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Research Article

Integral inequalities via general form of Riemann-Liouville fractional integrals

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Abstract

The primary goal of this paper is to obtain Hadamard type integral inequalities for a general fractional integral operator. New upper bounds of Hadamard type for a class of m-convex functions are gained with the help of the ω-Riemann-Liouville integral operator. For some special cases, the effectiveness of the procured results are demonstrated by obtaining some particular inequalities.

Keywords: Hermite-Hadamard inequality; m-convex functions; fractional integral operators.

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1. Introduction

In search of new results and possible generalizations in the theory of inequalities involving functions, some additional features of the considered functions are needed as well as some limitations in the conditions on functions may also be fruitful. When several functions exhibit similar properties, they may be considered in one class of functions, and it is also possible that such functions' class may be converted into another class of functions under special conditions. A similar remark applies to fractional integral operators. Many integral operators have been identified in the recent past that differ in terms of their properties and applications, and so they can be categorized into different classes. The definition of a convex function indicates a close connection between inequalities' theory and convex analysis, and it is given with the help of an inequality as follows.

Definition 1.1 (See [6]). *Let I be a subset of $\mathbb{R}$ and $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping. If f is convex function, then the following condition is valid:*

$$f\left(\tau u+(1-\tau)v\right) \leq \tau f(u)+(1-\tau)f(v)$$

for all $u, v \in I$ and $\tau \in [0,1]$.

The class of m-convex functions, defined by Toader [21], is the focus of many researchers.

Definition 1.2 (See [21]). *Assume that $m \in (0,1]$ and $f : [0,b] \to \mathbb{R}$. The function f is said to be m-convex if the following condition is valid:*

$$f\left(\tau u+m(1-\tau)v\right) \leq \tau f(u)+m(1-\tau)f(v)$$

for all $u, v \in [0,b]$ and $\tau \in [0,1]$.

Note that one obtains the definition of convexity when $m = 1$. Starting with the 17^{th} century, many mathematicians have been giving their own notation and approaches to the definitions of fractional derivatives and integrals. The existence of more than one definition requires researchers to use the most appropriate one according to the type of the considered problem. In recent years, fractional derivatives and integrals have been in a continuous development process with the theory of inequality as well as the closely related concept of convexity.

Now, we proceed with some of the fractional integral operators.

Definition 1.3 (See [18]). *Suppose that $f \in L_1[u,v]$ and $\alpha > 0$. The Riemann-Liouville fractional operator $J^{\alpha}_{u^+}f$ and $J^{\alpha}_{v^-}f$ are defined as*

$$J^{\alpha}_{u^+}f(\theta) = \frac{1}{\Gamma(\alpha)}\int_u^{\theta}(\theta-\tau)^{\alpha-1}f(\tau)\,d\tau, \quad \tau > u$$

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and

$$J_{v-}^{\alpha}f(\theta)=\frac{1}{\Gamma(\alpha)}\int_{\theta}^{v}(\tau-\theta)^{\alpha-1}f(\tau)\,d\tau,\quad \tau<v$$

respectively, where

$$\Gamma(\alpha)=\int_0^{\infty}e^{-t}t^{\alpha-1}dt.$$

In case of $\alpha=0$, it is obvious that $J_{u+}^{0}f(\theta)=J_{v-}^{0}f(\theta)=f(\theta)$.

We remark here that the definitions of the Riemann-Liouville fractional operator and standard integral coincide when $\alpha=1$. Using the Riemann-Liouville fractional operator, numerous integral inequalities and generalizations of the existing results for the classical integral have already been obtained. Some of the studies involving fractional integral operators and derivatives can be found in [1–5, 7, 9–17, 19, 22].

We now give the definition as well as some basic properties of the ω-Riemann-Liouville integral operator, which play a key role in obtaining the main results of the present study.

