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Research Article

Absolutely and conditionally zonal graphs

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Abstract

A zonal labeling of a plane graph G is an assignment of the two nonzero elements of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of G such that the sum of the labels of the vertices on the boundary of each region of G is the zero element of $\mathbb{Z}_3$. A plane graph possessing such a labeling is a zonal graph. A planar graph G is zonal if there exists a zonal planar embedding of G. If every planar embedding of G is zonal, then G is absolutely zonal. A zonal planar graph G is conditionally zonal if some planar embedding of G is not zonal. It is shown that there is a class of absolutely zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings as well as a class of conditionally zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings with prescribed irregularity and regularity properties.

Keywords: planar graph; graph embedding; zonal labeling; conditionally and absolutely zonal graph; irregularity and regularity.

2020 Mathematics Subject Classification: 05C10, 05C15, 05C78.

1. Introduction

In 2014 Cooroo Egan introduced a vertex labeling of plane graphs (planar graphs embedded in the plane) called a zonal labeling (see [2]). A *zonal labeling* ℓ of a plane graph G is an assignment of the two nonzero elements 1 and 2 of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of G such that the sum of the labels of the vertices on the boundary of each region (zone) of G, called the *value* of the region, is the zero element in $\mathbb{Z}_3$. If a connected plane graph G possesses a zonal labeling, then G is a *zonal graph*. The plane graph G_1 of Figure 1 is zonal and a zonal labeling of G_1 is given in that figure, while the plane graph G_2 of Figure 1 is not zonal.

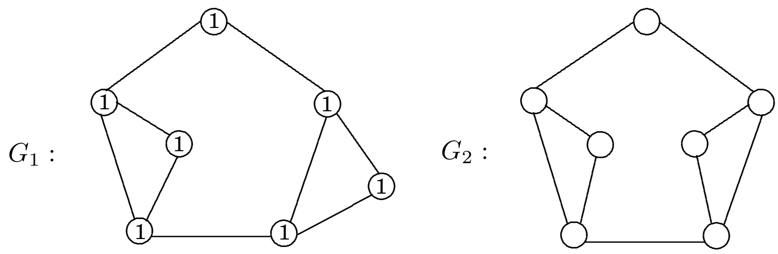

Figure 1: A zonal plane graph and a non-zonal plane graph.

A planar graph G is *zonal* if there exists a zonal planar embedding of G. If every planar embedding of G is zonal, then G is *absolutely zonal*. For example, if G is a maximal planar graph of order 3 or more embedded in the plane, then the boundary of every region of G is a triangle. Thus, the labeling that assigns the label 1 to every vertex of G is a zonal labeling. Therefore, every maximal planar graph of order 3 or more is absolutely zonal. A zonal planar graph G is *conditionally zonal* if some planar embedding of G is not zonal. Since the graphs G_1 and G_2 of Figure 1 are isomorphic, it follows that G_2 is a different planar embedding of G_1 and so the planar graph G_1 (or G_2) is conditionally zonal.

It is the goal of this paper to describe (1) a class of absolutely zonal graphs having an arbitrarily large number of distinct planar embeddings and (2) a class of conditionally zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings with prescribed irregularity or regularity properties. Before doing this, however, we review some information concerning zonal graphs, mentioned in [2], that illustrates some of the interest in studying zonal labelings. A *cubic map* is a connected bridgeless cubic plane graph. The following result was obtained in [2].

Theorem 1.1. *A connected cubic plane graph G is zonal if and only if G is bridgeless.*

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Consequently, by Theorem 1.1, the only zonal cubic plane graphs are cubic maps. That every cubic map is zonal was established in [2] with the aid of the Four Color Theorem (the chromatic number of every planar graph is at most 4). The converse of this statement is also true (see [2]).

Theorem 1.2. *If every cubic map is zonal, then the chromatic number of every planar graph is at most* 4*.*

Thus, if it could be shown that every cubic map is zonal without using the Four Color Theorem, then the Four Color Theorem would follow. This shows that studying zonal labelings of planar graphs are of interest, especially cubic planar graphs, and cubic maps in particular.

2. Absolutely zonal graphs

The following result was shown in [2].

Proposition 2.1. *Every nontrivial tree and every cycle is zonal.*

Since there is only one planar embedding of a nontrivial tree or a cycle, it follows that every nontrivial tree and every cycle is absolutely zonal. A nontrivial tree is connected but not 2-connected, while a cycle is 2-connected but not 3-connected. There is a class of 2-connected absolutely zonal graphs, as we show next.

Proposition 2.2. *Every* 2*-connected bipartite planar graph is absolutely zonal.*

Proof. Let G be a 2-connected bipartite plane graph with partite sets U and W. Define a labeling ℓ of G by assigning the label 1 to every vertex of U and the label 2 to every vertex of W. Let R be a region of G. Since G is a 2-connected bipartite plane graph, the boundary of R is an even cycle C. Thus, half of the vertices of C are labeled 1 and half are labeled 2. Hence, $\sum_{v\in V(C)} \ell(v) = 0$ in $\mathbb{Z}_3$ and so ℓ is a zonal labeling of G. Consequently, G is absolutely zonal. □

Whitney [3] obtained the following result on 3-connected planar graphs.

Theorem 2.1. **(Whitney's Theorem)** *Every* 3*-connected planar graph is uniquely embeddable in the plane.*

As a consequence of Theorem 2.1, every 3-connected planar graph is either absolutely zonal or non-zonal. Every wheel $W_n = C_n \vee K_1$ (the join of C_n and K_1), $n \geq 3$, is 3-connected. This observation gives rise to an infinite class of 3-connected absolutely zonal graphs and an infinite class of 3-connected non-zonal graphs. To establish this fact, we first state a definition. Let ℓ be a labeling of the vertices of a graph G with the labels 1 and 2 of $\mathbb{Z}_3$. The vertex labeling $\overline{\ell}$ of G defined by $\overline{\ell}(v) = 3 - \ell(v)$ for each vertex v of G is called the *complementary labeling* of G. The following is then immediate (see [2]).

Observation 2.1. *If* ℓ *is a zonal labeling of a connected plane graph, then so too is its complementary labeling* $\overline{\ell}$*.*

We now present the result on wheels indicated above.

Theorem 2.2. *For an integer* $n \geq 3$*, the wheel* $W_n = C_n \vee K_1$ *is zonal if and only if* $n \equiv 0 \pmod 3$*.*

Proof. Since W_n is a 3-connected planar graph, there is a unique planar embedding of W_n where, in a standard planar embedding of W_n, the boundary of every interior region of W_n is a triangle and the boundary of the exterior region is C_n. If $n \equiv 0 \pmod 3$, then the labeling of W_n that assigns the label 1 of $\mathbb{Z}_3$ to every vertex of W_n is a zonal labeling and so W_n is zonal. For the converse, let $n \geq 4$ be an integer such that $n \not\equiv 0 \pmod 3$. If there exists a zonal labeling ℓ of W_n, then ℓ must assign the same label to the three vertices of each triangle of W_n, implying that ℓ must assign the same label to all vertices of W_n. By Observation 2.1, we may assume that ℓ assigns the label 1 to every vertex of W_n. Hence, $\sum_{v\in V(C_n)} \ell(v) = n \neq 0$ in $\mathbb{Z}_3$. Since C_n is the boundary of a region of W_n, it follows that ℓ is not a zonal labeling and so W_n is not zonal. □

We mentioned that every connected bridgeless cubic planar graph is absolutely zonal. If a bridgeless cubic planar graph G is 3-connected, then it follows by Whitney's Theorem (Theorem 2.1) that there is a unique planar embedding of G. If a bridgeless cubic planar graph G is 2-connected but not 3-connected (that is, G has connectivity 2), then G may have two or more distinct planar embeddings, giving rise to distinct cubic maps. Since each cubic map is zonal, the graph G itself is absolutely zonal. In fact, there is a class of connected bridgeless cubic planar graphs having an arbitrarily large number of distinct planar embeddings.

