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Research Article

Fibonacci and Lucas identities derived via the golden ratio*

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Abstract

By expressing Fibonacci and Lucas numbers in terms of the powers of the golden ratio $\alpha = (1+\sqrt{5})/2$ and its inverse $\beta = -1/\alpha = (1-\sqrt{5})/2$, a multitude of Fibonacci and Lucas identities have been developed in the literature. In this paper, the reverse course is followed: numerous Fibonacci and Lucas identities are derived by making use of the well-known expressions for the powers of α and β in terms of Fibonacci and Lucas numbers.

Keywords: Fibonacci number; Lucas number; Gibonacci number; generating function; golden ratio.

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1. Introduction

For $n \in \mathbb{Z}$, the Fibonacci numbers F_n and the Lucas numbers L_n are defined through the recurrence relations

$$F_n = F_{n-1} + F_{n-2}, (n \geq 2), \quad F_0 = 0,\, F_1 = 1;$$

and

$$L_n = L_{n-1} + L_{n-2}, (n \geq 2), \quad L_0 = 2,\, L_1 = 1;$$

with

$$F_{-n} = (-1)^{n-1} F_n, \quad L_{-n} = (-1)^n L_n.$$

Throughout this paper, the golden ratio $(1+\sqrt{5})/2$ is denoted by α. Take $\beta = (1-\sqrt{5})/2$, then $\beta = -1/\alpha$, that is $\alpha\beta = -1$, and $\alpha + \beta = 1$. The following well-known algebraic properties of α and β can be proved directly from Binet's formula for the nth Fibonacci number or by induction:

$$\alpha^n = \alpha^{n-1} + \alpha^{n-2},$$

$$\beta^n = \beta^{n-1} + \beta^{n-2},$$

$$\alpha^n = \alpha F_n + F_{n-1}, \tag{1}$$

$$\alpha^n \sqrt{5} = \alpha^n(\alpha - \beta) = \alpha L_n + L_{n-1}, \tag{2}$$

$$\beta^n = \beta F_n + F_{n-1}, \tag{3}$$

$$\beta^n \sqrt{5} = \beta^n(\alpha - \beta) = -\beta L_n - L_{n-1}, \tag{4}$$

$$\beta^n = -\alpha F_n + F_{n+1}, \tag{5}$$

$$\beta^n \sqrt{5} = \alpha L_n - L_{n+1}, \tag{6}$$

$$\alpha^{-n} = (-1)^{n-1}\alpha F_n + (-1)^n F_{n+1},$$

and

$$\beta^{-n} = (-1)^n \alpha F_n + (-1)^n F_{n-1}.$$

Carlitz [4] (also Hoggatt et al. [11]) derived the identity $F_{k+t} = \alpha^k F_t + \beta^t F_k$, which can be put in the form

$$\alpha^s F_{k+t} = \alpha^{s+k} F_t + (-1)^t \alpha^{s-t} F_k, \tag{7}$$

*This paper is a slightly modified version of the preprint [2]
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or equivalently,

$$\alpha^s F_{k+t} = \alpha^{s+t} F_k + (-1)^k \alpha^{s-k} F_t. \tag{8}$$

As Koshy [14, p.93] noted, the two Binet formulas

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n,$$

expressing F_n and L_n in terms of α^n and β^n, can be used in tandem to derive an array of identities.

The aim of this paper is to derive numerous Fibonacci and Lucas identities by emphasizing identities (1)–(6), expressing α^n and β^n in terms of F_n and L_n. The method used in this paper for deriving the mentioned identities relies on the fact that α and β are irrational numbers. The following fact is used frequently in the remaining part of this paper: if a, b, c, and d are rational numbers, and if γ is an irrational number, then $a\gamma + b = c\gamma + d$ implies that $a = c$ and $b = d$; an observation that was used by also Griffiths [8, 9].

As a quick illustration of the above-mentioned method, take $x = \alpha F_p$ and $y = F_{p-1}$ in the binomial identity

$$\sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} = (x+y)^n,$$

to obtain

$$\sum_{j=0}^{n} \binom{n}{j} \alpha^j F_p^j F_{p-1}^{n-j} = \alpha^{np},$$

which, after multiplying both sides by α^q, can be written as

$$\sum_{j=0}^{n} \binom{n}{j} \alpha^{j+q} F_p^j F_{p-1}^{n-j} = \alpha^{np+q}, \tag{9}$$

which, by (1), gives

$$\alpha \sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q} + \sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q-1} = \alpha F_{np+q} + F_{np+q-1}. \tag{10}$$

Comparing the coefficients of α in (10), one finds

$$\sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} F_{j+q} = F_{np+q}\,, \tag{11}$$

which is valid for every non-negative integer n and for arbitrary integers p and q. Identity (11) contains many known identities as special cases. If one writes (9) as

$$\sum_{j=0}^{n} \binom{n}{j} \alpha^{j+q} \sqrt{5} F_p^j F_{p-1}^{n-j} = \alpha^{np+q} \sqrt{5}$$

and applies (2), then the Lucas version of (11) is obtained; namely,

$$\sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} L_{j+q} = L_{np+q}. \tag{12}$$