Definition 1.4 (See [20])**.** *Let (a,b), $-\infty\le a<b\le\infty$, be a subset in $\mathbb{R}$ and $\alpha>0$. Assume that $\omega(x):(a,b]\to\mathbb{R}^{+}$ is an increasing mapping and $\omega'(x)$ is continuous on (a,b). The following fractional integral operators:*

$$I_{a+}^{\alpha,\omega}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\omega'(t)\,(\omega(x)-\omega(t))^{\alpha-1}f(t)\,dt,\qquad x\ge a$$

and

$$I_{b-}^{\alpha,\omega}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\omega'(t)\,(\omega(x)-\omega(t))^{\alpha-1}f(t)\,dt,\qquad b\ge x$$

are called left- and right-sided ω-Riemann-Liouville fractional (ω-RLF) integrals of a function f with respect to another function ω on $[a,\ b]$, respectively.

The left- and right-sided ω-RLF integrals satisfy the following properties for $\beta>0$:

$$(i).\ I_{a+}^{\alpha,\omega}(\omega(x)-\omega(a))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\omega(x)-\omega(a))^{\beta+\alpha-1}.$$

$$(ii).\ I_{b-}^{\alpha,\omega}(\omega(b)-\omega(x))^{\beta-1}=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\omega(b)-\omega(x))^{\beta+\alpha-1}.$$

Lemma 1.1 (See [20])**.** *Let f be a continuous mapping, $\Re(\alpha)>0$ and $\Re(\beta)>0$. Then the following properties hold for ω-Riemann-Liouville fractional integrals:*

$$I_{a+}^{\alpha:\omega}I_{a+}^{\beta:\omega}f(x)=I_{a+}^{\alpha+\beta:\omega}f(x)$$

and

$$I_{b-}^{\alpha:\omega}I_{b-}^{\beta:\omega}f(x)=I_{b-}^{\alpha+\beta:\omega}f(x).$$

Remark 1.1. *We note that ω-RLF integrals are the generalizations of some other operators. Here, we mention some certain cases.*

(i). *Under the assumption $\omega(x)=x$, ω-RLF integrals and the left- and right-sided Riemann-Liouville fractional integrals coincide.*

(ii). *If we set $\omega(x)=\frac{x^{\rho}}{\rho}$, $\rho>0$, we get the Katugampola fractional integrals:*

$$\left({}^{\rho}I_{a+}^{\alpha}f\right)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{\tau^{\rho-1}f(\tau)}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}d\tau.$$

(iii) *If we take $\omega(x)=\ln x$, w obtain the Hadamard fractional operator:*

$$\left(\mathrm{I}_{a+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{\tau}\right)^{\alpha-1}f(\tau)\frac{d\tau}{\tau}.$$

In the next section, we obtain some Hadamard-type inequalities by using the ω-RLF operator and consider some special cases of the obtained results.

2. Main results

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}^+$ be an m-convex mapping and $\omega : [a,b] \to \mathbb{R}$ be a differentiable, strictly increasing and real valued mapping such that $\omega' \in L[a,b]$. Then, the following inequality holds:*

$$\begin{aligned}\Gamma(\alpha) I_{a^+}^{\alpha:\omega} f(mt) + \Gamma(\beta) I_{b^-}^{\beta:\omega} f(mt) \leq\ & \frac{(\omega(mt) - \omega(a))^{\alpha-1}}{mt-a}\Bigg[(mt-a)(\omega(mt) mf(t) - \omega(a) f(a)) \\ & - (mf(t) - f(a)) \int_a^{mt} \omega(k)\, dk\Bigg] \\ & + \frac{(\omega(b) - \omega(mt))^{\alpha-1}}{b-mt}\Bigg[(b-mt)(f(b)\omega(b) - mf(t)\omega(mt)) \\ & - (f(b) - mf(t)) \int_{mt}^{b} \omega(k)\, dk\Bigg]\end{aligned}$$

for ω-RLF integrals where $t \in [a,b]$, $m \in (0,1]$ and $\alpha, \beta \geq 1$.

