Research Article

Global regular solutions for the multi-dimensional Kuramoto-Sivashinsky equation posed on smooth domains

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(Received: 7 April 2022. Received in revised form: 6 June 2022. Accepted: 4 July 2022. Published online: 14 July 2022.)

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Abstract

Initial-boundary value problems for the \(n\)-dimensional Kuramoto-Sivashinsky equation posed on smooth bounded domains in \(\mathbb{R}^n\) are considered, where \(n\) is a natural number from the interval \([2, 7]\). The existence and uniqueness of global regular solutions as well as their exponential decay are established.

Keywords: global solutions; Kuramoto-Sivashinsky equation; decay in bounded domains.

2020 Mathematics Subject Classification: 35Q35, 35Q53.

1. Introduction

This work concerns the existence, uniqueness, regularity, and exponential decay rates of solutions to initial-boundary value problems for the \(n\)-dimensional Kuramoto-Sivashinsky (KS) equation

\[
\phi_t + \Delta^2 \phi + \Delta \phi + \frac{1}{2} |\nabla \phi|^2 = 0.
\]

(1)

Here \(n\) is a natural number from the interval \([2, 7]\), \(\Delta\) and \(\nabla\) are the Laplacian and the gradient in \(\mathbb{R}^n\). In [10], Kuramoto studied the turbulent phase waves and Sivashinsky in [17] obtained an asymptotic equation which simulated the evolution of a disturbed plane flame front (see also [7]). Mathematical results on initial and initial-boundary value problems for the one-dimensional KS equation (1) are presented in [3, 5, 6, 12, 14, 15, 19, 20]. The initial-value problem for the multi-dimensional KS type equations (1) was studied in [4, 5]. Two-dimensional periodic problems for the KS equation and its modifications posed on rectangles were examined in [2, 13, 14, 16, 19], where some results on the existence of weak solutions and nonlinear stability were established. In [11], initial-boundary value problems for the 3D Kuramoto-Sivashinsky equation were studied; the existence, uniqueness, and exponential decay of global regular solutions were proved. For \(n\) dimensions, \(x = (x_1, \ldots, x_n)\), \(n = 2, 3, 4, 5, 6, 7\), Equation (1) can be rewritten in the form of the following system:

\[
(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2} \sum_{i=1}^{n} (u_i)^2 x_j = 0, \quad j = 1, \ldots, n, \quad (2)
\]

\[
(u_i)_x_j = (u_j)_x_i, \quad j \neq i, \quad i, j = 1, \ldots, n. \quad (3)
\]

where \(u_j = (\phi)_{x_j}, \quad j = 1, \ldots, n\). Let \(\Omega_n = \prod_{i=1}^{n} (0, L_i)\) be the minimal \(n\)D parallelepiped containing a given smooth domain \(D_n\). The first essential problem that arises while one studies either (1) or (2)–(3), is concerned about the destabilizing effects of \(\Delta u_j\); they may be damped by dissipative terms \(\Delta^2 u_j\) provided \(D_n\) has some specific properties. In order to understand this, we use Steklov’s inequalities to estimate

\[
a \|u_j\|^2 \leq \|\nabla u_j\|^2, \quad a \|\nabla u_j\|^2 \leq \|\Delta u_j\|^2; \quad a = \sum_{i=1}^{n} \frac{\pi^2}{L_i^2}, \quad j = 1, \ldots, n.
\]

A simple analysis shows that if

\[
1 - \frac{1}{a} > 0,
\]

then \(\Delta^2 u_j\) damp \(\Delta u_j\). Naturally, here appear admissible domains where (4) is fulfilled; these are the so-called “thin domains”, where some \(L_i\) are sufficiently small while others \(L_j\) may be large \(i, j = 1, \ldots, 7; \quad i \neq j\).

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The second essential problem is the presence of semi-linear terms in (2) which are interconnected. This does not allow to obtain the first estimate independent of \( u_j \) and leads to a connection between \( L_i \) and \( u_j(0) \), \( i, j = 1, \ldots, 7 \).

