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Research Article

Convolutions involving Chebyshev polynomials

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Abstract

The generating function approach is utilized to establish several convolution formulae between Chebyshev polynomials and other well-known numbers and polynomials; for example, numbers/polynomials of Bernoulli/Euler and Fibonacci/Lucas numbers.

Keywords: generating function; Chebyshev polynomials; Bernoulli numbers/polynomials; Euler numbers/polynomials; Hermite polynomials.

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1. Introduction and motivation

The Chebyshev polynomials of the first kind and the second kind form important classes of special functions that have wide applications in pure mathematics and applied sciences. They are defined by

$$T_n(\cos\theta) = \cos(n\theta) \quad \text{and} \quad U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta},$$

and admit several useful properties (for example, see [17]):

- Recurrence relations ($n \geq 2$):

$$T_n(y) = 2yT_{n-1}(y) - T_{n-2}(y),$$
$$U_n(y) = 2yU_{n-1}(y) - U_{n-2}(y).$$

- Initial conditions:

$$T_0(y) = 1, \quad T_1(y) = y,$$
$$U_0(y) = 1, \quad U_1(y) = 2y.$$

- Ordinary generating functions:

$$\frac{1-\eta y}{1-2\eta y+\eta^2} = \sum_{n=0}^{\infty} T_n(y)\eta^n, \tag{1}$$

$$\frac{1}{1-2\eta y+\eta^2} = \sum_{n=0}^{\infty} U_n(y)\eta^n. \tag{2}$$

- Binet formulae:

$$\left\{ \begin{aligned} T_n(y) &= \frac{\alpha^n+\beta^n}{2} \\ U_n(y) &= \frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta} \end{aligned} \right\}, \quad \alpha,\beta = y \pm \sqrt{y^2-1}. \tag{3}$$

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- Exponential generating functions:

$$\frac{e^{\eta\alpha}+e^{\eta\beta}}{2}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}T_n(y), \tag{4}$$

$$\frac{\alpha e^{\eta\alpha}-\beta e^{\eta\beta}}{2\sqrt{y^2-1}}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}U_n(y). \tag{5}$$

- Fibonacci and Lucas numbers:

$$U_n(\tfrac{\mathbf{i}}{2})=F_{n+1}\mathbf{i}^n \quad\text{and}\quad T_n(\tfrac{\mathbf{i}}{2})=\frac{L_n}{2}\mathbf{i}^n.$$

The reader can find, in the recent papers [1,4,13,16], more formulae about trigonometric expressions, generating functions, and power sums as well as convolutions.

By means of the generating function approach (see [20]), we investigate convolution sums involving Chebyshev polynomials. In the next section, classical convolutions are examined through ordinary generating functions, that lead to several convolution identities including Catalan numbers, harmonic numbers and Fibonacci/Lucas numbers. Then in Section 3, by employing the exponential generating functions, we establish further convolution formulae of binomial type concerning numbers/polynomials of Bernoulli/Euler, Hermite polynomials, as well as Fibonacci/Lucas numbers. Among the identities presented in this paper, the following two unusual convolution formulae about Bernoulli numbers proposed by Frontczak [11] are contained as very particular cases:

$$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=\begin{cases}(1-n)B_nF_{n\lambda}, & n \text{ is even;}\\ -nB_{n-1}F_{n\lambda}, & n \text{ is odd;}\end{cases} \tag{6}$$

$$\sum_{k=0}^{n}\binom{n}{k}B'_kB'_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=\begin{cases}(1-n)B_nF_{n\lambda}, & n \text{ is even;}\\ 0, & n \text{ is odd;}\end{cases} \tag{7}$$

where $B'_n=B_n(2^{1-n}-1)$ and $\lambda,n\in\mathbb{N}$ with $\lambda\geq1$ and $n\geq3$.