Identities (11) and (12) are known in the literature; for example, see [19]. A more general identity that includes (11) and (12) as special cases is

$$\sum_{j=0}^{n} \binom{n}{j} F_p^j F_{p-1}^{n-j} G_{p+q} = G_{np+q},$$

where $(G_k)_{k\in\mathbb{Z}}$ is the Gibonacci sequence (a generalization of the Fibonacci sequence) whose initial terms G_0 and G_1 are given integers, not both zero, and

$$G_k = G_{k-1} + G_{k-2}, \quad G_{-k} = G_{-k+2} - G_{-k+1}. \tag{13}$$

Lemma 1.1. *The following properties hold for the rational numbers a, b, c, and d:*

$$a\alpha + b = c\alpha + d \iff a = c,\ b = d, \tag{P1}$$

$$a\beta + b = c\beta + d \iff a = c,\ b = d, \tag{P2}$$

$$\frac{1}{c\alpha+d}=\left(\frac{c}{c^2-d^2-cd}\right)\alpha-\left(\frac{c+d}{c^2-d^2-cd}\right), \tag{P3}$$

$$\frac{1}{c\beta+d}=\left(\frac{c}{c^2-d^2-cd}\right)\beta-\left(\frac{c+d}{c^2-d^2-cd}\right), \tag{P4}$$

$$\frac{a\alpha+b}{c\alpha+d}=\frac{cb-da}{c^2-d^2-cd}\alpha+\frac{ca-db-cb}{c^2-cd-d^2}, \tag{P5}$$

and

$$\frac{a\beta+b}{c\beta+d}=\frac{cb-da}{c^2-d^2-cd}\beta+\frac{ca-db-cb}{c^2-cd-d^2}. \tag{P6}$$

Properties P3 to P6 follow from properties P1 and P2. Observe that P3 is a special case of P5 and P4 is a special case of P6. The next section aims to re-discover some known identities, using the above-mentioned method, and to discover some new results that may be easily deduced from the known ones. Presumably new results are developed in Section 3.

2. Preliminary results

In this section, the method described in the preceding section is utilized to re-discover some known identities and to discover some new identities that may be easily deduced from the known ones. In establishing some of such identities, one requires the fundamental relations $F_{2n}=F_nL_n$, $L_n=F_{n-1}+F_{n+1}$, and $5F_n=L_{n-1}+L_{n+1}$.

Fibonacci and Lucas addition formulas

To derive the Fibonacci addition formula, use (1) to write the identity

$$\alpha^{p+q}=\alpha^p\alpha^q$$

as

$$\alpha F_{p+q}+F_{p+q-1}=(\alpha F_p+F_{p-1})(\alpha F_q+F_{q-1}).$$

By simplifying the right side of the last identity and then making use of (1), one gets

$$\begin{aligned}\alpha F_{p+q}+F_{p+q-1}&=\alpha(F_pF_q+F_pF_{q-1}+F_{p-1}F_q)+F_pF_q+F_{p-1}F_{q-1}\\&=\alpha(F_pF_{q+1}+F_{p-1}F_q)+F_pF_q+F_{p-1}F_{q-1}.\end{aligned} \tag{14}$$

By equating coefficients of α (property P1) from both sides of (14), one gets the well-known Fibonacci addition formula:

$$F_{p+q}=F_pF_{q+1}+F_{p-1}F_q. \tag{15}$$

A similar procedure using the identity

$$\alpha^p\beta^q=(-1)^q\alpha^{p-q} \tag{16}$$

produces the subtraction formula

$$(-1)^qF_{p-q}=F_pF_{q-1}-F_{p-1}F_q,$$

which may, of course, be obtained from (15) by changing q to $-q$. The Lucas counterpart of (15) is obtained by applying (1) and (2) to the identity

$$\alpha^{p+q}\sqrt{5}=(\alpha^p\sqrt{5}\,)\alpha^q,$$

and proceeding as in the Fibonacci case:

$$L_{p+q}=F_pL_{q+1}+F_{p-1}L_q.$$

Application of (1) to the right side and (2) to the left side of the identity

$$5\alpha^{p+q}=(\alpha^p\sqrt{5}\,)(\alpha^q\sqrt{5}\,)$$

produces

$$5F_{p+q}=L_pL_{q+1}+L_{p-1}L_q.$$

Fibonacci and Lucas multiplication formulas

By subtracting (7) from (8), one has

$$F_t\left(\alpha^{s+k}-(-1)^k\alpha^{s-k}\right)=F_k\left(\alpha^{s+t}-(-1)^t\alpha^{s-t}\right). \tag{17}$$

By applying (1) to (17) and equating coefficients of α, one obtains

$$F_t\left(F_{s+k}-(-1)^kF_{s-k}\right)=F_k\left(F_{s+t}-(-1)^tF_{s-t}\right),$$

which, upon setting $k=1$, gives

$$F_tL_s=F_{s+t}-(-1)^tF_{s-t}. \tag{18}$$

Writing (17) as

$$F_t\left(\alpha^{s+k}\sqrt{5}-(-1)^k\alpha^{s-k}\sqrt{5}\right)=F_k\left(\alpha^{s+t}\sqrt{5}-(-1)^t\alpha^{s-t}\sqrt{5}\right).$$