Proof. Since the mapping ω is differentiable and strictly increasing, we can write

$$\omega'(k)(\omega(mt) - \omega(k))^{\alpha-1} \leq \omega'(k)(\omega(mt) - \omega(a))^{\alpha-1} \tag{1}$$

for $t \in [a,b]$, $m \in (0,1]$, $k \in [a, mt]$, and $\alpha \geq 1$. Thus, if we use m-convexity of f, it is clear that

$$f(k) \leq \frac{mt-k}{mt-a} f(a) + \frac{k-a}{mt-a} mf(t). \tag{2}$$

By multiplying the inequalities (1) and (2) side by side, then by integrating with respect to k on $[a, mt]$, we get

$$\int_a^{mt} (\omega(mt) - \omega(k))^{\alpha-1} \omega'(k) f(k)\, dk \leq \frac{(\omega(mt) - \omega(a))^{\alpha-1}}{mt-a}\left[f(a)\int_a^{mt} (mt-k)\omega'(k)\, dk + mf(t)\int_a^{mt}(k-a)\omega'(k)\, dk\right].$$

By making use of the computations with taking into account the definition of ω-RLF integrals in (3), we get

$$\Gamma(\alpha) I_{a^+}^{\alpha:\omega} f(mt) \leq \frac{(\omega(mt) - \omega(a))^{\alpha-1}}{mt-a}\left[(mt-a)(\omega(mt) mf(t) - \omega(a) f(a)) - (mf(t) - f(a))\int_a^{mt}\omega(k)\, dk\right].$$

On the other hand, we have

$$\omega'(k)(\omega(k) - \omega(mt))^{\alpha-1} \leq \omega'(k)(\omega(b) - \omega(mt))^{\beta-1} \tag{3}$$

for $t \in [a,b]$, $k \in [mt, b]$ and $\beta \geq 1$. Since f is an m-convex function, we have

$$f(k) \leq \frac{b-k}{b-mt} mf(t) + \frac{k-mt}{b-mt} f(b). \tag{4}$$

Similarly, by multiplying the inequalities (3) and (4) side by side, and then by integrating with respect to k on $[mt, b]$, we deduce

$$\int_{mt}^{b} (\omega(k) - \omega(mt))^{\beta-1} \omega'(k) f(k)\, dk \leq \frac{(\omega(b) - \omega(mt))^{\beta-1}}{b-mt}\left[mf(t)\int_{mt}^{b}(b-k)\omega'(k)\, dk + f(b)\int_{mt}^{b}(k-mt)\omega'(k)\, dk\right].$$

After simple computations, we obtain

$$\Gamma(\beta) I_{b^-}^{\beta:\omega} f(mt) \leq \frac{(\omega(b) - \omega(mt))^{\beta-1}}{b-mt}\left[(b-mt)(\omega(b) f(b) - \omega(mt) mf(t)) - f(b) + mf(t)\int_{mt}^{b}\omega(k)\, dk\right]. \tag{5}$$

By adding (3) and (5), we get the required inequality. □

Remark 2.1. *In case of $m = 1$ in Theorem 2.1, we capture Theorem 1 of [8].*

Corollary 2.1. *Assume that all the assumptions of Theorem 2.1 are satisfied. If $\alpha=\beta$ in Theorem 2.1, we have the following new result:*

$$\begin{aligned}\Gamma(\alpha)\left(I_{a^+}^{\alpha:\omega}f(mt)+I_{b^-}^{\alpha:\omega}f(mt)\right) &\leq \frac{(\omega(mt)-\omega(a))^{\alpha-1}}{mt-a}\Bigg[(mt-a)(\omega(mt)mf(t)-\omega(a)f(a))\\ &\quad-(mf(t)-f(a))\int_a^{mt}\omega(k)\,dk\Bigg]\\ &\quad+\frac{(\omega(b)-\omega(mt))^{\alpha-1}}{b-mt}\Bigg[(b-mt)(f(b)\omega(b)-mf(t)\omega(mt))\\ &\quad-(f(b)-mf(t))\int_{mt}^{b}\omega(k)\,dk\Bigg].\end{aligned}$$

