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Research Article

Uniqueness of the tensor decomposition for tensors with small ranks over a field

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Abstract

We study the uniqueness of a decomposition of a tensor over a field as a sum of rank 1 tensors, when the tensor has low rank, up to 3. We put this in a more general framework (X-rank) and study two different definitions of decompositions over a given (not algebraically closed) field.

Keywords: Segre variety; tensor decomposition; perfect field.

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1. Introduction

Let K be a field. Let $\overline{K}$ be the algebraic closure of K. Unless otherwise stated we assume that K is a perfect field. We just mention that this assumption is satisfied if either K is a finite field or if $\mathrm{char}(K) = 0$. Let $X \subset \mathbb{P}^r$ be a geometrically integral subvariety defined over K and such that $X(\overline{K})$ is non-degenerate, i.e. no hyperplane of $\mathbb{P}^r(\overline{K})$ contains $X(\overline{K})$. Recall that for any $q \in \mathbb{P}^r(\overline{K})$ the $X(\overline{K})$-rank $r_{X(\overline{K})}(q)$ of q is the minimal cardinality of a finite set $A \subset X(\overline{K})$ such that $q \in \langle A\rangle_{\overline{K}}$, where $\langle\ \rangle_{\overline{K}}$ denotes the linear span over $\overline{K}$. The *solution set* $\mathcal{S}(X(\overline{K}), q)$ of q with respect to $X(\overline{K})$ is the set of all finite sets $A \subset X(\overline{K})$ such that $\#A = r_{X(\overline{K})}(q)$ and $q \in \langle A\rangle_{\overline{K}}$. This definition implies $\mathcal{S}(X(\overline{K}), q) \neq \emptyset$. If $\#\mathcal{S}(X(\overline{K}), q) = 1$ we say that q satisfies *uniqueness* or that it has *uniqueness with respect to* $X(\overline{K})$. Now assume $q \in \mathbb{P}^r(K)$. There are at least two very different ways to define the K-rank of q and each of these two ways gives a different definition of solution set. These definitions may give different ranks (Example 2.1) or the same rank, but different solution sets (Example 2.2).

Definition 1.1. *Let $r_{X(K)}(q)$ be the minimal cardinality of a set $A \subseteq X(K)$ such that A spans q with the convention $r_{X(K)}(q) = +\infty$ if there is no such set A exists, i.e. the set $X(K)$ is contained in a hyperplane not containing q.*

Definition 1.2. *The (X, K)-rank $r_{X,K}(q)$ of q is the minimal cardinality of a finite set $A \subset X(\overline{K})$ defined over K and whose linear span contains q (we do not require that all points of A are defined over q).*

If $r_{X(K)}(q) < +\infty$ let $\mathcal{S}(X(K), q)$ denote the set of all $A \subseteq X(K)$ spanning q and with $\#A = r_{X(K)}(q)$. The integer $r_{X(K)}(q)$ is often called the $X(K)$-rank of q.

Call $\mathcal{S}(X, K, q)$ the solution set of q for Definition 1.2, i.e., let $\mathcal{S}(X, K, q)$ denote the set of all $A \subset X(\overline{K})$ defined over K such that $\#A = r_{X,K}(q)$ and A spans q.

In the next two theorems $X(\overline{K}) \subset \mathbb{P}^r(\overline{K})$ is a Segre variety defined over K. In their statements $X(\overline{K})$ and $X(K)$ are the images by the Segre embedding ν of a multiprojective space

$$Y_K = \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$$

and conciseness over $\overline{K}$ means that there is no proper multiprojective space $Y'(\overline{K}) \subsetneq Y(\overline{K})$ such that $q \in \langle \nu(Y'(\overline{K}))\rangle_{\overline{K}}$.

Theorem 1.1. *Assume $\#K \geq 5$. Take*

$$X(\overline{K}) \cong \mathbb{P}^{n_1}(\overline{K}) \times \cdots \times \mathbb{P}^{n_k}$$

with $n_i > 0$ for all i and assume that this decomposition is defined over K. Fix $q \in \mathbb{P}^r(K)$ such that $r_{X(K)}(q) = 2$ (respectively $r_{X,K}(q) = 2$) and $X(\overline{K})$ is concise for q. Then $\#\mathcal{S}(X(K), q) > 1$ (respectively $\#\mathcal{S}(X, K, q) > 1$) if and only if $k = 2$ and $n_1 = n_2 = 1$. Moreover, the solution sets are infinite in each of these cases if K is infinite.

