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Research Article

Reciprocal degree distance of Eliasi-Taeri sums of graphs

K. Pattabiraman[1,*], Manzoor Ahmad Bhat[2]

[1]*Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, India*
[2]*Department of Mathematics, Annamalai University, Annamalainagar, India*

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Abstract

For a connected graph G, the reciprocal degree distance is defined as $RDD(G) = \sum_{u,v \in V(G)} \left(d_G(u) + d_G(v)\right)\left(2d_G(u,v)\right)^{-1}$, where $d_G(u)$ denotes the degree of a vertex u in G and $d_G(u,v)$ represents the distance between the vertices u and v in G. In this paper, upper bounds for the reciprocal degree distance of graphs, arising from four operations introduced by Eliasi and Taeri in [*Discrete Appl. Math.* **157** (2009) 794–803], are provided.

Keywords: distance; reciprocal degree distance; Eliasi-Taeri sums.

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1. Introduction

Throughout this paper, we assume that G is a finite, connected, undirected, and simple graph, with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex x in G is denoted by $d_G(x)$ and the distance between $u, v \in V(G)$ is denoted by $d_G(u,v)$.

Topological indices provide contemporary tools to predict diverse psychochemical features of chemical compounds. A lot of available research shows that topological indices are easy to compute and they efficiently encrypt significant structural information of chemical compounds. Distance-based topological indices form a vital class of indices and these indices have significantly better efficiency.

The Zagreb indices have been introduced around a half century ago by Gutman and Trinajestić in [6]. They are defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} (d_G(u) + d_G(v))$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v).$$

In the definitions of the Zagreb indices, if one takes the sums over the edges of the complement of G then the resulting quantities are the Zagreb coindices. More precisely, the first and second Zagreb coindices of G are defined as

$$\overline{M}_1(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)]$$

and

$$\overline{M}_2(G) = \sum_{uv \notin E(G)} d_G(u)d_G(v).$$

One of the oldest and well-studied topological indices is the *Wiener index*. The Wiener index of G is defined as the sum of the distances of all unordered vertex pairs in G, that is,

$$W(G) = \frac{1}{2} \sum_{u,v \in V(G)} d_G(u,v).$$

Another distance-based topological index, defined in a fully analogous manner to the Wiener index, is the *Harary index* which is equal to the sum of the the reciprocal distances overall all unordered vertex pairs in G, that is,

$$H(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}.$$

*Corresponding author (pramank@gmail.com).

Dobrynin and Kochetova [3], and Gutman [5] independently proposed a vertex-degree-weighted version of the Wiener index known as the *degree distance* (DD), which is defined as

$$DD(G) = \frac{1}{2} \sum_{u,v \in V(G)} (d_G(u) + d_G(v)) d_G(u,v).$$

The *reciprocal degree distance* (RDD) of a connected graph G is defined [1] as

$$RDD(G) = \frac{1}{2} \sum_{u,v \in V(G)} \frac{d_G(u) + d_G(v)}{d_G(u,v)}.$$

One can observe that the degree distance and reciprocal degree distance are defined by using the concepts of the first Zagreb index, Wiener index, and Harary index.

Chemical applications and mathematical properties of the reciprocal degree distance are well studied [1,8,12]. Hua and Zhang [7] obtained lower and upper bounds for the reciprocal degree distance of graphs in terms of other graph parameters. The mathematical behavior of the reciprocal degree distance of some composite graphs was analysed in [10, 11]. In the present paper, we obtain upper bounds for the reciprocal degree distance of F-sums of graphs introduced in [4].

2. Main results

In order to state and prove the main results, we need some definitions. In the following, we define four graphs generated from a given connected graph G.

(i). The graph formed by inserting a new vertex on every edge of G is known as the *subdivision graph* $S(G)$ of G.

(ii). The graph deduced from G by first inserting an additional vertex corresponding to every edge of G and then joining every newly added vertex to the end vertices of the corresponding edge is denoted by $R(G)$.

