**Electronic Journal of Mathematics**
www.shahindp.com/locate/ejm

*Electron. J. Math.* **4** (2022) 39–45
DOI: 10.47443/ejm.2022.032

*Research Article*

# A new asymptotic expansion and sharp inequality for the volume of the unit ball in $\mathbb{R}^n$

Xiao Zhang*, Chao-Ping Chen

*School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454003, Henan, China*

(Received: 29 July 2022. Received in revised form: 18 August 2022. Accepted: 23 August 2022. Published online: 26 August 2022.)



**Abstract**

For $n \in \{1,2,\ldots\}$, let $\Omega_n = \pi^{n/2}/\Gamma(\frac{n}{2}+1)$ be the volume of the unit ball in $\mathbb{R}^n$. In this paper, we give a new asymptotic expansion for $\Omega_n$. Based on the obtained result, we also establish a sharp double inequality for $\Omega_n$.

**Keywords:** asymptotic expansions; gamma function; inequalities; volume of the unit $n$-dimensional ball.

**2020 Mathematics Subject Classification:** 26D15, 33B15, 41A60.

## 1. Introduction

A considerable number of properties of the volume $\Omega_n$ of the unit ball in $\mathbb{R}^n$ have been reported by several researchers in the recent past, where

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}, \qquad n \in \mathbb{N} := \{1,2,\ldots\}.$$

For example, monotonicity properties of $\Omega_n$ can be found in [7, p. 264] and [4, 5, 11, 12]. For diverse sharp inequalities involving $\Omega_n$, the reader may consult [2, 3, 6, 8–10, 13–19]. In particular, for $n \in \mathbb{N}$, Chen and Lin [9, Theorem 3.4] showed that

$$\frac{1}{\sqrt{\pi(n+\theta)}}\left(\frac{2\pi e}{n}\right)^{n/2} \le \Omega_n < \frac{1}{\sqrt{\pi(n+\vartheta)}}\left(\frac{2\pi e}{n}\right)^{n/2}, \tag{1}$$

with the best possible constants

$$\theta = \frac{e}{2} - 1 = 0.3591409\ldots, \quad \vartheta = \frac{1}{3}.$$

From the right-hand side of (1), it follows that

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}, \qquad n \to \infty. \tag{2}$$

Chen and Paris [10, Equation (3.18)] developed (2) to produce a complete asymptotic expansion:

$$\begin{aligned}\Omega_n &\sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(\sum_{j=0}^{\infty}\frac{b_j}{n^j}\right)\\ &= \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1 - \frac{1}{36n^2} + \frac{23}{810n^3} - \frac{1}{864n^4} - \frac{5261}{204120n^5} + \frac{6889}{20995200n^6} + \frac{125549}{1632960n^7} - \cdots\right\}, \quad n\to\infty, \end{aligned}\tag{3}$$

where the coefficients $b_j$ are given by

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{k=1}^{j} k a_k b_{j-k}, \quad j \ge 1, \tag{4}$$

and $a_j$ are given by

$$a_j = \frac{(-1)^{j-1}}{2j\cdot 3^j} - \frac{2^j B_{j+1}}{j(j+1)}, \quad j \ge 1.$$

*Corresponding author (zhangxiao25733374@163.com).

Here, $B_n$ $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ are the Bernoulli numbers defined via the generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}, \qquad |t| < 2\pi. \tag{5}$$

By utilizing the Maple software, we find that

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}} \left(\frac{2\pi e}{n}\right)^{n/2} \left\{1 - \frac{\frac{1}{36}}{(n+\frac{23}{45})^2} + \frac{\frac{4007}{194400}}{(n+\frac{1865077}{3786615})^4} - \frac{\frac{22370792931839}{556505174736000}}{(n+\frac{633922198537106197381}{1270643701163933024775})^6} + \dots\right\}, \quad \text{as } n \to \infty. \tag{6}$$

The first aim of the present paper is to determine the constants $\lambda_\ell$ and $\mu_\ell$ such that

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}} \left(\frac{2\pi e}{n}\right)^{n/2} \left\{1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(n+\mu_\ell)^{2\ell}}\right\}, \quad n \to \infty.$$

