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*Research Article*

# The Hadamard product of series with Stirling numbers of the second kind and other special numbers

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**Abstract**

We evaluate in closed form a number of power series where the coefficients are products of Stirling numbers of the second kind and other special numbers or polynomials. The results include harmonic, hyperharmonic, derangement, Cauchy, Catalan numbers, zeta values, and also Bernoulli, Euler, and Laguerre polynomials.

**Keywords:** summation of series; Hadamard product of series; Stirling numbers of the second kind.

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## 1. Introduction

Let $a_n$ and $b_n, n = 0, 1, 2, \ldots,$ be the coefficients of two given power series $f(t)$ and $g(t)$, respectively. Then the power series of the form $\sum_{n\geq 0} a_n b_n z^n$ is known as the Hadamard product of the two series and gives the generating function for the product sequence $a_n b_n, n \geq 0$. It has the integral representation

$$\sum_{n=0}^{\infty} a_n b_n z^n = \frac{1}{2\pi i} \oint_L g\left(\frac{z}{\lambda}\right) \frac{f(\lambda)}{\lambda} d\lambda,$$

where $L$ is a small positively oriented circle centered at the origin. In some cases this integral can be manipulated to find interesting series transformation formulas or even to evaluate the Hadamard product in closed form - see [2] for details. In [3], by using a certain binomial series transformation the first author evaluated in closed form the series $\sum_{n\geq 0} C_n H_n z^n$, where $C_n$ are the Catalan numbers and $H_n$ are the harmonic numbers. In [7], for $p \geq 0$ an integer, several series of the form $\sum_{n\geq 0} \binom{n+p}{n} f(n) z^n$ were evaluated, where $f(n)$ is some function that involves values of the Riemann zeta function. In [6] the authors considered series where the coefficients are products of Hermite polynomials and harmonic numbers.

This paper is a continuation of our research started in [8]. We present a simple method for evaluating in closed form power series like

$$\sum_{n\geq 0} S(n,m)\, a_n\, z^n,$$

where $S(n, m)$ are the Stirling numbers of the second kind and $a_n$ are the coefficients of any power series ($a_n$ can be special numbers or polynomials). Using the theorem we derive ten corollaries where we evaluate a number of power series. Our examples include series with zeta values $\zeta(n)$, harmonic, hyperharmonic, derangement, Cauchy, and Catalan numbers, and also Bernoulli, Euler, and Laguerre polynomials. In particular, the following series are evaluated

$$\sum_{n=m}^{\infty} S(n,m)^2 \frac{z^n}{n!}, \quad \sum_{n=m}^{\infty} S(n,m)\zeta(n+1) z^n, \quad \sum_{n=m}^{\infty} S(n,m) H_n z^n, \quad \sum_{n=m}^{\infty} S(n,m) H_n \frac{z^n}{n!}.$$

## 2. Power series with Stirling numbers and other special numbers

The Stirling numbers of the second kind $S(n, m)$ originated in the works of James Stirling (see [4] and [22]). They have numerous important applications in analysis and combinatorics (for example, see [1, 5, 9, 10, 15, 19, 20]). In combinatorics

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†Statements and conclusions made in this article by R.F. are entirely those of the author. They do not necessarily reflect the views of LBBW.

$S(n,m)$ stands for the number of ways a set of $n$ elements can be partitioned into $m$ nonempty subsets. The numbers obey the recurrence relation

$$S(n+1,m) = mS(n,m) + S(n,m-1), \qquad m > 0, \tag{1}$$

with initial conditions $S(0,0) = 1$, $S(n,0) = 0$ for $n > 0$, $S(n,n) = 1$ and for $n \geq 1$, $S(n,1) = 1$.