Theorem 2.3. *For every positive integer* k*, there exists a connected bridgeless cubic planar graph having at least* k *distinct (zonal) planar embeddings.*

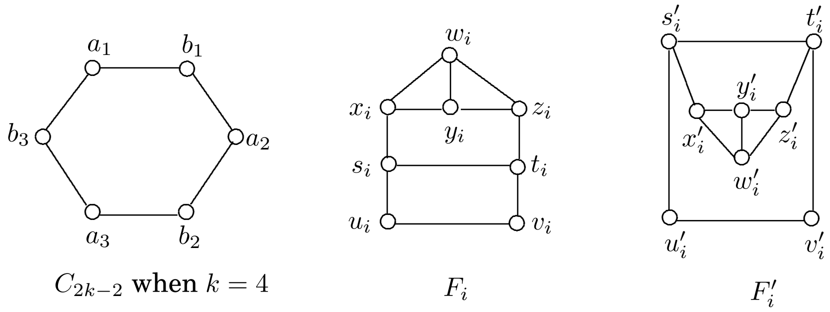

Figure 2: Three plane graphs.

Proof. For $k \geq 3$, ler $C_{2k-2} = (a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}, a_1)$ be a cycle of order $2k-2$. This cycle is shown in Figure 2 for $k = 4$. For $1 \leq i \leq k-1$, let F_i and F'_i be the planar embedding of a planar graph, also shown in Figure 2. Since the boundaries of the five regions of F_i and F'_i are the same, these are the same embeddings of a planar graph.

First, we construct a plane graph G_0 from C_{2k-2} and $F_1, F_2, \ldots, F_{k-1}$ by identifying the edge a_ib_i of C_{2k-2} with the edge u_iv_i of F_i for $1 \leq i \leq k-1$. Next, we embed G_0 in the plane such that (1) each interior region of G_0 is either an interior region of F_i for some integer i with $1 \leq i \leq k-1$ or the region whose boundary is C_{2k-2} and (2) the boundary of the exterior region of G_0 is the cycle of order $(2k-2) + 5(k-1) = 7k-7$. Hence, each F_i $(1 \leq i \leq k-1)$ lies outside of the cycle C_{2k-2}. The graph G_0 is shown in Figure 3 for $k = 4$. Observe that G_0 is a cubic map and so G_0 is zonal.

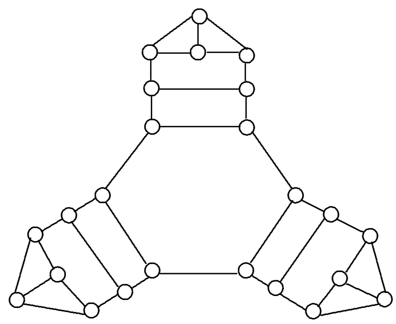

Figure 3: The plane graph G_0 for $k = 4$.

For an integer j with $1 \leq j \leq k-2$, we construct a graph G_j from C_{2k-2}, $F'_1, F'_2, \ldots, F'_j$ and $F_{j+1}, F_{j+2}, \ldots, F_{k-1}$ by identifying the edge $u'_iv'_i$ of F'_i with the edge a_ib_i of C_{2k-2} for $1 \leq i \leq j$ and the edge u_iv_i of F_i with the edge a_ib_i of C_{2k-2} for $j+1 \leq i \leq k-1$. We then embed G_j in the plane such that (1) each interior region of G_j is either an interior region of F'_i for $1 \leq i \leq j$, or an interior region of F_i for $j+1 \leq i \leq k-1$, or the region whose boundary is C_{2k-2} and (2) the boundary of the exterior region of G_j is the cycle of order $(2k-2) + 2j + 5(k-1-j) = 7k-7-3j$. Hence, each of F'_i $(1 \leq i \leq j)$ and F_i $(j+1 \leq i \leq k-1)$ lies outside of C_{2k-2}. The graph G_2 is shown in Figure 4 for $k = 4$. Observe that each plane graph G_j is a cubic map and so G_j is zonal for $1 \leq j \leq k-2$.

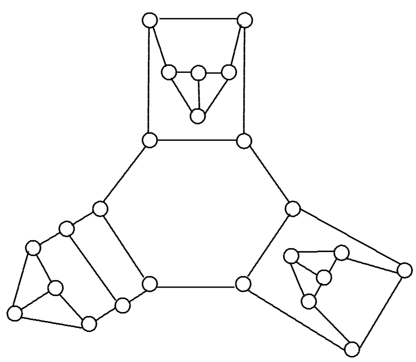

Figure 4: The plane graph G_2 for $k = 4$.

The plane graphs $G_0, G_1, \ldots, G_{k-1}$ are all cubic maps and are planar embeddings of the same graph. Since G_i, $0 \leq i \leq k-1$, has exactly $k-1-i$ regions whose boundary is a 4–cycle and whose largest boundary cycle has order $7k-7-3i$ for $0 \leq i \leq k-1$, these cubic maps are distinct planar embeddings of the same graph. $\square$

3. Conditionally zonal graphs

We now turn our attention to conditionally zonal graphs, that is, planar graphs having at least one zonal planar embedding and at least one non-zonal planar embedding. By Whitney's Theorem, necessarily each such graph must have connectivity less than 3. We now describe a class of connected bridgeless planar graphs. For integers $n \geq 3$ and $k \geq 3$, the (standard) *Dutch windmill graph* D_n^k is the graph obtained by taking k copies of the n-cycle C_n with a vertex in common. The Dutch windmill graph D_3^k is commonly called a *friendship graph* (every two vertices have a unique common neighbor). The Dutch windmill graphs D_3^3, D_4^3, and D_5^3 are shown in Figure 5, the first of which is a friendship graph. The planar embedding of D_n^k in which the boundary of each region is either C_n or D_n^k is called the *standard embedding* of D_n^k, as shown in Figure 5 for D_3^3, D_4^3 and D_5^3.

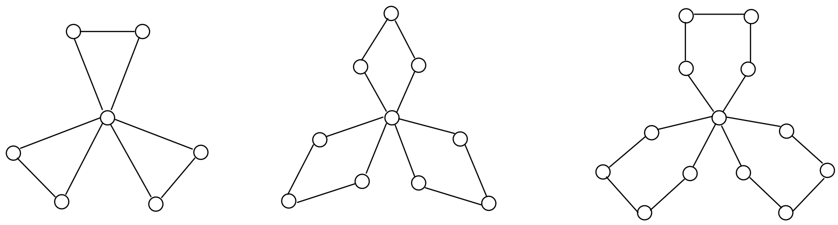

Figure 5: The Dutch windmill graphs D_3^3, D_4^3, and D_5^3.

For an integer $k \geq 2$, let S denote a multiset of k cycles and let $D(S)$ denote the Dutch windmill graph constructed from the cycles in S. The planar embedding of $D(S)$ in which the boundary of each region is either a cycle in S or $D(S)$ is called the *standard embedding* of $D(S)$. For example, if $S = \{C_3, C_3, C_4, C_6\}$, then the standard planar embedding of $D(S)$ is shown in Figure 6. This plane graph $D(S)$ is zonal and a zonal labeling of $D(S)$ is given in Figure 6.