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The next result only uses Definition 1.1.

Theorem 1.2. *Assume K infinite and $\mathrm{char}(K) \neq 2, 3$. Fix $q \in \mathbb{P}^r(K)$ which is concise over $\overline{K}$, i.e. there is no Segre variety $X'(\overline{K}) \subsetneq X(\overline{K})$ such that $q \in \langle X'(\overline{K})\rangle_{\overline{K}}$. Assume $r_{X(K)}(q) = 3$. We have $\#\mathcal{S}(X(K), q) > 1$ if and only if q and $Y_K = \mathbb{P}^{n_1}_K \times \cdots \mathbb{P}^{n_k}_K$ are in one of the following 6 cases (up to a permutation of the factors of Y):*

(1). *$k = 2$ and $n + 1 = n_2 = 1$;*

(2). *$k = 3$, $n_1 = n_2 = n_3 = 1$ and q is contained in the tangential variety of $X(\overline{K})$.*

(3). *$k = 4$, $n_1 = n_2 = n_3 = n_4 = 1$;*

(4). *$k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$;*

(5). *$k = 3$, $n_1 \in \{1, 2\}$, $n_2 \in \{1, 2\}$, $n_i = 1$ for all $i > 2$ and q is represented by a tensor which is the sum of a rank 1 tensor and a rank 2 tensor equivalent to a 2×2 matrix.*

(6). *$r_{X(\overline{K})}(q) = 2$, $\#\mathcal{S}(X(\overline{K}), q) = 1$ and $r_{X(K)}(q) = 3$.*

Moreover, $\mathcal{S}(X(K), q)$ is infinite in all these cases.

See Example 5.6 for case (6) of the list (of course, it does not occur for all K: it does not occur if $K = \overline{K}$). Case (6) does not occur for the (X, K)-rank by Lemma 2.1.

The first 5 items of the list are as the ones of [3, Theorem 7.1], except that case 4 covers two cases (case 4 and 5 of [3, Theorem 7.1]), because the integers k and n_i are the same and the thesis in both cases is that $\mathcal{S}(X(K), q)$ is infinite (see Example 5.5 for an explanation of the geometry involved). The last one is handled in End of Proof of Theorem 1.2 with a quotation to Proposition 4.2 proved in section 4.

A key tool for the proof of Theorem 1.1 is [3, Proposition 2.3]. A key tool for the proof of Theorem 1.2 is [3, Theorem 7.1], which is also listed in the introduction of [3]. To use [3, Proposition 2.3] it will be sufficient to quote it at a key point. The use of [3, Theorem 7.1] is more complicated, because as any reader of [3] can see it says that a concise tensor $q \in \mathbb{P}^r(\overline{K})$ such that $r_{X(\overline{K})}(q) = 3$ has $\#\mathcal{S}(X(\overline{K}), q) > 1$ if and only if q is as in 6 listed classes, with some of the classes described with the parameters of the concise Segre of q, the integer $\dim \mathcal{S}(X(\overline{K}), q)$ (which is always > 0) and, sometimes, the additional words: see Example so and so for a description of q and $\mathcal{S}(X(\overline{K}), q)$. In each case we will give all the details needed for our proofs over K (Examples 5.1, 5.2, 5.3, 5.4). Then in the end of proof of Theorem 1.2 we will connect the dots and explain the use of [3, Theorem 7.1] in the other cases, too.

2. Arbitrary X

In this section we only assume that $X \subset \mathbb{P}^r$ is a geometrically integral and defined over K and that $X(\overline{K})$ is non-degenerate. For any $q = (a_0 : \cdots : a_r) \in \mathbb{P}^r(\overline{K})$ let K_q be the subfield of $\overline{K}$ generated by K and all fractions a_i/a_j with $a_j \neq 0$. Note that for all $t \in \overline{K} \setminus \{0\}$ $(a_0 : \cdots ; a_r)$ and $(ta_0 : \cdots : ta_r)$ give the same ratios with non-zero denominators. The field K_q is invariant for the action of $GL(r+1, K)$ and it is often called the field generated by K and q. Since $\overline{K}$ is algebraic over K, the field K_q is a finite extension of q.

