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}$ *and* $\rho_{\mathcal{R}}^{-1}$ *is a function, then*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is left* $\mathrm{cl}_{\mathcal{R}}$*–seminormal ;* (2) $\mathrm{int}_{\mathcal{R}}$ *is right* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal .*

Moreover, as a partial converse of this corollary, we also prove the following result.

Theorem 39.3. *If any one of assertions*

(1) $\mathrm{cl}_{\mathcal{R}^{-1}}$ *is left* $\mathrm{cl}_{\mathcal{R}}$*–seminormal ,* (2) $\mathrm{int}_{\mathcal{R}}$ *is right* $\mathrm{int}_{\mathcal{R}^{-1}}$*–seminormal*

holds, then $\rho_{\mathcal{R}}^{-1}$ *is a function.*

Proof. By Theorem 39.1, we may suppose that assertion (1) holds. Then, by Definition 24.1, for any $A \subseteq X$ and $B \subseteq Y$,

$$A \subseteq \mathrm{cl}_{\mathcal{R}}(B) \quad \Longrightarrow \quad \mathrm{cl}_{\mathcal{R}^{-1}}(A) \subseteq B\,.$$

Thus, in particular, for any $x \in X$ and $y \in Y$

$$\{x\} \subseteq \mathrm{cl}_{\mathcal{R}}\big(\{y\}\big) \quad \Longrightarrow \quad \mathrm{cl}_{\mathcal{R}^{-1}}\big(\{x\}\big) \subseteq \{y\}\,.$$

Hence, since

$$\rho_{\mathcal{R}}(y) = \text{cl}_{\mathcal{R}}(\{y\}) \qquad \text{and} \qquad \rho_{\mathcal{R}}^{-1}(x) = \rho_{\mathcal{R}^{-1}}(x) = \text{cl}_{\mathcal{R}^{-1}}(\{x\}),$$

we infer that

$$y \in \rho_{\mathcal{R}}^{-1}(x) \implies x \in \rho_{\mathcal{R}}(y) \implies \{x\} \subseteq \rho_{\mathcal{R}}(y) \implies \rho_{\mathcal{R}}^{-1}(x) \subseteq \{y\}.$$

Therefore, $\rho_{\mathcal{R}}^{-1}(x)$ is at most a singleton for all $x \in X$, and thus $\rho_{\mathcal{R}}^{-1}$ is a function. ∎

Next, by using Corollary 39.1 and Theorem 39.3, we also easily establish the following result.

Theorem 39.4. *If $\mathcal{R}$ is uniformly simple, then the following assertions are equivalent:*

(1) $\rho_{\mathcal{R}}^{-1}$ *is a function;*

(2) $\text{cl}_{\mathcal{R}^{-1}}$ *is left* $\text{cl}_{\mathcal{R}}$*–seminormal,* (3) $\text{int}_{\mathcal{R}}$ *is right* $\text{int}_{\mathcal{R}^{-1}}$*–seminormal.*

Proof. Since $\mathcal{R}$ is uniformly simple, by Theorem 37.2 we have $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}$. Therefore, Corollary 39.1 and 39.3 can be applied to prove the equivalence of (1) and (2). ∎

40. Normality properties of closure relations

Combining the corresponding results of Sections 38 and 39, we establish the following four theorems.

Theorem 40.1. *The following assertions are equivalent:*

(1) $\text{cl}_{\mathcal{R}^{-1}}$ *is* $\text{cl}_{\mathcal{R}}$*–normal;* (2) $\text{int}_{\mathcal{R}}$ *is* $\text{int}_{\mathcal{R}^{-1}}$*–normal.*

Theorem 40.2. *If $\rho_{\mathcal{R}}^{-1} \in \mathcal{R}$ and $\rho_{\mathcal{R}}^{-1}$ is a function of X to Y, then*

(1) $\text{cl}_{\mathcal{R}^{-1}}$ *is* $\text{cl}_{\mathcal{R}}$*–normal;* (2) $\text{int}_{\mathcal{R}}$ *is* $\text{int}_{\mathcal{R}^{-1}}$*–normal.*

Theorem 40.3. *If $Y \neq \emptyset$, and any one of assertions*

(1) $\text{cl}_{\mathcal{R}^{-1}}$ *is* $\text{cl}_{\mathcal{R}}$*–normal,* (2) $\text{int}_{\mathcal{R}}$ *is* $\text{int}_{\mathcal{R}^{-1}}$*–normal*

holds, then $\rho_{\mathcal{R}}^{-1}$ is a function of X to Y.

