Absolutely and conditionally zonal graphs

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Abstract

A zonal labeling of a plane graph $G$ is an assignment of the two nonzero elements of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of $G$ such that the sum of the labels of the vertices on the boundary of each region of $G$ is the zero element of $\mathbb{Z}_3$. A plane graph possessing such a labeling is a zonal graph. A planar graph $G$ is zonal if there exists a zonal planar embedding of $G$. If every planar embedding of $G$ is zonal, then $G$ is absolutely zonal. A zonal planar graph $G$ is conditionally zonal if some planar embedding of $G$ is not zonal. It is shown that there is a class of absolutely zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings as well as a class of conditionally zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings with prescribed irregularity and regularity properties.

Keywords: planar graph; graph embedding; zonal labeling; conditionally and absolutely zonal graph; irregularity and regularity.

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1. Introduction

In 2014 Cooroo Egan introduced a vertex labeling of plane graphs (planar graphs embedded in the plane) called a zonal labeling (see [2]). A zonal labeling $\ell$ of a plane graph $G$ is an assignment of the two nonzero elements 1 and 2 of the ring $\mathbb{Z}_3$ of integers modulo 3 to the vertices of $G$ such that the sum of the labels of the vertices on the boundary of each region (zone) of $G$, called the value of the region, is the zero element in $\mathbb{Z}_3$. If a connected plane graph $G$ possesses a zonal labeling, then $G$ is a zonal graph. The plane graph $G_1$ of Figure 1 is zonal and a zonal labeling of $G_1$ is given in that figure, while the plane graph $G_2$ of Figure 1 is not zonal.

![Figure 1: A zonal plane graph and a non-zonal plane graph.](image)

A planar graph $G$ is zonal if there exists a zonal planar embedding of $G$. If every planar embedding of $G$ is zonal, then $G$ is absolutely zonal. For example, if $G$ is a maximal planar graph of order 3 or more embedded in the plane, then the boundary of every region of $G$ is a triangle. Thus, the labeling that assigns the label 1 to every vertex of $G$ is a zonal labeling. Therefore, every maximal planar graph of order 3 or more is absolutely zonal. A planar graph $G$ is conditionally zonal if some planar embedding of $G$ is not zonal. Since the graphs $G_1$ and $G_2$ of Figure 1 are isomorphic, it follows that $G_2$ is a different planar embedding of $G_1$ and so the planar graph $G_1$ (or $G_2$) is conditionally zonal.

It is the goal of this paper to describe (1) a class of absolutely zonal graphs having an arbitrarily large number of distinct planar embeddings and (2) a class of conditionally zonal graphs possessing an arbitrarily large number of distinct zonal planar embeddings with prescribed irregularity or regularity properties. Before doing this, however, we review some information concerning zonal graphs, mentioned in [2], that illustrates some of the interest in studying zonal labelings. A cubic map is a connected bridgeless cubic plane graph. The following result was obtained in [2].

**Theorem 1.1.** A connected cubic plane graph $G$ is zonal if and only if $G$ is bridgeless.

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Consequently, by Theorem 1.1, the only zonal cubic plane graphs are cubic maps. That every cubic map is zonal was established in [2] with the aid of the Four Color Theorem (the chromatic number of every planar graph is at most 4). The converse of this statement is also true (see [2]).

**Theorem 1.2.** If every cubic map is zonal, then the chromatic number of every planar graph is at most 4.

Thus, if it could be shown that every cubic map is zonal without using the Four Color Theorem, then the Four Color Theorem would follow. This shows that studying zonal labelings of planar graphs are of interest, especially cubic planar graphs, and cubic maps in particular.

### 2. Absolutely zonal graphs

The following result was shown in [2].

**Proposition 2.1.** Every nontrivial tree and every cycle is zonal.

Since there is only one planar embedding of a nontrivial tree or a cycle, it follows that every nontrivial tree and every cycle is absolutely zonal. A nontrivial tree is connected but not 2-connected, while a cycle is 2-connected but not 3-connected. There is a class of 2-connected absolutely zonal graphs, as we show next.

**Proposition 2.2.** Every 2-connected bipartite planar graph is absolutely zonal.

**Proof.** Let $G$ be a 2-connected bipartite plane graph with partite sets $U$ and $W$. Define a labeling $\ell$ of $G$ by assigning the label 1 to every vertex of $U$ and the label 2 to every vertex of $W$. Let $R$ be a region of $G$. Since $G$ is a 2-connected bipartite plane graph, the boundary of $R$ is an even cycle $C$. Thus, half of the vertices of $C$ are labeled 1 and half are labeled 2. Hence, $\sum_{v \in V(C)} \ell(v) = 0$ in $\mathbb{Z}_3$ and so $\ell$ is a zonal labeling of $G$. Consequently, $G$ is absolutely zonal. $\square$

Whitney [3] obtained the following result on 3-connected planar graphs.

**Theorem 2.1.** (Whitney’s Theorem) Every 3-connected planar graph is uniquely embeddable in the plane.

As a consequence of Theorem 2.1, every 3-connected planar graph is either absolutely zonal or non-zonal. Every wheel $W_n = C_n \vee K_1$ (the join of $C_n$ and $K_1$), $n \geq 3$, is 3-connected. This observation gives rise to an infinite class of 3-connected absolutely zonal graphs and an infinite class of 3-connected non-zonal graphs. To establish this fact, we first state a definition. Let $\ell$ be a labeling of the vertices of a graph $G$ with the labels 1 and 2 of $\mathbb{Z}_3$. The vertex labeling $\overline{\ell}$ of $G$ defined by $\overline{\ell}(v) = 3 - \ell(v)$ for each vertex $v$ of $G$ is called the complementary labeling of $G$. The following is then immediate (see [2]).

**Observation 2.1.** If $\ell$ is a zonal labeling of a connected plane graph, then so too is its complementary labeling $\overline{\ell}$.

We now present the result on wheels indicated above.

**Theorem 2.2.** For an integer $n \geq 3$, the wheel $W_n = C_n \vee K_1$ is zonal if and only if $n \equiv 0 \pmod{3}$.