2. Ordinary generating functions

For an arbitrary sequence $\{\Phi_n\}$, suppose that its ordinary generating function is given by the formal power series $\phi(\eta)$. Denote by $[\eta^n]\phi(\eta)$ the coefficient of η^n in $\phi(\eta)$. We have the following relations:

$$\phi(\eta)=\sum_{n=0}^{\infty}\Phi_n\eta^n \quad\text{if and only if}\quad \Phi_n=[\eta^n]\phi(\eta)\quad\text{for all}\quad n\in\mathbb{N}_0.$$

The main result of this section is the following theorem.

Theorem 2.1 ($\lambda\in\mathbb{N}$)**.** *Let $T_k(y)$ and $U_k(y)$ be Chebyshev polynomials. Then for an arbitrary sequence $\{\Phi_k\}$* (of numbers or polynomials)*, the following universal convolution formula holds*

$$\sum_{k=0}^{n}\Phi_k\Phi_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)}{2}\sum_{k=0}^{n}\Phi_k\Phi_{n-k}.$$

Proof. For a positive integer λ, rewrite the formal power series $\phi(\eta)$ by

$$\phi(\eta\alpha^\lambda)=\sum_{n=0}^{\infty}\Phi_n(\eta\alpha^\lambda)^n, \tag{8}$$

$$\phi(\eta\beta^\lambda)=\sum_{n=0}^{\infty}\Phi_n(\eta\beta^\lambda)^n. \tag{9}$$

Now, multiplying (8) + (9) by (8) − (9) and then extracting the coefficient of η^n, we get the equality

$$\sum_{k=0}^{n}\Phi_k\Phi_{n-k}(\alpha^{k\lambda}-\beta^{k\lambda})(\alpha^{n\lambda-k\lambda}+\beta^{n\lambda-k\lambda})=[\eta^n]\Big\{\phi^2(\eta\alpha^\lambda)-\phi^2(\eta\beta^\lambda)\Big\}. \tag{10}$$

Because the right member of (10) is substantially equal to $(\alpha^{n\lambda}-\beta^{n\lambda})$ times the coefficient $\phi^2(\eta)$, we have the equality

$$\sum_{k=0}^{n}\Phi_k\Phi_{n-k}(\alpha^{k\lambda}-\beta^{k\lambda})(\alpha^{n\lambda-k\lambda}+\beta^{n\lambda-k\lambda})=(\alpha^{n\lambda}-\beta^{n\lambda})[\eta^n]\phi^2(\eta). \tag{11}$$

Dividing the (11) by $\alpha-\beta$ and then writing in terms of Chebyshev polynomials, we confirm the convolution formula stated in Theorem 2.1. □

It should be noted that $U_{-1}(y)=0$ in Theorem 2.1, which can be obtained from the Binet formulae of Chebyshev polynomials. Letting $y=\frac{\mathrm{i}}{2}$ in Theorem 2.1, we get the following convolution formula involving Fibonacci and Lucas numbers.

Corollary 2.1 ($\lambda\in\mathbb{N}$)**.** *For an arbitrary sequence* $\{\Phi_k\}$ (of numbers or polynomials)*, we have the convolution formula*

$$\sum_{k=0}^{n}\Phi_k\Phi_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=F_{n\lambda}\sum_{k=0}^{n}\Phi_k\Phi_{n-k}.$$

By choosing properly the sequence $\{\Phi_k\}$ in Theorem 2.1 and Corollary 2.1 so that the convolution sums on the right hand sides can be evaluated in closed form. In this case, we would find closed expressions for the convolution sums on the left hand sides. Some interesting identities are exhibited below.

Catalan numbers

Recall the Catalan numbers (see [12, §5.4])

$$C_n=\frac{1}{n+1}\binom{2n}{n}$$

that are generated by the function

$$\frac{1-\sqrt{1-4\eta}}{2\eta}=\sum_{n=0}^{\infty}C_n\eta^n$$

and admit the following convolution formula

$$\sum_{k=0}^{n}C_kC_{n-k}=C_{n+1}.$$

By specifying $\Phi_n=C_n$ in Theorem 2.1 and Corollary 2.1, we derive the two identities as in the following proposition.