Theorem 40.4. *If $Y \neq \emptyset$ and $\mathcal{R}$ is uniformly simple, then the following assertions are equivalent:*

(1) $\rho_{\mathcal{R}}^{-1}$ *is a function of X to Y.*

(2) $\text{cl}_{\mathcal{R}^{-1}}$ *is* $\text{cl}_{\mathcal{R}}$*–normal,* (3) $\text{int}_{\mathcal{R}}$ *is* $\text{int}_{\mathcal{R}^{-1}}$*–normal.*

The next result is an immediate consequence of Theorem 40.4.

Theorem 40.5. *If there exists a function f of X to Y such that*

$$\mathcal{R}^* = \{f\}^*,$$

then

(1) $\text{cl}_{\mathcal{R}^{-1}}$ *is* $\text{cl}_{\mathcal{R}}$*–normal;* (2) $\text{int}_{\mathcal{R}}$ *is* $\text{int}_{\mathcal{R}^{-1}}$*–normal.*

Proof. By using Theorem 37.2, we infer that

$$\{f\}^* = \mathcal{R}^* = \left\{\rho_{\mathcal{R}}^{-1}\right\}^*.$$

Therefore, $\rho_{\mathcal{R}}^{-1} = f \in \mathcal{R}$, and thus Theorem 40.4 can be applied to obtain the required assertions. ∎

From this theorem, by using Theorem 29.1, we easily derive the following result.

Corollary 40.1. *If f is function of X to Y, then the super relation $f^{\triangleright}$ is $f^{-1\triangleright}$–normal.*

Proof. For the relator $\mathcal{R} = \{f\}$, we have $\mathcal{R}^* = \{f\}^*$. Moreover, by Definitions 17.1 and 24.1 and Theorems 29.1 and 40.5, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$f^{\triangleright}(A) \subseteq B \iff f[A] \subseteq B \iff \text{cl}_{f^{-1}}(A) \subseteq B \iff A \subseteq \text{cl}_{\mathcal{R}}(B) \iff A \subseteq f^{-1}[B] \iff A \subseteq f^{-1\triangleright}(B).$$

Therefore, by Definition 24.1, the required assertion is true. ∎

Now, by using this corollary and Theorem 40.3, we also prove the next theorem.

Theorem 40.6. *If $Y \neq \emptyset$, then for a relation R on X to Y, the following assertions are equivalent:*

(1) *R is a function of X to Y;*

(2) *$R^{\triangleright}$ is $R^{-1\triangleright}$–normal;*

(3) *$\operatorname{cl}_{R^{-1}}$ is cl_R–normal;*

(4) *$\operatorname{int}_{\mathcal{R}}$ is $\operatorname{int}_{\mathcal{R}^{-1}}$–normal.*

Proof. By Corollary 40.1, (1) implies (2). Moreover, if assertion (2) holds, then by Theorem 29.1 and Definition 24.1, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\operatorname{cl}_{R^{-1}}(A) \subseteq B \iff R[A] \subseteq B \iff R^{\triangleright}(A) \subseteq B \iff A \subseteq R^{-1\triangleright}(B) \iff A \subseteq R^{-1}[B] \iff A \subseteq \operatorname{cl}_R(B).$$

Therefore, by Definition 24.1, assertion (3) also holds.

Moreover, if assertion (3) holds, then by using the notation $\mathcal{R} = \{R\}$ we see that $\operatorname{cl}_{\mathcal{R}^{-1}}$ is $\operatorname{cl}_{\mathcal{R}}$–normal. Hence, by using Theorem 40.3, we infer that $\rho_{\mathcal{R}^{-1}}$ is a function of X to Y. Moreover, from Corollary 29.1, we see that $\rho_{\mathcal{R}^{-1}} = R$. Therefore, assertion (1) also holds. ∎

41. Some more important normality properties of closure relations

Analogously to Theorem 40.1, we prove following result.