**Proof.** Since $W_n$ is a 3-connected planar graph, there is a unique planar embedding of $W_n$ where, in a standard planar embedding of $W_n$, the boundary of every interior region of $W_n$ is a triangle and the boundary of the exterior region is $C_n$. If $n \equiv 0 \pmod{3}$, then the labeling of $W_n$ that assigns the label 1 of $\mathbb{Z}_3$ to every vertex of $W_n$ is a zonal labeling and so $W_n$ is zonal. For the converse, let $n \geq 4$ be an integer such that $n \not\equiv 0 \pmod{3}$. If there exists a zonal labeling $\ell$ of $W_n$, then $\ell$ must assign the same label to the three vertices of each triangle of $W_n$, implying that $\ell$ must assign the same label to all vertices of $W_n$. By Observation 2.1, we may assume that $\ell$ assigns the label 1 to every vertex of $W_n$. Hence, $\sum_{v \in V(C_n)} \ell(v) = n \not\equiv 0$ in $\mathbb{Z}_3$. Since $C_n$ is the boundary of a region of $W_n$, it follows that $\ell$ is not a zonal labeling and so $W_n$ is not zonal. $\square$

We mentioned that every connected bridgeless cubic planar graph is absolutely zonal. If a bridgeless cubic planar graph $G$ is 3-connected, then it follows by Whitney’s Theorem (Theorem 2.1) that there is a unique planar embedding of $G$. If a bridgeless cubic planar graph $G$ is 2-connected but not 3-connected (that is, $G$ has connectivity 2), then $G$ may have two or more distinct planar embeddings, giving rise to distinct cubic maps. Since each cubic map is zonal, the graph $G$ itself is absolutely zonal. In fact, there is a class of connected bridgeless cubic planar graphs having an arbitrarily large number of distinct planar embeddings.

**Theorem 2.3.** For every positive integer $k$, there exists a connected bridgeless cubic planar graph having at least $k$ distinct (zonal) planar embeddings.
For $k \geq 3$, let $C_{2k-2} = (a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1}, a_1)$ be a cycle of order $2k-2$. This cycle is shown in Figure 2 for $k = 4$. For $1 \leq i \leq k - 1$, let $F_i$ and $F_i'$ be the planar embedding of a planar graph, also shown in Figure 2. Since the boundaries of the five regions of $F_i$ and $F_i'$ are the same, these are the same embeddings of a planar graph.

First, we construct a plane graph $G_0$ from $C_{2k-2}$ and $F_1, F_2, \ldots, F_{k-1}$ by identifying the edge $a_ib_i$ of $C_{2k-2}$ with the edge $u_iv_i$ of $F_i$ for $1 \leq i \leq k - 1$. Next, we embed $G_0$ in the plane such that (1) each interior region of $G_0$ is either an interior region of $F_i$ for some integer $i$ with $1 \leq i \leq k - 1$ or the region whose boundary is $C_{2k-2}$ and (2) the boundary of the exterior region of $G_0$ is the cycle of order $(2k-2) + 5(k-1) = 7k - 7$. Hence, each $F_i$ (1 $\leq i \leq k - 1$) lies outside of the cycle $C_{2k-2}$. The graph $G_0$ is shown in Figure 3 for $k = 4$. Observe that $G_0$ is a cubic map and so $G_0$ is zonal.

For an integer $j$ with $1 \leq j \leq k - 2$, we construct a graph $G_j$ from $C_{2k-2}, F'_1, F'_2, \ldots, F'_j$ and $F_{j+1}, F_{j+2}, \ldots, F_{k-1}$ by identifying the edge $u'_iv'_i$ of $F'_i$ with the edge $a_ib_i$ of $C_{2k-2}$ for $1 \leq i \leq j$ and the edge $u_iv_i$ of $F_i$ with the edge $a_ib_i$ of $C_{2k-2}$ for $j + 1 \leq i \leq k - 1$. We then embed $G_j$ in the plane such that (1) each interior region of $G_j$ is either an interior region of $F'_i$ for $1 \leq i \leq j$, or an interior region of $F_i$ for $j + 1 \leq i \leq k - 1$, or the region whose boundary is $C_{2k-2}$ and (2) the boundary of the exterior region of $G_j$ is the cycle of order $(2k-2) + 2j + 5(k-1-j) = 7k - 7 - 3j$. Hence, each of $F'_i$ (1 $\leq i \leq j$) and $F_i$ $(j + 1 \leq i \leq k - 1)$ lies outside of $C_{2k-2}$. The graph $G_j$ is shown in Figure 4 for $k = 4$. Observe that each plane graph $G_j$ is a cubic map and so $G_j$ is zonal for $1 \leq j \leq k - 2$.

The plane graphs $G_0, G_1, \ldots, G_{k-1}$ are all cubic maps and are planar embeddings of the same graph. Since $G_i$, $0 \leq i \leq k - 1$, has exactly $k - 1 - i$ regions whose boundary is a 4-cycle and whose largest boundary cycle has order $7k - 7 - 3i$ for $0 \leq i \leq k - 1$, these cubic maps are distinct planar embeddings of the same graph.
3. Conditionally zonal graphs

We now turn our attention to conditionally zonal graphs, that is, planar graphs having at least one zonal planar embedding and at least one non-zonal planar embedding. By Whitney’s Theorem, necessarily each such graph must have connectivity less than 3. We now describe a class of connected bridgeless planar graphs. For integers $n \geq 3$ and $k \geq 3$, the (standard) Dutch windmill graph $D_n^k$ is the graph obtained by taking $k$ copies of the $n$-cycle $C_n$ with a vertex in common. The Dutch windmill graph $D_n^5$ is commonly called a friendship graph (every two vertices have a unique common neighbor). The Dutch windmill graphs $D_n^3$, $D_n^4$, and $D_n^5$ are shown in Figure 5, the first of which is a friendship graph. The planar embedding of $D_n^k$ in which the boundary of each region is either $C_n$ or $D_n^k$ is called the standard embedding of $D_n^k$, as shown in Figure 5 for $D_3^3$, $D_4^3$ and $D_5^3$.

![Figure 5: The Dutch windmill graphs $D_3^3$, $D_4^3$, and $D_5^3$.](image)

For an integer $k \geq 2$, let $S$ denote a multiset of $k$ cycles and let $D(S)$ denote the Dutch windmill graph constructed from the cycles in $S$. The planar embedding of $D(S)$ in which the boundary of each region is either a cycle in $S$ or $D(S)$ is called the standard embedding of $D(S)$. For example, if $S = \{C_3, C_3, C_4, C_5\}$, then the standard planar embedding of $D(S)$ is shown in Figure 6. This plane graph $D(S)$ is zonal and a zonal labeling of $D(S)$ is given in Figure 6.

![Figure 6: The Dutch windmill graph $D(S)$ for $S = \{C_3, C_3, C_4, C_5\}$.](image)

For a labeling $\ell : V(G) \to \{1, 2\}$ of a graph $G$ and a subgraph $H$ of $G$, let $\sum(\ell, H) = \sum_{x \in V(H)} \ell(x)$ in $\mathbb{Z}_3$. First, we show that there is a class of non-zonal Dutch windmill graphs.