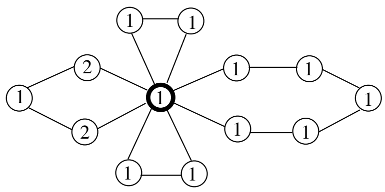

Figure 6: The Dutch windmill graph $D(S)$ for $S = \{C_3, C_3, C_4, C_6\}$.

For a labeling $\ell : V(G) \to \{1, 2\}$ of a graph G and a subgraph H of G, let $\sum(\ell, H) = \sum_{x \in V(H)} \ell(x)$ in $\mathbb{Z}_3$. First, we show that there is a class of non-zonal Dutch windmill graphs.

Proposition 3.1. *For every multiset S of two cycles, the Dutch windmill graph $D(S)$ is not zonal.*

Proof. Let $S = \{C, C'\}$ and let $D(S)$ be the Dutch windmill graph constructed from the two cycles C and C' in S. Assume, to the contrary, that $D(S)$ is zonal. Then there exists a planar embedding G of $D(S)$ having a zonal labeling ℓ. The plane graph G has three regions whose boundaries are C, C' and G. Since each of C and C' is the boundary of a region of G, it follows that $\sum(\ell, C) = \sum_{v \in V(C)} \ell(v) = 0$ and $\sum(\ell, C') = \sum_{v \in V(C')} \ell(v) = 0$. Let u be the vertex belonging to both C and C'. Then the value of the boundary G of the third region is $\left[\sum(\ell, C) + \sum(\ell, C')\right] - \ell(u) = 0 + 0 - \ell(u) \neq 0$ in $\mathbb{Z}_3$, which is a contradiction. □

Next, we show that for every multiset S of three or more cycles, the Dutch windmill graph $D(S)$ is conditionally zonal. It is convenient to introduce some additional notation. For $p \geq 2$ graphs $H_1, H_2, \ldots, H_p$, let v_i be the vertex of H_i for $1 \leq i \leq p$. Then $H_1 \star H_2 \star \cdots \star H_p$ denotes the plane graph constructed from $H_1, H_2, \ldots, H_p$ by identifying the p vertices $v_1, v_2, \ldots, v_p$ and denoting the identified vertex by v. For example, if $S_4 = \{C_3, C_3, C_4, C_6\}$, then $D(S_4) = C_3 \star C_3 \star C_4 \star C_6$ shown in Figure 6. The following elementary lemma will be useful to us.

Lemma 3.1. *Let X be a nonempty set of vertices of a graph.*

(1) *For each $i = 1, 2$, there is a labeling $\ell_i : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of X such that $\sum(\ell_i, X) = i$ in $\mathbb{Z}_3$.*

(2) *If $|X| \geq 2$, then there is a labeling $\ell_0 : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of X such that $\sum(\ell_0, X) = 0$ in $\mathbb{Z}_3$.*

Theorem 3.1. *For every multiset S of three or more cycles, the Dutch windmill graph $D(S)$ is conditionally zonal.*

Proof. Let S be a multiset of $k \geq 3$ cycles. We consider two cases, according to whether $k \equiv 1 \pmod 3$ or $k \not\equiv 1 \pmod 3$.

Case 1. $k \equiv 1 \pmod 3$. First, we show that $D(S)$ is zonal. Let G be the standard planar embedding of $D(S)$ such that the boundary of each region of G is either a cycle $C \in S$ or the graph G. We show that the plane graph G is zonal. Let u be the center of G, that is, $\deg_G u = 2k$. Since every cycle $C \in S$ is zonal, there is a zonal labeling $\ell_C : V(C) \to \{1, 2\}$ of C. By Observation 2.1, we may assume that $\ell_C(u) = 1$ for every cycle $C \in S$. Since $\sum(\ell_C, C) = 0$ and $\ell_C(u) = 1$ in $\mathbb{Z}_3$, where $C \in S$, it follows that $\sum(\ell_C, C - u) = 2$ in $\mathbb{Z}_3$. Define a labeling $\ell : V(G) \to \{1, 2\}$ of G by $\ell(v) = \ell_C(v)$ if v belongs to a cycle $C \in S$. Let B be the boundary of a region in G. If $B = C \in S$, then $\sum(\ell, C) = \sum(\ell_C, C) = 0$ in $\mathbb{Z}_3$. Thus, we may assume that $B = G$. Since $k \geq 4$ and $k \equiv 1 \pmod 3$, it follows that $k = 3t + 1$ for some integer $t \geq 1$. Hence,

$$\sum(\ell, B) = 1 + \sum_{C \in S} \sum(\ell_C, C - u) = 1 + 2k = 1 + 2(3t + 1) = 0 \text{ in } \mathbb{Z}_3.$$

Therefore, ℓ is a zonal labeling of G and so $D(S)$ is zonal.

Next, we show that $D(S)$ is conditionally zonal. Let C_1 and C_2 be two cycles in S and let H be the planar embedding of $D(S)$ by placing the cycle C_2 within the cycle C_1 of $D(S)$. This is illustrated in Figure 7. Thus, if B is the boundary of a region in H, then either $B = C \in S - \{C_1\}$, $B = C_1 \star C_2$ (consisting of C_1 and C_2 with common vertex u), or $B = H - V(C_2 - u)$.

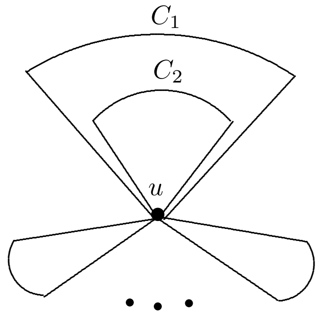

Figure 7: The planar embedding H of $D(S)$.

We claim that the plane graph H is not zonal, for suppose that H has a zonal labeling ℓ. By Observation 2.1, we may assume that $\ell(u) = 1$. This implies that (1) $\sum(\ell, C - u) = 2$ in $\mathbb{Z}_3$ for each cycle $C \in S - \{C_1\}$, (2) $\sum(\ell, C_1 - u) + \sum(\ell, C_2) = 0$ and so $\sum(\ell, C_1 - u) = 0$, and (3) $1 + \sum_{C \in S - \{C_2\}} \sum(\ell, C - u) = 0$. However, since

$$\sum(\ell, C_1 - u) = 0,$$

it follows that for the boundary $B = H - V(C_2 - u)$ of a region of H, we have

$$\sum(\ell, B) = 1 + \sum_{C \in S - \{C_2\}} \sum(\ell_C, C - u) = 1 + 2(k - 2) = 1 + 2(3t - 1) = 2 \text{ in } \mathbb{Z}_3,$$

which is a contradiction. Thus, H is not zonal and so $D(S)$ is conditionally zonal.

Case 2. $k \not\equiv 1 \pmod 3$. Let $S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\}$ be a set of k cycles of length n_i for $1 \leq i \leq k$. First, we show that $D(S)$ has a zonal planar embedding. We consider two subcases, according to whether k is odd or k is even.

Subcase 2.1. *$k \geq 3$ is odd.* Define the planar embedding $\tilde{D}(S)$ of $D(S)$ such that C_{n_i} is placed inside $C_{n_{i-1}}$ for $2 \leq i \leq k$. This is illustrated in Figure 8. Thus, if B is the boundary of a region of $\tilde{D}(S)$, then B is C_{n_1}, C_{n_k}, or $C_{n_i} \star C_{n_{i+1}}$ for $1 \leq i \leq k - 1$. Let u be the center of $\tilde{D}(S)$.