Theorem 2.2. *Suppose that $f,\omega:[a,b]\to\mathbb{R}$ are two functions such that $|f'|$ is m-convex, ω is differentiable, strictly increasing and real-valued function provided that $\omega'\in L[a,b]$. The following inequality holds for ω-RLF integrals:*

$$\begin{aligned}&\left|\Gamma(\alpha+1)I_{a^+}^{\alpha:\omega}f(mt)+\Gamma(\beta+1)I_{b^-}^{\beta:\omega}f(mt)-\left[f(a)(\omega(mt)-\omega(a))^{\alpha}+f(b)(\omega(b)-\omega(mt))^{\beta}\right]\right|\\ \leq\ &\frac{(\omega(mt)-\omega(a))^{\alpha}(mt-a)|f'(a)|+(\omega(b)-\omega(mt))^{\beta}(b-mt)|f'(b)|}{2}\\ &+\frac{m|f'(t)|}{2}\left((\omega(mt)-\omega(a))^{\alpha}(mt-a)+(\omega(b)-\omega(mt))^{\beta}(b-mt)\right)\end{aligned}$$

where $t\in[a,b]$, $m\in(0,1]$ and $\alpha,\beta>0$.

Proof. From the m-convexity of $|f'|$, it follows that

$$|f'(k)|\leq\frac{mt-k}{mt-a}|f'(a)|+\frac{k-a}{mt-a}m|f'(t)|. \tag{6}$$

Since ω is increasing, we have

$$(\omega(mt)-\omega(k))^{\alpha}\leq(\omega(mt)-\omega(a))^{\alpha} \tag{7}$$

for $mt\in[a,b]$, $k\in[a,mt]$ and $\alpha>0$. By multiplying the inequalities (6) and (7) side by side and integrating with respect to k on $[a,mt]$, we get

$$\begin{aligned}\int_a^{mt}(\omega(mt)-\omega(k))f'(k)\,dk &\leq\frac{(\omega(mt)-\omega(a))^{\alpha}}{mt-a}\left[|f'(a)|\int_a^{mt}(mt-k)\,dk+m|f'(t)|\int_a^{mt}(k-a)\,dk\right]\\ &=(\omega(mt)-\omega(a))^{\alpha}(mt-a)\left(\frac{|f'(a)|+m|f'(t)|}{2}\right).\end{aligned} \tag{8}$$

This completes the proof of the one side of the desired result. By applying the same process, we obtain

$$\begin{aligned}\int_a^{mt}(\omega(mt)-\omega(k))^{\alpha}f'(k)\,dk &= f(t)(\omega(mt)-\omega(k))^{\alpha}\Big|_a^{mt}+\alpha\int_a^{mt}(\omega(mt)-\omega(k))^{\alpha-1}f(k)\omega'(k)\,dk\\ &=\Gamma(\alpha+1)I_{a^+}^{\alpha:\omega}f(mt)-f(a)(\omega(mt)-\omega(a))^{\alpha}.\end{aligned} \tag{9}$$

From (8) and (9), we obtain

$$\Gamma(\alpha+1)I_{a^+}^{\alpha:\omega}f(mt)-f(a)(\omega(mt)-\omega(a))^{\alpha}\leq(mt-a)(\omega(mt)-\omega(a))^{\alpha}\left(\frac{|f'(a)|+m|f'(t)|}{2}\right). \tag{10}$$

By considering the inequality (6) again, we write

$$f'(k)\geq-\left(\frac{mt-k}{mt-a}|f'(a)|+\frac{k-a}{mt-a}m|f'(t)|\right). \tag{11}$$

By making use of the necessary computations for (11), we get

$$\left|\Gamma(\alpha+1)I_{a^+}^{\alpha:\omega}f(mt)-f(a)(\omega(mt)-\omega(a))^{\alpha}\right|\leq(mt-a)(\omega(mt)-\omega(a))^{\alpha}\left(\frac{|f'(a)|+m|f'(t)|}{2}\right). \tag{12}$$