The aim of this paper is to study \( n \)-dimensional initial-boundary value problems for (2)–(3) posed on smooth domains, where the existence and uniqueness of global regular solutions as well as their exponential decay of the \( H^2(D_n) \)-norm are established. A “smoothing effect” for solutions with respect to initial data is also obtained. Although, the cases \( n = 2, 3 \) are not new, we included them for the sake of generality.

The remaining part of this paper is organized as follows. Section 2 gives notations and auxiliary facts. In Section 3, formulation of an initial-boundary value problem in a smooth bounded domain \( D_n \) is given. The existence and uniqueness of global regular solutions, exponential decay of the \( H^2(D_n) \)-norm, and a “smoothing effect” are established also in Section 3. Section 4 consists of conclusion.

2. Notations and auxiliary facts

Let \( D_n \) be a sufficiently smooth domain in \( \mathbb{R}^n \), where \( n \in [2, 7] \) is a fixed natural number, satisfying the Cone condition (see [1]) and \( x = (x_1, \ldots, x_n) \in D_n \). We use the standard notations of Sobolev spaces \( W^{k,p} \), \( L^p \) and \( H^k \) for functions and the following notations for the norms [1] for scalar functions \( f(x,t) \):

\[
\|f\|^2 = \int_{D_n} |f|^2 \, dx,
\]

\[
\|f\|_{L^p(D_n)}^p = \int_{D_n} |f|^p \, dx,
\]

\[
\|f\|_{W^{k,p}(D_n)}^p = \sum_{0 \leq \alpha \leq k} \|D^\alpha f\|_{L^p(D_n)}^p,
\]

\[
\|f\|_{H^k(D_n)} = \|f\|_{W^{k,2}(D_n)}.
\]

When \( p = 2 \), \( W^{k,p}(D_n) = H^k(D_n) \) is a Hilbert space with the scalar product

\[
\langle (u,v) \rangle_{H^k(D_n)} = \sum_{|j| \leq k} \langle D^j u, D^j v \rangle, \quad \|u\|_{L^\infty(D_n)} = \text{ess sup}_{D_n} |u(x)|.
\]

We use the notation \( H^k_0(D_n) \) to represent the closure of \( C_0^\infty(D_n) \), the set of all \( C^\infty \) functions with compact support in \( D_n \), with respect to the norm of \( H^k(D_n) \).

**Lemma 2.1** (Steklov’s inequality [18]). Let \( v \in H^1_0(0, L) \). Then

\[
\frac{\pi^2}{L^2} \|v\|^2(t) \leq \|v_x\|^2(t).
\]

**Lemma 2.2** (Differential form of the Gronwall Inequality). Let \( I = [t_0, t_1] \). Suppose that functions \( a, b : I \to \mathbb{R} \) are integrable and a function \( a(t) \) may be of any sign. Let \( u : I \to \mathbb{R} \) be a differentiable function satisfying

\[
u_t(t) \leq a(t) u(t) + b(t), \quad \text{for } t \in I \text{ and } u(t_0) = u_0,
\]

then

\[
u(t) \leq u_0 e^{\int_{t_0}^t a(r) \, dr} + \int_{t_0}^t e^{\int_{r_0}^r a(r) \, dr} b(s) \, ds.
\]

*Proof.* Multiply (5) by the integrating factor \( e^{\int_{t_0}^r a(r) \, dr} \) and integrate from \( t_0 \) to \( t \). ☐

The next Lemmas will be used in estimates.

**Lemma 2.3** (See Theorem 9.1 in [8]). Let \( n \) be a natural number from the interval \( [2, 7] \). If \( D_n \) is a sufficiently smooth bounded domain in \( \mathbb{R}^n \) satisfying the cone condition and \( v \in H^4(D_n) \cap H^1_0(D_n) \), then

\[
\sup_{D_n} |v(x)| \leq C_n \|v\|_{H^4(D_n)}.
\]

The constant \( C_n \) depends on \( n \) and \( D_n \).
Lemma 2.4. Let \( f(t) \) be a continuous positive function and \( f'(t) \) be a measurable integrable function such that

\[
f'(t) + (\alpha - kf^n(t))f(t) \leq 0, \quad t > 0, \quad n \in \mathbb{N},
\]

\[
\alpha - kf^n(0) > 0, \quad k > 0,
\]

then

\[
f(t) < f(0)
\]

for all \( t > 0 \).