Proposition 2.1 ($n\in\mathbb{N}_0$)**.**

$$\text{(a)}\ \sum_{k=0}^{n}C_kC_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)C_{n+1}}{2},$$

$$\text{(b)}\ \sum_{k=0}^{n}C_kC_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=F_{n\lambda}C_{n+1}.$$

Harmonic numbers

The harmonic numbers are defined by (see [12, §6.3])

$$H_n=\sum_{k=1}^{n}\frac{1}{k}\quad\text{and}\quad H_n^{\langle 2\rangle}=\sum_{k=1}^{n}\frac{1}{k^2}.$$

Applying the generating function (see [2,5])

$$\sum_{n=0}^{\infty}H_n\eta^n=\frac{\ln(1-\eta)}{\eta-1},$$

we can express the convolution as

$$\sum_{k=0}^{n}H_kH_{n-k}=[\eta^n]\frac{\ln^2(1-\eta)}{(1-\eta)^2}.$$

By means of the MacLaurin series

$$\frac{1}{(1-\eta)^2}=\sum_{\ell=0}^{\infty}(\ell+1)\eta^\ell\quad\text{for}\quad\ln^2(1-\eta)=\sum_{i,j=1}^{\infty}\frac{\eta^{i+j}}{ij},$$

the rightmost double sum can be reformulated, under $i+j=k$, as

$$\sum_{k=2}^{\infty}\eta^k\sum_{i=1}^{k-1}\frac{1}{i(k-i)}=\sum_{k=2}^{\infty}\frac{\eta^k}{k}\sum_{i=1}^{k-1}\left\{\frac{1}{i}+\frac{1}{k-i}\right\}=2\sum_{k=2}^{\infty}\frac{H_{k-1}}{k}\eta^k.$$

Therefore we get the equality

$$\sum_{k=0}^{n}H_kH_{n-k}=2\sum_{k=2}^{n}\frac{1+n-k}{k}H_{k-1}=2(1+n)\sum_{k=2}^{n}\frac{H_{k-1}}{k}-2\sum_{k=2}^{n}H_{k-1}.$$

Evaluating the above two sums further

$$\sum_{k=2}^{n}\frac{H_{k-1}}{k}=\sum_{1\le i<k\le n}\frac{1}{ik}=\frac{1}{2}\left\{\sum_{k=1}^{n}\frac{1}{k}\right\}^2-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k^2}=\frac{H_n^2-H_n^{\langle 2\rangle}}{2},$$

$$\sum_{k=2}^{n}H_{k-1}=\sum_{k=2}^{n}\sum_{i=1}^{k-1}\frac{1}{i}=\sum_{i=1}^{n}\frac{1}{i}\sum_{k=i+1}^{n}1=\sum_{i=1}^{n}\frac{n-i}{i}=nH_n-n;$$

we find the following closed formula

$$\sum_{k=0}^{n}H_kH_{n-k}=(n+1)\Big\{H_n^2-H_n^{\langle 2\rangle}\Big\}-2nH_n+2n.$$

In Theorem 2.1 and Corollary 2.1, letting $\Phi_n=H_n$ yields immediately the following two convolution identities.

