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Research Article

Global regular solutions for the multi-dimensional Kuramoto-Sivashinsky equation posed on smooth domains

Nikolai A. Larkin*

Departamento de Matemática, Universidade Estadual de Maringá, Av. Colombo 5790: Agência UEM, 87020-900, Maringá, Parana, Brazil

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Abstract

Initial-boundary value problems for the n-dimensional Kuramoto-Sivashinsky equation posed on smooth bounded domains in $\mathbb{R}^n$ are considered, where n is a natural number from the interval $[2,7]$. The existence and uniqueness of global regular solutions as well as their exponential decay are established.

Keywords: global solutions; Kuramoto-Sivashinsky equation; decay in bounded domains.

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1. Introduction

This work concerns the existence, uniqueness, regularity, and exponential decay rates of solutions to initial-boundary value problems for the n-dimensional Kuramoto-Sivashinsky (KS) equation

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{1}$$

Here n is a natural number from the interval [2,7], Δ and ∇ are the Laplacian and the gradient in $\mathbb{R}^n$. In [10], Kuramoto studied the turbulent phase waves and Sivashinsky in [17] obtained an asymptotic equation which simulated the evolution of a disturbed plane flame front (see also [7]). Mathematical results on initial and initial-boundary value problems for the one-dimensional KS equation (1) are presented in [3,5,6,12,14,15,19,20]. The initial-value problem for the multi-dimensional KS type equations (1) was studied in [4,5]. Two-dimensional periodic problems for the KS equation and its modifications posed on rectangles were examined in [2,13,14,16,19], where some results on the existence of weak solutions and nonlinear stability were established. In [11], initial-boundary value problems for the 3D Kuramoto-Sivashinsky equation were studied; the existence, uniqueness, and exponential decay of global regular solutions were proved. For n dimensions, $x = (x_1, \dots, x_n)$, $n = 2,3,4,5,6,7$, Equation (1) can be rewritten in the form of the following system:

$$(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2}\sum_{i=1}^{n}(u_i)^2_{x_j} = 0,\ j = 1,\dots,n, \tag{2}$$

$$(u_i)_{x_j} = (u_j)_{x_i},\ j \neq i,\ \ i,j = 1,\dots,n. \tag{3}$$

where $u_j = (\phi)_{x_j},\ j = 1,\dots,n$. Let $\Omega_n = \prod_{i=1}^{n}(0, L_i)$ be the minimal nD parallelepiped containing a given smooth domain $\bar{D}_n$. The first essential problem that arises while one studies either (1) or (2)–(3), is concerned about the destabilizing effects of Δu_j; they may be damped by dissipative terms $\Delta^2 u_j$ provided D_n has some specific properties. In order to understand this, we use Steklov's inequalities to estimate

$$a\|u_j\|^2 \le \|\nabla u_j\|^2,\ a\|\nabla u_j\|^2 \le \|\Delta u_j\|^2;\ a = \sum_{i=1}^{n}\frac{\pi^2}{L_i^2},\ \ j = 1,\dots,n.$$

A simple analysis shows that if

$$1 - \frac{1}{a} > 0, \tag{4}$$

then $\Delta^2 u_j$ damp Δu_j. Naturally, here appear admissible domains where (4) is fulfilled; these are the so-called "thin domains", where some L_i are sufficiently small while others L_j may be large $i,j = 1,\dots,7;\ i \neq j$.

*E-mail addresses: nlarkine@uem.br, nlarkine@yahoo.com.br

The second essential problem is the presence of semi-linear terms in (2) which are interconnected. This does not allow to obtain the first estimate independent of u_j and leads to a connection between L_i and $u_j(0),\ i,j=1,\dots,7.$

The aim of this paper is to study n-dimensional initial-boundary value problems for (2)–(3) posed on smooth domains, where the existence and uniqueness of global regular solutions as well as their exponential decay of the $H^2(D_n)$-norm are established. A "smoothing effect" for solutions with respect to initial data is also obtained. Although, the cases $n=2,3$ are not new, we included them for the sake of generality.

The remaining part of this paper is organized as follows. Section 2 gives notations and auxiliary facts. In Section 3, formulation of an initial-boundary value problem in a smooth bounded domain D_n is given. The existence and uniqueness of global regular solutions, exponential decay of the $H^2(D_n)$-norm, and a "smoothing effect" are established also in Section 3. Section 4 consists of conclusion.