By applying (2) and equating coefficients of α, one gets

$$F_t\left(L_{s+k}-(-1)^kL_{s-k}\right)=F_k\left(L_{s+t}-(-1)^tL_{s-t}\right),$$

which, upon setting $k=1$, gives

$$5F_tF_s=L_{s+t}-(-1)^tL_{s-t}. \tag{19}$$

By adding (7) and (8), making use of (1), and then equating the coefficients of α, one obtains

$$2F_{k+t}F_s=F_t\left(F_{s+k}+(-1)^kF_{s-k}\right)+F_k\left(F_{s+t}+(-1)^tF_{s-t}\right),$$

which, at $k=t$ reduces to

$$L_tF_s=F_{s+t}+(-1)^tF_{s-t}. \tag{20}$$

Identities (18), (19), and (20) are already known, see [14, p. 118, Formulas 55–57]. Similarly, by adding (7) and (8), multiplying with $\sqrt{5}$, making use of (2), and then equating the coefficients of α, one has

$$2F_{k+t}L_s=F_t\left(L_{s+k}+(-1)^kL_{s-k}\right)+F_k\left(L_{s+t}+(-1)^tL_{s-t}\right),$$

which, at $k=t$ reduces to

$$L_tL_s=L_{s+t}+(-1)^tL_{s-t}. \tag{21}$$

Identities (20) and (21), first reported by Carlitz [4, Identities (10) and (15)], are identities (15a) and (17a) of [18].

Cassini's identity

Since

$$\alpha^n\beta^n=(\alpha\beta)^n=(-1)^n; \tag{22}$$

applying identities (1) and (5) to the left hand side of the above identity gives

$$\begin{aligned}\alpha^n\beta^n&=(\alpha F_n+F_{n-1})(-\alpha F_n+F_{n+1})\\&=-\alpha^2F_n^2+\alpha(F_nF_{n+1}-F_nF_{n-1})+F_{n-1}F_{n+1}\\&=\alpha(-F_n^2+F_n^2)-F_n^2+F_{n-1}F_{n+1}.\end{aligned}$$

Thus, according to (22), we have

$$\alpha(-F_n^2+F_n^2)-F_n^2+F_{n-1}F_{n+1}=(-1)^n.$$

Comparing coefficients of α^0 from both sides gives Cassini's identity:

$$F_{n-1}F_{n+1}=F_n^2+(-1)^n. \tag{23}$$

To derive the Lucas version of (23), write

$$(\alpha^n\sqrt{5}\,)(\beta^n\sqrt{5}\,)=(-1)^n5\,;$$

apply (2) and (6) to the left hand side, multiply out and equate coefficients, obtaining

$$L_{n-1}L_{n+1}-L_n^2=(-1)^{n-1}5.$$

General Fibonacci and Lucas addition formulas and Catalan's identity

From (7), we can derive an addition formula that includes (15) as a particular case.

Using (1) to write the left hand side (lhs) and the right hand side (rhs) of (7), we have

$$\text{lhs of (7)} = \alpha F_s F_{k+t} + F_{s-1} F_{k+t} \tag{24}$$

and

$$\begin{aligned}\text{rhs of (7)} &= \alpha F_{s+k} F_t + F_{s+k-1} F_t + \alpha (-1)^t F_{s-t} F_k + (-1)^t F_{s-t-1} F_k \\ &= \alpha (F_{s+k} F_t + (-1)^t F_{s-t} F_k) + (F_{s+k-1} F_t + (-1)^t F_{s-t-1} F_k).\end{aligned} \tag{25}$$

Comparing the coefficients of α from (24) and (25), we find

$$F_s F_{k+t} = F_{s+k} F_t + (-1)^t F_{s-t} F_k, \tag{26}$$

of which (15) is a particular case.

Setting $t = s - k$ in (26) produces Catalan's identity:

$$F_s^2 = F_{s+k} F_{s-k} + (-1)^{s+k} F_k^2.$$

Multiplying through (7) by $\sqrt{5}$ and performing similar calculations to above produces

$$L_s F_{k+t} = L_{s+k} F_t + (-1)^t L_{s-t} F_k, \tag{27}$$

which at $t = s - k$ gives

$$F_{2s} = L_{s+k} F_{s-k} + (-1)^{s+k} F_{2k}.$$

Identities (26) and (27) appeared as problem proposals in The Fibonacci Quarterly [17, Problems B-460, B-461].

Sums of Fibonacci and Lucas numbers with subscripts in arithmetic progression

Setting $x = \alpha^p$ in the geometric sum identity

$$\sum_{j=0}^{n} x^j = \frac{1 - x^{n+1}}{1 - x} \tag{28}$$

and multiplying through by α^q gives

$$\sum_{j=0}^{n} \alpha^{pj+q} = \frac{\alpha^q - \alpha^{pn+p+q}}{1 - \alpha^p}. \tag{29}$$

Thus, we have

$$\sum_{j=0}^{n} \alpha^{pj+q} = \frac{\alpha (F_{pn+p+q} - F_q) + F_{pn+p+q-1} - F_{q-1}}{\alpha F_p + F_{p-1} - 1},$$

from which, with the use of (1) and property P5, we find

$$\sum_{j=0}^{n} F_{pj+q} = \frac{F_p (F_{pn+p+q-1} - F_{q-1}) - (F_{p-1} - 1)(F_{pn+p+q} - F_q)}{L_p - 1 + (-1)^{p-1}}, \tag{30}$$

valid for all integers p, q and n. The derivation here is considerably simpler than the one involving the direct use of Binet's formula; as done, for example, by Koshy [14, p. 104, Theorem 5.10] and Freitag [7] or by Siler [16] who first derived (30).