Proposition 2.2 ($n\in\mathbb{N}_0$).

$$\text{(a)}\ \sum_{k=0}^{n}H_kH_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)}{2}\Big\{(n+1)\big(H_n^2-H_n^{\langle 2\rangle}\big)-2nH_n+2n\Big\},$$

$$\text{(b)}\ \sum_{k=0}^{n}H_kH_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=F_{n\lambda}\Big\{(n+1)\big(H_n^2-H_n^{\langle 2\rangle}\big)-2nH_n+2n\Big\}.$$

Chebyshev polynomials

For the Chebyshev polynomials defined in (1) and (2), by manipulating their generating functions

$$\frac{2(1-x\eta)^2}{(1-2x\eta+\eta^2)^2}=\frac{1}{1-2x\eta+\eta^2}+(1+\eta\mathfrak{D}_\eta)\frac{1-x\eta}{1-2x\eta+\eta^2},$$

$$\frac{2(1-x^2)}{(1-2x\eta+\eta^2)^2}=\frac{1}{1-2x\eta+\eta^2}+(1-2x+\eta)\mathfrak{D}_\eta\frac{1-x\eta}{1-2x\eta+\eta^2};$$

and then making use of the recurrence relation

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x)$$

we can evaluate, in closed form, the following two convolutions

$$\sum_{k=0}^{n}T_k(x)T_{n-k}(x)=\frac{U_n(x)+(n+1)T_n(x)}{2},$$

$$\sum_{k=0}^{n}U_k(x)U_{n-k}(x)=\frac{U_n(x)-(n+1)T_{n+2}(x)}{2(1-x^2)}.$$

Now, specializing $\Phi_n=T_n(x)$ and $\Phi_n=U_n(x)$ in Theorem 2.1 and Corollary 2.1, we establish two pairs of convolution identities below.

Proposition 2.3 ($n\in\mathbb{N}_0$).

$$\text{(a)}\ \sum_{k=0}^{n}T_k(x)T_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)\{U_n(x)+(n+1)T_n(x)\}}{4},$$

(b) $\sum_{k=0}^{n} U_k(x)U_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = \frac{U_{n\lambda-1}(y)\{U_n(x)-(n+1)T_{n+2}(x)\}}{4(1-x^2)}$;

(c) $\sum_{k=0}^{n} T_k(x)T_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = \frac{F_{n\lambda}\{U_n(x)+(n+1)T_n(x)\}}{2}$,

(d) $\sum_{k=0}^{n} U_k(x)U_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = \frac{F_{n\lambda}\{U_n(x)-(n+1)T_{n+2}(x)\}}{2(1-x^2)}$.

Fibonacci numbers and Lucas numbers

According to the convolution formulae (see [14])

$$\sum_{k=0}^{n} F_kF_{n-k} = \frac{(n+1)L_n - 2F_{n+1}}{5},$$

$$\sum_{k=0}^{n} L_kL_{n-k} = (n+1)L_n + 2F_{n+1};$$

we get, by putting $\Phi_n = F_n$ and $\Phi_n = L_n$ in Theorem 2.1 and Corollary 2.1, two pairs of identities as in the following proposition.

Proposition 2.4 ($n \in \mathbb{N}_0$)**.**

(a) $\sum_{k=0}^{n} F_kF_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = \frac{U_{n\lambda-1}(y)}{10}\Big\{(n+1)L_n - 2F_{n+1}\Big\}$,

(b) $\sum_{k=0}^{n} L_kL_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = \frac{U_{n\lambda-1}(y)}{2}\Big\{(n+1)L_n + 2F_{n+1}\Big\}$,

(c) $\sum_{k=0}^{n} F_kF_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = \frac{F_{n\lambda}}{5}\Big\{(n+1)L_n - 2F_{n+1}\Big\}$,

(d) $\sum_{k=0}^{n} L_kL_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}\Big\{(n+1)L_n + 2F_{n+1}\Big\}$.

We remark that these four identities can also be obtained as consequences of Proposition 2.3 under $x = \frac{\mathrm{i}}{2}$.

3. Exponential generating functions

The convolution identities produced in the previous section admit binomial convolution counterparts. This can be fulfilled by employing exponential generating functions. Suppose that $\{\Psi_n\}$ is a sequence whose exponential generating function is the formal power series $\psi(\eta)$. Then there are similar relations:

$$\psi(\eta) = \sum_{n=0}^{\infty} \frac{\Psi_n}{n!}\eta^n \quad \text{if and only if} \quad \Psi_n = n![\eta^n]\psi(\eta) \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Now performing the substitutions $\Phi_k \to \Psi_k/k!$ in Theorem 2.1 and then making some routine simplification, we deduce the following binomial convolution formula.