In view of (6), it is natural to ask: what is the smallest value of $\alpha$ and what is the largest value of $\beta$ such that the inequality

$$\frac{1}{\sqrt{\pi(n+\frac{1}{3})}} \left(\frac{2\pi e}{n}\right)^{n/2} \left(1 - \frac{1}{36(n+\alpha)^2}\right) \le \Omega_n \le \frac{1}{\sqrt{\pi(n+\frac{1}{3})}} \left(\frac{2\pi e}{n}\right)^{n/2} \left(1 - \frac{1}{36(n+\beta)^2}\right)$$

is valid for every $n \in \mathbb{N}$? Answering this question is the second aim of the present paper.

The gamma function $\Gamma(x)$ is one of the most important functions in mathematical analysis and has applications in many diverse areas. The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function. It is known that

$$\Gamma(x+1) = x\Gamma(x) \quad \text{and} \quad \psi(x+1) = \psi(x) + \frac{1}{x}.$$

The following inequalities are needed in the present study:

$$\frac{1}{12x} - \frac{1}{360x^3} < \ln\Gamma(x+1) - \ln\sqrt{2\pi x} - x\ln x + x < \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5} \tag{7}$$

and

$$\frac{1}{12x^2} - \frac{1}{120x^4} < \ln x + \frac{1}{2x} - \psi(x+1) < \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^6} \tag{8}$$

where $x > 0$. The inequalities (7) and (8) can be found in [10]. We remark that the inequalities (7) and (8) follow from Theorem 8 of [1].

We end this introductory section with the remark that the numerical values reported in this paper are calculated by using MAPLE 11 (a software).

## 2. Main results

**Theorem 2.1.** *As $n \to \infty$, we have*

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}} \left(\frac{2\pi e}{n}\right)^{n/2} \left\{1 + \sum_{\ell=1}^{\infty} \frac{\lambda_\ell}{(n+\mu_\ell)^{2\ell}}\right\}, \tag{9}$$

*where the constants $\lambda_\ell$ and $\mu_\ell$ are given by a pair of recurrence relations as follows:*

$$\lambda_\ell = b_{2\ell} - \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k}, \quad \ell \ge 2 \tag{10}$$

*and*

$$\mu_\ell = -\frac{1}{2\ell\lambda_\ell} \left\{ b_{2\ell+1} + \sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} \right\}, \quad \ell \ge 2, \tag{11}$$

*with*

$$\lambda_1 = -\frac{1}{36} \quad \text{and} \quad \mu_1 = \frac{23}{45}.$$

*Here $b_j$ are given in (4).*

*Proof.* In view of (6), we can assume that

$$\Omega_n \sim \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(n+\mu_\ell)^{2\ell}}\right\}$$

as $n\to\infty$, where $\lambda_\ell$ and $\mu_\ell$ are real numbers to be determined. This can be written as follows:

$$\frac{\Omega_n}{\frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}}-1\sim\sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j}}\left(1+\frac{\mu_j}{n}\right)^{-2j},\quad n\to\infty.$$

Direct computation yields

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j}}\left(1+\frac{\mu_j}{n}\right)^{-2j}&=\sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j}}\sum_{k=0}^{\infty}\binom{-2j}{k}\frac{\mu_j^k}{n^k}\\&=\sum_{j=1}^{\infty}\frac{\lambda_j}{n^{2j}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j-1}{k}\frac{\mu_j^k}{n^k}\\&=\sum_{j=0}^{\infty}\frac{\lambda_{j+1}}{n^{2j+2}}\sum_{k=0}^{\infty}(-1)^k\binom{k+2j+1}{k}\frac{\mu_{j+1}^k}{n^k}\\&=\sum_{j=0}^{\infty}\sum_{k=0}^{j}\lambda_{k+1}(-1)^{j-k}\binom{j+k+1}{j-k}\frac{\mu_{k+1}^{j-k}}{n^{j+k+2}}\\&=\sum_{j=0}^{\infty}\sum_{k=1}^{j+1}\lambda_k\mu_k^{j-k+1}(-1)^{j-k+1}\binom{j+k}{j-k+1}\frac{1}{n^{j+k+1}}\\&=\sum_{\ell=2}^{\infty}\left\{\sum_{k=1}^{\lfloor\ell/2\rfloor}\lambda_k\mu_k^{\ell-2k}(-1)^{\ell}\binom{\ell-1}{\ell-2k}\right\}\frac{1}{n^{\ell}}.\end{aligned}$$