2. Notations and auxiliary facts

Let D_n be a sufficiently smooth domain in $\mathbb{R}^n$, where $n\in[2,7]$ is a fixed natural number, satisfying the Cone condition (see [1]) and $x=(x_1,\dots,x_n)\in D_n$. We use the standard notations of Sobolev spaces $W^{k,p}$, L^p and H^k for functions and the following notations for the norms [1] for scalar functions $f(x,t)$:

$$\|f\|^2=\int_{D_n}|f|^2dx,$$

$$\|f\|^p_{L^p(D_n)}=\int_{D_n}|f|^p\,dx,$$

$$\|f\|^p_{W^{k,p}(D_n)}=\sum_{0|\le\alpha|\le k}\|D^\alpha f\|^p_{L^p(D_n)},$$

$$\|f\|_{H^k(D_n)}=\|f\|_{W^{k,2}(D_n)}.$$

When $p=2$, $W^{k,p}(D_n)=H^k(D_n)$ is a Hilbert space with the scalar product

$$((u,v))_{H^k(D_n)}=\sum_{|j|\le k}(D^ju,D^jv),\ \ \|u\|_{L^\infty(D_n)}=ess\ sup_{D_n}|u(x)|.$$

We use the notation $H^k_0(D_n)$ to represent the closure of $C_0^\infty(D_n)$, the set of all C^∞ functions with compact support in D_n, with respect to the norm of $H^k(D_n)$.

Lemma 2.1 (Steklov's inequality [18]). *Let $v\in H^1_0(0,L)$. Then*

$$\frac{\pi^2}{L^2}\|v\|^2(t)\le\|v_x\|^2(t).$$

Lemma 2.2 (Differential form of the Gronwall Inequality). *Let $I=[t_0,t_1]$. Suppose that functions $a,b:I\to\mathbb{R}$ are integrable and a function $a(t)$ may be of any sign. Let $u:I\to\mathbb{R}$ be a differentiable function satisfying*

$$u_t(t)\le a(t)u(t)+b(t),\ \text{for } t\in I \text{ and }\ u(t_0)=u_0, \tag{5}$$

then

$$u(t)\le u_0e^{\int_{t_0}^t a(t)\,dt}+\int_{t_0}^t e^{\int_{t_0}^s a(r)\,dr}b(s)\,ds.$$

.

Proof. Multiply (5) by the integrating factor $e^{\int_{t_0}^s a(r)\,dr}$ and integrate from t_0 to t. □

The next Lemmas will be used in estimates.

Lemma 2.3 (See Theorem 9.1 in [8]). *Let n be a natural number from the interval $[2,7]$. If D_n is a sufficiently smooth bounded domain in $\mathbb{R}^n$ satisfying the cone condition and $v\in H^4(D_n)\cap H^1_0(D_n)$, then*

$$\sup_{D_n}|v(x)|\le C_n\|v\|_{H^4(D_n)}.$$

The constant C_n depends on n and D_n.

Lemma 2.4. *Let $f(t)$ be a continuous positive function and $f'(t)$ be a measurable integrable function such that*

$$f'(t) + (\alpha - kf^n(t))f(t) \le 0,\ t > 0,\ n \in \mathbb{N}, \tag{6}$$

$$\alpha - kf^n(0) > 0,\ k > 0, \tag{7}$$

then

$$f(t) < f(0) \tag{8}$$

for all $t > 0$.

3. KS equation posed on smooth domains

Let Ω_n be the minimal nD-parallelepiped containing a given bounded smooth domain $\bar{D_n} \in \mathbb{R}^n,\ n = 1, \dots, 7$:

$$\Omega_n = \{x \in \mathbb{R}^n; x_i \in (0, L_i)\},\ u_i = (\phi)_{x_i},\ i = 1, \dots, n.$$