**Proposition 3.1.** For every multiset $S$ of two cycles, the Dutch windmill graph $D(S)$ is not zonal.

**Proof.** Let $S = \{C, C'\}$ and let $D(S)$ be the Dutch windmill graph constructed from the two cycles $C$ and $C'$ in $S$. Assume, to the contrary, that $D(S)$ is zonal. Then there exists a planar embedding $G$ of $D(S)$ having a zonal labeling $\ell$. The plane graph $G$ has three regions whose boundaries are $C$, $C'$ and $G$. Since each of $C$ and $C'$ is the boundary of a region of $G$, it follows that $\sum(\ell, C) = \sum_{v \in V(C)} \ell(v) = 0$ and $\sum(\ell, C') = \sum_{v \in V(C')} \ell(v) = 0$. Let $u$ be the vertex belonging to both $C$ and $C'$. Then the value of the boundary $G$ of the third region is $\left[ \sum(\ell, C) + \sum(\ell, C') \right] - \ell(u) = 0 + 0 - \ell(u) \neq 0$ in $\mathbb{Z}_3$, which is a contradiction. \hfill \Box

Next, we show that for every multiset $S$ of three or more cycles, the Dutch windmill graph $D(S)$ is conditionally zonal. It is convenient to introduce some additional notation. For $p \geq 2$ graphs $H_1, H_2, \ldots, H_p$, let $v_i$ be the vertex of $H_i$ for $1 \leq i \leq p$. Then $H_1 * H_2 * \cdots * H_p$ denotes the plane graph constructed from $H_1, H_2, \ldots, H_p$ by identifying the $p$ vertices $v_1, v_2, \ldots, v_p$ and denoting the identified vertex by $v$. For example, if $S_4 = \{C_3, C_3, C_4, C_5\}$, then $D(S_4) = C_3 * C_3 * C_4 * C_5$ shown in Figure 6. The following elementary lemma will be useful to us.

**Lemma 3.1.** Let $X$ be a nonempty set of vertices of a graph.

1. For each $i = 1, 2$, there is a labeling $\ell_i : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of $X$ such that $\sum(\ell_i, X) = i$ in $\mathbb{Z}_3$.

2. If $|X| \geq 2$, then there is a labeling $\ell_0 : X \to \{1, 2\} \subseteq \mathbb{Z}_3$ of $X$ such that $\sum(\ell_0, X) = 0$ in $\mathbb{Z}_3$.

**Theorem 3.1.** For every multiset $S$ of three or more cycles, the Dutch windmill graph $D(S)$ is conditionally zonal.
Proof. Let \( S \) be a multiset of \( k \geq 3 \) cycles. We consider two cases, according to whether \( k \equiv 1 \pmod{3} \) or \( k \not\equiv 1 \pmod{3} \).

Case 1. \( k \equiv 1 \pmod{3} \). First, we show that \( D(S) \) is zonal. Let \( G \) be the standard planar embedding of \( D(S) \) such that the boundary of each region of \( G \) is either a cycle \( C \in S \) or the graph \( G \). We show that the plane graph \( G \) is zonal. Let \( u \) be the center of \( G \), that is, \( \deg_G u = 2k \). Since every cycle \( C \in S \) is zonal, there is a zonal labeling \( \ell_C : V(C) \to \{1,2\} \) of \( C \). By Observation 2.1, we may assume that \( \ell_C(u) = 1 \) for every cycle \( C \in S \). Since \( \sum(\ell_C,C) = 0 \) and \( \ell_C(u) = 1 \) in \( \mathbb{Z}_3 \), where \( C \in S \), it follows that \( \sum(\ell_C,C-u) = 2 \) in \( \mathbb{Z}_3 \). Define a labeling \( \ell : V(G) \to \{1,2\} \) of \( G \) by \( \ell(v) = \ell_C(v) \) if \( v \) belongs to a cycle \( C \in S \). Let \( B \) be the boundary of a region in \( G \). If \( B = C \in S \), then \( \sum(\ell,C) = \sum(\ell_C,C) = 0 \) in \( \mathbb{Z}_3 \). Thus, we may assume that \( B = G \). Since \( k \geq 4 \) and \( k \equiv 1 \pmod{3} \), it follows that \( k = 3t + 1 \) for some integer \( t \geq 1 \). Hence,

\[
\sum(\ell,B) = 1 + \sum_{C \in S} \sum(\ell_C,C-u) = 1 + 2k - 1 + 2(3t + 1) = 0 \quad \text{in} \quad \mathbb{Z}_3.
\]

Therefore, \( \ell \) is a zonal labeling of \( G \) and so \( D(S) \) is zonal.

Next, we show that \( D(S) \) is conditionally zonal. Let \( C_1 \) and \( C_2 \) be two cycles in \( S \) and let \( H \) be the planar embedding of \( D(S) \) by placing the cycle \( C_2 \) within the cycle \( C_1 \) of \( D(S) \). This is illustrated in Figure 7. Thus, if \( B \) is the boundary of a region in \( H \), then either \( B = C \in S - \{C_1\} \), \( B = C_1 \ast C_2 \) (consisting of \( C_1 \) and \( C_2 \) with common vertex \( u \)), or \( B = H - V(C_2-u) \).

![Figure 7: The planar embedding \( H \) of \( D(S) \).](image)

We claim that the plane graph \( H \) is not zonal, for suppose that \( H \) has a zonal labeling \( \ell \). By Observation 2.1, we may assume that \( \ell(u) = 1 \). This implies that (1) \( \sum(\ell,C-u) = 2 \) in \( \mathbb{Z}_3 \) for each cycle \( C \in S - \{C_1\} \), (2) \( \sum(\ell,C_1-u) + \sum(\ell,C_2) = 0 \) and so \( \sum(\ell,C_1-u) = 0 \), and (3) \( 1 + \sum_{C \in S - \{C_2\}} \sum(\ell,C-u) = 0 \). However, since

\[
\sum(\ell,C_1-u) = 0,
\]

it follows that for the boundary \( B = H - V(C_2-u) \) of a region of \( H \), we have

\[
\sum(\ell,B) = 1 + \sum_{C \in S - \{C_2\}} \sum(\ell_C,C-u) = 1 + 2(k-2) = 1 + 2(3t - 1) = 2 \quad \text{in} \quad \mathbb{Z}_3,
\]

which is a contradiction. Thus, \( H \) is not zonal and so \( D(S) \) is conditionally zonal.

Case 2. \( k \not\equiv 1 \pmod{3} \). Let \( S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_k}\} \) be a set of \( k \) cycles of length \( n_i \) for \( 1 \leq i \leq k \). First, we show that \( D(S) \) has a zonal planar embedding. We consider two subcases, according to whether \( k \) is odd or \( k \) is even.