(iii). The graph deduced from G by first inserting an additional vertex into every edge of G and then joining with edges the pairs of new vertices on adjacent edges of G is denoted by $Q(G)$.

(iv). The *total graph* $T(G)$ of G has the vertex set consisting of the vertices and edges of G. Two vertices in $T(G)$ are adjacent if and only if they are either adjacent or incident in G.

For $F \in \{S, R, Q, T\}$, the F-sum or Eliasi-Taeri sum of the graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G_1 +_F G_2$ and is defined as the graph with the vertex set $V(G_1 +_F G_2) = (V_1 \cup E_1) \times V_2$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if either $u_1 = v_1 \in V_1$ and $u_2 v_2 \in E_2$ or $u_2 = v_2$ and $u_1 v_1 \in E(F(G_1))$.

The following known lemmas are used in the proofs of our main results.

Lemma 2.1 (see [13]). *For a graph G and $v, v' \in V(G)$, we have*

$$\frac{1}{2} d_{S(G)}(v, v') = d_{T(G)}(v, v') = d_{R(G)}(v, v') = d_{Q(G)}(v, v') - 1 = d_G(v, v').$$

Lemma 2.2 (see [13]). *For an edge $e \in E(G)$ and a vertex $v \in V(G)$, we have*

$$\frac{1}{2}(d_{S(G)}(e, v) + 1) = d_{T(G)}(e, v) = d_{R(G)}(e, v) = d_{Q(G)}(e, v).$$

Lemma 2.3 (see [4]). *Let G_1 and G_2 be two connected graphs and $v = (v_1, v_2)$ be a vertex of $G_1 +_F G_2$. Then*

(i). *If $v_1 \notin E(G_1)$, then for every $u = (u_1, u_2) \in V(G_1 +_F G_2)$ we have*

$$d_{G_1 +_F G_2}(u, v) = d_{F(G_1)}(u_1, v_1) + d_{G_2}(u_2, v_2).$$

(ii). *If $v_1 \in E(G_1)$, then for every $u = (u_1, u_2) \in V(G_1 +_F G_2)$ with $u_2 \neq v_2$, $u_1 = u_1' v_1' \in E(G_1)$ and $u_1', v_1' \in V(G_1)$ we have*

$$d_{G_1 +_F G_2}(u, v) = 1 + d_{G_2}(u_2, v_2) + \min\left\{d_{F(G_1)}(u_1', v_1), d_{F(G_1)}(v_1', v_1)\right\}.$$

(iii). *If $v_1 \in E(G_1)$, then for every $u = (u_1, u_2) \in V(G_1 +_F G_2)$, where $u_2 = v_2$, and $u_1 \in E(G_1)$, we have*

$$d_{G_1 +_F G_2}(u, v) = d_{F(G_1)}(u_1, v_1) + d_{G_2}(u_2, v_2) = d_{F(G_1)}(u_1, v_1).$$

Lemma 2.4 (see [4]). *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E(G_1), u_2, v_2 \in V(G_2)$, and $F \in \{S, R\}$. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 +_F G_2)$, with $u_2 \neq v_2$, we have*

$$d_{G_1+_F G_2}(u,v) = \begin{cases} 2 + d_{G_2}(u_2, v_2) & \text{if } u_1 = v_1, \\ d_{F(G_1)}(u_1, v_1) + d_{G_2}(u_2, v_2) & \text{if } u_1 \neq v_1. \end{cases}$$

Lemma 2.5 (see [4]). *Let G_1 and G_2 be two connected graphs, $u_1, v_1 \in E(G_1), u_2, v_2 \in V(G_2)$, and $F \in \{Q, T\}$. Then for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $V(G_1 +_F G_2)$, with $u_2 \neq v_2$, we have*

$$d_{G_1+_F G_2}(u,v) = \begin{cases} 2 + d_{G_2}(u_2, v_2) & \text{if } u_1 = v_1 \\ 1 + d_{F(G_1)}(u_1, v_1) + d_{G_2}(u_2, v_2) & \text{if } u_1 \neq v_1,\ u_2 \neq v_2. \end{cases}$$