Theorem 3.1 ($\lambda \in \mathbb{N}$)**.** *Let $T_k(y)$ and $U_k(y)$ be Chebyshev polynomials. Then for an arbitrary sequence $\{\Psi_k\}$* (of numbers or polynomials)*, the following universal convolution formula holds*

$$\sum_{k=0}^{n} \binom{n}{k} \Psi_k\Psi_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = \frac{U_{n\lambda-1}(y)}{2}\sum_{k=0}^{n} \binom{n}{k} \Psi_k\Psi_{n-k}.$$

Analogously, letting $y = \frac{\mathrm{i}}{2}$ in the above theorem yields the formula below about Fibonacci and Lucas numbers.

Corollary 3.1 ($\lambda \in \mathbb{N}$)**.** *For an arbitrary sequence $\{\Psi_k\}$* (of numbers or polynomials)*, we have the following convolution formula*

$$\sum_{k=0}^{n} \binom{n}{k} \Psi_k\Psi_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}\sum_{k=0}^{n} \binom{n}{k} \Psi_k\Psi_{n-k}.$$

As applications of Theorem 3.1 and Corollary 3.1 to convolution identities, we show five examples by specifying sequence Ψ_n with known polynomial and number sequences.

Bernoulli polynomials and numbers

Bernoulli polynomials and numbers are defined by (see [6] and [9, §1.14])

$$\frac{\eta e^{x\eta}}{e^{\eta}-1}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}B_n(x) \quad \text{and} \quad \frac{\eta}{e^{\eta}-1}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}B_n.$$

By applying the following known convolution formulae (see [3, 8] and [10, §24.14(i)])

$$\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(y)=(1-n)B_n(x+y)+n(x+y-1)B_{n-1}(x+y),$$

$$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}=(1-n)B_n-nB_{n-1};$$

and then specifying $\Psi_n=B_n(x)$ and $\Psi_n=B_n$ in Theorem 3.1 and Corollary 3.1, we derive two pairs of identities as in the proposition below.

Proposition 3.1 ($n\in\mathbb{N}_0$).

(a) $$\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)}{2}\Big\{(1-n)B_n(2x)+n(2x-1)B_{n-1}(2x)\Big\},$$

(b) $$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=\frac{U_{n\lambda-1}(y)}{2}\Big\{(1-n)B_n-nB_{n-1}\Big\},$$

(c) $$\sum_{k=0}^{n}\binom{n}{k}B_k(x)B_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda}=F_{n\lambda}\Big\{(1-n)B_n(2x)+n(2x-1)B_{n-1}(2x)\Big\},$$

(d) $$\sum_{k=0}^{n}\binom{n}{k}B_kB_{n-k}F_{k\lambda}L_{n\lambda-k\lambda}=F_{n\lambda}\Big\{(1-n)B_n-nB_{n-1}\Big\}.$$

In particular, the case $x=0$ of identity (c) (equivalently, the case $y=\frac{\mathrm{i}}{2}$ of identity (b)) recovers the identity (6) proposed by Frontczek [11].

Alternatively, letting $x=\frac{1}{2}$ in the first identity (a) of Proposition 3.1 and keeping in mind that $B_n(1)=(-1)^nB_n$, we get the following interesting formula.

Corollary 3.2 ($n\in\mathbb{N}_0$).

$$\sum_{k=0}^{n}\binom{n}{k}B_k(\tfrac{1}{2})B_{n-k}(\tfrac{1}{2})U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=(-1)^n\frac{1-n}{2}B_nU_{n\lambda-1}(y).$$

Taking into account the following further fact

$$B_n(\tfrac{1}{2})=B_n'=(2^{1-n}-1)B_n$$

we can rewrite the last identity as

$$\sum_{k=0}^{n}\binom{n}{k}B_k'B_{n-k}'U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y)=(-1)^n\frac{1-n}{2}B_nU_{n\lambda-1}(y).$$

Finally by letting $y=\frac{\mathrm{i}}{2}$, we recover another formula (7) due to Frontczek [11].