Similarly, we have

$$\left|f'(k)\right| \leq \frac{b-k}{b-mt} m\left|f'(t)\right| + \frac{k-mt}{b-mt}\left|f'(b)\right| \tag{13}$$

and

$$\left(\omega(k)-\omega(mt)\right)^{\beta} \leq \left(\omega(b)-\omega(mt)\right)^{\beta}. \tag{14}$$

By proceeding with a similar argument, we obtain

$$\left|\Gamma(\beta+1) I_{b^-}^{\beta:\omega} f(mt) - f(b)\left(\omega(b)-\omega(mt)\right)^{\beta}\right| \leq (b-mt)\left(\omega(b)-\omega(mt)\right)^{\beta}\left(\frac{\left|f'(b)\right| + m\left|f'(t)\right|}{2}\right). \tag{15}$$

By using the triangle inequality for (12) and (15), we get the result. □

Corollary 2.2. *Assume that all the assumptions of Theorem 2.2 are satisfied. If $\alpha=\beta$ in Theorem 2.2, we have the following new inequality:*

$$\begin{aligned}
&\left|\Gamma(\alpha+1)\left(I_{a^+}^{\alpha:\omega} f(mt) + I_{b^-}^{\alpha:\omega} f(mt)\right) - \left[f(a)\left(\omega(mt)-\omega(a)\right)^{\alpha} + f(b)\left(\omega(b)-\omega(mt)\right)^{\alpha}\right]\right| \\
\leq\ & \left(\omega(mt)-\omega(a)\right)^{\alpha}(mt-a)\left(\frac{\left|f'(a)\right| + m\left|f'(t)\right|}{2}\right) + \left(\omega(b)-\omega(mt)\right)^{\alpha}(b-mt)\left(\frac{\left|f'(b)\right| + m\left|f'(t)\right|}{2}\right)
\end{aligned}$$

Remark 2.2. *In case of $m=1$ in Theorem 2.2, we obtain Theorem 2 of [8].*

Theorem 2.3. *Suppose that $f,\omega : [u,v] \to \mathbb{R}$ are two functions such that $|f'|$ is m-convex, ω is differentiable, strictly increasing and real-valued function provided that $\omega' \in L[u,v]$. The following inequality holds for ω-RLF integrals:*

$$\begin{aligned}
f\left(\frac{u+mv}{2}\right) &\leq \frac{\Gamma(\alpha+1)}{2(mv-u)^{\alpha}}\left[I_{\omega^{-1}(u)^+}^{\alpha,\omega}(f\circ\omega)\left(\omega^{-1}(mv)\right) + m^{\alpha+1} I_{\omega^{-1}(v)^-}^{\alpha,\omega}(f\circ\omega)\left(\omega^{-1}\left(\frac{u}{m}\right)\right)\right] \\
&\leq \frac{\alpha}{2(\alpha+1)} f(u) - \frac{m^2}{2}\left(\frac{2\alpha+1}{\alpha+1}\right) f\left(\frac{u}{m^2}\right) + \frac{m}{2} f(v)
\end{aligned} \tag{16}$$

where $0 \leq u \leq mv$, $\alpha > 0$ and $m \in (0,1]$.

Proof. Using the Hadamard inequality for the m-convexity, we get

$$f\left(\frac{\zeta+m\xi}{2}\right) \leq \frac{f(\zeta)+mf(\xi)}{2} \tag{17}$$

for $\zeta,\xi \in [u,mv]$ and $\tau \in [0,1]$. By changing of the variables as $\zeta = \tau u + m(1-\tau)v$ and $\xi = \tau v + \frac{1}{m}(1-\tau)u$, it yields

$$2f\left(\frac{u+mv}{2}\right) \leq f\left(\tau u + m(1-\tau)v\right) + mf\left(\tau v + \frac{1}{m}(1-\tau)u\right). \tag{18}$$