Fix a natural number $n \in [2, 7]$ and consider in $Q_n = D_n \times (0, t)$ the following initial-boundary value problem:

$$(u_j)_t + \Delta^2 u_j + \Delta u_j + \frac{1}{2}\sum_{i=1}^n (u_i^2)_{x_j} = 0,\ j = 1, \dots, n, \tag{9}$$

$$(u_i)_{x_j} = (u_j)_{x_i},\ j \ne i,\ \ i, j = 1, \dots, n; \tag{10}$$

$$u_j|_{\partial D_n} = \Delta u_j|_{\partial D_n} = 0,\ t > 0, \tag{11}$$

$$u_j(x, 0) = u_{j0}(x),\ j = 1, \dots, n,\ \ x \in D_n. \tag{12}$$

Lemma 3.1. *If $f \in H^4(D_n) \cap H_0^1(D_n)$ and $\Delta f|_{\partial D_n} = 0$, then*

$$a\|f\|^2 \le \|\nabla f\|^2,$$

$$a^2\|f\|^2 \le \|\Delta f\|^2,$$

$$a\|\nabla f\|^2 \le \|\Delta f\|^2,$$

$$a^2\|\Delta f\|^2 \le \|\Delta^2 f\|^2,$$

$$\|\Delta \nabla f\|^2 \le \|\Delta^2 f\|\|\Delta f\| \le \frac{1}{a}\|\Delta^2 f\|^2.$$

where

$$a = \sum_{i=1}^n \frac{\pi^2}{L_i^2},$$

and

$$\|f\|^2 = \int_{D_n} f^2(x)dx.$$

Proof. We have

$$\|\nabla f\|^2 = \sum_{i=1}^n \|f_{x_i}\|^2.$$

Define

$$\tilde{f}(x, t) = \begin{cases} f(x, t) & \text{if } x \in D_n; \\ 0 & \text{if } x \in \Omega_n / D_n. \end{cases} \tag{13}$$

Making use of Steklov's inequalities for $\tilde{f}(x, t)$ and taking into account that $\|\nabla f\| = \|\nabla \tilde{f}\|$, we get

$$\|\nabla f\|^2 \ge a\|f\|^2,\ \text{where } a = \sum_{i=1}^n \frac{\pi^2}{L_i^2}.$$

On the other hand,

$$a\|f\|^2 \le \|\nabla f\|^2 = -\int_{D_n} f\Delta f dx \le \|\Delta f\|\|f\|.$$

This implies

$$a\|f\| \le \|\Delta f\| \text{ and } a^2\|f\|^2 \le \|\Delta f\|^2.$$

Consequently, $a\|\nabla f\|^2 \le \|\Delta f\|^2$. Similarly,

$$\|\Delta f\|^2 = \int_{D_n} f\Delta^2 f dx \le \|\Delta^2 f\|\|f\| \le \frac{1}{a}\|\Delta^2 f\|\|\Delta f\|.$$

Hence, $a\|\Delta f\| \le \|\Delta^2 f\|$. Moreover,

$$\|\Delta\nabla f\|^2 = -\int_{D_n} \Delta^2 f \Delta f dx \le \|\Delta^2 f\|\|\Delta f\| \le \frac{1}{a}\|\Delta^2 f\|^2.$$

□

Remark 3.1. *Assertions of Lemma 3.1 are true if the function f is replaced respectively by $u_j,\ j = 1, \dots, n$.*

Lemma 3.2. *In conditions of Lemma 3.1, the following inequalities hold*

$$\|f\|^2(t)_{H^2(D_n)} \le 3\|\Delta f\|^2(t), \tag{14}$$

$$\|f\|^2(t)_{H^4(D_n)} \le 5\|\Delta^2 f\|^2(t), \tag{15}$$

$$\sup_{D_n} |f(x)| \le C_s\|\Delta^2 f\|, \text{ where } C_s = 5C_n. \tag{16}$$

Proof. To prove (15), we make use of Lemma 3.1 and find

$$\|f\|^2_{H^4(D_n)} = \|f\|^2 + \|\nabla f\|^2 + \|\Delta f\|^2 + \|\nabla\Delta f\|^2 + \|\Delta^2 f\|^2$$
$$\le \left(\frac{1}{a^4} + \frac{1}{a^3} + \frac{1}{a^2} + \frac{1}{a} + 1\right)\|\Delta^2 f\|^2.$$

Since $a > 1$, then (15) follows. Similarly, (14) can be proved. Moreover, taking into account Lemma 2.3, we get (16). □

Theorem 3.1 (Special basis)**.** *Let $n \in \{2, 3, \dots, 7\}$ and $D_n \in \mathbb{R}^n$ be a bounded smooth domain satisfying the Cone condition. Let Ω_n be a minimal nD-parallelepiped containing $\bar{D_n}$ and*

$$\theta = 1 - \frac{1}{a} = 1 - \frac{1}{\sum_{i=1}^n \frac{\pi^2}{L_i^2}} > 0. \tag{17}$$