Subcase 2.1. \( k \geq 3 \) is odd. Define the planar embedding \( \tilde{D}(S) \) of \( D(S) \) such that \( C_{n_i} \) is placed inside \( C_{n_{i-1}} \) for \( 2 \leq i \leq k \). This is illustrated in Figure 8. Thus, if \( B \) is the boundary of a region of \( \tilde{D}(S) \), then \( B \) is \( C_{n_1}, C_{n_k}, \) or \( C_{n_i} \ast C_{n_{i+1}} \) for \( 1 \leq i \leq k-1 \). Let \( u \) be the center of \( \tilde{D}(S) \).

![Figure 8: A planar embedding \( \tilde{D}(S) \) of \( D(S) \) in Case 1.](image)
We show that $\tilde{D}(S)$ is zonal. For $1 \leq i \leq k$, let $Q_i = C_n_i - u$ be the path of order $n_i - 1 \geq 2$. Define a labeling $\ell : V(\tilde{D}(S)) \to \{1,2\}$ such that $\ell(u) = 1$ and

(a) $\sum(\ell, Q_i) = 2$ in $Z_3$ for each odd integers $i$ with $1 \leq i \leq k$ and

(b) $\sum(\ell, Q_i) = 0$ in $Z_3$ for each even integer $i$ with $2 \leq i \leq k - 1$.

Such a labeling in (a) and (b) is guaranteed by Lemma 3.1. Let $B$ be the boundary of a region of $\tilde{D}(S)$. If $B = C_{n_i}$ for $i = 1, k$, then $\sum(\ell, B) = 1 + 2 = 0$ in $Z_3$. If $B = C_{n_i} \ast C_{n_{i+1}}$ for $1 \leq i \leq k - 1$, then $\sum(\ell, B) = 1 + 2 = 0$ in $Z_3$. Consequently, $\ell$ is a zonal labeling of $\tilde{D}(S)$.

Subcase 2.2. $k \geq 6$ is even. Let $S_0 = \{C_{n_5}, C_{n_6}, \ldots, C_{n_k}\}$ be the subset of $k - 4$ cycles in $S$ and let $\tilde{D}(S_0)$ be the planar embedding of $D(S_0)$ such that $C_{n_i+1}$ is placed inside $C_{n_i}$ for $i = 5, 6, \ldots, k - 1$. Thus, if $B$ is the boundary of a region of $\tilde{D}(S_0)$, then $B = C_{n_3}, C_{n_4}$, or $C_{n_i} \ast C_{n_{i+1}}$ for $5 \leq i \leq k - 1$. Then $\tilde{D}(S) = C_{n_1} \ast C_{n_2} \ast C_{n_3} \ast C_{n_4} \ast \tilde{D}(S_0)$ is the planar embedding obtained by identifying the center of $\tilde{D}(S_0)$ and a vertex in $C_{n_i}$ for $i = 1, 2, 3, 4$. This identified vertex $u$ is then the center of $\tilde{D}(S)$. This is illustrated in Figure 9. Thus, if $B$ is the boundary of a region of $\tilde{D}(S)$, then $B = C_{n_i}$ for $i \in \{1, 2, 3, 4, k\}$, or $B = C_{n_i} \ast C_{n_{i+1}}$ for $5 \leq i \leq k - 1$, or $B = C_{n_1} \ast C_{n_2} \ast C_{n_3} \ast C_{n_4} \ast C_{n_5}$ where in this case $B$ is the boundary of the exterior region of $\tilde{D}(S)$.

Figure 9: A planar embedding $\tilde{D}(S)$ of $D(S)$ in Case 2.

We show that $\tilde{D}(S)$ is zonal. For $1 \leq i \leq k$, let $Q_i = C_{n_i} - u$ be the path of order $n_i - 1 \geq 2$. Define a labeling $\ell : V(\tilde{D}(S)) \to \{1,2\}$ such that $\ell(u) = 1$ and

(a) $\sum(\ell, Q_i) = 2$ in $Z_3$ for $1 \leq i \leq 4$ and for each even integer $i$ with $6 \leq i \leq k$ and

(b) $\sum(\ell, Q_i) = 0$ in $Z_3$ for each odd integer $i$ with $5 \leq i \leq k - 1$.

Again, such a labeling in (a) and (b) is guaranteed by Lemma 3.1. Let $B$ be the boundary of a region of $\tilde{D}(S)$. If $B = C_{n_i}$ for $i \in \{1, 2, 3, 4, k\}$, then $\sum(\ell, B) = 1 + 2 = 0$ in $Z_3$. If $B = C_{n_i} \ast C_{n_{i+1}}$ for $5 \leq i \leq k - 1$, then $\sum(\ell, B) = 1 + 2 = 0$ in $Z_3$. $\tilde{D}(S)$ is conditionally zonal. Let $G$ be the standard planar embedding of $D(S)$ such that the boundary of each region of $G$ is either a cycle $C \in S$ or the graph $G$. We show that the plane graph $G$ is not zonal. Assume, to the contrary, that there is a zonal labeling $\ell : V(G) \to \{1,2\}$ of $G$. We may assume that $\ell(u) = 1$ by Lemma 3.1. Since $\sum(\ell, C) = 0$ for each $C \in S$ and $\ell(u) = 1$ in $Z_3$, it follows that $\sum(\ell, C - u) = 2$ in $Z_3$. Let $B = G$ be the boundary of the exterior region of $G$, then $\sum(\ell, B) = 1 + 2k$ in $Z_3$. Since $k \neq 1 \mod 3$, it follows that $\sum(\ell, B) \neq 0$ in $Z_3$, which is a contradiction. Therefore, $\ell$ is not a zonal labeling of $G$ and so $D(S)$ is conditionally zonal.

4. Irregular Dutch windmill graphs

In this and the next section, we study Dutch windmill graphs with an irregularity or regularity property (see [1] for a discussion of irregularity in graphs). A Dutch windmill graph is irregular if no two cycles in the graph have the same length. Thus, if $S$ is a set of $k \geq 3$ distinct cycles, then the Dutch windmill graph $D(S)$ is irregular. By Theorem 3.1, for every multiset $S$ of three or more cycles, the graph $D(S)$ is conditionally zonal. For example, if $S = \{C_3, C_4, C_5, C_6\}$, then $D(S)$, which is the standard plane embedding, is irregular and zonal. A zonal labeling of $D(S)$ is shown in Figure 10.
Subsets is constructed from these definitions and notation and preliminary observations.