Multiplying through (29) by $\sqrt{5}$ gives

$$\sum_{j=0}^{n} \alpha^{pj+q} \sqrt{5} = \frac{\alpha^q \sqrt{5} - \alpha^{pn+p+q} \sqrt{5}}{1 - \alpha^p},$$

from which, by identities (2), (1), and properties P5, P1, we find

$$\sum_{j=0}^{n} L_{pj+q} = \frac{F_p (L_{pn+p+q-1} - L_{q-1}) - (F_{p-1} - 1)(L_{pn+p+q} - L_q)}{L_p - 1 + (-1)^{p-1}}. \tag{31}$$

Identity (31) was first derived by Zeitlin [20] who established a generalization of Siler's result.

Equations (30) and (31) can be summarized as

$$\sum_{j=0}^{n} G_{pj+q} = \frac{F_p (G_{pn+p+q-1} - G_{q-1}) - (F_{p-1} - 1)(G_{pn+p+q} - G_q)}{L_p - 1 - (-1)^{p-1}};$$

which is a special case of (2.11) in [13] and (2) in [3].

Generating functions of Fibonacci and Lucas numbers with indices in arithmetic progression

Setting $x = y\alpha^p$ in the identity

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x}$$

and multiplying through by α^q gives

$$\sum_{j=0}^{\infty} \alpha^{pj+q} y^j = \frac{\alpha^q}{1-\alpha^p y} = \frac{F_q\alpha + F_{q-1}}{-yF_p\alpha + 1 - yF_{p-1}}.$$

Application of (1) and properties P5 and P1 then produces

$$\sum_{j=0}^{\infty} F_{pj+q} y^j = \frac{F_q + (-1)^q F_{p-q} y}{1 - L_p y + (-1)^p y^2}. \tag{32}$$

To find the corresponding Lucas result, we write

$$\sum_{j=0}^{\infty} \alpha^{pj+q}\sqrt{5} y^j = \frac{\alpha^q\sqrt{5}}{1-\alpha^p y}$$

and use (2) and properties P5 and P1, obtaining

$$\sum_{j=0}^{\infty} L_{pj+q} y^j = \frac{L_q - (-1)^q L_{p-q} y}{1 - L_p y + (-1)^p y^2}. \tag{33}$$

Identities (32) and (33) were first derived by Zeitlin [20]. Identity (32), but not (33), was reported by Koshy [14, Identity 18, p.245]. The case $q = 0$ in (32) was also obtained by Hoggatt [10] while the case $p = 1$ in (33) is also found by Koshy [14, Identity (13.13), p.246]. Identity (33) is a special case of the generating function of k-Lucas numbers with indices in arithmetic progression derived by Falcon [6].

3. Main results

Fibonacci relations involving several subscripts

Theorem 3.1. *The following identities hold for integers p, q and r:*

$$F_{p+q+r} = F_{p+1}F_{q+1}F_{r+1} + F_pF_qF_r - F_{p-1}F_{q-1}F_{r-1}, \tag{34}$$

$$L_{p+q+r} = F_{p+1}F_{q+1}L_{r+1} + F_pF_qL_r - F_{p-1}F_{q-1}L_{r-1}, \tag{35}$$

$$5F_{p+q+r} = L_{p+1}L_{q+1}F_{r+1} + L_pL_qF_r - F_{p-1}F_{q-1}F_{r-1}, \tag{36}$$

$$5L_{p+q+r} = L_{p+1}L_{q+1}L_{r+1} + L_pL_qL_r - L_{p-1}L_{q-1}L_{r-1}. \tag{37}$$

Proof. Identities (34) – (37) are derived by applying (1) and (2) to the following identities:

$$\alpha^{p+q+r} = \alpha^p\alpha^q\alpha^r,$$

$$\alpha^{p+q+r}\sqrt{5} = (\alpha^p\sqrt{5})\alpha^q\alpha^r,$$

$$5\alpha^{p+q+r} = (\alpha^p\sqrt{5})(\alpha^q\sqrt{5})\alpha^r,$$

$$5\alpha^{p+q+r}\sqrt{5} = (\alpha^p\sqrt{5})(\alpha^q\sqrt{5})(\alpha^r\sqrt{5});$$

replacing α^3 with $\alpha^2 + \alpha$, replacing α^2 with $\alpha + 1$ and making use of property P1.

□

Identity (34) appeared as problem H-4, proposed by Ruggles [15]. The identities (34) and (35) can be generalized as

$$G_{p+q+r} = F_{p+1}F_{q+1}G_{r+1} + F_pF_qG_r - F_{p-1}F_{q-1}G_{r-1}.$$

Similarly, identities (36) and (37) can be generalized as

$$5G_{p+q+r} = L_{p+1}L_{q+1}G_{r+1} + L_pL_qG_r - L_{p-1}L_{q-1}G_{r-1}.$$

Theorem 3.2. *The following identities hold for integers p, q, r, s and t:*

$$F_{p+q-r}F_{t-s+r} + F_{p+q-r-1}F_{t-s+r-1} = F_{p-s}F_{t+q} + F_{p-s-1}F_{t+q-1}, \tag{38}$$

$$F_{p+q-r}L_{t-s+r} + F_{p+q-r-1}L_{t-s+r-1} = F_{p-s}L_{t+q} + F_{p-s-1}L_{t+q-1}, \tag{39}$$

and

$$L_{p+q-r}L_{t-s+r} + L_{p+q-r-1}L_{t-s+r-1} = L_{p-s}L_{t+q} + L_{p-s-1}L_{t+q-1}. \tag{40}$$