3. KS equation posed on smooth domains

Let \( \Omega_n \) be the minimal nD-parallelepiped containing a given bounded smooth domain \( \bar{D}_n \in \mathbb{R}^n, \ n = 1, \ldots, 7 \):

\[
\Omega_n = \{ x \in \mathbb{R}^n; x_i \in (0, L_i) \}, \ u_i = (\phi)_x, \ i = 1, \ldots, n.
\]

Fix a natural number \( n \in [2, 7] \) and consider in \( Q_n = D_n \times (0, t) \) the following initial-boundary value problem:

\[
(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2} \sum_{i=1}^{n} (u_i^2)_{x_j} = 0, \quad j = 1, \ldots, n,
\]

(9)

\[
(u_i)_{x_j} = (u_j)_{x_i}, \quad j \neq i, \quad i, j = 1, \ldots, n;
\]

(10)

\[
u_j|_{\partial D_n} = \Delta u_j|_{\partial D_n} = 0, \quad t > 0,
\]

(11)

\[
u_j(x, 0) = u_{j0}(x), \quad j = 1, \ldots, n, \quad x \in D_n.
\]

(12)

Lemma 3.1. If \( f \in H^4(D_n) \cap H^1_0(D_n) \) and \( \Delta f|_{\partial D_n} = 0 \), then

\[
a||f||^2 \leq ||\nabla f||^2,
\]

\[
a^2 ||f||^2 \leq ||\Delta f||^2,
\]

\[
a||\nabla f||^2 \leq ||\Delta f||^2,
\]

\[
a^2 ||\Delta f||^2 \leq ||\Delta^2 f||^2,
\]

\[
||\Delta \nabla f||^2 \leq ||\Delta^2 f|| ||\Delta f|| \leq \frac{1}{a} ||\Delta^2 f||^2.
\]

where

\[
a = \sum_{i=1}^{n} \frac{\pi^2}{L_i^2},
\]

and

\[
||f||^2 = \int_{D_n} f^2(x)dx.
\]

Proof. We have

\[
||\nabla f||^2 = \sum_{i=1}^{n} ||f_{x_i}||^2.
\]

Define

\[
\tilde{f}(x, t) = \begin{cases} f(x, t) & \text{if } x \in D_n; \\ 0 & \text{if } x \in \Omega_n \setminus D_n. \end{cases}
\]

(13)

Making use of Steklov's inequalities for \( \tilde{f}(x, t) \) and taking into account that \( ||\nabla f|| = ||\tilde{f}|| \), we get

\[
||\nabla f||^2 \geq a ||f||^2, \quad \text{where} \quad a = \sum_{i=1}^{n} \frac{\pi^2}{L_i^2}.
\]

On the other hand,

\[
a ||f||^2 \leq ||\nabla f||^2 = - \int_{D_n} f \Delta f dx \leq ||\Delta f|| ||f||.
\]
This implies
\[ a\|f\| \leq \|\Delta f\| \text{ and } a^2\|f\|^2 \leq \|\Delta f\|^2. \]

Consequently, \( a\|\nabla f\|^2 \leq \|\Delta f\|^2 \). Similarly,
\[
\|\Delta f\|^2 = \int_{D_n} f\Delta^2 f \, dx \leq \|\Delta^2 f\|\|f\| \leq \frac{1}{a}\|\Delta^2 f\|\|\Delta f\|.
\]

Hence, \( a\|\Delta f\| \leq \|\Delta^2 f\| \). Moreover,
\[
\|\Delta \nabla f\|^2 = -\int_{D_n} \Delta^2 f \Delta f \, dx \leq \|\Delta^2 f\|\|\Delta f\| \leq \frac{1}{a}\|\Delta^2 f\|^2.
\]

\[ \square \]

**Remark 3.1.** Assertions of Lemma 3.1 are true if the function \( f \) is replaced respectively by \( u_j, j = 1, \ldots, n \).