We then obtain

$$\frac{\Omega_n}{\frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}}-1\sim\sum_{j=2}^{\infty}\left\{\sum_{k=1}^{\lfloor j/2\rfloor}\lambda_k\mu_k^{j-2k}(-1)^{j}\binom{j-1}{j-2k}\right\}\frac{1}{n^{j}},\tag{12}$$

as $n\to\infty$. On the other hand, it follows from (3) that

$$\frac{\Omega_n}{\frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}}-1\sim\sum_{j=2}^{\infty}\frac{b_j}{n^j}\tag{13}$$

as $n\to\infty$, where $b_j$ are given in (4). By equating coefficients of the term $n^{-j}$ on the right-hand sides of (12) and (13), we obtain

$$b_j=\sum_{k=1}^{\lfloor j/2\rfloor}\lambda_k\mu_k^{j-2k}(-1)^{j}\binom{j-1}{j-2k},\quad j\geq 2.\tag{14}$$

By setting $j=2\ell$ and $j=2\ell+1$ in (14), respectively, we find

$$b_{2\ell}=\sum_{k=1}^{\ell}\lambda_k\mu_k^{2\ell-2k}\binom{2\ell-1}{2\ell-2k},\quad \ell\geq 1\tag{15}$$

and

$$b_{2\ell+1}=-\sum_{k=1}^{\ell}\lambda_k\mu_k^{2\ell-2k+1}\binom{2\ell}{2\ell-2k+1},\quad \ell\geq 1.\tag{16}$$

From (15) and (16) we obtain for $\ell=1$,

$$\lambda_1=b_2=-\frac{1}{36}\quad\text{and}\quad\mu_1=-\frac{b_3}{2\lambda_1}=\frac{23}{45}.$$

Also, for $\ell \geq 2$, we have

$$\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k} \binom{2\ell-1}{2\ell-2k} + \lambda_\ell = b_{2\ell}$$

and

$$-\sum_{k=1}^{\ell-1} \lambda_k \mu_k^{2\ell-2k+1} \binom{2\ell}{2\ell-2k+1} - 2\ell\lambda_\ell\mu_\ell = b_{2\ell+1}.$$

Consequently, we arrive at the recurrence relations (10) and (11).

□

Here, we give explicit numerical values of some first terms of $\lambda_\ell$ and $\mu_\ell$ by using the formulas (10) and (11). This shows how easily we can determine the constants $\lambda_\ell$ and $\mu_\ell$ in Theorem 2.1. We see from (3) that

$$b_2 = -\frac{1}{36}, \quad b_3 = \frac{23}{810}, \quad b_4 = -\frac{1}{864}, \quad b_5 = -\frac{5261}{204120}, \quad b_6 = \frac{6889}{20995200}, \quad b_7 = \frac{125549}{1632960}.$$

We obtain from (10) and (11) that

$$\lambda_1 = b_2 = -\frac{1}{36}, \qquad \mu_1 = -\frac{b_3}{2\lambda_1} = \frac{23}{45},$$

$$\lambda_2 = b_4 - 3\lambda_1\mu_1^2 = \frac{4007}{194400}, \qquad \mu_2 = -\frac{b_5 + 4\lambda_1\mu_1^3}{4\lambda_2} = \frac{1865077}{3786615},$$

$$\lambda_3 = b_6 - 5\lambda_1\mu_1^4 - 10\lambda_2\mu_2^2 = -\frac{22370792931839}{556505174736000}, \qquad \mu_3 = -\frac{b_7 + 6\lambda_1\mu_1^5 + 20\lambda_2\mu_2^3}{6\lambda_3} = \frac{633922198537106197381}{1270643701163933024775}.$$

We note that the values of $\lambda_\ell$ and $\mu_\ell$ (for $\ell = 1, 2, 3$), given above, are equal to the constants appearing in (6). From a computational viewpoint, the formula (6) improves the formula (3).