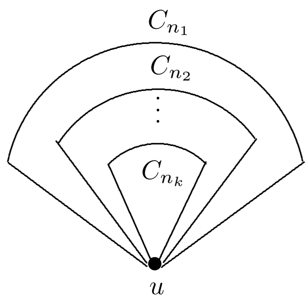

Figure 8: A planar embedding $\tilde{D}(S)$ of $D(S)$ in Case 1.

We show that $\tilde{D}(S)$ is zonal. For $1 \le i \le k$, let $Q_i = C_{n_i} - u$ be the path of order $n_i - 1 \ge 2$. Define a labeling $\ell : V(\tilde{D}(S)) \to \{1, 2\}$ such that $\ell(u) = 1$ and

(a) $\sum(\ell, Q_i) = 2$ in $\mathbb{Z}_3$ for each odd integers i with $1 \le i \le k$ and

(b) $\sum(\ell, Q_i) = 0$ in $\mathbb{Z}_3$ for each even integer i with $2 \le i \le k-1$.

Such a labeling in (a) and (b) is guaranteed by Lemma 3.1. Let B be the boundary of a region of $\tilde{D}(S)$. If $B = C_{n_i}$ for $i = 1, k$, then $\sum(\ell, B) = \ell(u) + \sum(\ell, Q_i) = 1 + 2 = 0$ in $\mathbb{Z}_3$. If $B = C_{n_i} \star C_{n_{i+1}}$ for $1 \le i \le k-1$, then $\sum(\ell, B) = \ell(u) + \sum(\ell, Q_i) + \sum(\ell, Q_{i+1}) = 1 + 2 + 0 = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of $\tilde{D}(S)$.

Subcase 2.2. *$k \ge 6$ is even.* Let $S_0 = \{C_{n_5}, C_{n_6}, \ldots, C_{n_k}\}$ be the subset of $k-4$ cycles in S and let $\tilde{D}(S_0)$ be the planar embedding of $D(S_0)$ such that $C_{n_{i+1}}$ is placed inside C_{n_i} for $i = 5, 6, \ldots, k-1$. Thus, if B is the boundary of a region of $\tilde{D}(S_0)$, then B is C_{n_5}, C_{n_k}, or $C_{n_i} \star C_{n_{i+1}}$ for $5 \le i \le k-1$. Then $\tilde{D}(S) = C_{n_1} \star C_{n_2} \star C_{n_3} \star C_{n_4} \star \tilde{D}(S_0)$ is the planar embedding obtained by identifying the center of $\tilde{D}(S_0)$ and a vertex in C_{n_i} for $i = 1, 2, 3, 4$. This identified vertex u is then the center of $\tilde{D}(S)$. This is illustrated in Figure 9. Thus, if B is the boundary of a region of $\tilde{D}(S)$, then $B = C_{n_i}$ for $i \in \{1, 2, 3, 4, k\}$, or $B = C_{n_i} \star C_{n_{i+1}}$ for $5 \le i \le k-1$, or $B = C_{n_1} \star C_{n_2} \star C_{n_3} \star C_{n_4} \star C_{n_5}$ where in this case B is the boundary of the exterior region of $\tilde{D}(S)$.

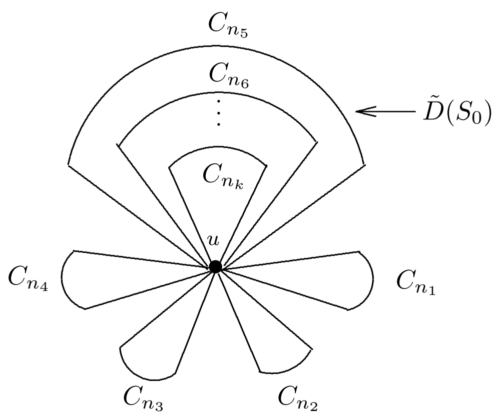

Figure 9: A planar embedding $\tilde{D}(S)$ of $D(S)$ in Case 2.

We show that $\tilde{D}(S)$ is zonal. For $1 \le i \le k$, let $Q_i = C_{n_i} - u$ be the path of order $n_i - 1 \ge 2$. Define a labeling $\ell : V(\tilde{D}(S)) \to \{1, 2\}$ such that $\ell(u) = 1$ and

(a) $\sum(\ell, Q_i) = 2$ in $\mathbb{Z}_3$ for $1 \le i \le 4$ and for each even integer i with $6 \le i \le k$ and

(b) $\sum(\ell, Q_i) = 0$ in $\mathbb{Z}_3$ for each odd integer i with $5 \le i \le k-1$.

Again, such a labeling in (a) and (b) is guaranteed by Lemma 3.1. Let B be the boundary of a region of $\tilde{D}(S)$. If $B = C_{n_i}$ for $i \in \{1, 2, 3, 4, k\}$, then $\sum(\ell, B) = \ell(u) + \sum(\ell, Q_i) = 1 + 2 = 0$ in $\mathbb{Z}_3$. If $B = C_{n_i} \star C_{n_{i+1}}$ for $5 \le i \le k-1$, then $\sum(\ell, B) = \ell(u) + \sum(\ell, Q_i) + \sum(\ell, Q_{i+1}) = 1 + 2 + 0 = 0$ in $\mathbb{Z}_3$. If $B = C_{n_1} \star C_{n_2} \star C_{n_3} \star C_{n_4} \star C_{n_5}$, then $\sum(\ell, B) = \ell(u) + \sum_{i=1}^{4} \sum(\ell, Q_i) + \sum(\ell, Q_5) = 1 + 4 \cdot 2 + 0 = 0$ in $\mathbb{Z}_3$. Consequently, ℓ is a zonal labeling of $\tilde{D}(S)$.

It remains to show that $D(S)$ is conditionally zonal. Let G be the standard planar embedding of $D(S)$ such that the boundary of each region of G is either a cycle $C \in S$ or the graph G. We show that the plane graph G is not zonal. Assume, to the contrary, that there is a zonal labeling $\ell : V(G) \to \{1, 2\}$ of G. We may assume that $\ell(u) = 1$ by Lemma 3.1. Since $\sum(\ell, C) = 0$ for each $C \in S$ and $\ell(u) = 1$ in $\mathbb{Z}_3$, it follows that $\sum(\ell, C - u) = 2$ in $\mathbb{Z}_3$. Let $B = G$ be the boundary of the exterior region of G, then $\sum(\ell, B) = 1 + \sum_{C \in S} \sum(\ell, C - u) = 1 + 2k$ in $\mathbb{Z}_3$. Since $k \not\equiv 1 \pmod 3$, it follows that $\sum(\ell, B) \ne 0$ in $\mathbb{Z}_3$, which is a contradiction. Therefore, ℓ is not a zonal labeling of G and so $D(S)$ is conditionally zonal. □

4. Irregular Dutch windmill graphs

In this and the next section, we study Dutch windmill graphs with an irregularity or regularity property (see [1] for a discussion of irregularity in graphs). A Dutch windmill graph is *irregular* if no two cycles in the graph have the same length. Thus, if S is a set of $k \ge 3$ distinct cycles, then the Dutch windmill graph $D(S)$ is irregular. By Theorem 3.1, for every multiset S of three or more cycles, the graph $D(S)$ is conditionally zonal. For example, if $S = \{C_3, C_4, C_5, C_6\}$, then $D(S)$, which is the standard plane embedding, is irregular and zonal. A zonal labeling of $D(S)$ is shown in Figure 10.

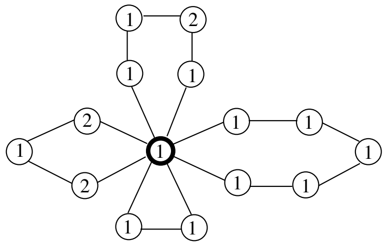

Figure 10: A zonal labeling of an irregular Dutch windmill plane graph.