Lemma 2.6 (see [2]). *Let G be a graph.*

(i). *If $u_1 \in V(G)$, then $d_{F(G_1)}(u_1) = k\, d_{G_1}(u_1)$, where*

$$k = \begin{cases} 1 & \text{if } F \in \{S, Q\}, \\ 2 & \text{if } F \in \{R, T\}. \end{cases}$$

(ii). *If $u = u_1' u_1'' \in E(G)$, then we have $d_{S(G)}(u_1) = d_{R(G)}(u_1) = 2$ and $d_{Q(G)}(u_1) = d_{T(G)}(u_1) = d_{S(G)}(u_1) = d_{L(G)}(u_1) + 2$, where $d_{L(G)}(u_1) = d_{(G)}(u_1') + d_{(G)}(u_1'') - 2$.*

Lemma 2.7 (see [9]). *Let g be a convex function on the interval Z and $z_1, z_2, \ldots, z_n \in Z$. Then*

$$g\Big(\frac{z_1 + z_2 + \ldots + z_n}{n}\Big) \leq \frac{g(z_1) + g(z_2) + \ldots + g(z_n)}{n}$$

with equality if and only if $z_1 = z_2 = \ldots = z_n$.

Now, we are able to state and prove the first main result of this paper.

Theorem 2.1. *For $i = 1, 2$, let G_i be a connected graph with n_i vertices and m_i edges, and take $F \in \{S, R\}$. Then*

$$\begin{aligned} RDD(G_1 +_F G_2) \leq{}& \frac{n_2^2}{4} RDD(F(G_1)) + \frac{n_1(n_1 + m_1)}{4} RDD(G_2) + n_2 m_2 H(F(G_1)) + \frac{n_2(n_2 - 1)k}{8}(M_1(G_1) - 2m_1) \\ &+ \frac{n_2^2(m_1 - 1)}{4} M_1(G_1) + \frac{H(G_2)}{4}\Big[(3k + 2m_1 - 2)M_1(G_1) + k\overline{M}_1(G_1) + 4m_1(k(m_1 - 1) + n_1)\Big] \\ &+ \frac{n_2^2 m_1(k m_1 + n_1)}{2}. \end{aligned}$$

Proof. Let $G = G_1 +_F G_2$. By the definition of the reciprocal degree distance, we have

$$\begin{aligned} RDD(G) &= \frac{1}{2} \sum_{(u_1,u_2),(v_1,v_2) \in V(G_1+_F G_2)} \frac{d_G(u_1, u_2) + d_G(v_1, v_2)}{d_G((u_1, u_2), (v_1, v_2))}, \\ &= \frac{1}{2} \sum_{(u_1,u_2),(v_1,v_2) \in V(G_1) \times V(G_2)} \frac{d_G(u_1, u_2) + d_G(v_1, v_2)}{d_G((u_1, u_2), (v_1, v_2))} \\ &\quad + \sum_{(u_1,u_2) \in V(G_1) \times V(G_2),(v_1,v_2) \in E(G_1) \times V(G_2)} \frac{d_G(u_1, u_2) + d_G(v_1, v_2)}{d_G((u_1, u_2), (v_1, v_2))} \\ &\quad + \frac{1}{2} \sum_{(u_1,u_2)(v_1,v_2) \in E(G_1), u_1 = v_1, u_2 \neq v_2} \frac{d_G(u_1, u_2) + d_G(v_1, v_2)}{d_G((u_1, u_2), (v_1, v_2))} \\ &\quad + \frac{1}{2} \sum_{(u_1,u_2)(v_1,v_2) \in E(G_1) \times V(G_2), u_1 \neq v_1} \frac{d_G(u_1, u_2) + d_G(v_1, v_2)}{d_G((u_1, u_2), (v_1, v_2))}, \\ &\quad = S_1 + S_2 + S_3 + S_4, \end{aligned}$$

where S_1, S_2, S_3, S_4 are the sums in order.