By multiplying the both sides of (18) with $\tau^{\alpha-1}$ and integrating on $[0,1]$, we get

$$\frac{2}{\alpha} f\left(\frac{u+mv}{2}\right) \leq \int_0^1 \tau^{\alpha-1} f\left(\tau u + m(1-\tau)v\right)d\tau + m\int_0^1 \tau^{\alpha-1} f\left(\tau v + \frac{1}{m}(1-\tau)u\right)d\tau \tag{19}$$

Thus,

$$\begin{aligned}
&\frac{\Gamma(\alpha)}{(mv-u)^{\alpha}}\left[I_{\omega^{-1}(u)^+}^{\alpha,\omega}(f\circ\omega)\left(\omega^{-1}(mv)\right) + m^{\alpha+1} I_{\omega^{-1}(v)^-}^{\alpha,\omega}(f\circ\omega)\left(\omega^{-1}\left(\frac{u}{m}\right)\right)\right] \\
=\ & \frac{\Gamma(\alpha)}{(mv-u)^{\alpha}}\left[\frac{1}{\Gamma(\alpha)}\int_{\omega^{-1}(u)}^{\omega^{-1}(mv)} \frac{mv-\omega(x)}{mv-u} f(\omega(x))\frac{\omega'(x)}{mv-u}dx\right. \\
&\left. + \frac{m^2}{\Gamma(\alpha)}\int_{\omega^{-1}\left(\frac{u}{m}\right)}^{\omega^{-1}(v)}\left(\frac{\omega(x)-\frac{u}{m}}{v-\frac{u}{m}}\right)^{\alpha-1} f(\omega(x))\frac{\omega'(x)}{mv-u}dx\right] \\
=\ & \int_0^1 \tau^{\alpha-1} f\left(\tau u + m(1-\tau)v\right)d\tau + m\int_0^1 \tau^{\alpha-1} f\left(\tau v + \frac{1}{m}(1-\tau)u\right)d\tau \\
\geq\ & \frac{2}{\alpha} f\left(\frac{u+mv}{2}\right).
\end{aligned}$$

This completes the proof of the left hand side of the (16). From Definition 1.2, it follows that

$$f\left(\tau u+m\left(1-\tau\right)v\right)\leq\tau f\left(u\right)+m\left(1-\tau\right)f\left(v\right)$$

and

$$mf\left(\tau v+\frac{1}{m}\left(1-\tau\right)u\right)\leq m\tau f\left(v\right)+m^{2}\left(1-\tau\right)f\left(\frac{u}{m^{2}}\right).$$

By adding these two inequalities, we get

$$f\left(\tau u+m\left(1-\tau\right)v\right)+mf\left(\tau v+\frac{1}{m}\left(1-\tau\right)u\right)\leq\tau f\left(u\right)+m\left(1-\tau\right)f\left(v\right)+m\tau f\left(v\right)+m^{2}\left(1-\tau\right)f\left(\frac{u}{m^{2}}\right).$$

Now, by multiplying the this last inequality with $\frac{\alpha}{2}\tau^{\alpha-1}$ and then by integrating on $[0,1]$, we obtain

$$\begin{aligned}&\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha-1}f\left(\tau u+m\left(1-\tau\right)v\right)d\tau+\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha-1}mf\left(\tau v+\frac{1}{m}\left(1-\tau\right)u\right)d\tau\\ \leq\ &\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha}f\left(u\right)d\tau+m\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha-1}\left(1-\tau\right)f\left(v\right)d\tau+m\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha}f\left(v\right)d\tau+m^{2}\frac{\alpha}{2}\int_{0}^{1}\tau^{\alpha-1}\left(1-\tau\right)f\left(\frac{u}{m^{2}}\right)d\tau.\end{aligned}$$