There is a class of irregular Dutch windmill graphs none of which is absolutely zonal but having an arbitrarily large number of distinct zonal planar embeddings all of which have a similar structure. First, we introduce some additional definitions and notation and preliminary observations.

Let $A = \{C^1, C^2, \ldots, C^p\}$ be a set of $p \geq 2$ cycles, let $D(A)$ denote the planar embedding of the Dutch windmill graph constructed from these p cycles in A by placing $C^1, C^2, \cdots, C^{p-1}$ inside C^p. For example, if $A = \{C^1, C^2, C^3, C^4\}$ is a set of 4 cycles, then $D(A)$ is shown in Figure 11, where the three cycles C^1, C^2, C^3 are placed inside the cycle C^4. Thus, the boundary of a region in $D(A)$ is either C^i for $i = 1, 2, 3, 4$ or $D(A)$.

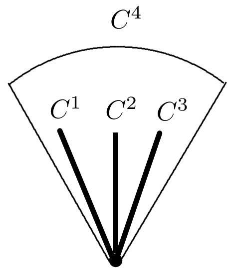

Figure 11: A plane graph $D(A)$ for $A = \{C^1, C^2, C^3, C^4\}$.

For a set S of $p \geq 4$ cycles, let $\Pi = \{S_1, S_2, \ldots, S_q\}$ be a partition of S into $q \geq 2$ subsets $S_1, S_2, \ldots, S_q$. For $1 \leq i \leq q$, let $D(S_i)$ be the Dutch windmill plane graph with center u_i. The plane graph $D(\Pi) = D(S_1) \star D(S_2) \star \cdots \star D(S_q)$ is constructed from the q plane graphs $D(S_1)$, $D(S_2)$, $\ldots$, $D(S_q)$ by identifying their centers $u_1, u_2, \ldots, u_q$ and denoting the identified vertex by u. For example, let $S = \{C^1, C^2, \ldots, C^{16}\}$ and let $\Pi = \{S_1, S_2, S_3, S_4\}$ be a partition of S into four subsets, where $S_1 = \{C^1, C^2, C^3, C^4\}$, $S_2 = \{C^5, C^6, C^7, C^8\}$, $S_3 = \{C^9, C^{10}, C^{11}, C^{12}\}$, and $S_4 = \{C^{13}, C^{14}, C^{15}, C^{16}\}$. Then $D(\Pi)$ is shown in Figure 12. In this example, if B is the boundary of a region of $D(\Pi)$, then either $B = C^i$ for some $C^i \in S - \{C^4, C^8, C^{12}, C^{16}\}$, or $B = D(S_i)$ for $1 \leq i \leq 4$, or $B = C^4 \star C^8 \star C^{12} \star C^{16}$ having center u where the region is the exterior region of $D(S)$.

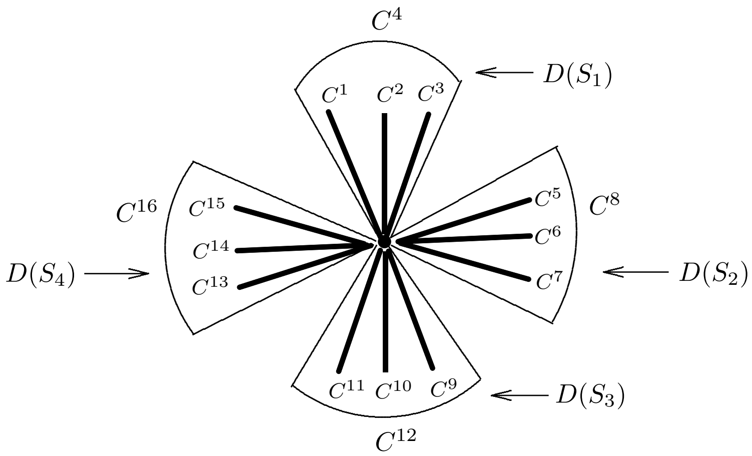

Figure 12: A planar embedding of a graph.

We now present the following lemma.

Lemma 4.1. *Let S be a set with $4k$ elements for some positive integer k. The number of partitions of S into four k-element subsets is*

$$\prod_{i=1}^{4} \binom{ik-1}{k-1} = \binom{4k-1}{k-1}\binom{3k-1}{k-1}\binom{2k-1}{k-1}. \qquad (1)$$

Proof. For $k = 1$, the expression in (1) is 1, which is correct since there is only one way to partition a set with 4 elements into four subsets. Thus, we may assume that $k \geq 2$. Suppose that $S = \{a_1, a_2, \ldots, a_{4k}\}$. In any partition of S into four k-element subsets, the element a_1 must belong to a k-element subset of S in this partition. The number of ways to choose a k-element subset of S containing a_1 is $\binom{4k-1}{k-1}$. Once such a k-element subset S_1 of S in the partition is given, let a_i be an element of $S - S_1$. The number of ways to choose a k-element subset of $S - S_1$ containing a_i is $\binom{3k-1}{k-1}$. Once two such disjoint k-element subsets S_1 and S_2 of S in the partition are given, let a_j be an element of $S - (S_1 \cup S_2)$. The number of ways to chose a k-element subset of $S - (S_1 \cup S_2)$ containing a_j is $\binom{2k-1}{k-1}$. Once three such pairwise disjoint k-element subsets S_1, S_2, and S_3 of S in the partition are given, only k elements remain in $S - (S_1 \cup S_2 \cup S_3)$, which constitutes the fourth k-element subset of S in the partition. Therefore, the total number of such partitions is $\binom{4k-1}{k-1}\binom{3k-1}{k-1}\binom{2k-1}{k-1}$. □

For $1 \leq i \leq 4k$, let $n_i = 10^i + 1$. If $k = 4$, then $n_1 = 11$, $n_2 = 101$, $n_3 = 1001$, $n_4 = 10001$, and $n_{16} = 10\cdots01$ where there are fifteen 0s between the two 1s. In general, for $1 \leq i \leq 16$, the first and last digits of n_i are 1 and the remaining $i - 1$ digits of n_i are 0 (where there is a total $i - 1$ 0s between the two 1s in n_i).

Theorem 4.1. *There is an irregular Dutch windmill graph that has an arbitrarily large number of distinct zonal planar embeddings.*

Proof. Let $k \geq 4$ be an integer such that $k \equiv 1 \pmod 3$ and let $\Pi(k) = \prod_{i=1}^{4}\binom{ik-1}{k-1}$. We show that there is an irregular Dutch windmill graph having at least $\Pi(k)$ distinct zonal planar embeddings. For $1 \leq i \leq 4k$, let $n_i = 10^i + 1$ and let $S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{4k}}\}$ be the set of $4k$ cycles of length n_i for $1 \leq i \leq 4k$. By Lemma 4.1, the number of partitions of S into four k-element subsets is $\Pi(k)$. For $1 \leq j \leq \Pi(k)$, let Π_j be a partition of S into four k-element subsets and so $D(\Pi_j)$ is a planar embedding of the Dutch windmill graph $D(S)$. We show that $D(\Pi_1), D(\Pi_2), \ldots, D(\Pi_{\Pi(k)})$ are $\Pi(k)$ distinct zonal planar embeddings of $D(S)$.