Proof. Identity (38) is proved by applying (1) to the identity

$$\alpha^{p+q-r}\alpha^{t-s+r} = \alpha^{p-s}\alpha^{t+q},$$

multiplying out the products and applying property P1. Identity (39) is derived by writing

$$\alpha^{p+q-r}(\alpha^{t-s+r}\sqrt{5}) = \alpha^{p-s}(\alpha^{t+q}\sqrt{5})$$

and applying identities (1) and (2) and property P1. Finally (40) is derived from

$$(\alpha^{p+q-r}\sqrt{5})(\alpha^{t-s+r}\sqrt{5}) = (\alpha^{p-s}\sqrt{5})(\alpha^{t+q}\sqrt{5}).$$

□

Identities (38) and (40) can be generalized to

$$G_{p+q-r}G_{t-s+r} + G_{p+q-r-1}G_{t-s+r-1} = G_{p-s}G_{t+q} + G_{p-s-1}G_{t+q-1}.$$

Binomial summation identities

Lemma 3.1. *The following identities hold for positive integer n and arbitrary x and y:*

$$\sum_{j=0}^{n}\binom{n}{j}y^jx^{n-j} = (x+y)^n, \tag{41}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(x+y)^jx^{n-j} = (-1)^ny^n, \tag{42}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}y^j(x+y)^{n-j} = x^n, \tag{43}$$

$$\sum_{j=0}^{n}\binom{n}{j}jy^{j-1}x^{n-j} = n(x+y)^{n-1}, \tag{44}$$

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}j(x+y)^{j-1}x^{n-j} = (-1)^nny^{n-1} \tag{45}$$

and

$$\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}y^{j-1}j(x+y)^{n-j} = nx^{n-1}. \tag{46}$$

Proof. Identities (42) and (43) are obtained from (41) by obvious transformations. Identity (44) is obtained by differentiating the identity

$$\sum_{j=0}^{n}\binom{n}{j}y^je^{jz}x^{n-j} = (x+ye^z)^n$$

with respect to z and then setting z to zero. More generally,

$$\sum_{j=0}^{n}\binom{n}{j}j^ry^jx^{n-j} = \frac{d^r}{dz^r}(x+ye^z)^n\bigg|_{z=0}.$$

Identities (45) and (46) are obtained from (44) by transformations.

□

Theorem 3.3. *The following identities hold for integers k, t, s and positive integer n:*

$$\sum_{j=0}^{n}(-1)^{tj}\binom{n}{j}F_k^jF_t^{n-j}F_{(s+k)n-(t+k)j}=F_{t+k}^nF_{sn},\tag{47}$$

$$\sum_{j=0}^{n}(-1)^{tj}\binom{n}{j}F_k^jF_t^{n-j}L_{(s+k)n-(t+k)j}=F_{t+k}^nL_{sn},\tag{48}$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}F_{k+t}^jF_t^{n-j}F_{(s+k)n-kj}=(-1)^{n(t+1)}F_k^nF_{n(s-t)},\tag{49}$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}F_{k+t}^jF_t^{n-j}L_{(s+k)n-kj}=(-1)^{n(t+1)}F_k^nL_{n(s-t)},\tag{50}$$

$$\sum_{j=0}^{n}(-1)^{(t+1)j}\binom{n}{j}F_k^jF_{k+t}^{n-j}F_{sn-tj}=F_t^nF_{n(s+k)},\tag{51}$$

$$\sum_{j=0}^{n}(-1)^{(t+1)j}\binom{n}{j}F_k^jF_{k+t}^{n-j}L_{sn-tj}=F_t^nL_{n(s+k)},\tag{52}$$

$$\sum_{j=1}^{n}(-1)^{tj}\binom{n}{j}jF_k^{j-1}F_t^{n-j}F_{(k+s)n+t-s-(k+t)j}=(-1)^tnF_{k+t}^{n-1}F_{s(n-1)},\tag{53}$$

$$\sum_{j=1}^{n}(-1)^{tj}\binom{n}{j}jF_k^{j-1}F_t^{n-j}L_{(k+s)n+t-s-(k+t)j}=(-1)^tnF_{k+t}^{n-1}L_{s(n-1)},\tag{54}$$

$$\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}jF_{k+t}^{j-1}F_t^{n-j}F_{(k+s)n-s-kj}=(-1)^{n(t+1)+t}nF_k^{n-1}F_{(s-t)(n-1)},\tag{55}$$

$$\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}jF_{k+t}^{j-1}F_t^{n-j}L_{(k+s)n-s-kj}=(-1)^{n(t+1)+t}nF_k^{n-1}L_{(s-t)(n-1)},\tag{56}$$

$$\sum_{j=1}^{n}(-1)^{(t+1)j}\binom{n}{j}jF_k^{j-1}F_{k+t}^{n-j}F_{s(n-1)+t-tj}=(-1)^{t+1}nF_t^{n-1}F_{(s+k)(n-1)}\tag{57}$$

and

$$\sum_{j=1}^{n}(-1)^{(t+1)j}\binom{n}{j}jF_k^{j-1}F_{k+t}^{n-j}L_{s(n-1)+t-tj}=(-1)^{t+1}nF_t^{n-1}L_{(s+k)(n-1)}.\tag{58}$$