Let $A = \{C^1, C^2, \ldots, C^p\}$ be a set of $p \geq 2$ cycles, let $D(A)$ denote the planar embedding of the Dutch windmill graph constructed from these $p$ cycles in $A$ by placing $C^1, C^2, \ldots, C^{p-1}$ inside $C^p$. For example, if $A = \{C^1, C^2, C^3, C^4\}$ is a set of 4 cycles, then $D(A)$ is shown in Figure 11, where the three cycles $C^1, C^2, C^3$ are placed inside the cycle $C^4$. Thus, the boundary of a region in $D(A)$ is either $C^i$ for $i = 1, 2, 3, 4$ or $D(A)$.

For a set $S$ of $p \geq 4$ cycles, let $\Pi = \{S_1, S_2, \ldots, S_q\}$ be a partition of $S$ into $q \geq 2$ subsets $S_1, S_2, \ldots, S_q$. For $1 \leq i \leq q$, let $D(S_i)$ be the Dutch windmill plane graph with center $u_i$. The plane graph $D(\Pi) = D(S_1) \ast D(S_2) \ast \cdots \ast D(S_q)$ is constructed from the $q$ plane graphs $D(S_1), D(S_2), \ldots, D(S_q)$ by identifying their centers $u_1, u_2, \ldots, u_q$ and denoting the identified vertex by $u$. For example, let $S = \{C^1, C^2, \ldots, C^{16}\}$ and let $\Pi = \{S_1, S_2, S_3, S_4\}$ be a partition of $S$ into four subsets, where $S_1 = \{C^1, C^2, C^3, C^4\}, S_2 = \{C^5, C^6, C^7, C^8\}, S_3 = \{C^9, C^{10}, C^{11}, C^{12}\}$, and $S_4 = \{C^{13}, C^{14}, C^{15}, C^{16}\}$. Then $D(\Pi)$ is shown in Figure 12. In this example, if $B$ is the boundary of a region of $D(\Pi)$, then either $B = C^i$ for some $C^i \in S - \{C^4, C^8, C^{12}, C^{16}\}$, or $B = D(S_i)$ for $1 \leq i \leq 4$, or $B = C^4 \ast C^8 \ast C^{12} \ast C^{16}$ having center $u$ where the region is the exterior region of $D(S)$.

We now present the following lemma.

**Lemma 4.1.** Let $S$ be a set with $4k$ elements for some positive integer $k$. The number of partitions of $S$ into four $k$-element subsets is
\[ \prod_{i=1}^{4} \binom{4k-1}{k-1} = \binom{4k-1}{k-1} \binom{3k-1}{k-1} \binom{2k-1}{k-1}. \]
Proof. For $k = 1$, the expression in (1) is 1, which is correct since there is only one way to partition a set with 4 elements into four subsets. Thus, we may assume that $k \geq 2$. Suppose that $S = \{a_1, a_2, \ldots, a_{4k}\}$. In any partition of $S$ into four $k$-element subsets, the element $a_1$ must belong to a $k$-element subset of $S$ in this partition. The number of ways to choose a $k$-element subset of $S$ containing $a_1$ is $\binom{4k-1}{k-1}$. Once such a $k$-element subset $S_1$ of $S$ in the partition is given, let $a_i$ be an element of $S - S_1$. The number of ways to choose a $k$-element subset of $S - S_1$ containing $a_i$ is $\binom{4k-1-k}{k-1}$. Once two such disjoint $k$-element subsets $S_1$ and $S_2$ of $S$ in the partition are given, let $a_j$ be an element of $S - (S_1 \cup S_2)$. The number of ways to choose a $k$-element subset of $S - (S_1 \cup S_2)$ containing $a_j$ is $\binom{4k-1-k-k}{k-1}$. Once three such pairwise disjoint $k$-element subsets $S_1, S_2,$ and $S_3$ of $S$ in the partition are given, only $k$ elements remain in $S - (S_1 \cup S_2 \cup S_3)$, which constitutes the fourth $k$-element subset of $S$ in the partition. Therefore, the total number of such partitions is $\binom{4k-1}{k-1} \binom{4k-1-k}{k-1} \binom{4k-1-k-k}{k-1}$.

For $1 \leq i \leq 4k$, let $n_i = 10^i + 1$. If $k = 4$, then $n_1 = 11, n_2 = 101, n_3 = 1001, n_4 = 10001$, and $n_{16} = 10\ldots\ldots01$ where there are fifteen $0$s between the two $1$s. In general, for $1 \leq i \leq 16$, the first and last digits of $n_i$ are $1$ and the remaining $i - 1$ digits of $n_i$ are $0$ (where there is a total $i - 1$ $0$s between the two $1$s in $n_i$).

**Theorem 4.1.** There is an irregular Dutch windmill graph that has an arbitrarily large number of distinct zonal planar embeddings.

Proof. Let $k \geq 4$ be an integer such that $k \equiv 1 \pmod{3}$ and let $\Pi(k) = \prod_{i=1}^{\ell} \binom{4k-1}{k-1}$. We show that there is an irregular Dutch windmill graph having at least $\Pi(k)$ distinct zonal planar embeddings. For $1 \leq i \leq 4k$, let $n_i = 10^i + 1$ and let $S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{4k}}\}$ be the set of $4k$ cycles of length $n_i$ for $1 \leq i \leq 4k$. By Lemma 4.1, the number of partitions of $S$ into four $k$-element subsets is $\Pi(k)$. For $1 \leq j \leq \Pi(k)$, let $\Pi_j$ be a partition of $S$ into four $k$-element subsets and so $D(\Pi_j)$ is a planar embedding of the Dutch windmill graph $D(S)$. We show that $D(\Pi_1), D(\Pi_2), \ldots, D(\Pi_{\Pi(k)})$ are $\Pi(k)$ distinct zonal planar embeddings of $D(S)$.

First, we make an observation concerning the structural property of $D(\Pi_j)$ where $1 \leq j \leq \Pi(k)$. For example, let $\Pi_1 = \{S_1, S_2, S_3, S_4\}$ be the partition of $S$ into $k$-element subsets where

$$S_1 = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{k-1}}\},$$

$$S_2 = \{C_{n_{k+1}}, C_{n_{k+2}}, \ldots, C_{n_{2k}}\},$$

$$S_3 = \{C_{n_{2k+1}}, C_{n_{2k+2}}, \ldots, C_{n_{3k}}\},$$

$$S_4 = \{C_{n_{3k+1}}, C_{n_{3k+2}}, \ldots, C_{n_{4k}}\}.$$

If $k = 4$ and $C_{n_i} = C_{n_i}$ for $1 \leq i \leq 4$, then $D(\Pi_1)$ is shown in Figure 12. For the plane graph $D(\Pi_1)$, there are $4k + 1$ regions $R_1, R_2, \ldots, R_{4k+1}$ of $D(\Pi_1)$, where