First, we compute S_1. By Lemma 2.3, we have

$$S_1 = \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G)} \frac{d_{F(G_1)}(u_1)+d_{G_2}(u_2)+d_{F(G_1)}(v_1)+d_{G_2}(v_2)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)},$$
$$= \frac{1}{2}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left[\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}+\frac{d_{(G_2)}(u_2)+d_{G_2}(v_2)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}\right].$$

By Lemma 2.7, we have,

$$\frac{1}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)} \le \frac{1}{4d_{F(G_1)}(u_1,v_1)}+\frac{1}{4d_{G_2}(u_2,v_2)},$$

with equality if and only if $d_{F(G_1)}(u_1,v_1)=d_{G_2}(u_2,v_2)$. Therefore,

$$\begin{aligned}
S_1 &\le \frac{1}{8}\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\Bigg[\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)\left(\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{d_{G_2}(u_2,v_2)}\right)\\
&\quad+\Big(d_{G_2}(u_2)+d_{G_2}(v_2)\Big)\left(\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{d_{G_2}(u_2,v_2)}\right)\Bigg]\\
&= \frac{1}{8}\Bigg[\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left(\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}\right)+\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left(\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{G_2}(u_2,v_2)}\right)\\
&\quad+\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left(\frac{d_{G_2}(u_2)+d_{G_2}(v_2)}{d_{F(G_1)}(u_1,v_1)}\right)+\sum_{u_1,v_1\in V(G_1)}\sum_{u_2,v_2\in V(G_2)}\left(\frac{d_{G_2}(u_2)+d_{G_2}(v_2)}{d_{G_2}(u_2,v_2)}\right)\Bigg].
\end{aligned}$$

By the definitions of the Harary index and reciprocal degree distance, we obtain

$$\begin{aligned}
S_1 &\le \frac{n_2^2}{8}\sum_{u_1,v_1\in V(G_1)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{4}H(G_2)\sum_{u_1,v_1\in V(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)\\
&\quad+\frac{1}{2}n_2m_2\sum_{u_1,v_1\in V(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{n_1^2}{4}RDD(G_2).
\end{aligned}$$

For any vertex $(u,v)\in V(G_1+_F G_2)$, we have $d_{G_1+_F G_2}(u)=d_{F(G_1)}(u_1)+d_{G_2}(u_2)$ and $d_{G_1+_F(G_2)}(v)=d_{F(G_1)}(v_1)$.

$$S_2 = \sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{F(G_1)}(u_1)+d_{G_2}(u_2)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}.$$

By Lemma 2.3, we obtain

$$S_2 = \sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\left[\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}+\frac{d_{G_2}(u_2)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}\right].$$

By Jensen's inequality, we have,

$$\frac{1}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)} \le \frac{1}{4d_{F(G_1)}(u_1,v_1)}+\frac{1}{4d_{G_2}(u_2,v_2)},$$

with equality if and only if $d_{F(G_1)}(u_1,v_1)=d_{G_2}(u_2,v_2)$. Thus

$$\begin{aligned}
S_2 &\le \frac{1}{4}\sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\Bigg[\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)\left(\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{d_{G_2}(u_2,v_2)}\right)\\
&\quad+d_{G_2}(u_2)\left(\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{d_{G_2}(u_2,v_2)}\right)\Bigg]\\
&\le \frac{1}{4}\Bigg[\sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\left(\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}\right)\\
&\quad+\sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]}{d_{G_2}(u_2,v_2)}
\end{aligned}$$

$$+\sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{G_2}(u_2)}{d_{F(G_1)}(u_1,v_1)}+\sum_{u_2,v_2\in V(G_2)}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{G_2}(u_2)}{d_{G_2}(u_2,v_2)}\Bigg]$$