The exponential generating function for these numbers is

$$\sum_{n=0}^{\infty} S(n,m)\frac{z^n}{n!} = \frac{1}{m!}(e^z - 1)^m, \tag{2}$$

where $m \geq 0$ and the series converges everywhere. The summation on the left hand side starts, in fact, from $n = m$, as $S(n,m) = 0$ for $n < m$. The ordinary generating function is

$$\sum_{n=m}^{\infty} S(n,m)z^n = \frac{z^m}{(1-z)(1-2z)\cdots(1-mz)} \tag{3}$$

with convergence for $|z| < 1/m$. For large $n$ the Stirling numbers of the second kind have the asymptotic behavior $S(n,m) \sim \frac{m^n}{m!}$.

Our results are based on the following theorem.

**Theorem 2.1.** *Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$ be a function analytic on the disk $|z| \leq R$. Then for every integer $m > 0$ we have*

$$\sum_{n=m}^{\infty} S(n,m)\, a_n z^n = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k f(kz), \tag{4}$$

*with convergence in the disk $|z| < R/m$.*

*Proof.* Let $L$ be the circle $|\lambda| = R$. For every $n \geq 0$ we have

$$a_n = \frac{1}{2\pi i} \oint_L \frac{f(\lambda)d\lambda}{\lambda^{n+1}}$$

and from here (with $|z| < R/m$), by changing the order of summation and integration

$$\begin{aligned}
\sum_{n=0}^{\infty} S(n,m)\, a_n z^n &= \frac{1}{2\pi i} \oint_L \left( \sum_{n=0}^{\infty} S(n,m) \left(\frac{z}{\lambda}\right)^n \right) \frac{f(\lambda)d\lambda}{\lambda} \\
&= \frac{z^m}{2\pi i} \oint_L \left( \left(1 - \frac{z}{\lambda}\right)\left(1 - \frac{2z}{\lambda}\right) \cdots \left(1 - \frac{mz}{\lambda}\right) \right)^{-1} \frac{f(\lambda)d\lambda}{\lambda^{m+1}} \\
&= \frac{z^m}{2\pi i} \oint_L \left((\lambda - z)(\lambda - 2z)\cdots(\lambda - mz)\right)^{-1} \frac{f(\lambda)d\lambda}{\lambda} \\
&= \frac{1}{2\pi i} \oint_L \left( \left(\frac{\lambda}{z} - 1\right)\left(\frac{\lambda}{z} - 2\right) \cdots \left(\frac{\lambda}{z} - m\right) \right)^{-1} \frac{f(\lambda)d\lambda}{\lambda}.
\end{aligned}$$

With the substitution $\lambda = \mu z$ this becomes

$$\sum_{n=0}^{\infty} S(n,m)\, a_n z^n = \frac{1}{2\pi i} \oint_M \frac{f(\mu z)d\mu}{\mu(\mu-1)(\mu-2)\cdots(\mu-m)},$$

where $M$ is now a circle centered at the origin and containing the numbers $1, 2, \ldots, m$. Note that for $|z| < R/m$ we have $|\mu| = |\lambda/z| = R/|z| > m$. By the Nørlund-Rice formula (for details see [16]) we have

$$\frac{1}{2\pi i} \oint_M \frac{f(\mu z)d\mu}{\mu(\mu-1)(\mu-2)\cdots(\mu-m)} = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k f(kz).$$

The proof is completed. □

**Remark 2.1.** *The above theorem can also be proved as follows: We use the well-known representation of the Stirling numbers (see [4] for instance)*

$$S(n,m) = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k k^n. \tag{5}$$

*Let $p$ be a positive integer. We multiply both sides in the above equation by $a_n z^n$ and sum for $n = 0, 1, 2, \dots, p$ to get*

$$\sum_{n=0}^{p} S(n,m) a_n z^n = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \left( \sum_{n=0}^{p} a_n (kz)^n \right).$$

*When $|z| < R/m$ and $p \to \infty$ the right hand side will converge to the right hand side in (4). Therefore, the partial sums on the left hand side will also converge and passing to limits we come to equation (4).*

For the function $f(z) = 1 + z + z^2 + \dots = \frac{1}{1-z}$, equation (4) becomes

$$\sum_{n=m}^{\infty} S(n,m) z^n = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{1-kz}, \tag{6}$$

which shows the decomposition in partial fractions of the ordinary generating function

$$\frac{z^m}{(1-z)(1-2z)\cdots(1-mz)} = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{1-kz}. \tag{7}$$

## 3. Applications of the theorem

In our first application, we present a “Stirling double series” by applying Theorem 2.1 to the generating function (2).