- the $k$ regions $R_1, R_2, \ldots, R_k$ have the boundaries $C_{n_1}, C_{n_2}, \ldots, C_{n_{k-1}}$ and $D(S_1)$, respectively;
- the $k$ regions $R_{k+1}, R_{k+2}, \ldots, R_{2k}$ have the boundaries $C_{n_{k+1}}, C_{n_{k+2}}, \ldots, C_{n_{2k-1}}$ and $D(S_2)$, respectively;
- the $k$ regions $R_{2k+1}, R_{2k+2}, \ldots, R_{3k}$ have boundaries $C_{n_{2k+1}}, C_{n_{2k+2}}, \ldots, C_{n_{3k-1}}$ and $D(S_3)$, respectively;
- the $k$ regions $R_{3k+1}, R_{3k+2}, \ldots, R_{4k}$ have boundaries $C_{n_{3k+1}}, C_{n_{3k+2}}, \ldots, C_{n_{4k-1}}$ and $D(S_4)$, respectively;
- the exterior region $R_{4k+1}$ has the boundary $C_{n_k} \ast C_{n_{2k}} \ast C_{n_{3k}} \ast C_{n_{4k}}$.

In particular, the boundary of $R_k$ is $D(S_k)$ which has order $b_{1,k} = 10^{n_1} + 10^{n_2} + \cdots + 10^{n_k} + 1$, the boundary of $R_{2k}$ is $D(S_2)$ which has order $b_{1,2k} = 10^{n_{k+1}} + 10^{n_{k+2}} + \cdots + 10^{n_{2k}} + 1$, the boundary of $R_{3k}$ is $D(S_3)$ which has order $b_{1,3k} = 10^{n_{2k+1}} + 10^{n_{2k+2}} + \cdots + 10^{n_{3k}} + 1$, and the boundary of $R_{4k}$ is $D(S_4)$ which has order $b_{1,4k} = 10^{n_{3k+1}} + 10^{n_{3k+2}} + \cdots + 10^{n_{4k}} + 1$.

First, we show that the planar embeddings $D(\Pi_1), D(\Pi_2), \ldots, D(\Pi_{\Pi(k)})$ of $D(S)$ are distinct. Let $i_1, i_2 \in \{1, 2, \ldots, \Pi(k)\}$ such that $i_1 \neq i_2$. From the way in which the plane graphs $D(\Pi_j)$ where $1 \leq j \leq \Pi(k)$ are constructed, it follows that

$$\{b_{i_1,1,k}, b_{i_1,2k}, b_{i_1,3k}, b_{i_1,4k}\} \neq \{b_{i_2,1,k}, b_{i_2,2k}, b_{i_2,3k}, b_{i_2,4k}\}.$$

Hence, $D(\Pi_1)$ and $D(\Pi_2)$ are distinct. Consequently, $D(\Pi_1), D(\Pi_2), \ldots, D(\Pi_{\Pi(k)})$ are distinct planar embeddings of $D(S)$.

Next, we show that each $D(\Pi_j)$ is zonal for $1 \leq j \leq \Pi(k)$. It suffices to show that $D(\Pi_1)$ is zonal since the argument for showing that $D(\Pi_j)$ is zonal for $2 \leq j \leq \Pi(k)$ is similar. By Theorem 3.1, the plane graph $D(S_1)$ is zonal for $i = 1, 2, 3, 4$ and so there is a zonal labeling $\ell_i$ of $D(S_i)$. Let $u$ be the center of $D(\Pi_1)$. By Observation 2.1, we may assume that $\ell_i(u) = 1$ for $1 \leq i \leq 4$. Define a labeling $\ell$ of $D(\Pi_1)$ by $\ell(v) = \ell_i(v)$ if $v$ belongs to $D(S_i)$ for $1 \leq i \leq 4$. We show that $\ell$ is a zonal labeling of $D(\Pi_1)$. Let $B$ be the boundary of a region $R$ of $D(\Pi_1)$. If $R$ is an interior region of $D(\Pi_1)$, then $R$ is a region of $D(S_i)$ for some integer $i$ with $1 \leq i \leq 4$ and so $\sum(\ell, B) = \sum(\ell, B) = 0$. Thus, we may assume that $R$ is the exterior region of $D(\Pi_1)$ and so $B = C_{n_k} \ast C_{n_{2k}} \ast C_{n_{3k}} \ast C_{n_{4k}}$ whose center is $u$. Since $\sum(\ell, C_{n_k}) = 0$ in $Z_3$ (that is, the value of the boundary of the exterior region of $D(S_i)$ is 0) and $\ell_i(u) = 1$ for $1 \leq i \leq 4$, it follows that $\sum(\ell, C_{n_{ik}} - u) = 2$ in $Z_3$. A. Bowling and P. Zhang / Electron. J. Math. 4 (2022) 1–11
Hence, $\sum(\ell, B) = 1 + \sum_{i=1}^{k} (\ell_i, C_{n_{ik}} - u) = 1 + 4 \cdot 2 = 0$ in $\mathbb{Z}_3$. Therefore, $D(P_1)$ is zonal. Consequently, $D(P_1), D(P_2), \ldots, D(P_{n(k)})$ are distinct zonal planar embeddings of $D(S)$.

Let $N$ be an arbitrarily large positive integer. Since $\lim_{k \to \infty} II(k) = \infty$, it follows that there is an integer $k_0$ such that $k_0 \equiv 1 \pmod{3}$ and $II(k_0) > N$. Let $S = \{C_{n_1}, C_{n_2}, \ldots, C_{n_{k_0}}\}$. Then the Dutch windmill graph $D(S)$ has at least $II(k_0) > N$ distinct zonal planar embeddings.

For an integer $k \geq 4$ and $k \equiv 1 \pmod{3}$, the irregular Dutch windmill graph used to verify Theorem 4.1 has order $(10^{4k+1} - 1)/9$. An irregular Dutch windmill graph of smaller order can be used to verify Theorem 4.1 by changing the base integer of each integer $n_i$ $(1 \leq i \leq 4k)$ from 10 to a smaller base. For example, if we let $n_i = 2^i + 1$ (using base 2), the same proof applies and the order of the irregular Dutch windmill graph is $2^{4k+1} - 1$.

5. A special class of Dutch windmill graphs

In the proof of Theorem 4.1, no two cycles in the irregular Dutch windmill graph have the same length and we were able to obtain an arbitrarily large number of planar embeddings of the graph such that the structure of these embeddings are similar. If the cycles of a Dutch windmill graph all have the same length (and is consequently a regular Dutch windmill graph), then this proof does not provide the desired result. In this case, however, by varying the embedding of the Dutch windmill graph, the same conclusion can be obtained.