$$=\frac{n_2^2}{4}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]}{d_{F(G_1)}(u_1,v_1)}$$

$$+\frac{1}{2}H(G_2)\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)+\frac{1}{2}n_2m_2\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}$$

$$+\frac{1}{4}m_1n_1RDD(G_2).$$

In order to calculate S_3 and S_4, we assume that u and v are the vertices such that $u\in E(G_1)\times V(G_2)$ and $v\in E(G_1)\times V(G_2)$. By Lemma 2.4, we have

$$\begin{aligned}
S_3 &= \frac{1}{2}\sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{2d_{F(G_1)}(u_1)}{(2+d_{G_2}(u_2,v_2))}\\
&\le \sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{1}{4}\left[\frac{d_{F(G_1)}(u_1)}{2}+\frac{d_{F(G_1)}(u_1)}{d_{G_2}(u_2,v_2)}\right]\ \text{(with equality if } d_{G_2}(u_2,v_2)=2)\\
&= \sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)}{8}+\frac{1}{4}\sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)}{d_{G_2}(u_2,v_2)}\\
&= \frac{1}{8}(n_2^2-n_2)\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1)+\frac{1}{2}H(G_2)\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1).
\end{aligned}$$

By using Lemma 2.4, we have

$$S_4=\frac{1}{2}\sum_{u_2,v_2\in V(G_2)}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}.$$

By Jensen's inequality, we have

$$\frac{1}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}\le\frac{1}{4d_{F(G_1)}(u_1,v_1)}+\frac{1}{4d_{G_2}(u_2,v_2)},$$

with equality if and only if $d_{F(G_1)}(u_1,v_1)=d_{G_2}(u_2,v_2)$. Thus,

$$\begin{aligned}
S_4 &= \frac{1}{8}\Bigg[\sum_{u_2,v_2\in V(G_2)}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)(u_1,v_1)}}\\
&\quad+\sum_{u_2,v_2\in V(G_2)}\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{G_2}(u_2,v_2)}\Bigg]\\
&= \frac{1}{8}\sum_{u_2,v_2\in V(G_2)}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)(u_1,v_1)}}\\
&\quad+\frac{1}{4}H(G_2)\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)).
\end{aligned}$$

From the obtained values and inequalities of S_1, S_2, S_3, and S_4, we get

$$\begin{aligned}
RDD(G) &= S_1+S_2+S_3+S_4\\
&\le \frac{n_2^2}{8}\sum_{u_1,v_1\in V(G_1)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{4}H(G_2)\sum_{u_1,v_1\in V(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)\\
&\quad+\frac{1}{2}n_2m_2\sum_{u_1,v_1\in V(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{n_1^2}{4}RDD(G_2)+\frac{n_2^2}{4}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}\\
&\quad+\frac{1}{2}H(G_2)\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)+\frac{1}{2}n_2m_2\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}
\end{aligned}$$

$$+\frac{1}{4}m_1n_1RDD(G_2)+\frac{1}{8}(n_2^2-n_2)\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1)+\frac{1}{2}H(G_2)\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1)$$
$$+\frac{n_2^2}{8}\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}$$
$$+\frac{1}{4}H(G_2)\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)). \quad (1)$$

Now, we calculate the sums given in (1) separately.

(i). By Lemma 2.6, one has

$$\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}[d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)]=\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}[k\,d_{G_1}(u_1)+2]=2km_1^2+2m_1n_1.$$

(ii). By the definition of the Harary index, we get

$$\begin{aligned}\sum_{u_1,v_1\in V(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}+\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)} &= 2W(F(G_1))-\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}\frac{1}{d_{F(G_1)}(u_1,v_1)}\\ &\quad -\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}\\ &\le 2H(F(G_1)).\end{aligned}$$