**Theorem 2.2.** *For $n \in \mathbb{N}$, the following double inequality holds:*

$$\frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{36(n+\alpha)^2}\right) < \Omega_n \leq \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left(1 - \frac{1}{36(n+\beta)^2}\right), \tag{17}$$

*where the constants*

$$\alpha = \frac{23}{45} = 0.51111111\ldots \quad \text{and} \quad \beta = \frac{\sqrt{3}e^{\frac{1}{4}}}{6(3\sqrt{e} - 2\sqrt{6})^{\frac{1}{2}}} - 1 = 0.70641286\ldots$$

*are the best possible.*

*Proof.* First of all, we show that the double inequality (17) with $\alpha = \frac{23}{45}$ and

$$\beta = \frac{\sqrt{3}e^{\frac{1}{4}}}{6(3\sqrt{e} - 2\sqrt{6})^{\frac{1}{2}}} - 1$$

is valid for $n = 1, 2, 3$. For $n \in \mathbb{N}$, let

$$L_n = \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1 - \frac{1}{36(n+\alpha)^2}\right\},$$

$$U_n = \frac{1}{\sqrt{\pi(n+\frac{1}{3})}}\left(\frac{2\pi e}{n}\right)^{n/2}\left\{1 - \frac{1}{36(n+\beta)^2}\right\}.$$

Direct computation yields

$$L_1 = 1.9946\ldots, \quad \Omega_1 = 2, \quad U_1 = 2,$$

$$L_2 = 3.1402\ldots, \quad \Omega_2 = \pi = 3.1415\ldots, \quad U_2 = 3.1477\ldots,$$

$$L_3 = 4.1882\ldots, \quad \Omega_3 = \frac{4\pi}{3} = 4.1887\ldots, \quad U_3 = 4.1924\ldots.$$

Now, it is clear that the double inequality (17) with $\alpha = \frac{23}{45}$ and

$$\beta = \frac{\sqrt{3}e^{\frac{1}{4}}}{6(3\sqrt{e} - 2\sqrt{6})^{\frac{1}{2}}} - 1$$

is valid for $n = 1, 2$ and $3$. For $n = 1$, the equality sign on the right-hand side of (17) holds. We now prove that the double inequality (17) with $\alpha = \frac{23}{45}$ and

$$\beta = \frac{\sqrt{3}e^{\frac{1}{4}}}{6(3\sqrt{e} - 2\sqrt{6})^{\frac{1}{2}}} - 1$$

is valid for $n \geq 4$. It suffices to show that for $x \geq 2$,

$$\frac{1}{\sqrt{\pi(2x + \frac{1}{3})}} \left(\frac{\pi e}{x}\right)^x \left(1 - \frac{1}{36(2x + \alpha)^2}\right) < \Omega_{2x} \leq \frac{1}{\sqrt{\pi(2x + \frac{1}{3})}} \left(\frac{\pi e}{x}\right)^x \left(1 - \frac{1}{36(2x + \beta)^2}\right), \tag{18}$$

where

$$\Omega_x = \frac{\pi^{x/2}}{\Gamma(\frac{x}{2} + 1)}.$$

The double inequality (18) can be written as

$$\frac{1}{\sqrt{2\pi x(1 + \frac{1}{6x})}} \left(\frac{e}{x}\right)^x \left(1 - \frac{1}{36(2x + \alpha)^2}\right) < \frac{1}{\Gamma(x+1)} \leq \frac{1}{\sqrt{2\pi x(1 + \frac{1}{6x})}} \left(\frac{e}{x}\right)^x \left(1 - \frac{1}{36(2x + \beta)^2}\right). \tag{19}$$

In order to prove the double inequality (19) for $x \geq 2$, it suffices to show that

$$f(x) > 0 \quad \text{and} \quad g(x) < 0 \quad \text{for} \quad x \geq 2,$$

where

$$f(x) = x \ln x - x + \ln(\sqrt{2\pi x}) - \ln \Gamma(x+1) + \frac{1}{2} \ln\left(1 + \frac{1}{6x}\right) - \ln\left(1 - \frac{1}{36(2x + \frac{23}{45})^2}\right)$$

and

$$g(x) = x \ln x - x + \ln(\sqrt{2\pi x}) - \ln \Gamma(x+1) + \frac{1}{2} \ln\left(1 + \frac{1}{6x}\right) - \ln\left(1 - \frac{1}{36(2x + \beta)^2}\right).$$