Before presenting the next result, we construct a sequence $F_1, F_2, F_3, \ldots$ of plane graphs recursively as follows. The plane graph $F_1 = D^4_3$ consists of a triangle $T_1$ within which are three triangles, as indicated in Figure 13. The vertex of degree 8 in $F_1$ is the center of $F_1$. The plane graph $F_2$ is constructed from three copies of $F_1$ and a triangle $T_2$ by placing the three copies of $F_1$ inside $T_2$ and identifying their centers with a vertex of $T_2$. Thus, the identified vertex is the center of $F_2$ and has degree 26 in $F_2$, as indicated in Figure 13. For $k \geq 3$, the plane graph $F_k$ is constructed by placing three copies of $F_{k-1}$ inside a triangle $T_k$ and identifying their centers with a vertex of $T_k$. The plane graph $F_k$ is shown in Figure 13 whose center has degree 80. Observe that the boundary of every region of $F_k$ either has order 3 or 9 for all $k \geq 1$. Therefore, if every vertex of $F_k$ were to be labeled 1 in $\mathbb{Z}_3$, then each region would have the label 0 in $\mathbb{Z}_3$. The region of $F_k$ whose boundary is the triangle $T_k$ is referred to as the exterior region of $F_k$ and each of the other regions is referred to as an interior region of $F_k$.

![Figure 13: The plane graphs $F_1 = D^4_3, F_2 = D^4_1, F_3 = D^4_2$.](image)

For a positive integer $k$, let $t_k$ denote the number of triangles in $F_k$. Then

$$t_k = 1 + 3 + 3^2 + 3^3 + \cdots + 3^k = \frac{3^{k+1} - 1}{2}.$$ 

For example, $t_1 = 4, t_2 = 13$, and $t_3 = 40$. For $k \geq 1$, the plane graph $F_k$ is a specific planar embedding of the Dutch windmill graph $D^4_3$ of order $2t_k + 1 = 3^k+1$. Furthermore, $F_k$ is a planar embedding of the friendship graph $(t_k K_2) \vee K_1$. The graph $F_k$ therefore has one vertex of degree $2t_k$ and $2t_k$ vertices of degree 2.

Proposition 5.1. For a positive integer $k$, the plane graph $F_k$ is zonal.

Proof. For $k \geq 1$, define a labeling $\ell_k : V(F_k) \to \{1, 2\}$ by $\ell_k(x) = 1$ for every vertex $x$ of $F_k$. Since the boundary $B$ of a region of $F_k$ has order 3 or 9, it follows that $\sum(\ell, B) = 0$ in $\mathbb{Z}_3$.

For a positive integer $k$, the integer $\pi(k)$ is defined as

$$\pi(k) = t_1t_2\cdots t_k = \prod_{i=1}^{k} \left(\frac{3^{i+1} - 1}{2}\right) = \frac{1}{2^k} \prod_{i=1}^{k} (3^{i+1} - 1).$$
Thus, \( \pi(1) = t_1 = 4, \pi(2) = t_1t_2 = 4 \cdot 13 = 52, \) and \( \pi(3) = t_1t_2t_3 = 4 \cdot 13 \cdot 40 = 2080. \) Since \( 3^{i+1} \equiv 0 \pmod{3}, \) where \( 1 \leq i \leq k, \) it follows that \( 3^{i+1} - 1 \equiv 2 \pmod{3} \) and so
\[
\frac{3^{i+1} - 1}{2} \equiv 1 \pmod{3}.
\]
Therefore, \( \pi(k) \equiv 1 \pmod{3} \) and \( 2\pi(k) + 1 \equiv 0 \pmod{3} \) for every positive integer \( k. \) By Theorem 3.1, the Dutch windmill graph \( D_3^{\pi(k)} \) is zonal. For \( 1 \leq i \leq k, \) let
\[
s_i = \frac{\pi(k)}{t_i}.
\]
Then \( s_i \equiv 1 \pmod{3} \) for \( 1 \leq i \leq k \) and so \( 2s_i + 1 \equiv 0 \pmod{3}. \) Since \( t_1 < t_2 < \ldots < t_k \) for a fixed positive integer \( k, \) it follows that \( s_1 > s_2 > \ldots > s_k. \)

For a Dutch windmill plane graph \( H, \) let \( H_1, H_2, \ldots, H_p \) be \( p \) copies of \( H \) and let \( v_i \) be the center of \( H_i \) for \( 1 \leq i \leq p. \) Then \( p \ast H \) denotes the Dutch windmill plane graph constructed from \( H_1, H_2, \ldots, H_p \) by identifying the \( p \) vertices \( v_1, v_2, \ldots, v_p \) and denoting the identified vertex by \( v. \) For example, let \( F_2 \) be the planar embedding of the Dutch windmill graph \( D_3^2 \) shown in Figure 13 and \( p = 4, \) then the plane graph \( 4 \ast F_2 \) is shown in Figure 14. Thus, \( 4 \ast F_2 \) is a planar embedding of the Dutch windmill graph \( D_3^2. \)

**Figure 14:** The plane graph \( 4 \ast F_2. \)

### Theorem 5.1
There is a regular Dutch windmill graph that has an arbitrarily large number of distinct zonal planar embeddings.

**Proof.** Let \( k \) be a positive integer. We show that the Dutch windmill graph \( D_3^{\pi(k)} \) has at least \( k \) distinct zonal planar embeddings. For \( k = 1, \) the zonal planar embedding of \( D_3^1 = D_3^{\pi(1)} \) shown in Figure 15 is zonal. Hence, we may assume that \( k \geq 2. \) Here we show that there are \( k \) distinct zonal planar embeddings of \( D_3^{\pi(k)} \). Let \( G_{k,1} \) be the planar embedding of \( D_3^{\pi(k)} \) such that the boundary of each interior region is a 3-cycle and the boundary of the exterior region is \( D_3^{\pi(k)}. \) Since \( D_3^{\pi(k)} \) has order \( 2\pi(k) + 1 \equiv 0 \pmod{3}, \) the labeling that assigns the label 1 in \( \mathbb{Z}_3 \) to each vertex of \( G_{k,1} \) is a zonal labeling. For \( 2 \leq i \leq k, \) let \( s_i = \frac{\pi(k)}{t_i} \) and let \( G_{k,i} = t_{i-1} \ast F_{i-1} \) be the planar embedding of \( D_3^{\pi(k)}. \) We show that \( G_{k,1}, G_{k,2}, \ldots, G_{k,k} \) are \( k \) distinct zonal planar embeddings of \( D_3^{\pi(k)}. \) Observe that if \( B \) is the boundary of a region of \( G_{k,i}, \) where \( 2 \leq i \leq k, \) then the order of \( B \) is 3, 9 or \( 2s_{i-1} + 1, \) where \( 2s_{i-1} + 1 \equiv 0 \pmod{3}. \) Since \( \pi(k) > s_1 > s_2 > \ldots > s_{k-1}, \) these planar embeddings are distinct. By assigning the label 1 in \( \mathbb{Z}_3 \) to each vertex of \( G_{k,i} \) for \( 2 \leq i \leq k, \) we see that the plane graph \( G_{k,i} \) is zonal. 