Proof. Choosing $x=\alpha^{s+k}F_t$ and $y=(-1)^t\alpha^{s-t}F_k$ in (41) and taking note of (7), we have

$$\sum_{j=0}^{n}(-1)^{tj}\binom{n}{j}F_k^jF_t^{n-j}\alpha^{(s+k)n-(t+k)j}=F_{k+t}^n\alpha^{ns}.\tag{59}$$

Application of (1) and property P1 to (59) produces (47). To prove (48), multiply through (59) by $\sqrt{5}$ to obtain

$$\sum_{j=0}^{n}(-1)^{tj}\binom{n}{j}F_k^jF_t^{n-j}\alpha^{(s+k)n-(t+k)j}\sqrt{5}=F_{k+t}^n\alpha^{ns}\sqrt{5}.\tag{60}$$

Use of (2) and property P1 in (60) gives (48). Identities (49) – (58) are derived in a similar fashion. □

Identities related to or equivalent to some of the identities listed in Theorem 3.3 were also derived by Carlitz [5] and Griffiths [9].

We list below the Gibonacci versions of identities (47) – (58).

$$\sum_{j=0}^{n}(-1)^{tj}\binom{n}{j}F_k^jF_t^{n-j}G_{(s+k)n-(t+k)j}=F_{t+k}^nG_{sn},\tag{61}$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}F_{k+t}^jF_t^{n-j}G_{(s+k)n-kj}=(-1)^{n(t+1)}F_k^nG_{n(s-t)},\tag{62}$$

$$\sum_{j=0}^{n}(-1)^{(t+1)j}\binom{n}{j}F_k^jF_{k+t}^{n-j}G_{sn-tj}=F_t^nG_{n(s+k)}, \tag{63}$$

$$\sum_{j=1}^{n}(-1)^{tj}\binom{n}{j}jF_k^{j-1}F_t^{n-j}G_{(k+s)n+t-s-(k+t)j}=(-1)^tnF_{k+t}^{n-1}G_{s(n-1)}, \tag{64}$$

$$\sum_{j=1}^{n}(-1)^{j}\binom{n}{j}jF_{k+t}^{j-1}F_t^{n-j}G_{(k+s)n-s-kj}=(-1)^{n(t+1)+t}nF_k^{n-1}G_{(s-t)(n-1)}, \tag{65}$$

and

$$\sum_{j=1}^{n}(-1)^{(t+1)j}\binom{n}{j}jF_k^{j-1}F_{k+t}^{n-j}G_{s(n-1)+t-tj}=(-1)^{t+1}nF_t^{n-1}G_{(s+k)(n-1)}. \tag{66}$$

Lemma 3.2. *The following identities hold true for integer r, non-negative integer n, and arbitrary x and y:*

$$\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(xy)^{r-j}(x-y)^{2j+1}=x^{r+n+1}y^{r-n}-y^{r+n+1}x^{r-n}, \tag{67}$$

$$\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(x(x-y))^{r-j}y^{2j+1}=x^{r+n+1}(x-y)^{r-n}-(x-y)^{r+n+1}x^{r-n} \tag{68}$$

and

$$\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(y(y-x))^{r-j}x^{2j+1}=y^{r+n+1}(y-x)^{r-n}-(y-x)^{r+n+1}y^{r-n}. \tag{69}$$

Proof. Jennings [12, Lemma (i)] derived an identity equivalent to the following:

$$\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}\left(\frac{z^2-1}{z}\right)^{2j}=\frac{z^2z^{2n}-z^{-2n}}{z^2-1}.$$

Setting $z^2=x/y$ in the above identity and clearing fractions gives (67). Identity (68) is obtained by replacing y with $x-y$ in (67). Identity (69) is obtained by interchanging x with y in (68).

□

Theorem 3.4. *The following identities hold for non-negative integer n and integers s, k, r and t:*

$$\begin{aligned}&\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_{k+t}F_t)^{r-j}F_k^{2j+1}F_{r(2s+k)+s-t-(2t+k)j}\\&\quad=(-1)^tF_{k+t}^{r+n+1}F_t^{r-n}F_{s(r+n+1)+(s+k)(r-n)}-(-1)^tF_t^{r+n+1}F_{k+t}^{r-n}F_{(s+k)(r+n+1)+s(r-n)},\end{aligned} \tag{70}$$

$$\begin{aligned}&\sum_{j=0}^{n}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_{k+t}F_t)^{r-j}F_k^{2j+1}L_{r(2s+k)+s-t-(2t+k)j}\\&\quad=(-1)^tF_{k+t}^{r+n+1}F_k^{r-n}L_{s(r+n+1)+(s+k)(r-n)}-(-1)^tF_t^{r+n+1}F_{k+t}^{r-n}L_{(s+k)(r+n+1)+s(r-n)},\end{aligned} \tag{71}$$