By (7), we obtain

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

By differentiating $f(x)$ and applying the right-hand side of (8), we obtain for $x \geq 2$,

$$\begin{aligned} f'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) - \frac{2916000x^3 + 2721600x^2 + 632070x + 43493}{2x(32400x^2 + 16560x + 1891)(90x + 23)(6x + 1)} \\ &< \frac{1}{12x^2} - \frac{1}{120x^4} + \frac{1}{252x^4} - \frac{2916000x^3 + 2721600x^2 + 632070x + 43493}{2x(32400x^2 + 16560x + 1891)(90x + 23)(6x + 1)} \\ &= -\frac{P_5(x-2)}{2520x^6(32400x^2 + 16560x + 1891)(90x + 23)(6x + 1)}, \end{aligned}$$

where

$$P_5(x) = 227196900x^5 + 2204257890x^4 + 8399943708x^3 + 15618718041x^2 + 14027927868x + 4782272866.$$

Hence, $f'(x) < 0$ for $x \geq 2$. So, $f(x)$ is strictly decreasing for $x \geq 2$ and we have

$$f(x) > \lim_{t\to\infty} f(t) = 0, \quad x \geq 2.$$

Therefore, the left-hand side of (17) is valid for $n \in \mathbb{N}$.

Now, by differentiating $g(x)$ and applying the left-hand side of (8), we obtain for $x \geq 2$,

$$\begin{aligned} g'(x) &= \ln x + \frac{1}{2x} - \psi(x+1) - \frac{1}{2x(6x+1)} - \frac{4}{(144x^2+144\beta x+36\beta^2-1)(2x+\beta)} \\ &> \frac{1}{12x^2} - \frac{1}{120x^4} - \frac{1}{2x(6x+1)} - \frac{4}{\left(144x^2+144\cdot\frac{7}{10}\cdot x+36\cdot\left(\frac{7}{10}\right)^2-1\right)(2x+\frac{7}{10})} \\ &= \frac{29658+310451(x-2)+387220(x-2)^2+170750(x-2)^3+25500(x-2)^4}{120x^4(6x+1)(20x+7)(450x^2+315x+52)} > 0. \end{aligned}$$

Hence, $g(x)$ is strictly increasing for $x \geq 2$, and we have

$$g(x) < \lim_{t\to\infty} g(t) = 0, \quad x \geq 2.$$

Therefore, the right-hand side of (17) holds for $n \in \mathbb{N}$. If we write the double inequality (17) as

$$\alpha < x_n \leq \beta,$$

where

$$x_n = \frac{1}{6\left[1 - \frac{\sqrt{\pi(n+\frac{1}{3})}\left(\frac{n}{2e}\right)^{n/2}}{\Gamma(\frac{n}{2}+1)}\right]^{1/2}} - n, \tag{20}$$

we find that

$$x_1 = \frac{\sqrt{3}e^{\frac{1}{4}}}{6\left(3\sqrt{e}-2\sqrt{6}\right)^{\frac{1}{2}}} - 1$$

and

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty}\left\{\frac{1}{6\left[1 - \frac{\sqrt{\pi(n+\frac{1}{3})}\left(\frac{n}{2e}\right)^{n/2}}{\Gamma(\frac{n}{2}+1)}\right]^{1/2}} - n\right\} \\ &\overset{x=n/2}{=\!=\!=} \frac{1}{6}\lim_{x\to\infty}\left\{\frac{1}{\left[1 - \frac{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x}{\Gamma(x+1)}\left(1+\frac{1}{6x}\right)^{1/2}\right]^{1/2}} - 12x\right\} \\ &= \frac{1}{6}\lim_{x\to\infty}\left\{\frac{1}{\left[\frac{1}{144x^2} - \frac{23}{6480x^3} + \frac{1}{13824x^4} + O(x^{-5})\right]^{1/2}} - 12x\right\} \\ &= \frac{1}{6}\lim_{x\to\infty}\left\{\frac{1}{\frac{1}{12x} - \frac{23}{1080x^2} - \frac{3557}{1555200x^3} + O(x^{-4})} - 12x\right\} \\ &= \frac{1}{6}\lim_{x\to\infty}\left\{\frac{46}{15} + O\left(\frac{1}{x}\right)\right\} = \frac{23}{45}. \end{aligned}$$