Given

$$u_{j0}(D_n) \in H^2(D_n) \cap H^1_0(D_n),\ j = 1, \dots, n$$

such that

$$\theta - \frac{2C_s^2 7^3}{a\theta}\left(\sum_{j=1}^n \|\Delta u_j\|^2(0)\right) > 0, \tag{18}$$

then there exists a unique global regular solution to (9)–(12):

$$u_j \in L^\infty(\mathbb{R}^+; H^2(D_n)) \cap L^2(\mathbb{R}^+; H^4(D_n) \cap H^1_0(D_n));$$

$$u_{jt} \in L^2(\mathbb{R}^+; L^2(D_n)),\ j = 1, \dots, n.$$

Moreover,

$$\sum_{j=1}^n \|\Delta u_j\|^2(t) \le \left(\sum_{j=1}^n \|\Delta u_{j0}\|^2\right)\exp\{-a^2 t\theta/2\} \tag{19}$$

and

$$\sum_{i=1}^n \|\Delta u_i\|^2(t) + \int_0^t \sum_{i=1}^n \|\Delta^2 u_i\|^2(\tau)d\tau \le C\sum_{i=1}^n \|\Delta u_{i0}\|^2,\ t > 0.$$

Remark 3.2. *In Theorem 3.1, there are two types of restrictions: the first one is pure geometrical,*

$$1-\frac{1}{a}>0$$

which is needed to eliminate destabilizing effects of the terms Δu_j in (9):

$$\|\Delta u_j\|^2-\|\nabla u_j\|^2.$$

It is clear that

$$\lim_{L_i\to 0} a=\sum_{i=1}^n \frac{\pi^2}{L_i^2}=+\infty,$$

hence to achieve (17), it is possible to decrease $L_i,\ i=1,\dots,n$ allowing other $L_j,\ j\neq i$ to grow. The situation with condition (18) is more complicated: if initial data are not small, then it is possible either to decrease $L_i,\ i=1,\dots,n$, to fulfill this condition or for fixed $L_i,\ i=1,\dots,n$ to decrease initial data $\|u_{j0}\|$.

Proof of Theorem 3.1. It is possible to construct Galerkin‘s approximations to (9)–(12) by the following way. Let $w_j(x)$ be eigenfunctions of the problem:

$$\Delta^2 w_j-\lambda_j w_j=0 \text{ in } D_n;\quad w_j|_{\partial D_n}=\Delta w_j|_{\partial D_n}=0, j=1,2,\dots.$$

Define

$$u_j^N(x,t)=\sum_{k=1}^N g_k^j(t)w_j(x).$$

Unknown functions $g_i^j(t)$ satisfy the following initial problems:

$$\left(\frac{d}{dt}u_j^N,w_j\right)(t)+\left(\Delta^2 u_j^N,w_j\right)(t)+\left(\Delta u_j^N,w_j\right)(t)+\frac{1}{2}\left(\sum_{i=1}^n (u_i^N)^2_{x_j},w_j\right)(t)=0,$$

$$g_k^j(0)=g_{0k}^j,\quad j=1,\dots,n,\quad k=1,2,\dots.$$

The estimates that follow may be established on Galerkin's approximations (see [5, 6]), but it is more explicitly to prove them on smooth solutions of (9)-(12).

Estimate I: $u\in L^\infty(\mathbb{R}^+;H^2(D_n)\cap H_0^1(D_n))\cap L^2(\mathbb{R}^+;H^4(D_n)\cap H_0^1(D_n))$.
For any natural number $n\in[2,7]$, multiply (9) by $2\Delta^2 u_j$ to obtain

$$\frac{d}{dt}\|\Delta u_j\|^2(t)+2\|\Delta^2 u_j\|^2(t)+2\|\Delta^2 u\|(t)\|\Delta u_j\|(t)+2\sum_{i=1}^n (u_i(u_i)_{x_j},\Delta^2 u_j)(t)=0. \tag{20}$$

Making use of (15) and Lemmas 2.3, 3.1, 3.2, we write

$$\begin{aligned}\frac{d}{dt}\|\Delta u_j\|^2(t)+2\theta\|\Delta^2 u_j\|^2(t) &\le 2\left[\sum_{i=1}^n \sup_{D_n}|u_i(x,t)|\|\nabla u_i\|(t)\right]\|\Delta^2 u_j\|(t)\\ &\le 2\left[C_s\sum_{i=1}^n \|\Delta^2 u_i\|(t)\|\nabla u_i\|(t)\right]\|\Delta^2 u_j\|(t); j=1,\dots,n.\end{aligned} \tag{21}$$