Euler polynomials and numbers

Euler polynomials and numbers have the following generating functions (see [9, §1.14] and [7, 19])

$$\frac{2e^{x\eta}}{1+e^{\eta}}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}E_n(x) \quad \text{and} \quad \frac{2e^{\eta}}{1+e^{2\eta}}=\sum_{n=0}^{\infty}\frac{\eta^n}{n!}E_n.$$

By letting $\Psi_n = E_n(x)$ and $\Psi_n = E_n$ Theorem 3.1 and Corollary 3.1, and then appealing to the convolution formulae (see [8] and [10, §24.14(i)])

$$\sum_{k=0}^{n}\binom{n}{k}E_k(x)E_{n-k}(y) = 2E_{n+1}(x+y) - 2(x+y-1)E_n(x+y),$$

$$\sum_{k=0}^{n}\binom{n}{k}E_kE_{n-k} = 2^{n+2}\frac{E_{n+2}}{n+2}(2^{n+2}-1);$$

we establish two pairs of identities as in the following proposition.

Proposition 3.2 ($n \in \mathbb{N}_0$).

(a) $$\sum_{k=0}^{n}\binom{n}{k}E_k(x)E_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = U_{n\lambda-1}(y)\Big\{E_{n+1}(2x) - 2xE_n(2x) + E_n(2x)\Big\},$$

(b) $$\sum_{k=0}^{n}\binom{n}{k}E_kE_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = 2^{n+1}U_{n\lambda-1}(y)\frac{E_{n+2}}{n+2}(2^{n+2}-1),$$

(c) $$\sum_{k=0}^{n}\binom{n}{k}E_k(x)E_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}\Big\{2E_{n+1}(2x) - 4xE_n(2x) + 2E_n(2x)\Big\},$$

(d) $$\sum_{k=0}^{n}\binom{n}{k}E_kE_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = 2^{n+2}F_{n\lambda}\frac{E_{n+2}}{n+2}(2^{n+2}-1).$$

Hermite polynomials

Hermite polynomials $H_n(x)$ are defined (see Rainville [18, §103]) by the exponential generating function

$$\exp(2x\eta - \eta^2) = \sum_{n=0}^{\infty}H_n(x)\frac{\eta^n}{n!}$$

which satisfy the convolution equation

$$\sum_{k=0}^{n}\binom{n}{k}H_k(x)H_{n-k}(y) = H_n(x+y).$$

By letting $\Psi_n = H_n(x)$ in Theorem 3.1 and Corollary 3.1, we find the following two respective identities.

Proposition 3.3 ($n \in \mathbb{N}_0$).

(a) $$\sum_{k=0}^{n}\binom{n}{k}H_k(x)H_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = \frac{U_{n\lambda-1}(y)}{2}H_n(2x),$$

(b) $$\sum_{k=0}^{n}\binom{n}{k}H_k(x)H_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}H_n(2x).$$

Chebyshev polynomials

By combining the exponential generating functions (4–5) with the Binet formulae (3), we can compute without difficulty the convolutions of Chebyshev polynomials