**Corollary 3.1.** *For any two integers $m, p \geq 0$ and every $z$ we have*

$$\sum_{n=\max\{m,p\}}^{\infty} S(n,m)\, S(n,p) \frac{z^n}{n!} = \frac{(-1)^m}{p!m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k (e^{kz} - 1)^p. \tag{8}$$

*When $m = p$ the series takes the form*

$$\sum_{n=0}^{\infty} S^2(n,p) \frac{z^n}{n!} = \frac{(-1)^p}{(p!)^2} \sum_{k=0}^{p} \binom{p}{k} (-1)^k (e^{kz} - 1)^p. \tag{9}$$

Next, we consider the Stirling numbers of the first kind $s(n,p)$ with exponential generating function

$$\sum_{n=0}^{\infty} s(n,p) \frac{z^n}{n!} = \frac{1}{p!} \ln^p(1+z), \tag{10}$$

where the radius of convergence is $R = 1$ (see [9, 10, 15, 20, 22]). These numbers are dual to $S(n,m)$ in the sense that

$$\sum_{k=0}^{n} S(n,k) s(k,j) = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases} \quad \text{and} \quad \sum_{k=0}^{n} s(n,k) S(k,j) = \begin{cases} 1 & n = j \\ 0 & n \neq j \end{cases}.$$

With the help of Theorem 2.1 we construct a series involving the Stirling numbers of both kinds and evaluate it in closed form.

**Corollary 3.2.** *For any two integers $m > 0, p \geq 0$we have*

$$\sum_{n=0}^{\infty} S(n,m) s(n,p) \frac{z^n}{n!} = \frac{(-1)^m}{p!m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \ln^p(1+kz), \tag{11}$$

*true for $|z| < 1/m$. Again, the summation on the left hand side starts actually from $n = \max\{m,p\}$, as $s(n,p) = 0$ when $n < p$.*

Our next example involves zeta values. Consider the Hurwitz zeta function

$$\zeta(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s}, \qquad Re(s) > 1, a > 0,$$

where $\zeta(s,1) = \zeta(s)$ is Riemann zeta function. The sequence $\zeta(n,a)/n, n = 1, 2, \dots,$ has the generating function (see [20])

$$\sum_{n=1}^{\infty} \zeta(n,a) \frac{z^n}{n} = \ln\Gamma(a-z) - \ln\Gamma(a) + z\psi(a), \quad |z| < a,$$

where $\Gamma(z)$ is the Gamma function and $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$ is the digamma function. Differentiating this representation we also have

$$\sum_{n=1}^{\infty}\zeta(n+1,a)z^n = -\psi(a-z)+\psi(a), \qquad |z|<a$$

and in particular, for $a=1$

$$\sum_{n=1}^{\infty}\zeta(n+1)z^n = -\psi(1-z)+\psi(1), \qquad |z|<1.$$

This leads to the next result which generalizes the above series representations.

**Corollary 3.3.** *For every integer $m>1$ and every $a>0$ we have the representations*

$$\begin{aligned}\sum_{n=m}^{\infty}S(n,m)\,\zeta(n,a)\frac{z^n}{n} &= \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\left(\ln\Gamma(a-kz)-\ln\Gamma(a)+kz\psi(a)\right)\\ &= \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\ln\Gamma(a-kz)\end{aligned}$$

$$\sum_{n=m}^{\infty}S(n,m)\,\zeta(n+1,a)z^n = \frac{(-1)^{m-1}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\psi(a-kz) \tag{12}$$

*with convergence for $|z|<a/m$. Also, with $a=1$*

$$\sum_{n=m}^{\infty}S(n,m)\,\zeta(n+1)z^n = \frac{(-1)^{m-1}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\psi(1-kz) \tag{13}$$