Summing over $j=1,\dots,n$ and making use of Lemma 3.1, we rewrite (20) in the form:

$$\begin{aligned}\frac{d}{dt}\sum_{j=1}^n\|\Delta u_j\|^2(t)+2\theta\sum_{j=1}^n\|\Delta^2 u_j\|(t) &\le 2C_s n\left(\sum_{j=1}^n\|\nabla u_j\|(t)\right)\left[\sum_{j=1}^n\|\Delta^2 u_j\|^2(t)\right]\\ &\le\left[\frac{\theta}{2}+\frac{2C_s^2 n^2}{\theta}\left(\sum_{j=1}^n\|\nabla u_j\|(t)\right)^2\right]\sum_{j=1}^n\|\Delta^2 u_j\|^2(t)\\ &\le\left[\frac{\theta}{2}+\frac{2C_s^2 n^3}{\theta}\left(\sum_{j=1}^n\|\nabla u_j\|^2(t)\right)\right]\sum_{j=1}^n\|\Delta^2 u_j\|^2(t)\end{aligned}$$

$$\leq \left[\frac{\theta}{2} + \frac{2C_s^2 n^3}{a\theta}\left(\sum_{j=1}^{n}\|\Delta u_j\|^2(t)\right)\right]\sum_{j=1}^{n}\|\Delta^2 u_j\|^2(t).$$

Taking this into account, we transform (20) in the form

$$\frac{d}{dt}\sum_{j=1}^{n}\|\Delta u_j\|^2(t) + \frac{\theta}{2}\sum_{j=1}^{n}\|\Delta^2 u_j\|^2(t) + \left[\theta - \frac{2C_s^2 n^3}{a\theta}\left(\sum_{j=1}^{n}\|\Delta u_j\|^2(t)\right)\right]\sum_{j=1}^{n}\|\Delta^2 u_j\|^2(t) \leq 0. \tag{22}$$

Condition (18) and Lemma 2.4 guarantee that

$$\theta - \frac{2C_s^2 n^3}{a\theta}\left(\sum_{j=1}^{n}\|\Delta u_j\|^2(t)\right) > 0, \quad t > 0.$$

Hence, (21) can be rewritten as

$$\frac{d}{dt}\sum_{j=1}^{n}\|\Delta u_j\|^2(t) + \frac{a^2\theta}{2}\sum_{j=1}^{n}\|\Delta u_j\|^2(t) \leq 0. \tag{23}$$

Integrating, we get

$$\sum_{i=1}^{n}\|\Delta u_j\|^2(t) \leq \sum_{j=1}^{n}\|\Delta u_{j0}\|^2\exp\{-a^2\theta t/2\}. \tag{24}$$

Consequently, we find

$$\sum_{i=1}^{n}\|\Delta u_i\|^2(t) + \int_0^t\sum_{i=1}^{n}\|\Delta^2 u_i\|^2(\tau)d\tau \leq C\sum_{i=1}^{n}\|\Delta u_{i0}\|^2. \tag{25}$$

Finally, directly from (9), we obtain

$$(u_j)_t \in L^2(\mathbb{R}^+; L^2(D_n)), \ j = 1, \ldots, n.$$

Since these inclusions, estimates (24), (25) and Lemma 3.2 do not depend on N, the same estimates are valid also for $u_j^N(x,t)$. Hence, it is possible to pass to the limit as $N \to +\infty$ in

$$u_j^N(x,t) = \sum_{k=1}^{N} g_k^j(t) w_j(x)$$

and to prove the existence part of Theorem 3.1.