Thus, the double inequality (17) holds for $n \in \mathbb{N}$, and the constants $\alpha = \frac{23}{45}$ and $\beta = \frac{\sqrt{3}e^{\frac{1}{4}}}{6\left(3\sqrt{e}-2\sqrt{6}\right)^{\frac{1}{2}}} - 1$ are the best possible.

□

**Remark 2.1.** *Suppose that the sequence $\{x_n\}$ is defined via (20). In order to prove Theorem 2.2, it suffices to show that the sequence $\{x_n\}$ is strictly decreasing for $n \in \mathbb{N}$.*

## Acknowledgement

This work was supported by the Key Science Research Project in the Universities of the Henan Province (20B110007) and by the Fundamental Research Funds for the Universities of the Henan Province (NSFRF210446).

## References

[1] H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comput.* **66** (1997) 373–389.
[2] H. Alzer, Inequalities for the volume of the unit ball in $\mathbb{R}^n$, *J. Math. Anal. Appl.* **252** (2000) 353–363.
[3] H. Alzer, Inequalities for the volume of the unit ball in $\mathbb{R}^n$, II, *Mediterr. J. Math.* **5** (2008) 395–413.
[4] G. D. Anderson, S.-L. Qiu, A monotoneity property of the gamma function, *Proc. Amer. Math. Soc.* **125** (1997) 3355–3362.
[5] G. D. Anderson, M. K. Vamanamurthy, M. Vuorinen, Special functions of quasiconformal theory, *Expo. Math.* **7** (1989) 97–136.
[6] T. Ban, C.-P. Chen, New inequalities for the volume of the unit ball in $\mathbb{R}^n$, *J. Math. Inequal.* **11** (2017) 527–542.
[7] J. Böhm, E. Hertel, Polyedergeometrie in $n$-dimensionalen Räumen konstanter Krümmung, Birkhäuser, Basel, 1981.
[8] K. H. Borgwardt, *The Simplex Method*, Springer, Berlin, 1987.
[9] C.-P. Chen, L. Lin, Inequalities for the volume of the unit ball in $\mathbb{R}^n$, *Mediterr. J. Math.* **11** (2014) 299–314.
[10] C.-P. Chen, R. B. Paris, Inequalities and asymptotic expansions related to the volume of the unit ball in $\mathbb{R}^n$, *Results Math.* **74** (2019) #44.
[11] B.-N. Guo, F. Qi, Monotonicity and logarithmic convexity relating to the volume of the unit ball, *Optim. Lett.* **7** (2013) 1139–1153.
[12] D. A. Klain, G.-C. Rota, A continuous analogue of Sperner's theorem, *Comm. Pure Appl. Math.* **50** (1997) 205–223.
[13] D. Lu, P. Zhang, A new general asymptotic formula and inequalities involving the volume of the unit ball, *J. Number Theory* **170** (2017) 302–314.
[14] M. Merkle, Gurland's ratio for the gamma function, *Comput. Math. Appl.* **49** (2005) 389–406.
[15] C. Mortici, Monotonicity properties of the volume of the unit ball in $\mathbb{R}^n$, *Optim. Lett.* **4** (2010) 457–464.
[16] C. Mortici, Estimates of the function and quotient by Minc-Sathre, *Appl. Math. Comput.* **253** (2015) 52–60.
[17] C. Mortici, Series associated to some expressions involving the volume of the unit ball and applications, *Appl. Math. Comput.* **294** (2017) 121–138.
[18] L. Yin, L.-G. Huang, Some inequalities for the volume of the unit ball, *J. Class. Anal.* **6** (2015) 39–46.
[19] H. Zhang, New bounds and asymptotic expansions for the volume of the unit ball in $\mathbb{R}^n$ based on Padé approximation, *Results Math.* **77** (2022) #116.