(iii). By Lemma 2.6, we have

$$\begin{aligned}\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1) &= \sum_{u_1\in E(G_1)}k\,d_{G_1}(u_1)\\ &= k\sum_{u_1\in E(G_1)}d_{L(G_1)}(u_1),\\ &= k\sum_{u_1=u_1'u_1''\in E(G_1)}(d_{G_1}(u_1')+d_{G_1}(u_1'')-2)\\ &= k[M_1(G_1)-2m_1].\end{aligned}$$

(iv).

$$\begin{aligned}\sum_{u_1,v_1\in V(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big) &= \sum_{u_1,v_1\in E(G_1)}\Big(k\,d_{G_1}(u_1)+k,\ d_{G_1}(V_1)\Big)\\ &\quad+\sum_{u_1,v_1\notin E(G_1)}\Big(k\,d_{G_1}(u_1)+k,\ d_{G_1}(V_1)\Big)\\ &= k[M_1(G_1)+\overline{M}_1(G_1)].\end{aligned}$$

(v).

$$\begin{aligned}\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)) &= \sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1)+4)\\ &= \sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(d_{L(G_1)}(u_1)+d_{L(G_1)}(v_1))\\ &\quad+4\sum_{u_1,v_1\in E(G_1);u_1\neq v_1}(1)\\ &= 2(m_1-1)(M_1(G_1))-2m_1+4(m_1^2-m_1)\\ &= 2(m_1-1)M_1(G_1).\end{aligned}$$

(vi). We note that $\left(d_{F(G_1)}(u_1,v_1)\right)^{-1} \leq 1$ with equality if $d_{F(G_1)}(u_1,v_1)=1$. By (i), we have

$$\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1))}{d_{F(G_1)}(u_1,v_1)} \leq \sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)) = 2km_1^2+2m_1n_1.$$

Now, by making use of (i)–(vi) in (1), we get

$$\begin{aligned} RDD(G) &\leq \frac{n_2^2}{4}RDD(F(G_1))+\frac{n_1(n_1+m_1)}{4}RDD(G_2)+n_2m_2H(F(G_1))+\frac{n_2(n_2-1)k}{8}(M_1(G_1)-2m_1) \\ &+\frac{n_2^2(m_1-1)}{4}M_1(G_1)+\frac{H(G_2)}{4}\Big[(3k+2m_1-2)M_1(G_1)+k\overline{M}_1(G_1)+4m_1(k(m_1-1)+n_1)\Big] \\ &+\frac{n_2^2m_1(km_1+n_1)}{2}. \end{aligned}$$

□

Next, we state and prove the second main result of this paper.

Theorem 2.2. *Suppose that G_i is a connected graph with n_i vertices and m_i edges, $i=1,2$, and take $F\in\{Q,T\}$. Then*

$$\begin{aligned} RDD(G_1+_F G_2) &\leq \frac{n_2^2}{4}RDD(F(G_1))+\frac{n_1(n_1+m_1)}{4}RDD(G_2)+2n_2m_2H(F(G_1)) \\ &+\frac{H_(G_2)}{2}\left[\frac{k}{2}(M_1+\overline{M}_1(G_1))22m-1(km_1+n_1)+\frac{M_1(G_1)(m_1-1)}{4}\right] \\ &+\frac{n_2}{16}M_1(G_1)\Big((21n_2-1)(m_1-1)+2k(n_2-1)\Big)-\frac{km_1n_2(n_2-1)}{4}. \end{aligned}$$

Proof. Let $G=G_1+_F G_2$. By the definition of the RDD index, we have

$$\begin{aligned} RDD(G) &= \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G_1+_F G_2)}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))}, \\ &= \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G_1)\times V(G_2)}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))} \\ &\quad \sum_{(u_1,u_2)\in V(G_1)\times V(G_2),(v_1,v_2)\in E(G_1)\times V(G_2)}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))} \\ &\quad +\frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in E(G_1),u_1=v_1,u_2\neq v_2}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))} \\ &\quad +\frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in E(G_1)\times V(G_2),u_2=v_2;u_1\neq v_1}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))} \\ &\quad +\frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in E(G_1)\times V(G_2),u_2\neq v_2;u_1\neq v_1}\frac{d_G(u_1,u_2)+d_G(v_1,v_2)}{d_G((u_1,u_2),(v_1,v_2))} \\ &= S_1+S_2+S_3+S_4+S_5, \end{aligned}$$