**Figure 15:** The zonal plane graph \( D_3^4. \)

Let \( N \) be an arbitrary positive integer. Since \( \lim_{k \to \infty} \pi(k) = \infty, \) it follows that there is an integer \( k_0 \) such that \( k_0 \equiv 1 \pmod{3} \) and \( \pi(k_0) > N. \) Then the Dutch windmill graph \( D_3^{\pi(k_0)} \) has at least \( \pi(k_0) > N \) distinct zonal planar embeddings.
We illustrate the proof of Theorem 5.1 for $k = 2, 3$.

* If $k = 2$, then $\pi(2) = 52$ and $t_1 = 4$. Thus, $s_1 = 52/4 = 13$. Let $G_{2, 1}$ be the planar embedding of $D_3^{52}$, where the boundary of each interior region is a 3-cycle and the boundary of the exterior region is $D_3^{13}$ and let $G_{2, 2} = s_1 \ast F_1 = 13 \ast F_1$. Here, $G_{2, 1}$ and $G_{2, 2}$ are distinct planar embeddings of $D_3^{52}$.

* If $k = 3$, then $\pi(3) = 2080$, $t_1 = 4$, and $t_2 = 13$. Thus, $s_1 = 2080/4 = 520$ and $s_2 = \pi(3)/t_2 = 2080/13 = 160$. Let $G_{3, 1}$ be the planar embedding of $D_3^{2080}$ where the boundary of each interior region is a 3-cycle and the boundary of the exterior region is $D_3^{160}$, let $G_{3, 2} = s_1 \ast F_1 = 520 \ast F_1$, and let $G_{3, 3} = s_2 \ast F_2 = 160 \ast F_2$. Here, $G_{3, 1}, G_{3, 2},$ and $G_{3, 3}$ are distinct planar embeddings of $D_3^{2080}$.

As we saw in the proof of Theorem 5.1, for a given positive integer $k$, the graph $G = D_3^{\mu(k)}$ has $k$ distinct zonal planar embeddings. Since the graph $G$ has order $2\pi(k) + 1$, which is large for a large positive integer $k$, a question here is whether there is a graph of smaller order with this property. We now discuss this question.

For a positive integer $k$, let $\mu(k) = \text{lcm}\{t_1, t_2, \ldots, t_k\}$ be the least common multiple of $t_1, t_2, \ldots, t_k$. Since $t_i = \frac{3^{i+1}-1}{2}$ (mod 3) for $1 \leq i \leq k$, it follows that $3 \mid t_i$ and so $3 \mid \mu(k)$. Thus, $\mu(k) \equiv 1$ (mod 3) or $\mu(k) \equiv 2$ (mod 3). If $\mu(k) \equiv 2$ (mod 3), then $2\mu(k) \equiv 1$ (mod 3). Hence, $\mu(k) \equiv 1$ (mod 3) or $2\mu(k) \equiv 1$ (mod 3). Consequently, either $\frac{\mu(k)}{k} \equiv 1$ (mod 3) for $i \in \{1, 2, \ldots, k\}$ or $2\mu(k) \equiv 1$ (mod 3) for $i \in \{1, 2, \ldots, k\}$. If $k$ is sufficiently large, then $\pi(k)$ is substantially larger than $\mu(k)$ or $2\mu(k)$. Applying the argument in the proof of Theorem 5.1, we have the following result.

**Proposition 5.2.** Let $k$ be a positive integer.

* If $\mu(k) \equiv 1$ (mod 3), then the zonal Dutch windmill graph $D_3^{\mu(k)}$ is conditionally zonal and has at least $k$ distinct zonal planar embeddings.

* If $\mu(k) \equiv 2$ (mod 3), then the zonal Dutch windmill graph $D_3^{2\mu(k)}$ is conditionally zonal and has at least $k$ distinct zonal planar embeddings.

We now illustrate Proposition 5.2 for integers $k$ with $1 \leq k \leq 12$. In this case, expressing each integer $t_k$ and $\mu(k)$ as a product of primes, we have the following, where each underlined integer is congruent to 2 modulo 3.

| $t_1 = 4 = 2^2$ | $\mu(1) = 2^4$ |
| $t_2 = 13$ | $\mu(2) = 2^2 \cdot 13$ |
| $t_3 = 40 = 2^3 \cdot 5$ | $\mu(3) = 2^3 \cdot 5 \cdot 13$ |
| $t_4 = 121 = 11^2$ | $\mu(4) = 2^4 \cdot 5 \cdot 11^2 \cdot 13$ |
| $t_5 = 364 = 2^2 \cdot 7 \cdot 13$ | $\mu(5) = 2^3 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13$ |
| $t_6 = 733$ | $\mu(6) = 2^4 \cdot 5 \cdot 7 \cdot 11^2 \cdot 13$ |
| $t_7 = 2200 = 2^3 \cdot 5^2 \cdot 11$ | $\mu(7) = 2^3 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13$ |
| $t_8 = 6601 = 7 \cdot 23 \cdot 41$ | $\mu(8) = 2^3 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13$ |
| $t_9 = 19,804 = 2^2 \cdot 4951$ | $\mu(9) = 2^4 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13$ |
| $t_{10} = 59,413 = 19 \cdot 53 \cdot 59$ | $\mu(10) = 2^4 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23$ |
| $t_{11} = 178,240 = 2^6 \cdot 5 \cdot 557$ | $\mu(11) = 2^6 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23 \cdot 41$ |
| $t_{12} = 534,721 = 11 \cdot 48611$ | $\mu(12) = 2^{10} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 23 \cdot 41$ |

Hence, $\mu(k) \equiv 1$ (mod 3) for $1 \leq k \leq 6$, while $\mu(k) \equiv 2$ (mod 3) for $7 \leq k \leq 11$ and $\mu(12) \equiv 1$ (mod 3). Therefore, if $1 \leq k \leq 6$ or $k = 12$, then the zonal Dutch windmill graph $D_3^{\mu(k)}$ is conditionally zonal and has at least $k$ distinct zonal planar embeddings; while if $7 \leq k \leq 11$, then the zonal Dutch windmill graph $D_3^{2\mu(k)}$ is conditionally zonal and has at least $k$ distinct zonal planar embeddings.

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**References**