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{tj}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_{k+t}F_t)^{r-j}F_t^{2j+1}F_{r(2s-t)+s+k+(2k+t)j}\\&\quad=(-1)^{tn}F_{k+t}^{r+n+1}F_k^{r-n}F_{s(2r+1)-t(r-n)}-(-1)^{tn+t}F_k^{r+n+1}F_{k+t}^{r-n}F_{s(2r+1)-t(r+n+1)},\end{aligned} \tag{72}$$

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{tj}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_{k+t}F_t)^{r-j}F_t^{2j+1}L_{r(2s-t)+s+k+(2k+t)j}\\&\quad=(-1)^{tn}F_{k+t}^{r+n+1}F_k^{r-n}L_{s(2r+1)-t(r-n)}-(-1)^{tn+t}F_k^{r+n+1}F_{k+t}^{r-n}L_{s(2r+1)-t(r+n+1)},\end{aligned} \tag{73}$$

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^{(t-1)j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_tF_k)^{r-j}F_{k+t}^{2j+1}F_{s(2r+1)+(k-t)r-(k-t)j}\\&\quad=(-1)^{(t-1)n}F_t^{r+n+1}F_k^{r-n}F_{s(2r+1)+k(r+n+1)+t(n-r)}\\&\quad\quad-(-1)^{(t-1)(n+1)}F_k^{r+n+1}F_t^{r-n}F_{s(2r+1)-k(n-r)-t(r+n+1)}\end{aligned} \tag{74}$$

and

$$\begin{aligned}\sum_{j=0}^n (-1)^{(t-1)j}\frac{2n+1}{n+j+1}\binom{n+j+1}{2j+1}(F_tF_k)^{r-j}F_{k+t}^{2j+1}L_{s(2r+1)+(k-t)r-(k-t)j}\\ =(-1)^{(t-1)n}F_t^{r+n+1}F_k^{r-n}L_{s(2r+1)+k(r+n+1)+t(n-r)}\\ -(-1)^{(t-1)(n+1)}F_k^{r+n+1}F_t^{r-n}L_{s(2r+1)-k(n-r)-t(r+n+1)}.\end{aligned}\tag{75}$$

Proof. Each of the identities (70) – (75) is proved by setting $x=\alpha^{s+k}F_t$ and $y=(-1)^t\alpha^{s-t}F_k$ in the identities of Lemma 3.2 and taking note of (7) while making use of (1) and (2) and property P1. □

Summation identities not involving binomial coefficients

Lemma 3.3 (See [1, Lemma 1]). *Let (X_t) and (Y_t) be any two sequences such that X_t and Y_t, $t\in\mathbb{Z}$, are connected by a three-term recurrence relation $hX_t=f_1X_{t-a}+f_2Y_{t-b}$, where h, f_1 and f_2 are arbitrary non-vanishing complex functions, not dependent on t, and a and b are integers. Then, the following identity holds for integer n:*

$$f_2\sum_{j=0}^n f_1^{n-j}h^jY_{t-na-b+aj}=h^{n+1}X_t-f_1^{n+1}X_{t-(n+1)a}.$$

Theorem 3.5. *The following identities hold for integers n, k, s and t:*

$$F_k\sum_{j=0}^n(-1)^{kj}F_{n(s-k)+s-t+2kj}=(-1)^{nk}F_{k(n+1)}F_{s(n+1)-t},\tag{76}$$

$$F_k\sum_{j=0}^n(-1)^{kj}L_{n(s-k)+s-t+2kj}=(-1)^{nk}F_{k(n+1)}L_{s(n+1)-t}.\tag{77}$$

Proof. Write (7) as $\alpha^{s+k}F_t=\alpha^sF_{t+k}+(-1)^{t-1}\alpha^{s-t}F_k$ and identify $h=\alpha^{s+k}$, $f_1=\alpha^s$, $f_2=F_k$, $a=-k$, $b=0$, $X_t=F_t$ and $Y_t=(-1)^{t-1}\alpha^{s-t}$ in Lemma 3.3. Application of (1) to the resulting summation identity yields

$$(-1)^{nk+t-1}F_k\sum_{j=0}^n(-1)^{kj}F_{n(s-k)+s-t+2kj}=F_tF_{(s+k)(n+1)}-F_{t+(n+1)k}F_{s(n+1)},$$

which, in view of the identity (see [18, Formula (20a)])

$$F_aF_b-F_{a+c}F_{b-c}=(-1)^{b-c}F_cF_{a+c-b}$$

gives (76). Multiplying the $\alpha-$sum by $\sqrt5$ and using (2) gives

$$(-1)^{nk+t-1}F_k\sum_{j=0}^n(-1)^{kj}L_{n(s-k)+s-t+2kj}=F_tL_{(s+k)(n+1)}-F_{t+(n+1)k}L_{s(n+1)},$$

which with the use of the identity (see [18, Formula (19b)])

$$F_aL_b-F_{a+c}L_{b-c}=(-1)^{b-c-1}F_cL_{a+c-b}$$

gives (77). □

Lemma 3.4. *The following identities hold for integers r and n and arbitrary x and y:*

$$(x-y)\sum_{j=0}^n y^{r-j}x^j=y^{r-n}x^{n+1}-y^{r+1},\tag{78}$$

$$x\sum_{j=0}^n y^{r-j}(x+y)^j=y^{r-n}(x+y)^{n+1}-y^{r+1}\tag{79}$$

and

$$(x-y)\sum_{j=0}^n x^{r-j}y^j=x^{r+1}-x^{r-n}y^{n+1}.\tag{80}$$

Proof. Identity (78) is obtained by replacing x with x/y in (28). Identity (79) is obtained by replacing x with $x+y$ in (78). Finally, identity (80) is obtained by interchanging x and y in (78).