where $S_1,S_2,\cdots,S_5$ are the sums in order.
First, we compute S_1 and S_2. By Lemma 2.4 and a similar way as used in Theorem 2.1, we obtain

$$\begin{aligned} S_1 &= \frac{1}{2}\sum_{(u_1,u_2),(v_1,v_2)\in V(G)}\frac{d_{F(G_1)}(u_1)+d_{G_2}(u_2)+d_{F(G_1)}(v_1)+d_{G_2}(v_2)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}. \\ &\leq \frac{n_2^2}{8}\sum_{u_1,v_1\in V(G_1)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{4}H(G_2)\sum_{u_1,v_1\in V(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big) \\ &\quad +\frac{1}{2}n_2m_2\sum_{u_1,v_1\in V(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{n_1^2}{4}RDD(G_2). \end{aligned}$$

Similarly, we have

$$\begin{aligned}
S_2 &= \sum_{u_2,v_2\in V_2}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{F(G_1)}(u_1)+d_{G_2}(u_2)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2)}\\
&\le \frac{n_2^2}{4}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{2}H(G_2)\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\Big(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)\Big)\\
&\quad+\frac{n_2m_2}{2}\sum_{u_1\in V(G_1)}\sum_{v_1\in E(G_1)}\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{m_1n_1}{4}RDD(G_2).
\end{aligned}$$

To calculate S_3 and S_4, we assume that u and v are the vertices such that $u\in E(G_1)\times V(G_2)$ and $v\in E(G_1)\times V(G_2)$. By Lemma 2.4, we have

$$\begin{aligned}
S_3 &= \frac{1}{2}\sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{2d_{F(G_1)}(u_1)}{2+d_{G_2}(u_2,v_2)}\\
&\le \sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{1}{4}\left[\frac{d_{F(G_1)}(u_1)}{2}+\frac{d_{F(G_1)}(u_1)}{d_{G_2}(u_2,v_2)}\right]\quad \text{(with equality if } d_{G_2}(u_2,v_2)=2)\\
&= \sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)}{8}+\frac{1}{4}\sum_{u_1\in E(G_1)}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)}{d_{G_2}(u_2,v_2)}\\
&= \frac{n_2^2-n_2}{8}\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1)+\frac{H(G_2)}{2}\sum_{u_1\in E(G_1)}d_{F(G_1)}(u_1).
\end{aligned}$$

By Lemma 2.5, we get

$$\begin{aligned}
S_4 &= \frac{1}{2}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2)}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)},\\
&= \frac{n_2}{2}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}.
\end{aligned}$$

Finally, the quantity of S_5 is computed as follows.

$$S_5 = \frac{1}{2}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{(1+d_{F(G_1)}(u_1,v_1)+d_{G_2}(u_2,v_2))}.$$

By using Lemma 2.4 and Jenson's inequality, we have

$$\begin{aligned}
S_5 &\le \frac{1}{8}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1))\left[\frac{1}{d_{F(G_1)}(u_1,v_1)}+\frac{1}{d_{G_2}(u_2,v_2)+1}\right]\\
&\quad \text{(with equality if } d_{F(G_1)}(u_1,v_1)=d_{(G_2)}(u_2,v_2)+1)\\
&\le \frac{1}{8}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)(u_1,v_1)}}\\
&\quad+\frac{1}{32}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\left[\frac{1}{d_{G_2}(u_2,v_2)}+1\right]\quad \text{(with equality if } d_{G_2}(u_2,v_2)=1)\\
&= \frac{1}{32}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}(d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1))\\
&\quad+\frac{1}{8}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1,v_1)}\\
&\quad+\frac{1}{32}\sum_{u_1,v_1\in E(G_1);\ u_1\neq v_1}\sum_{u_2,v_2\in V(G_2);u_2\neq v_2}\frac{d_{F(G_1)}(u_1)+d_{F(G_1)}(v_1)}{d_{G_2}(u_2,v_2)}
\end{aligned}$$