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}T_k(y)T_{n-k}(y) &= n![\eta^n]\frac{(e^{\eta\alpha}+e^{\eta\beta})^2}{4}\\ &= n![\eta^n]\frac{e^{2y\eta}}{2} + n![\eta^n]\frac{e^{2\eta\alpha}+e^{2\eta\beta}}{4}\\ &= 2^{n-1}y^n + 2^{n-2}(\alpha^n+\beta^n),\end{aligned}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}U_k(y)U_{n-k}(y) = n![\eta^n]\frac{(\alpha e^{\eta\alpha}-\beta e^{\eta\beta})^2}{4(y^2-1)}$$

$$= n![\eta^n]\frac{e^{2y\eta}}{2(1-y^2)} - n![\eta^n]\frac{\alpha^2 e^{2\eta\alpha}+\beta^2 e^{2\eta\beta}}{4(1-y^2)}$$

$$= \frac{2^{n-1}y^n}{1-y^2} - \frac{2^{n-2}}{1-y^2}(\alpha^{n+2}+\beta^{n+2});$$

that result in the following closed formulae

$$\sum_{k=0}^{n}\binom{n}{k}T_k(y)T_{n-k}(y) = 2^{n-1}\{y^n + T_n(y)\},$$

$$\sum_{k=0}^{n}\binom{n}{k}U_k(y)U_{n-k}(y) = \frac{2^{n-1}}{1-y^2}\{y^n - T_{n+2}(y)\}.$$

According to Theorem 3.1 and Corollary 3.1, by assigning Ψ_n to $T_n(x)$ and $U_n(x)$, we deduce two pairs of convolution identities as in the following proposition.

Proposition 3.4 ($n \in \mathbb{N}_0$).

$$\text{(a)} \sum_{k=0}^{n}\binom{n}{k}T_k(x)T_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = 2^{n-2}U_{n\lambda-1}(y)\Big\{x^n + T_n(x)\Big\},$$

$$\text{(b)} \sum_{k=0}^{n}\binom{n}{k}U_k(x)U_{n-k}(x)U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = 2^{n-2}\frac{U_{n\lambda-1}(y)}{1-x^2}\Big\{x^n - T_{n+2}(x)\Big\},$$

$$\text{(c)} \sum_{k=0}^{n}\binom{n}{k}T_k(x)T_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = 2^{n-1}F_{n\lambda}\Big\{x^n + T_n(x)\Big\},$$

$$\text{(d)} \sum_{k=0}^{n}\binom{n}{k}U_k(x)U_{n-k}(x)F_{k\lambda}L_{n\lambda-k\lambda} = 2^{n-1}\frac{F_{n\lambda}}{1-x^2}\Big\{x^n - T_{n+2}(x)\Big\}.$$

Fibonacci numbers and Lucas numbers

Finally, specializing Ψ_n by F_n and L_n in Theorem 3.1 and Corollary 3.1, and then making use of the following closed formulae (see [15, P235])

$$\sum_{k=0}^{n}\binom{n}{k}F_kF_{n-k} = \frac{2^n}{5}L_n - \frac{2}{5},$$

$$\sum_{k=0}^{n}\binom{n}{k}L_kL_{n-k} = 2^nL_n + 2;$$

we find two further pairs of convolution identities, that can also be obtained by letting $x = \frac{\mathrm{i}}{2}$ in Proposition 3.4.

Proposition 3.5 ($n \in \mathbb{N}_0$).

$$\text{(a)} \sum_{k=0}^{n}\binom{n}{k}F_kF_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = U_{n\lambda-1}(y)\left\{\frac{2^n}{10}L_n - \frac{1}{5}\right\},$$

$$\text{(b)} \sum_{k=0}^{n}\binom{n}{k}L_kL_{n-k}U_{k\lambda-1}(y)T_{n\lambda-k\lambda}(y) = U_{n\lambda-1}(y)\left\{1 + 2^{n-1}L_n\right\},$$

$$\text{(c)} \sum_{k=0}^{n}\binom{n}{k}F_kF_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}\left\{\frac{2^n}{5}L_n - \frac{2}{5}\right\},$$

$$\text{(d)} \sum_{k=0}^{n}\binom{n}{k}L_kL_{n-k}F_{k\lambda}L_{n\lambda-k\lambda} = F_{n\lambda}\{2 + 2^nL_n\}.$$

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