After some necessary computations, we deduce

$$\begin{aligned}&\frac{\Gamma\left(\alpha+1\right)}{2\left(mv-u\right)^{\alpha}}\left[I_{\omega^{-1}(u)^{+}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(mv\right)\right)+m^{\alpha+1}I_{\omega^{-1}(v)^{-}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(\frac{u}{m}\right)\right)\right]\\ \leq\ &\frac{\alpha}{2\left(\alpha+1\right)}f\left(u\right)-\frac{m^{2}}{2}\left(\frac{2\alpha+1}{\alpha+1}\right)f\left(\frac{u}{m^{2}}\right)+\frac{m}{2}f\left(v\right)\end{aligned}$$

which is the right hand side of (16). □

Lemma 2.1. *Suppose that $f:[u,mv]\to\mathbb{R}$ is a positive differentiable function such that $f'\in L[u,mv]$. Let ω be a positive and strictly increasing function. If $\omega'(x)$ is a continuous differentiable function on (u,mv), then the following identity holds:*

$$\begin{aligned}K&=\frac{f\left(u\right)+f\left(mv\right)}{2}-\frac{\Gamma\left(\alpha+1\right)}{2\left(mv-u\right)^{\alpha}}\times\left[I_{\omega^{-1}(u)^{+}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(mv\right)\right)+I_{\omega^{-1}(mv)^{-}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(u\right)\right)\right]\\ &=\frac{1}{2\left(mv-u\right)^{\alpha}}\int_{\omega^{-1}(u)}^{\omega^{-1}(mv)}\left[\left(\omega\left(\tau\right)-u\right)^{\alpha}-\left(mv-\omega\left(\tau\right)\right)^{\alpha}\right]\omega'\left(\tau\right)\left(f'\circ\omega\right)(\tau)d\tau\end{aligned}$$

for $\alpha>0$.

Proof. Assume that

$$T_{1}=\frac{\Gamma\left(\alpha+1\right)}{2\left(mv-u\right)^{\alpha}}I_{\omega^{-1}(u)^{+}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(mv\right)\right)$$

and

$$T_{2}=\frac{\Gamma\left(\alpha+1\right)}{2\left(mv-u\right)^{\alpha}}I_{\omega^{-1}(mv)^{-}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(u\right)\right).$$

Then, we obtain

$$\begin{aligned}T_{1}&=\frac{\Gamma\left(\alpha+1\right)}{2\left(mv-u\right)^{\alpha}}I_{\omega^{-1}(u)^{+}}^{\alpha,\omega}\left(f\circ\omega\right)\left(\omega^{-1}\left(mv\right)\right)\\ &=\frac{\alpha}{2\left(mv-u\right)^{\alpha}}\int_{\omega^{-1}(u)}^{\omega^{-1}(mv)}\left(mv-\omega\left(\tau\right)\right)^{\alpha-1}\omega'\left(\tau\right)\left(f\circ\omega\right)(\tau)d\tau\\ &=-\frac{1}{2\left(mv-u\right)^{\alpha}}\int_{\omega^{-1}(u)}^{\omega^{-1}(mv)}\left(f\circ\omega\right)\left(\tau\right)d\left(mv-\omega\left(\tau\right)\right)^{\alpha}\\ &=\frac{1}{2\left(mv-u\right)^{\alpha}}\left[\left(mv-u\right)^{\alpha}\right]f(u) \qquad (20)\\ &\quad+\int_{\omega^{-1}(u)}^{\omega^{-1}(mv)}\left(mv-\omega\left(\tau\right)\right)^{\alpha}\omega'\left(\tau\right)\left(f'\circ\omega\right)(\tau)d\tau.\end{aligned}$$