Theorem 41.1. *The following assertions are equivalent:*

(1) *$\operatorname{cl}_{\mathcal{R}^{-1}}$ is $\operatorname{int}_{\mathcal{R}}$–normal;*

(2) *$\operatorname{cl}_{\mathcal{R}}$ is $\operatorname{int}_{\mathcal{R}^{-1}}$–normal.*

Proof. By Corollary 26.2, we have

$$\operatorname{cl}_{\mathcal{R}^{-1}} \text{ is } \operatorname{int}_{\mathcal{R}}\text{–normal} \iff \operatorname{int}^{\star}_{\mathcal{R}} \text{ is } \operatorname{cl}^{\star}_{\mathcal{R}^{-1}}\text{–normal} .$$

Moreover, by Theorem 29.1 and Definition 19.1, we have

$$\operatorname{int}^{\star}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}}, \qquad \text{and thus} \qquad \operatorname{cl}^{\star}_{\mathcal{R}^{-1}} = \operatorname{int}_{\mathcal{R}^{-1}} .$$

Therefore, assertions (1) and (2) are equivalent. ∎

Remark 41.1. If R is a relation on X to Y such that

$$\mathcal{R}^{\#} = \{R\}^{\#},$$

then by using Theorem 34.2, we see that

$$\mathcal{R}^{\wedge} = \mathcal{R}^{\#\wedge} = \{R\}^{\#\wedge} = \{R\}^{\wedge}.$$

Hence, by using Corollary 34.1, we infer that

$$\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_R \qquad \text{and} \qquad \operatorname{int}_{\mathcal{R}} = \operatorname{int}_R .$$

By using Theorem 34.3, we also see that

$$\left(R^{-1}\right)^{\#} = \left(R^{\#}\right)^{-1} = \left(\{R\}^{\#}\right)^{-1} = \left(\{R\}^{-1}\right)^{\#} = \left(\{R^{-1}\}\right)^{\#}.$$

Therefore, quite similarly as above, we also state that $\left(\mathcal{R}^{-1}\right)^{\wedge} = \{R^{-1}\}^{\wedge}$, and thus

$$\operatorname{cl}_{\mathcal{R}^{-1}} = \operatorname{cl}_{R^{-1}} \qquad \text{and} \qquad \operatorname{int}_{\mathcal{R}^{-1}} = \operatorname{int}_{R^{-1}} .$$

Now, as a counterpart of Theorem 40.5, we also prove the following result.

Theorem 41.2. *If $\mathcal{R}$ is proximally simple, then*

(1) *$\operatorname{cl}_{\mathcal{R}^{-1}}$ is $\operatorname{int}_{\mathcal{R}}$–normal;*

(2) *$\operatorname{cl}_{\mathcal{R}}$ is $\operatorname{int}_{\mathcal{R}^{-1}}$–normal.*

Proof. By Definition 37.1, there exists a relation R on X to Y such $\mathcal{R}^{\#} = \{R\}^{\#}$. Therefore, by Remark 41.1 and Theorem 34.1, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\operatorname{cl}_{\mathcal{R}^{-1}}(A) \subseteq B \iff \operatorname{cl}_{R^{-1}}(A) \subseteq B \iff R[A] \subseteq B \iff A \in \operatorname{Int}_R(B) \iff A \subseteq \operatorname{int}_R(B) \iff A \subseteq \operatorname{int}_{\mathcal{R}}(B).$$

Hence, by Definition 24.1, we see that assertion (1) is true.

Moreover, from Theorem 41.1, we know that assertions (1) and (2) are equivalent. ∎

Remark 41.2. Recall that, by Theorem 37.3, for any relation R on X to Y we have

$$\mathcal{R}^{\#} = \{R\}^{\#} \quad \Longleftrightarrow \quad \big(\, R \in \mathcal{R}^{\#}\,, \qquad \mathcal{R} \subseteq \{R\}^{*}\,\big)\,.$$

Now, as an important particular case of Theorem 41.2, we state the following result.

Corollary 41.1. *If R is a relation on X to Y, then*

(1) $\mathrm{cl}_{R^{-1}}$ *is* int_{R}*–normal ;* (2) cl_{R} *is* $\mathrm{int}_{R^{-1}}$*–normal .*

Proof. The relator $\mathcal{R} = \{R\}$ is proximally simply. Moreover

$$\mathrm{cl}_{R} = \mathrm{cl}_{\{\mathcal{R}\}} = \mathrm{cl}_{\mathcal{R}} \qquad \text{and} \qquad \mathrm{int}_{R^{-1}} = \mathrm{int}_{\{R^{-1}\}} = \mathrm{int}_{\{R\}^{-1}} = \mathrm{int}_{\mathcal{R}^{-1}}\,.$$

Therefore, Theorem 41.2 can be applied to obtain assertion (2). ■

Remark 41.3. The above $\big(\mathrm{cl}_{R^{-1}},\ \mathrm{int}_{R}\big)$ Galois connection was first considered in [71], with reference to [13, Exercise 7.18]. However, it importance could become completely clear only from the results of [84].

42. Normality properties of upper bound relations

The following theorem can be proved analogously to Theorem 41.2. However, it is now more interesting to show that it can also be derived from Theorem 41.2.