**Lemma 3.2.** In conditions of Lemma 3.1, the following inequalities hold
\[
\|f\|^2(t)_{H^2(D_n)} \leq 3\|\Delta f\|^2(t), \tag{14}
\]
\[
\|f\|^2(t)_{H^4(D_n)} \leq 5\|\Delta^2 f\|^2(t), \tag{15}
\]
\[
\sup_{D_n} |f(x)| \leq C_s\|\Delta^2 f\|, \text{ where } C_s = 5C_n. \tag{16}
\]

**Proof.** To prove (15), we make use of Lemma 3.1 and find
\[
\|f\|^2_{H^1(D_n)} = \|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 + \|\nabla \Delta f\|^2 + \|\Delta^2 f\|^2 \leq \left( \frac{1}{a^4} + \frac{1}{a^3} + \frac{1}{a^2} + \frac{1}{a} + 1 \right) \|\Delta^2 f\|^2.
\]

Since \( a > 1 \), then (15) follows. Similarly, (14) can be proved. Moreover, taking into account Lemma 2.3, we get (16). \[ \square \]

**Theorem 3.1 (Special basis).** Let \( n \in \{2, 3, \ldots, 7\} \) and \( D_n \subset \mathbb{R}^n \) be a bounded smooth domain satisfying the Cone condition. Let \( \Omega_n \) be a minimal nd-parallelepiped containing \( D_n \) and
\[
\theta = 1 - \frac{1}{a} = 1 - \frac{1}{\sum_{i=1}^{n} \frac{1}{L_i^2}} > 0. \tag{17}
\]

Given
\[
u_{j0}(D_n) \in H^2(D_n) \cap H^1_0(D_n), \quad j = 1, \ldots, n
\]

such that
\[
\theta - \frac{2C_n^2\theta^3}{a\theta} \left( \sum_{j=1}^{n} \|\Delta u_j\|^2(0) \right) > 0, \tag{18}
\]

then there exists a unique global regular solution to (9)–(12):
\[
u_j \in L^\infty(\mathbb{R}^+; H^2(D_n)) \cap L^2(\mathbb{R}^+; H^4(D_n) \cap H^1_0(D_n)); \]
\[
u_{jt} \in L^2(\mathbb{R}^+; L^2(D_n)), \quad j = 1, \ldots, n.
\]

Moreover,
\[
\sum_{j=1}^{n} \|\Delta u_j\|^2(t) \leq \left( \sum_{j=1}^{n} \|\Delta u_{j0}\|^2 \right) \exp\left\{-a^2\theta t/2\right\} \tag{19}
\]

and
\[
\sum_{i=1}^{n} \|\Delta u_i\|^2(t) + \int_{0}^{t} \sum_{i=1}^{n} \|\Delta^2 u_i\|^2(\tau) \, d\tau \leq C \sum_{i=1}^{n} \|\Delta u_{i0}\|^2, \quad t > 0.
\]
Remark 3.2. In Theorem 3.1, there are two types of restrictions: the first one is pure geometrical,
\[ 1 - \frac{1}{a} > 0 \]
which is needed to eliminate destabilizing effects of the terms \( \Delta u_j \) in (9):
\[ \|\Delta u_j\|^2 - \|\nabla u_j\|^2. \]

It is clear that
\[ \lim_{L_i \to 0} a = \sum_{i=1}^{n} \frac{\pi^2}{L_i^2} = +\infty, \]
hence to achieve (17), it is possible to decrease \( L_i, \ i = 1, \ldots, n \) allowing other \( L_i, \ j \neq i \) to grow. The situation with condition (18) is more complicated: if initial data are not small, then it is possible either to decrease \( L_i, \ i = 1, \ldots, n \), to fulfill this condition or for fixed \( L_i, \ i = 1, \ldots, n \) to decrease initial data \( \|u_{j0}\| \).