First, we make an observation concerning the structural property of $D(\Pi_j)$ where $1 \leq j \leq \Pi(k)$. For example, let $\Pi_1 = \{S_1, S_2, S_3, S_4\}$ be the partition of S into k-element subsets where

$$S_1 = \{C_{n_1}, C_{n_2}, \ldots C_{n_k}\},\ S_2 = \{C_{n_{k+1}}, C_{n_{k+2}}, \ldots, C_{n_{2k}}\},$$
$$S_3 = \{C_{n_{2k+1}}, C_{n_{2k+2}}, \ldots, C_{n_{3k}}\}, \text{ and } S_4 = \{C_{n_{3k+1}}, C_{n_{3k+2}}, \ldots, C_{n_{4k}}\}.$$

If $k = 4$ and $C^i = C_{n_i}$ for $1 \leq i \leq 4$, then $D(\Pi_1)$ is shown in Figure 12. For the plane graph $D(\Pi_1)$, there are $4k + 1$ regions $R_1, R_2, \ldots, R_{4k+1}$ of $D(\Pi_1)$, where

- the k regions $R_1, R_2, \ldots, R_k$ have the boundaries $C_{n_1}, C_{n_2}, \ldots, C_{n_{k-1}}$ and $D(S_1)$, respectively;
- the k regions $R_{k+1}, R_{k+2}, \ldots, R_{2k}$ have the boundaries $C_{n_{k+1}}, C_{n_{k+2}}, \ldots, C_{n_{2k-1}}$ and $D(S_2)$, respectively;
- the k regions $R_{2k+1}, R_{2k+2}, \ldots, R_{3k}$ have boundaries $C_{n_{2k+1}}, C_{n_{2k+2}}, \ldots, C_{n_{3k-1}}$ and $D(S_3)$, respectively;
- the k regions $R_{3k+1}, R_{3k+2}, \ldots, R_{4k}$ have the boundaries $C_{n_{3k+1}}, C_{n_{3k+2}}, \ldots, C_{n_{4k-1}}$ and $D(S_4)$, respectively;
- the exterior region R_{4k+1} has the boundary $C_{n_k} \star C_{n_{2k}} \star C_{n_{3k}} \star C_{n_{4k}}$.

In particular, the boundary of R_k is $D(S_1)$ which has order $b_{1,k} = 10^{n_1} + 10^{n_2} + \cdots + 10^{n_k} + 1$, the boundary of R_{2k} is $D(S_2)$ which has order $b_{1,2k} = 10^{n_{k+1}} + 10^{n_{k+2}} + \cdots + 10^{n_{2k}} + 1$, the boundary of R_{3k} is $D(S_3)$ which has order $b_{1,3k} = 10^{n_{2k+1}} + 10^{n_{2k+2}} + \cdots + 10^{n_{3k}} + 1$, and the boundary of R_{4k} is $D(S_4)$ which has order $b_{1,4k} = 10^{n_{3k+1}} + 10^{n_{3k+2}} + \cdots + 10^{n_{4k}} + 1$.

First, we show that the planar embeddings $D(\Pi_1)$, $D(\Pi_2)$, …, $D(\Pi_{\Pi(k)})$ of $D(S)$ are distinct. Let $i_1, i_2 \in \{1, 2, \ldots, \Pi(k)\}$ such that $i_1 \neq i_2$. From the way in which the plane graphs $D(\Pi_j)$ where $1 \leq j \leq \Pi(k)$ are constructed, it follows that

$$\{b_{i_1,k}, b_{i_1,2k}, b_{i_1,3k}, b_{i_1,4k}\} \neq \{b_{i_2,k}, b_{i_2,2k}, b_{i_2,3k}, b_{i_2,4k}\}.$$

Hence, $D(\Pi_{i_1})$ and $D(\Pi_{i_2})$ are distinct. Consequently, $D(\Pi_1)$, $D(\Pi_2)$, …, $D(\Pi_{\Pi(k)})$ are distinct planar embeddings of $D(S)$.

Next, we show that each $D(\Pi_i)$ is zonal for $1 \leq j \leq \Pi(k)$. It suffices to show that $D(\Pi_1)$ is zonal since the argument for showing that $D(\Pi_j)$ is zonal for $2 \leq j \leq \Pi(k)$ is similar. By Theorem 3.1, the plane graph $D(S_i)$ is zonal for $i = 1, 2, 3, 4$ and so there is a zonal labeling ℓ_i of $D(S_i)$. Let u be the center of $D(\Pi_1)$. By Observation 2.1, we may assume that $\ell_i(u) = 1$ for $1 \leq i \leq 4$. Define a labeling ℓ of $D(\Pi_1)$ by $\ell(v) = \ell_i(v)$ if v belongs to $D(S_i)$ for $1 \leq i \leq 4$. We show that ℓ is a zonal labeling of $D(\Pi_1)$. Let B be the boundary of a region R of $D(\Pi_1)$. If R is an interior region of $D(\Pi_1)$, then R is a region of $D(S_i)$ for some integer i with $1 \leq i \leq 4$ and so $\sum(\ell, B) = \sum(\ell_i, B) = 0$. Thus, we may assume that R is the exterior region of $D(\Pi_1)$ and so $B = C_{n_k} \star C_{n_{2k}} \star C_{n_{3k}} \star C_{n_{4k}}$ whose center is u. Since $\sum(\ell_i, C_{n_{ik}}) = 0$ in $\mathbb{Z}_3$ (that is, the value of the boundary of the exterior region of $D(S_i)$ is 0) and $\ell_i(u) = 1$ for $1 \leq i \leq 4$, it follows that $\sum(\ell_i, C_{n_{ik}} - u) = 2$ in $\mathbb{Z}_3$.

Hence, $\sum(\ell, B) = 1 + \sum_{i=1}^{4} \sum(\ell_i, C_{n_{ik}} - u) = 1 + 4 \cdot 2 = 0$ in $\mathbb{Z}_3$. Therefore, $D(\Pi_1)$ is zonal. Consequently, $D(\Pi_1)$, $D(\Pi_2)$, $\ldots$, $D(\Pi_{\Pi(k)})$ are distinct zonal planar embeddings of $D(S)$.

Let N be an arbitrarily large positive integer. Since $\lim_{k\to\infty} \Pi(k) = \infty$, it follows that there is an integer k_0 such that $k_0 \equiv 1 \pmod 3$ and $\Pi(k_0) > N$. Let $S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{4k_0}}\}$. Then the Dutch windmill graph $D(S)$ has at least $\Pi(k_0) > N$ distinct zonal planar embeddings. $\square$

For an integer $k \geq 4$ and $k \equiv 1 \pmod 3$, the irregular Dutch windmill graph used to verify Theorem 4.1 has order $(10^{4k+1} - 1)/9$. An irregular Dutch windmill graph of smaller order can be used to verify Theorem 4.1 by changing the base integer of each integer n_i ($1 \leq i \leq 4k$) from 10 to a smaller base. For example, if we let $n_i = 2^i + 1$ (using base 2), the same proof applies and the order of the irregular Dutch windmill graph is $2^{4k+1} - 1$.

5. A special class of Dutch windmill graphs

In the proof of Theorem 4.1, no two cycles in the irregular Dutch windmill graph have the same length and we were able to obtain an arbitrarily large number of planar embeddings of the graph such that the structure of these embeddings are similar. If the cycles of a Dutch windmill graph all have the same length (and is consequently a regular Dutch windmill graph), then this proof does not provide the desired result. In this case, however, by varying the embedding of the Dutch windmill graph, the same conclusion can be obtained.