*with convergence for $|z|<1/m$.*

Note that the first sum in the corollary was simplified because for $m>1$ by the binomial theorem

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k=0, \qquad \sum_{k=0}^{m}\binom{m}{k}(-1)^k k=0.$$

If we set $a=m$ in (12) we obtain an interesting power series converging in the disk $|z|<1$

$$\sum_{n=m}^{\infty}S(n,m)\,\zeta(n+1,m)z^n = \frac{(-1)^{m-1}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\psi(m-kz). \tag{14}$$

For example, we have the curious identity

$$\sum_{n=m}^{\infty}\frac{1}{2^n}S(n,m)\,\zeta(n+1,m) = \frac{(-1)^{m-1}}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k\psi\Big(m-\frac{k}{2}\Big).$$

Now we consider the hyperharmonic numbers $h_n^{(p)}$ (also denoted by $H_{n,p}$). These numbers first appeared in the book of Conway and Guy [11] and generalize the harmonic numbers $H_n$. They were studied recently in [12–14]. Hyperharmonic numbers can be defined by the formula

$$h_n^{(p+1)} = \sum_{k=0}^{n}\binom{n+p}{n}\left(H_{n+p}-H_p\right)$$

for $n,p=0,1,2,\ldots$, where $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}, H_0=0$, are the harmonic numbers. For $p=0$, we have $h_n^{(1)}=H_n$. The ordinary generating functions for the hyperharmonic numbers and the harmonic numbers are given by

$$\sum_{n=0}^{\infty}h_n^{(p)}z^n = \frac{-\ln(1-z)}{(1-z)^p}, \qquad \sum_{n=0}^{\infty}H_nz^n = \frac{-\ln(1-z)}{1-z},$$

convergent for $|z|<1$. We will also use the exponential generating function for the harmonic numbers (entry (5.13.13) in Hansen's table [17])

$$\sum_{n=0}^{\infty}H_n\frac{z^n}{n!} = e^z Ein(z),$$

where $Ein(z)$ is the exponential integral function (an entire function)

$$Ein(z) = \int_0^z\frac{1-e^{-t}}{t}dt = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^n}{n!n}.$$

Theorem 2.1 implies the following representations.

**Corollary 3.4.** *For all integers $m, p > 0$ and for all $|z| < 1/m$ we have*

$$\sum_{n=0}^{\infty} S(n,m) h_n^{(p)} z^n = \frac{(-1)^{m-1}}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{\ln(1-kz)}{(1-kz)^p}, \tag{15}$$

$$\sum_{n=0}^{\infty} S(n,m) H_n z^n = \frac{(-1)^{m-1}}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \frac{\ln(1-kz)}{1-kz}, \tag{16}$$

*and for all $z$*

$$\sum_{n=0}^{\infty} S(n,m) H_n \frac{z^n}{n!} = \frac{(-1)^{m-1}}{m!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k e^{kz} Ein(kz). \tag{17}$$

*Integrating identity (16) we find also*

$$\sum_{n=0}^{\infty} S(n,m) \frac{H_n}{n+1} z^{n+1} = \frac{(-1)^m}{m!} \frac{1}{2} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^k}{k} \ln^2(1-kz). \tag{18}$$