Proof of Theorem 3.1. It is possible to construct Galerkin’s approximations to (9)–(12) by the following way. Let \( w_j(x) \) be eigenfunctions of the problem:
\[ \Delta^2 w_j - \lambda_j w_j = 0 \text{ in } D_n; \quad w_j|_{\partial D_n} = \Delta w_j|_{\partial D_n} = 0, j = 1, 2, \ldots. \]

Define
\[ u^N_j(x, t) = \sum_{k=1}^{N} g^N_{ij}(t)w_j(x). \]

Unknown functions \( g^N_{ij}(t) \) satisfy the following initial problems:
\[ \left( \frac{d}{dt} u^N_j, w_j \right)(t) + (\Delta^2 u^N_j, w_j)(t) + (\Delta u^N_j, w_j)(t) + \frac{1}{2} \left( \sum_{i=1}^{n} (u^N_i)^2, w_j \right)(t) = 0, \]
\[ g^N_{ij}(0) = g_{0ij}, \ j = 1, \ldots, n, \ k = 1, 2, \ldots. \]

The estimates that follow may be established on Galerkin’s approximations (see [5, 6]), but it is more explicitly to prove them on smooth solutions of (9)-(12).

**Estimate I:** \( u \in L^\infty(\mathbb{R}^+; H^2(D_n) \cap H_0^1(D_n)) \cap L^2(\mathbb{R}^+; H^4(D_n) \cap H_0^1(D_n)). \)

For any natural number \( n \in [2, 7], \) multiply (9) by \( 2\Delta^2 u_j \) to obtain
\[ \frac{d}{dt} \|\Delta u_j\|^2(t) + 2\|\Delta^2 u_j\|^2(t) + 2\|\Delta^2 u_j(t)\|\|\Delta u_j(t)\| + 2\sum_{i=1}^{n} (u_i(u_{ij}x_j, \Delta^2 u_j)(t) = 0. \]  

Making use of (15) and Lemmas 2.3, 3.1, 3.2, we write
\[ \frac{d}{dt} \|\Delta u_j\|^2(t) + 2\theta \|\Delta^2 u_j\|^2(t) \leq 2 \left( \sum_{i=1}^{n} \sup_{D_n} |u_i(x, t)| \|\nabla u_i\| \right) \|\Delta^2 u_j\| \|\Delta u_j\| \]
\[ \leq 2 \left( C \sum_{i=1}^{n} \|\Delta^2 u_i\|(t) \|\nabla u_i\|(t) \right) \|\Delta^2 u_j\|(t); j = 1, \ldots, n. \]  

Summing over \( j = 1, \ldots, n \) and making use of Lemma 3.1, we rewrite (20) in the form:
\[ \frac{d}{dt} \sum_{j=1}^{n} \|\Delta u_j\|^2(t) + 2\theta \sum_{j=1}^{n} \|\Delta^2 u_j\|(t) \leq 2C_{a,n} \left( \sum_{j=1}^{n} \|\nabla u_j\|(t) \right) \left( \sum_{j=1}^{n} \|\Delta^2 u_j\|^2(t) \right) \]
\[ \leq \left( \frac{\theta}{2} + \frac{2C_{a}^2n^2}{\theta} \left( \sum_{j=1}^{n} \|\nabla u_j\|(t) \right)^2 \right) \sum_{j=1}^{n} \|\Delta^2 u_j\|^2(t) \]
\[ \leq \left( \frac{\theta}{2} + \frac{2C_{a}^2n^3}{\theta} \left( \sum_{j=1}^{n} \|\nabla u_j\|^2(t) \right) \right) \sum_{j=1}^{n} \|\Delta^2 u_j\|^2(t) \]
Lemma 3.3. A regular solution of Theorem 3.1 is uniquely defined.