Before presenting the next result, we construct a sequence $F_1, F_2, F_3, \ldots$ of plane graphs recursively as follows. The plane graph $F_1 = D_3^4$ consists of a triangle T_1 within which are three triangles, as indicated in Figure 13. The vertex of degree 8 in F_1 is the center of F_1. The plane graph F_2 is constructed from three copies of F_1 and a triangle T_2 by placing the three copies of F_1 inside T_2 and identifying their centers with a vertex of T_2. Thus, the identified vertex is the center of F_2 and has degree 26 in F_2, as indicated in Figure 13. For $k \geq 3$, the plane graph F_k is constructed by placing three copies of F_{k-1} inside a triangle T_k and identifying their centers with a vertex of T_k. The plane graph F_3 is shown in Figure 13 whose center has degree 80. Observe that the boundary of every region of F_k either has order 3 or 9 for all $k \geq 1$. Therefore, if every vertex of F_k were to be labeled 1 in $\mathbb{Z}_3$, then each region would have the label 0 in $\mathbb{Z}_3$. The region of F_k whose boundary is the triangle T_k is referred to as the exterior region of F_k and each of the other regions is referred to as an interior region of F_k.

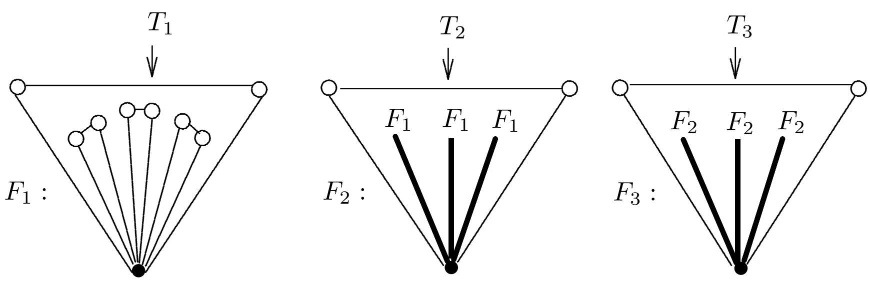

Figure 13: The plane graphs $F_1 = D_3^4, F_2 = D_3^{13}, F_3 = D_3^{40}$.

For a positive integer k, let t_k denote the number of triangles in F_k. Then

$$t_k = 1 + 3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 1}{2}.$$

For example, $t_1 = 4$, $t_2 = 13$, and $t_3 = 40$. For $k \geq 1$, the plane graph F_k is a specific planar embedding of the Dutch windmill graph $D_3^{t_k}$ of order $2t_k + 1 = 3^{k+1}$. Furthermore, F_k is a planar embedding of the friendship graph $(t_k K_2) \vee K_1$. The graph F_k therefore has one vertex of degree $2t_k$ and $2t_k$ vertices of degree 2.

Proposition 5.1. *For a positive integer k, the plane graph F_k is zonal.*

Proof. For $k \geq 1$, define a labeling $\ell_k : V(F_k) \to \{1, 2\}$ by $\ell_k(x) = 1$ for every vertex x of F_k. Since the boundary B of a region of F_k has order 3 or 9, it follows that $\sum(\ell, B) = 0$ in $\mathbb{Z}_3$. $\square$

For a positive integer k, the integer $\pi(k)$ is defined as

$$\pi(k) = t_1 t_2 \cdots t_k = \prod_{i=1}^{k} \left(\frac{3^{i+1} - 1}{2}\right) = \frac{1}{2^k} \prod_{i=1}^{k} (3^{i+1} - 1).$$

Thus, $\pi(1) = t_1 = 4$, $\pi(2) = t_1 t_2 = 4 \cdot 13 = 52$, and $\pi(3) = t_1 t_2 t_3 = 4 \cdot 13 \cdot 40 = 2080$. Since $3^{i+1} \equiv 0 \pmod 3$, where $1 \le i \le k$, it follows that $3^{i+1} - 1 \equiv 2 \pmod 3$ and so

$$\frac{3^{i+1}-1}{2} \equiv 1 \pmod 3.$$

Therefore, $\pi(k) \equiv 1 \pmod 3$ and $2\pi(k) + 1 \equiv 0 \pmod 3$ for every positive integer k. By Theorem 3.1, the Dutch windmill graph $D_3^{\pi(k)}$ is zonal. For $1 \le i \le k$, let

$$s_i = \frac{\pi(k)}{t_i}.$$

Then $s_i \equiv 1 \pmod 3$ for $1 \le i \le k$ and so $2s_i + 1 \equiv 0 \pmod 3$. Since $t_1 < t_2 < \ldots < t_k$ for a fixed positive integer k, it follows that $s_1 > s_2 > \ldots > s_k$.

For a Dutch windmill plane graph H, let $H_1, H_2, \ldots, H_p$ be p copies of H and let v_i be the center of H_i for $1 \le i \le p$. Then $p \star H$ denotes the Dutch windmill plane graph constructed from $H_1, H_2, \ldots, H_p$ by identifying the p vertices $v_1, v_2, \ldots, v_p$ and denoting the identified vertex by v. For example, let F_2 be the planar embedding of the Dutch windmill graph D_3^{13} shown in Figure 13 and $p = 4$, then the plane graph $4 \star F_2$ is shown in Figure 14. Thus, $4 \star F_2$ is a planar embedding of the Dutch windmill graph D_3^{52}.

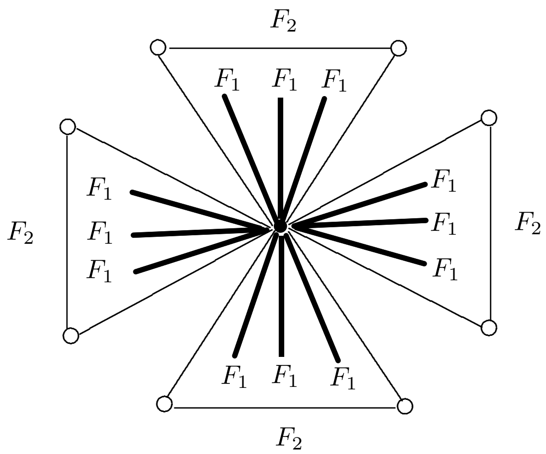

Figure 14: The plane graph $4 \star F_2$.

Theorem 5.1. *There is a regular Dutch windmill graph that has an arbitrarily large number of distinct zonal planar embeddings.*

Proof. Let k be a positive integer. We show that the Dutch windmill graph $D_3^{\pi(k)}$ has at least k distinct zonal planar embeddings. For $k = 1$, the zonal planar embedding of $D_3^4 = D_3^{\pi(1)}$ shown in Figure 15 is zonal. Hence, we may assume that $k \ge 2$. Here we show that there are k distinct zonal planar embeddings of $D_3^{\pi(k)}$. Let $G_{k,1}$ be the planar embedding of $D_3^{\pi(k)}$ such that the boundary of each interior region is a 3-cycle and the boundary of the exterior region is $D_3^{\pi(k)}$. Since $D_3^{\pi(k)}$ has order $2\pi(k) + 1 \equiv 0 \pmod 3$, the labeling that assigns the label 1 in $\mathbb{Z}_3$ to each vertex of $G_{k,1}$ is a zonal labeling. For $2 \le i \le k$, let $s_i = \frac{\pi(k)}{t_i}$ and let $G_{k,i} = s_{i-1} \star F_{i-1}$ be the planar embedding of $D_3^{\pi(k)}$. We show that $G_{k,1}, G_{k,2}, \ldots, G_{k,k}$ are k distinct zonal planar embeddings of $D_3^{\pi(k)}$. Observe that if B is the boundary of a region of $G_{k,i}$, where $2 \le i \le k$, then the order of B is 3, 9 or $2s_{i-1} + 1$, where $2s_{i-1} + 1 \equiv 0 \pmod 3$. Since $\pi(k) > s_1 > s_2 > \cdots > s_{k-1}$, these planar embeddings are distinct. By assigning the label 1 in $\mathbb{Z}_3$ to each vertex of $G_{k,i}$ for $2 \le i \le k$, we see that the plane graph $G_{k,i}$ is zonal.