**Remark 3.1.** *The identity (18) contains a special case of identity (11), namely the series with $s(n,2)$ via*

$$s(n,2) = (-1)^n (n-1)! H_{n-1}$$

*(see [20]) and the recurrence (1):*

$$\begin{aligned}
\sum_{n=0}^{\infty} S(n+1,m) \frac{H_n}{n+1} (-z)^{n+1} &= \sum_{n=0}^{\infty} \Big( mS(n,m) + S(n,m-1) \Big) \frac{H_n}{n+1} (-z)^{n+1} \\
&= m \frac{(-1)^m}{m!} \frac{1}{2} \sum_{k=1}^{m} \binom{m}{k} \frac{(-1)^k}{k} \ln^2(1+kz) + \frac{(-1)^{m-1}}{(m-1)!} \frac{1}{2} \sum_{k=1}^{m-1} \binom{m-1}{k} \frac{(-1)^k}{k} \ln^2(1+kz) \\
&= \frac{(-1)^m}{m!} \frac{1}{2} \sum_{k=1}^{m-1} \left( \binom{m}{k} - \binom{m-1}{k} \right) \frac{m}{k} (-1)^k \ln^2(1+kz) + \frac{1}{m!} \frac{1}{2} \ln^2(1+mz) \\
&= \frac{(-1)^m}{m!} \frac{1}{2} \sum_{k=1}^{m-1} \binom{m-1}{k-1} \frac{m}{k} (-1)^k \ln^2(1+kz) + \frac{1}{m!} \frac{1}{2} \ln^2(1+mz) \\
&= \frac{(-1)^m}{m!} \frac{1}{2} \sum_{k=1}^{m} \binom{m}{k} (-1)^k \ln^2(1+kz) \\
&= \sum_{n=0}^{\infty} S(n,m) s(n,2) \frac{z^n}{n!}.
\end{aligned}$$

Next, we apply Theorem 2.1 to the case when $a_n = B_n(x)$, the Bernoulli polynomials. The Bernoulli polynomials have the exponential generating function

$$\sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}, \qquad |z| < 2\pi,$$

Here, $B_n(0) = B_n$ are the Bernoulli numbers (see [10, 20]).

**Corollary 3.5.** *For every integer $m > 0$ and every $|z| < 2\pi/m$*

$$\sum_{n=0}^{\infty} S(n,m) B_n(x) \frac{z^n}{n!} = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k kze^{kxz}}{e^{kz} - 1}, \tag{19}$$

$$\sum_{n=0}^{\infty} S(n,m) B_n \frac{z^n}{n!} = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k kz}{e^{kz} - 1}. \tag{20}$$

We also list the similar result for the Euler polynomials $E_n(x)$ and the Euler numbers $E_n = 2^n E_n(1/2)$. They are defined correspondingly by the exponential generating functions

$$\sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} = \frac{2e^{xz}}{e^z + 1}, \quad \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = \frac{1}{\cosh(z)},$$

convergent in $|z| < \pi$ (see [10, 20]). Theorem 2.1 implies the following corollary.

**Corollary 3.6.** *For every integer $m > 0$ and every $|z| < \pi/m$ we have*

$$\sum_{n=0}^{\infty} S(n,m)E_n(x)\frac{z^n}{n!} = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k e^{kxz}}{e^{kz}+1} \tag{21}$$

*and*

$$\sum_{n=0}^{\infty} S(n,m)E_n\frac{z^n}{n!} = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\cosh(kz)}. \tag{22}$$

Next, in the line are the derangement numbers $D_n, n = 0, 1, \ldots,$ defined by

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right).$$

They give the number of permutations of $\{1, 2, \ldots, n\}$ with no fixed points (see [15, 21]). The generating function of the derangement numbers is given by

$$\sum_{n=0}^{\infty} D_n\frac{z^n}{n!} = \frac{e^{-z}}{1-z}, \quad |z| < 1.$$

According to our theorem we have the series identity

**Corollary 3.7.** *For every integer $m > 0$ and every $|z| < 1/m$*

$$\sum_{n=0}^{\infty} S(n,m)D_n\frac{z^n}{n!} = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k e^{-kz}}{1-kz}. \tag{23}$$

The Cauchy numbers of the first kind $c_n$ and second kind $d_n$ have exponential generating functions

$$\sum_{n=0}^{\infty} c_n\frac{z^n}{n!} = \frac{z}{\ln(1+z)}, \qquad \sum_{n=0}^{\infty} d_n\frac{z^n}{n!} = \frac{-z}{(1-z)\ln(1-z)},$$

convergent for $|z| < 1$ (see [10]). Therefore the main theorem immediately gives the next series expressions.