Lemma 3.3. *A regular solution of Theorem 3.1 is uniquely defined.*

Proof. Let u_j and v_j, $j = 1, \ldots, n$, be two distinct solutions to (9)–(12). Denoting $w_j = u_j - v_j$, we come to the following system:

$$(w_j)_t + \Delta^2 w_j + \Delta w_j + \frac{1}{2}\sum_{i=1}^{n}\left(u_i^2 - v_i^2\right)_{x_j} = 0, \tag{26}$$

$$(w_j)_{x_i} = (w_i)_{x_j}, \ i \neq j, \tag{27}$$

$$w_j|_{\partial D_n} = \Delta w_j|_{\partial D_n} = 0, \ t > 0; \tag{28}$$

$$w_j(x,0) = 0 \text{ in } D, \quad j = 1, \ldots, n. \tag{29}$$

Multiply (26) by $2w_j$, we obtain

$$\frac{d}{dt}\|w_j\|^2(t) + 2\|\Delta w_j\|^2(t) - 2\|\nabla w_j\|^2(t) - \sum_{i=1}^{n}\left(\{u_i + v_i\}w_i, (w_j)_{x_j}\right)(t) = 0, \ j = 1, \ldots, n. \tag{30}$$

Making use of Lemmas 2.3 and 3.1, 3.2, we estimate

$$\begin{aligned} I &= \sum_{i=1}^{n} \left(\{u_i + v_i\} w_i, (w_j)_{x_j} \right) \\ &\leq \frac{\epsilon}{2} \|\nabla w_j\|^2 + \frac{1}{2\epsilon} \left(\sum_{i=1}^{n} \|\{u_i + v_i\} w_i\| \right)^2 \\ &\leq \frac{\epsilon}{2a} \|\Delta w_j\|^2 + \frac{2}{\epsilon} \sum_{i=1}^{n} \sup_{D_n} \{u_i^2(x,t)_i + v_i^2(x,t)\} \|w_i^2\|(t) \\ &\leq \frac{\epsilon}{2a} \|\Delta w_j\|^2 + \frac{2nC_s^2}{\epsilon} \left[\sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2\} \right] \sum_{j=1}^{n} \|w_j\|^2. \end{aligned}$$

Here, ϵ is an arbitrary positive number. Substituting I into (30), we get

$$\frac{d}{dt} \|w_j\|^2(t) + (2 - \frac{2}{a} - \frac{\epsilon}{2a}) \|\Delta w_j\|^2(t) \leq \frac{2nC_s^2}{\epsilon} \left[\sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2\} \right] \sum_{j=1}^{n} \|w_j\|^2. \tag{31}$$

Taking $\epsilon = \frac{\theta}{2}$ and summing up over $j = 1, \dots, n$, we transform (31) as follows:

$$\frac{d}{dt} \sum_{j=1}^{n} \|w_j\|^2(t) \leq C \left[\sum_{i=1}^{n} \{\|\Delta^2 u_i\|^2 + \|\Delta^2 v_i\|^2 + \|u_i\|^2(t) + \|v_i\|^2(t)\} \right] \sum_{j=1}^{n} \|w_j\|^2, \ i = 1, \dots, n.$$

By (25) and Lemma 3.1, we have

$$\|\Delta^2 u_i\|^2(t), \|\Delta^2 v_i\|^2(t) \in L^1(\mathbb{R}^+)$$

and

$$\|u_i\|(t), \|v_i\|^2(t) \in L^1(\mathbb{R}^+), \ i = 1, \dots, n,$$

thence by Lemma 2.2, it holds that

$$\|w_j\|^2(t) \equiv 0 \ \ j = 1, \dots, n, \ \text{for all} \ \ t > 0.$$

Therefore,

$$u_j(x,t) \equiv v_j(x,t); j = 1, \dots, n.$$

This completes the proofs of Lemma 3.3 and Theorem 3.1. □

□

4. Conclusion

This paper is concerned with the formulation and solvability of initial-boundary value problems for the n-dimensional Kuramoto-Sivashinsky system (9)–(10) posed on smooth bounded domains, where $n \in \{2, 3, \dots, 7\}$. Theorem 3.1 contains results on existence and uniqueness of global regular solutions as well as exponential decay of the $H^2(D_n)$-norm, where D_n is a smooth bounded domain in $\mathbb{R}^n$. We define a set of admissible domains, where destabilizing effects of terms Δu_j are damped by dissipativity of $\Delta^2 u_j$ due to condition (17). This set contains "thin domains", see [9, 13, 16], where some dimensions of D_n are small while others may be large. Since initial-boundary value problems studied in this paper do not admit a priori estimate independent of t, u_j, in order to prove the existence of global regular solutions, we put conditions (18) connecting geometrical properties of D_n with initial data u_{j0}. Moreover, Theorem 3.1 provides "smoothing effect": initial data $u_{j0} \in H^2(D_n) \cap H_0^1(D_n)$ imply that $u_j \in L^2(\mathbb{R}^; H^4(D_n))$.

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