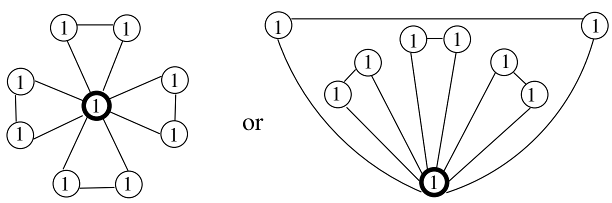

Figure 15: The zonal plane graph D_3^4.

Let N be an arbitrary positive integer. Since $\lim_{k\to\infty} \pi(k) = \infty$, it follows that there is an integer k_0 such that $k_0 \equiv 1 \pmod 3$ and $\pi(k_0) > N$. Then the Dutch windmill graph $D_3^{\pi(k_0)}$ has at least $\pi(k_0) > N$ distinct distinct zonal planar embeddings. $\square$

We illustrate the proof of Theorem 5.1 for $k = 2, 3$.

- ⋆ If $k = 2$, then $\pi(2) = 52$ and $t_1 = 4$. Thus, $s_1 = 52/4 = 13$. Let $G_{2,1}$ be the planar embedding of D_3^{52}, where the boundary of each interior region is a 3-cycle and the boundary of the exterior region is D_3^{52} and let $G_{2,2} = s_1 \star F_1 = 13 \star F_1$. Here, $G_{2,1}$ and $G_{2,2}$ are distinct planar embeddings of D_3^{52}.
- ⋆ If $k = 3$, then $\pi(3) = 2080$, $t_1 = 4$, and $t_2 = 13$. Thus, $s_1 = 2080/4 = 520$ and $s_2 = \pi(3)/t_2 = 2080/13 = 160$. Let $G_{3,1}$ be the planar embedding of D_3^{2080} where the boundary of each interior region is a 3-cycle and the boundary of the exterior region is D_3^{2080}, let $G_{3,2} = s_1 \star F_1 = 520 \star F_1$, and let $G_{3,3} = s_2 \star F_2 = 160 \star F_2$. Here, $G_{3,1}$, $G_{3,2}$, and $G_{3,3}$ are distinct planar embeddings of D_3^{2080}.

As we saw in the proof of Theorem 5.1, for a given positive integer k, the graph $G = D_3^{\pi(k)}$ has k distinct zonal planar embeddings. Since the graph G has order $2\pi(k) + 1$, which is large for a large positive integer k, a question here is whether there is a graph of smaller order with this property. We now discuss this question.

For a positive integer k, let $\mu(k) = \mathrm{lcm}\{t_1, t_2, \ldots, t_k\}$ be the least common multiple of t_1, t_2, …, t_k. Since $t_i = \frac{3^{i+1}-1}{2} \equiv 1 \pmod 3$ for $1 \le i \le k$, it follows that $3 \nmid t_i$ and so $3 \nmid \mu(k)$. Thus, $\mu(k) \equiv 1 \pmod 3$ or $\mu(k) \equiv 2 \pmod 3$. If $\mu(k) \equiv 2 \pmod 3$, then $2\mu(k) \equiv 1 \pmod 3$. Hence, $\mu(k) \equiv 1 \pmod 3$ or $2\mu(k) \equiv 1 \pmod 3$. Consequently, either $\frac{\mu(k)}{t_i} \equiv 1 \pmod 3$ for $i \in \{1, 2, \ldots, k\}$ or $\frac{2\mu(k)}{t_i} \equiv 1 \pmod 3$ for $i \in \{1, 2, \ldots, k\}$. If k is sufficiently large, then $\pi(k)$ is substantially larger than $\mu(k)$ or $2\mu(k)$. Applying the argument in the proof of Theorem 5.1, we have the following result.

Proposition 5.2. *Let k be a positive integer.*

- ⋆ *If $\mu(k) \equiv 1 \pmod 3$, then the zonal Dutch windmill graph $D_3^{\mu(k)}$ is conditionally zonal and has at least k distinct zonal planar embeddings.*
- ⋆ *If $\mu(k) \equiv 2 \pmod 3$, then the zonal Dutch windmill graph $D_3^{2\mu(k)}$ is conditionally zonal and has at least k distinct zonal planar embeddings.*

We now illustrate Proposition 5.2 for integers k with $1 \le k \le 12$. In this case, expressing each integer t_k and $\mu(k)$ as a product of primes, we have the following, where each underlined integer is congruent to 2 modulo 3.

$t_1 = 4 = 2^2$	$\mu(1) = 2^2$
$t_2 = 13$	$\mu(2) = 2^2 \cdot 13$
$t_3 = 40 = 2^3 \cdot 5$	$\mu(3) = 2^3 \cdot 5 \cdot 13$
$t_4 = 121 = 11^2$	$\mu(4) = 2^3 \cdot 5 \cdot 11^2 \cdot 13$
$t_5 = 364 = 2^2 \cdot 7 \cdot 13$	$\mu(5) = 2^3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13$
$t_6 = 733,$	$\mu(6) = 2^3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13 \cdot 733$
$t_7 = 2200 = 2^3 \cdot 5^2 \cdot 11$	$\underline{\mu(7)} = \underline{2^3} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 733$
$t_8 = 6601 = 7 \cdot 23 \cdot 41$	$\underline{\mu(8)} = \underline{2^3} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot \underline{23} \cdot \underline{41} \cdot 733$
$t_9 = 19,804 = 2^2 \cdot 4951$	$\underline{\mu(9)} = \underline{2^3} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot \underline{23} \cdot \underline{41} \cdot 733 \cdot 4951$
$t_{10} = 59,413 = 19 \cdot 53 \cdot 59$	$\underline{\mu(10)} = \underline{2^3} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 19 \cdot \underline{23} \cdot \underline{41} \cdot \underline{53} \cdot \underline{59} \cdot 733 \cdot 4951$
$t_{11} = 178,240 = 2^6 \cdot 5 \cdot 557$	$\underline{\mu(11)} = 2^6 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 19 \cdot \underline{23} \cdot \underline{41} \cdot \underline{53} \cdot \underline{59} \cdot \cdot \underline{557} \cdot 733 \cdot 4951$
$t_{12} = 534,721 = 11 \cdot 48611$	$\underline{\mu(12)} = 2^6 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 19 \cdot \underline{23} \cdot \underline{41} \cdot \underline{53} \cdot \underline{59} \cdot \cdot \underline{557} \cdot 733 \cdot 4951 \cdot \underline{48611}$

Hence, $\mu(k) \equiv 1 \pmod 3$ for $1 \le k \le 6$, while $\mu(k) \equiv 2 \pmod 3$ for $7 \le k \le 11$ and $\mu(12) \equiv 1 \pmod 3$. Therefore, if $1 \le k \le 6$ or $k = 12$, then the zonal Dutch windmill graph $D_3^{\mu(k)}$ is conditionally zonal and has at least k distinct zonal planar embeddings; while if $7 \le k \le 11$, then the zonal Dutch windmill graph $D_3^{2\mu(k)}$ is conditionally zonal and has at least k distinct zonal planar embeddings.

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References

[1] A. Ali, G. Chartrand, P. Zhang, *Irregularity in Graphs*, Springer, New York, 2021.
[2] G. Chartrand, C. Egan, P. Zhang, *How to Label a Graph*, Springer, New York, 2019.
[3] H. Whitney, Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932) 150–168.