□

Theorem 3.6. *The following identities hold for integers r, n, s, k and t:*

$$F_k\sum_{j=0}^{n}F_{k+t}^{j}F_t^{r-j}F_{r(s+k)+s-t-kj}=(-1)^tF_t^{r-n}F_{k+t}^{n+1}F_{s(r+1)+k(r-n)}-(-1)^tF_t^{r+1}F_{(s+k)(r+1)},\tag{81}$$

$$F_k\sum_{j=0}^{n}F_{k+t}^{j}F_t^{r-j}L_{r(s+k)+s-t-kj}=(-1)^tF_t^{r-n}F_{k+t}^{n+1}L_{s(r+1)+k(r-n)}-(-1)^tF_t^{r+1}L_{(s+k)(r+1)},\tag{82}$$

$$F_t\sum_{j=0}^{n}(-1)^{tj}F_k^{r-j}F_{k+t}^{j}F_{r(s-t)+s+k+tj}=(-1)^{rt}F_k^{r-n}F_{k+t}^{n+1}F_{r(s-t)+tn+s}-(-1)^tF_k^{r+1}F_{(s-t)(r+1)},\tag{83}$$

$$F_t\sum_{j=0}^{n}(-1)^{tj}F_k^{r-j}F_{k+t}^{j}L_{r(s-t)+s+k+tj}=(-1)^{rt}F_k^{r-n}F_{k+t}^{n+1}L_{r(s-t)+tn+s}-(-1)^tF_k^{r+1}L_{(s-t)(r+1)},\tag{84}$$

$$F_k\sum_{j=0}^{n}F_{k+t}^{r-j}F_t^{j}F_{s(r+1)+s-t+kj}=(-1)^tF_{k+t}^{r+1}F_{s(r+1)}-(-1)^tF_{k+t}^{r-n}F_t^{n+1}F_{s(r+1)+k(n+1)},\tag{85}$$

and

$$F_k\sum_{j=0}^{n}F_{k+t}^{r-j}F_t^{j}L_{s(r+1)+s-t+kj}=(-1)^tF_{k+t}^{r+1}L_{s(r+1)}-(-1)^tF_{k+t}^{r-n}F_t^{n+1}L_{s(r+1)+k(n+1)}.\tag{86}$$

Proof. Identities (81) and (82) and identities (85) and (86) are obtained by setting $x=\alpha^sF_{k+t}$ and $y=\alpha^{s+k}F_t$ in identities (78) and (80) while taking note of (7). Identities (83) and (84) are derived by setting $x=\alpha^{s+k}F_t$ and $y=(-1)^t\alpha^{s-t}F_k$ in (79).

□

Theorem 3.7. *The following identities hold for integers p, q and n:*

$$\begin{aligned}\sum_{j=0}^{n}jF_{pj+q}&=(n+1)\frac{F_pF_{p(n+1)+q-1}-(F_{p-1}-1)F_{p(n+1)+q}}{L_p-1+(-1)^{p-1}}\\&\quad+\frac{(F_{2p}-2F_p)(F_{p(n+2)+q-1}-F_{p+q-1})}{(F_{2p-1}-2F_{p-1}+1)(F_{2p+1}-2F_{p+1}+1)-(F_{2p}-2F_p)^2}\\&\quad-\frac{(F_{2p-1}-2F_{p-1}+1)(F_{p(n+2)+q}-F_{p+q})}{(F_{2p-1}-2F_{p-1}+1)(F_{2p+1}-2F_{p+1}+1)-(F_{2p}-2F_p)^2}\end{aligned}$$

and

$$\begin{aligned}\sum_{j=0}^{n}jL_{pj+q}&=(n+1)\frac{F_pL_{p(n+1)+q-1}-(F_{p-1}-1)L_{p(n+1)+q}}{L_p-1+(-1)^{p-1}}\\&\quad+\frac{(F_{2p}-2F_p)(L_{p(n+2)+q-1}-L_{p+q-1})}{(F_{2p-1}-2F_{p-1}+1)(F_{2p+1}-2F_{p+1}+1)-(F_{2p}-2F_p)^2}\\&\quad-\frac{(F_{2p-1}-2F_{p-1}+1)(L_{p(n+2)+q}-L_{p+q})}{(F_{2p-1}-2F_{p-1}+1)(F_{2p+1}-2F_{p+1}+1)-(F_{2p}-2F_p)^2}.\end{aligned}$$

Proof. Differentiating (28) with respect to x and multiplying through by x gives

$$\sum_{j=0}^{n}jx^j=(n+1)\frac{x^{n+1}}{x-1}-\frac{x^{n+2}-x}{(x-1)^2}.$$

Setting $x=\alpha^p$ and multiplying through by α^q produces

$$\sum_{j=0}^{n}j\alpha^{pj+q}=\frac{(n+1)\alpha^{p(n+1)+q}}{\alpha^p-1}+\frac{\alpha^{p(n+2)+q}-\alpha^{p+q}}{2\alpha^p-\alpha^{2p}-1},$$

from which the results follow after the use of the identities (1) and (2) and properties P1 and P5.

□

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