Also, we have

$$\begin{aligned} T_2 &= \frac{\Gamma(\alpha+1)}{2(mv-u)^{\alpha}} I^{\alpha,\omega}_{\omega^{-1}(mv)^-}(f\circ\omega)\left(\omega^{-1}(u)\right) \\ &= \frac{\alpha}{2(mv-u)^{\alpha}} \int_{\omega^{-1}(u)}^{\omega^{-1}(mv)} (\omega(\tau)-u)^{\alpha-1}\omega'(\tau)(f\circ\omega)(\tau)d\tau \\ &= \frac{1}{2(mv-u)^{\alpha}} \int_{\omega^{-1}(u)}^{\omega^{-1}(mv)} (f\circ\omega)(\tau)\, d(\omega(\tau)-u)^{\alpha} \\ &= \frac{1}{2(mv-u)^{\alpha}} \left[(mv-u)^{\alpha} f(mv)\right] \qquad (21) \\ &\quad - \int_{\omega^{-1}(u)}^{\omega^{-1}(mv)} (\omega(\tau)-u)^{\alpha}\omega'(\tau)(f'\circ\omega)(\tau)d\tau \end{aligned}$$

By adding (20) and (21), we get the required result. □

Theorem 2.4. *Suppose that $f:[u,mv]\to\mathbb{R}$ is a positive differentiable function such that $f'\in L[u,mv]$. Also, let ω be a positive and strictly increasing function, and $\omega'(x)$ be a continuous differentiable function on (u,mv). If $|f'|$ is an m-convex function, then the following inequality holds:*

$$|K| \le \frac{mv-u}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|f'(u)|+m|f'(v)|\right]$$

for $\alpha>0$.

Proof. From Lemma 2.1, it follows that

$$|K| \le \frac{1}{2(mv-u)^{\alpha}} \times \int_{\omega^{-1}(u)}^{\omega^{-1}(mv)} \left|(\omega(\tau)-u)^{\alpha}-(mv-\omega(\tau))^{\alpha}\right| |(f'\circ\omega)(\tau)| d\omega(\tau).$$

If we consider $\tau\in\left(\omega^{-1}(u),\omega^{-1}(mv)\right)$, then we have $u\le\omega(\tau)\le mv$. By taking

$$t=\frac{mv-\omega(\tau)}{mv-u},$$

we have $\omega(\tau)=ut+m(1-t)v$. By using the m-convexity of $|f'|$, we get

$$\begin{aligned} |K| &\le \frac{mv-u}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right| |f'(ut+m(1-t)v)|dt \\ &\le \frac{mv-u}{2}\int_0^1 \left|(1-t)^{\alpha}-t^{\alpha}\right| \left|tf'(u)+m(1-t)f'(v)\right|dt \\ &= \frac{mv-u}{2}\int_0^{\frac{1}{2}} \left((1-t)^{\alpha}-t^{\alpha}\right) \left|tf'(u)+m(1-t)f'(v)\right|dt \\ &\quad +\frac{mv-u}{2}\int_{\frac{1}{2}}^1 \left(t^{\alpha}-(1-t)^{\alpha}\right) \left|tf'(u)+m(1-t)f'(v)\right|dt \\ &= \frac{mv-u}{2}(S_1+S_2), \end{aligned}$$

where

$$S_1 = f'(u)\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]+mf'(v)\left[\frac{1}{(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]$$

and

$$S_2 = f'(u)\left[\frac{1}{(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right]+mf'(v)\left[\frac{1}{(\alpha+1)(\alpha+2)}-\frac{1}{2^{\alpha+1}(\alpha+1)}\right].$$

This completes the proof. □

Remark 2.3. *If we take $m=1$ in Theorem 2.4, then we obtain Theorem 3.4 of [15].*

3. Conclusion

One of the main problems in the field of inequality theory is to optimize bounds. In this direction, by using a generalized version of the Riemann-Liouville fractional integral operator (which is one of the most important fractional integral operators known in the field of fractional analysis), some integral inequalities have been obtained in this paper. By choosing a class of functions that generate efficient approaches, such as m-convex functions, one can obtain new and optimal integral bounds from the procured inequalities. Some interesting results have been obtained with the help of m-convex functions and ω-RLF integrals. It is observed that some results available in the literature can be obtained as special cases of the results reported in this paper. Additional results similar to the ones given in this paper can be obtained also for different classes of convex functions. Furthermore, the obtained results can be reduced to inequalities involving different kinds of fractional integral operators for the special cases of the ω-RLF integral operator.

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