Theorem 42.1. *If $\mathcal{R}$ is $\text{\textcircled{\#}}$–simple, then for any $A \subseteq X$ and $B \subseteq Y$, we have*

$$A \subseteq \mathrm{lb}_{\mathcal{R}}(B) \quad \Longleftrightarrow \quad B \subseteq \mathrm{ub}_{\mathcal{R}}(A)\,.$$

Proof. By Definition 37.1, there exists a relation R on X to Y such that

$$\mathcal{R}^{\text{\textcircled{\#}}} = \{R\}^{\text{\textcircled{\#}}}\,.$$

Thus, by Definition 36.1, we have

$$\mathcal{R}^{\,c\,\#\,c} = \{R\}^{\,c\,\#\,c}\,.$$

Hence, by using elementwise complementation, we infer that

$$\big(\mathcal{R}^{\,c}\big)^{\#} = \big(\{R\}^{\,c}\big)^{\#} = \{\,R^{\,c}\}^{\#}\,.$$

This shows that the relator $\mathcal{R}^{c}$ is proximally simple. Therefore, by Theorem 41.2, we state that

$$\mathrm{cl}_{\,(\mathcal{R}^{\,c})^{-1}} \quad \text{is} \quad \mathrm{int}_{\mathcal{R}^{\,c}}\text{–normal}\,.$$

Thus, since $(\mathcal{R}^{\,c})^{-1} = \big(\mathcal{R}^{-1}\big)^{\,c}$, we also state that

$$\mathrm{cl}_{\,(\mathcal{R}^{-1})^{c}} \quad \text{is} \quad \mathrm{int}_{\,\mathcal{R}^{\,c}}\text{–normal}\,.$$

Hence, by Definition 24.1, we see that

$$\mathrm{cl}_{\,(R^{-1})^{c}}(A) \subseteq B^{\,c} \quad \Longleftrightarrow \quad A \subseteq \mathrm{int}_{\,\mathcal{R}^{\,c}}(B^{c})\,.$$

Thus, by taking complement with respect to Y, we also state that

$$B \subseteq \mathrm{cl}_{\,(R^{-1})^{c}}(A)^{c} \quad \Longleftrightarrow \quad A \subseteq \mathrm{int}_{\,\mathcal{R}^{\,c}}(\,B^{\,c})\,.$$

Hence, by using Theorem 32.2, we already infer that

$$B \subseteq \mathrm{lb}_{\mathcal{R}^{-1}}(A) \quad \Longleftrightarrow \quad A \subseteq \mathrm{lb}_{\mathcal{R}}(B)\,.$$

Thus, since $\mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^{-1}}$, the required equivalence is also true. ■

Remark 42.1. Note that, by Theorem 37.3, for any relation R on X to Y, we also have

$$\mathcal{R}^{\text{\textcircled{\#}}} = \{R\}^{\text{\textcircled{\#}}} \quad \Longleftrightarrow \quad \big(\, R \in \mathcal{R}^{\text{\textcircled{\#}}}\,, \qquad \mathcal{R} \subseteq \{R\}^{\circledast}\,\big)\,.$$

Remark 42.2. Note that if $\mathcal{R}$ is $\oplus$–simple, then by defining

$$F(A) = \operatorname{ub}_{\mathcal{R}}(A) \qquad \text{and} \qquad G(B) = \operatorname{lb}_{\mathcal{R}}(B),$$

for all $A \subseteq X$ and $B \subseteq Y$, we easily see that

$$F(A) \subseteq' B \iff B \subseteq F(A) \iff B \subseteq \operatorname{ub}_{\mathcal{R}}(A) \iff A \subseteq \operatorname{lb}_{\mathcal{R}}(B) \iff A \subseteq G(B)$$

for all $A \subseteq X$ and $B \subseteq Y$. Therefore, by Definition 24.1, we also state that F is a G–normal function of the poset $\mathcal{P}(X)$ to the dual of the poset $\mathcal{P}(Y)$.

The next result is an immediate consequence of Theorem 42.1, which has been formerly proved more directly.

Corollary 42.1. *If R is a relation on X to Y, then for any $A \subseteq X$ and $B \subseteq Y$, we have*

$$A \subseteq \operatorname{lb}_R(B) \iff B \subseteq \operatorname{ub}_R(A).$$

Proof. The relator $\mathcal{R} = \{R\}$ is $\oplus$–simple. Moreover, we have

$$\operatorname{ub}_R = \operatorname{ub}_{\{R\}} = \operatorname{ub}_{\mathcal{R}} \qquad \text{and} \qquad \operatorname{lb}_R = \operatorname{lb}_{\{R\}} = \operatorname{lb}_{\mathcal{R}}.$$

Therefore, Theorem 42.1 can be applied to obtain the required assertion. ∎

From Theorem 42.1, by using Corollary 27.2, we also derive the following result.