Let $A \subset \mathbb{P}^r(\overline{K})$ be a finite set. Let $K'_A \subseteq \overline{K}$ be the subfield generated by $\cup_{q \in A} K_q$. The field K'_A will be called the subfield of $\overline{K}$ generated by the points of A. Since K is a perfect field, there is a finite extension K_1 of K'_A such that the extension K_1/K is Galois, say with Galois group G. Set $H := \{g \in G \mid g(A) = A\}$ and $K_A := K_1^H$ (the fixed field). The field K_A is called the Galois subfield of $\overline{K}$ generated by A. If $K_A = K$ we say that A is defined over K. Fix any $q \in \mathbb{P}^r(K)$. The (X, K)-rank $r_{X,K}(q)$ of q is the minimal cardinality of a finite set $A \subset \mathbb{P}^r(\overline{K})$ defined over K and spanning q. We always have $r_{X,K}(q) < +\infty$. Obviously

$$r_{X(\overline{K})}(q) \le r_{X,K}(q) \le r_{X(K)}(q) \tag{1}$$

Recall that $\mathcal{S}(X, K, q)$ denotes the set of all finite sets $S \subset Y(\overline{K})$ such that S is defined over K (but we are not assuming that all points of S are defined over q), $\#S = r_{X,K}(q)$ and $q \in \langle \nu(S)\rangle_{\overline{K}}$.

For any field $L \supseteq K$ and any finite set $S \subseteq X(L)$ let $\langle S\rangle_L$ denote the linear span of S in $\mathbb{P}^r(L)$. For any $q \in \mathbb{P}^r(K)$ and any $S \in \mathcal{S}(X(\overline{K}), q)$ there is a finite extension L of K such that $q \in \mathcal{S}(X(L), q)$. The field L depends on S. If $\mathcal{S}(X(\overline{K}), q)$ is infinite there should not be, in general, a finite extension L of K such that $\mathcal{S}(X(\overline{K}), q) = \mathcal{S}(X(L), q)$ (but it may exist, e.g. for $K = \mathbb{R}$, we may take as L the field $\mathbb{C}$).

Example 2.1. Take $K = \mathbb{R}$ and $\overline{K} = \mathbb{C} = \mathbb{R}(i)$. Let $C \subset \mathbb{P}^2$ be a smooth curve defined over $\mathbb{R}$ and with $C(\mathbb{R}) \neq \emptyset$. All $q \in \mathbb{P}^2(\mathbb{R}) \setminus C(\mathbb{R})$ have $r_{X(\mathbb{C})}(q) = 2$, but there are many example in [4, §3] of pairs (X, q) with $r_{C(\mathbb{R})}(q) = 3$.

Example 2.2. Take $K = \mathbb{R}$ and $\overline{K} = \mathbb{C} = \mathbb{R}(i)$. Let $C \subset \mathbb{P}^2$ be a real smooth conic with $C(\mathbb{R}) \neq \emptyset$. Up to a real change of variables we may take $C = \{x^2 + y^2 - z^2 = 0\}$, where x, y, z are homogeneous coordinates. Fix $q \in \mathbb{P}^2(\mathbb{R}) \setminus C(\mathbb{R})$. Since $q \notin C(\mathbb{C})$, $r_{X(\mathbb{C})}(q) = 2$. There are 2 tangent lines of $C(\mathbb{R})$ passing through q. Call o_1, o_2 the points of $C(\mathbb{R})$ whose tangent lines contain q. For any real line $L(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ containing q and not intersecting $\{o_1, o_2\}$ the set $L(\mathbb{R}) \cap C(\mathbb{R})$ is formed by two distinct points of $C(\mathbb{C})$ and the set $L(\mathbb{R}) \cap C(\mathbb{R})$ is invariant for the complex conjugation. Thus $r_{X,\mathbb{R}}(q) = 2$ and $\mathcal{S}(X, \mathbb{R}, q)$ is a real $\mathbb{P}^1(\mathbb{R})$ (the real pencil of all lines through q) minus 2 points. Thus topologically $\mathcal{S}(X, \mathbb{R}, q)$ is the union of 2 disjoint circles. In the same way $r_{X(\mathbb{C})}(q) = 2$ and that $\mathcal{S}(C(\mathbb{C}), q)$ is a complex $\mathbb{P}^1(\mathbb{C})$ minus 2 points. Let $\ell_q : \mathbb{P}^2(\mathbb{C}) \setminus \{q\} \longrightarrow \mathbb{P}^1(\mathbb{C})$ denote the linear projection from q. Since $q \notin C(\mathbb{C})$, $\ell := \ell_{q|C(\mathbb{C})} : C(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ is a degree 2 surjection. Since $q \in \mathbb{P}^2(\mathbb{R})$, ℓ_q and ℓ are defined over $\mathbb{R}$. The set $C(\mathbb{R})$ is a circle, while $C(\mathbb{R}) \setminus \{o_1, o_2\}$ is the union of two disjoint intervals. To see that $\mathcal{S}(C(\mathbb{R}), q) \subsetneq \mathcal{S}(C, \mathbb{R}, q)$ for some q it is sufficient to dehomogenize the equation of C and take as q a point outside the circle $\{x^2 + y^2 = 1\} \subset \mathbb{R}^2$, i.e. to take $q = (x : y : 1)$ with $x^2 + y^2 > 1$.