$$= \frac{n_2^2 - n_2}{32} \sum_{u_1, v_1 \in E(G_1);\ u_1 \neq v_1} (d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1))$$
$$+ \frac{n_2^2 - n_2}{8} \sum_{u_1, v_1 \in E(G_1);\ u_1 \neq v_1} \frac{d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)}{d_{F(G_1)}(u_1, v_1)}$$
$$+ \frac{H(G_2)}{16} \sum_{u_1, v_1 \in E(G_1);\ u_1 \neq v_1} (d_{F(G_1)}(u_1) + d_{F(G_1)}(v_1)).$$

By using the obtained values and inequalities of $S_1, S_2, \cdots, S_5$, and an argument similar to the one used in the proof of Theorem 2.1, we obtain

$$\begin{aligned} RDD(G) \leq \ & \frac{n_2^2}{4} RDD(F(G_1)) + \frac{n_1(n_1 + m_1)}{4} RDD(G_2) + 2n_2 m_2 H(F(G_1)) \\ & + \frac{H(G_2)}{2} \left[\frac{k}{2} (M_1 + \overline{M}_1(G_1)) 22m - 1(km_1 + n_1) + \frac{M_1(G_1)(m_1 - 1)}{4} \right] \\ & + \frac{n_2}{16} M_1(G_1) \Big((21n_2 - 1)(m_1 - 1) + 2k(n_2 - 1) \Big) - \frac{km_1 n_2(n_2 - 1)}{4}. \end{aligned}$$

□

We end this paper by posing the following open problem which is related to our main results.

Problem 2.1. *Characterize the graphs attaining the upper bounds given in Theorems 2.1 and 2.2.*

References

[1] Y. Alizadeh, A. Iranmanesh, T. Došlić, Additively weighted Harary index of some composite graphs, *Discrete Math.* **313** (2013) 26–34.
[2] M. An, L. Xiong, K. C. Das, Two upper bounds for the degree distances of four sums of graphs, *Filomat* **28** (2014) 579–590.
[3] A. A. Dobrynin, A. A. Kochetova, Degree distance of a graph: A degree analogue of the Wiener index, *J. Chem. Inf. Comput. Sci.* **34** (1994) 1082–1086.
[4] M. Eliasi, B. Taeri, Four new sums of graphs and their Wiener indices, *Discrete Appl. Math.* **157** (2009) 794–803.
[5] I. Gutman, Selected properties of the Schultz molecular topological index, *J. Chem. Inf. Comput. Sci.* **34** (1994) 1087–1089.
[6] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) 535–538.
[7] H. Hua, S. Zhang, On the reciprocal degree distance of graphs, *Discrete Appl. Math.* **160** (2012) 1152–1163.
[8] S. C. Li, X. Meng, Four edge-grafting theorems on the reciprocal degree distance of graphs and their applications, *J. Comb. Optim.* **30** (2015) 468–488.
[9] C. Niculescu, L. E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, Springer, New York, 2006.
[10] K. Pattabiraman, M. Vijayaragavan, Reciprocal degree distance of some graph operations, *Trans. Comb.* **2** (2013) 13–24.
[11] K. Pattabiraman, M. Vijayaragavan, Reciprocal degree distance of product graphs, *Discrete Appl. Math.* **179** (2014) 201–213.
[12] G. F. Su, L. Xiong, X. F. Su, X. L. Chen, Some results on the reciprocal sum-degree distance of graphs, *J. Comb. Optim.* **30** (2015) 435–446.
[13] W. Yan, B. Y. Yang, Y. N. Yeh, The behavior of Wiener indices and polynomials of graphs under five graph decorations, *Appl. Math. Lett.* **20** (2007) 290–295.