Theorem 42.2. *If $\mathcal{R}$ is $\oplus$–simple, then*

(1) $\operatorname{ub}^c_{\mathcal{R}}$ *is* $\operatorname{lb}_{\mathcal{R}} \circ \mathcal{C}_Y$*–normal;* (2) $\operatorname{lb}^c_{\mathcal{R}}$ *is* $\operatorname{ub}_{\mathcal{R}} \circ \mathcal{C}_X$*–normal.*

In particular, we also state the next result.

Corollary 42.2. *If R is a relation on X to Y, then*

(1) ub^c_R *is* $\operatorname{lb}_R \circ \mathcal{C}_Y$*–normal;* (2) lb^c_R *is* $\operatorname{ub}_R \circ \mathcal{C}_X$*–normal.*

Analogously to Theorem 42.1, we also prove the following theorem.

Theorem 42.3. *If there exists a function f of X to Y such that*

$$\mathcal{R}^{\circledast} = \{f^c\}^{\circledast},$$

then

(1) $\operatorname{ub}^c_{\mathcal{R}}$ *is* $\operatorname{lb}^c_{\mathcal{R}}$*–normal;* (2) $\operatorname{lb}_{\mathcal{R}} \circ \mathcal{C}_Y$ *is* $\operatorname{ub}_{\mathcal{R}} \circ \mathcal{C}_X$*–normal.*

Proof. By Definition 36.1, we have

$$\mathcal{R}^{c*c} = \{f^c\}^{c*c}.$$

By using elementwise complementation, we infer that

$$(\mathcal{R}^c)^* = \{f\}^*.$$

Therefore, by Theorem 40.5, we state that

$$\operatorname{int}_{\mathcal{R}^c} \text{ is } \operatorname{int}_{(\mathcal{R}^c)^{-1}}\text{–normal}.$$

Hence, by using the fact that $(\mathcal{R}^c)^{-1} = (\mathcal{R}^{-1})^c$, we infer that

$$\operatorname{int}_{\mathcal{R}^c} \text{ is } \operatorname{int}_{(\mathcal{R}^{-1})^c}\text{–normal}$$

Thus, by Definition 24.1, for any $A \subseteq X$ and $B \subseteq Y$ we have

$$\operatorname{int}_{\mathcal{R}^c}(B^c) \subseteq A^c \iff B^c \subseteq \operatorname{int}_{(\mathcal{R}^{-1})^c}(A^c).$$

Hence, by using ordinary complementation, we infer that

$$A \subseteq \operatorname{int}_{\mathcal{R}^c}(B^c)^c \iff \operatorname{int}_{(\mathcal{R}^{-1})^c}(A^c)^c \subseteq B.$$

This equivalences can be reformulated in the form

$$\left(\operatorname{int}_{(\mathcal{R}^{-1})^c} \circ \mathcal{C}_X\right)^c(A) \subseteq B \iff A \subseteq \left(\operatorname{int}_{\mathcal{R}^c} \circ \mathcal{C}_Y\right)^c(B).$$

Thence, by using Theorem 32.2, we infer that

$$\mathrm{lb}^{c}_{\mathcal{R}^{-1}}(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq \mathrm{lb}^{c}_{\mathcal{R}}(B)\,.$$

By using the fact $\mathrm{ub}_{\mathcal{R}} = \mathrm{lb}_{\mathcal{R}^{-1}}$, we note that

$$\mathrm{ub}^{c}_{\mathcal{R}}(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq \mathrm{lb}^{c}_{\mathcal{R}}(B)\,.$$

Thus, by Definition 24.1 , assertion (1) is true. Consequently, by using Corollary 26.1, we conclude that assertion (2) is also true. ■

Remark 42.3. To obtain some more instructive reformulations of Theorems 42.2 and 42.3, note that

$$\mathrm{ub}^{c}_{\mathcal{R}} = \mathcal{C}_Y \circ \mathrm{ub}_{\mathcal{R}} \qquad \text{and} \qquad \mathrm{lb}^{c}_{\mathcal{R}} = \mathcal{C}_X \circ \mathrm{lb}_{\mathcal{R}}\,.$$

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