Proof. Let \( u_j \) and \( v_j, j = 1, \ldots, n \), be two distinct solutions to (9)–(12). Denoting \( w_j = u_j - v_j \), we come to the following system:

\[
(w_j)_t + \Delta^2 w_j + \Delta w_j + \frac{1}{2} \sum_{i=1}^{n} (u_i^2 - v_i^2)_{x_j} = 0,
\]

\[
(w_j)_{x_i} = (w_i)_{x_j}, \quad i \neq j,
\]

\[
w_j|_{\partial D_n} = \Delta w_j|_{\partial D_n} = 0, \quad t > 0;
\]

\[
w_j(x, 0) = 0 \text{ in } D, \quad j = 1, \ldots, n.
\]

Multiply (26) by \( 2w_j \), we obtain

\[
\frac{d}{dt} \|w_j\|^2(t) + 2\|\Delta w_j\|^2(t) - 2\|\nabla w_j\|^2(t) - \sum_{i=1}^{n} \left\langle (u_i + v_i) w_i, (w_j)_{x_j} \right\rangle (t) = 0, \quad j = 1, \ldots, n.
\]
Making use of Lemmas 2.3 and 3.1, 3.2, we estimate
\[ I = \sum_{i=1}^{n} \left( \{u_i + v_i\} w_i, (w_j)_{x_j} \right) \]
\[ \leq \frac{\epsilon}{2} \|\nabla w_j\|^2 + \frac{1}{2\epsilon} \left( \sum_{i=1}^{n} \|\{u_i + v_i\} w_i\| \right)^2 \]
\[ \leq \frac{\epsilon}{2a} \|\Delta w_j\|^2 + \frac{2nC^2_s}{\epsilon} \left( \sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2\} \right) \sum_{j=1}^{n} \|w_j\|^2. \]

Here, \( \epsilon \) is an arbitrary positive number. Substituting \( I \) into (30), we get
\[ \frac{d}{dt} \|w_j\|^2(t) + (2 - \frac{2}{a} - \frac{\epsilon}{2a}) \|\Delta w_j\|^2(t) \leq \frac{2nC^2_s}{\epsilon} \left( \sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2\} \right) \sum_{j=1}^{n} \|w_j\|^2. \]
Taking \( \epsilon = \frac{\theta}{2} \) and summing up over \( j = 1, \ldots, n \), we transform (31) as follows:
\[ \frac{d}{dt} \sum_{j=1}^{n} \|w_j\|^2(t) \leq C \left[ \sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2 + \|u_i\|^2(t) + \|v_i\|^2(t)\} \right] \sum_{j=1}^{n} \|w_j\|^2, \quad i = 1, \ldots, n. \]

By (25) and Lemma 3.1, we have
\[ \|\Delta^2 u_i\|^2(t), \|\Delta^2 v_i\|^2(t) \in L^1(\mathbb{R}^+) \]
and
\[ \|u_i\|^2(t), \|v_i\|^2(t) \in L^1(\mathbb{R}^+), \quad i = 1, \ldots, n, \]
then by Lemma 2.2, it holds that
\[ \|w_j\|^2(t) \equiv 0 \quad j = 1, \ldots, n, \quad \text{for all} \quad t > 0. \]
Therefore,
\[ u_j(x,t) \equiv v_j(x,t); \quad j = 1, \ldots, n. \]
This completes the proofs of Lemma 3.3 and Theorem 3.1.

4. Conclusion

This paper is concerned with the formulation and solvability of initial-boundary value problems for the \( n \)-dimensional Kuramoto-Sivashinsky system (9)–(10) posed on smooth bounded domains, where \( n \in \{2,3,\ldots,7\} \). Theorem 3.1 contains results on existence and uniqueness of global regular solutions as well as exponential decay of the \( H^2(D_n) \)-norm, where \( D_n \) is a smooth bounded domain in \( \mathbb{R}^n \). We define a set of admissible domains, where destabilizing effects of terms \( \Delta u_j \) are damped by dissipativity of \( \Delta^2 u_j \) due to condition (17). This set contains “thin domains”, see [9, 13, 16], where some dimensions of \( D_n \) are small while others may be large. Since initial-boundary value problems studied in this paper do not admit a priori estimate independent of \( t, u_j \), in order to prove the existence of global regular solutions, we put conditions (18) connecting geometrical properties of \( D_n \) with initial data \( u_{j0} \). Moreover, Theorem 3.1 provides “smoothing effect”: initial data \( u_{j0} \in H^2(D_n) \cap H^1_0(D_n) \) imply that \( u_j \in L^2(\mathbb{R}; H^4(D_n)) \).

Acknowledgement

The author appreciates very much anonymous referees for their helpful comments.
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