**Corollary 3.8.** *For every integer $m > 0$ and every $|z| < 1/m$*

$$\sum_{n=0}^{\infty} S(n,m)c_n\frac{z^n}{n!} = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k kz}{\ln(1+kz)}, \tag{24}$$

*and*

$$\sum_{n=0}^{\infty} S(n,m)d_n\frac{z^n}{n!} = \frac{(-1)^{m-1}}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k kz}{(1-kz)\ln(1-kz)}. \tag{25}$$

Consider now the binomial series. For any real number $\alpha$ and $|z| < 1$

$$(1+z)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}z^n.$$

For $\alpha = -1/2$ we have $\binom{-1/2}{n} = \frac{1}{4^n}\binom{2n}{n}$ and therefore, the central binomial numbers $\binom{2n}{n}$ have the generating function

$$\sum_{n=0}^{\infty}\binom{2n}{n}z^n = \frac{1}{\sqrt{1-4z}}.$$

By integration, the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ have generating function (see [3])

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z} = \frac{2}{1+\sqrt{1-4z}}.$$

Theorem 2.1 leads to the next corollary.

**Corollary 3.9.** *For every integer $m > 0$, every real number $\alpha$ and every $|z| < 1/m$*

$$\sum_{n=0}^{\infty} S(n,m)\binom{\alpha}{n}z^n = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k(1+kz)^\alpha \tag{26}$$

*and for every $|z| < 1/4m$*

$$\sum_{n=0}^{\infty} S(n,m)\binom{2n}{n}z^n = \frac{(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{\sqrt{1-4kz}}, \tag{27}$$

$$\sum_{n=0}^{\infty} S(n,m)C_n z^n = \frac{2(-1)^m}{m!}\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{1+\sqrt{1-4kz}}. \tag{28}$$

As our last example, and one involving classical orthogonal polynomials, we consider the generalized Laguerre polynomials $L_n^{\alpha}(x)$. The generalized Laguerre polynomials are defined for all $\alpha$ with $Re(\alpha) \geq -1$ by

$$L_n^{\alpha}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1(-n; \alpha+1; x),$$

with ${}_1F_1(a; b; x)$ being the confluent hypergeometric function [20]. When $\alpha = 0$, these polynomials reduce to the classical Laguerre polynomials given by

$$L_n(x) = \frac{e^x}{n!} \left(\frac{d}{dx}\right)^n (x^n e^{-x}), \; n \geq 0, \tag{29}$$

or by the explicit binomial representation

$$L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}. \tag{30}$$

Generalized Laguerre polynomials $L_n^{\alpha}(x)$ have the generating function [18]

$$\sum_{n=0}^{\infty} L_n^{\alpha}(x) z^n = \frac{1}{(1-z)^{\alpha}} \exp\left(\frac{xz}{1-z}\right), \qquad |z| < 1.$$

Theorem 2.1 implies the next result.

**Corollary 3.10.** *For every $m > 0$ and every $|z| < 1/m$ we have the representation*

$$\sum_{n=0}^{\infty} S(n,m) L_n^{\alpha}(x) z^n = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \frac{1}{(1-kz)^{\alpha}} \exp\left(\frac{kxz}{1-kz}\right). \tag{31}$$

*In particular,*

$$\sum_{n=0}^{\infty} S(n,m) L_n(x) z^n = \frac{(-1)^m}{m!} \sum_{k=0}^{m} \binom{m}{k} \exp\left(\frac{kxz}{1-kz}\right). \tag{32}$$

## 4. Conclusion

This article was concerned with the closed form evaluation of a number of power series where the coefficients are products of Stirling numbers of the second kind and other important numbers or polynomials. The list of examples that was presented is, of course, not complete and some more series could be derived straightforwardly from Theorem 2.1. We mention Chebyshev polynomials, Fibonacci (Lucas) polynomials, Hermite polynomials and generalized Bernoulli polynomials as four important examples that were not stated explicitly. We leave them to the interested reader.

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