Lemma 2.1. *Fix $q \in \mathbb{P}^r(K)$ such that $\#\mathcal{S}(X(\overline{K}), q) = 1$. Then:*

(1). $r_{X,K} = r_{X(\overline{K})}(q)$ *and* $\mathcal{S}(X(\overline{K}), q) = \mathcal{S}(X, K, q)$.

(2). *If* $r_{X(K)} = r_{X,K}(q)$, *then* $\mathcal{S}(X(\overline{K}), q) = \mathcal{S}(X(K), q)$.

Proof. Write $\mathcal{S}(X(\overline{K}), q) = \{A\}$ for some $A \in X(\overline{K})$. Since K is perfect, there is a finite Galois extension K' of K such that each point of A is defined over K'. Call G the Galois group of the extension K'/K. Fix $g \in G$. Since $q \in \mathbb{P}^r(K) \cap \langle A \rangle_{\overline{K}}$ and $g_{|K}$ is the identity map, $q \in \langle g(A) \rangle_{\overline{K}}$. Thus $g(A) = A$. Thus $A \in \mathcal{S}(X, K, q)$ and $r_{X,K}(q) = \#A$. Now assume $r_{X(K)} = r_{X(\overline{K})}(q)$. Since $\mathcal{S}(X(K), q) \neq \emptyset$, $\mathcal{S}(X(K), q) = \{A\}$. □

Example 2.1 shows that in part (2) of Lemma 2.1 the assumption " $r_{X(K)} = r_{X,K}(q)$ " is not always satisfied.

3. Segre varieties: notation and preliminaries

Remember that K is a perfect field. We call $\mathbb{P}^n_K$ an n-dimensional projective space defined over K. Note that we impose in the definition of $\mathbb{P}^n_K$ that the degree 1 line bundle is defined over K. For all fields $L \supseteq K$ let $\mathbb{P}^n(L)$ denote the set of all L-points of $\mathbb{P}^n_K$. Fix positive integers k and n_i, $1 \leq i \leq k$ and set $Y_K := \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$ (or just Y since K is fixed). We impose that Y_K splits over K as a product of k projective spaces, each of them defined over K. For any multiprojective space Y let ν denote its Segre embedding. Thus if $Y = \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K$, ν is an embedding $\nu : Y \longrightarrow \mathbb{P}^r_K$, $r = -1 + \prod_{i=1}^k (n_i + 1)$, defined over K. For instance, for $k = 2$ and $n_1 = n_2 = 1$ the scheme $\nu(Y(K)) \subset \mathbb{P}^3(K)$ is projectively equivalent to the smooth hyperbolic quadric surface. For many K there are non-hyperbolic smooth quadric surfaces. The non-hyperbolic smooth quadric surfaces are not counterexamples to many of the statement of this paper, because our assumptions prevent such objects as subjects of the theorems. See [6] for a description of the Segre varieties over a finite field. For any field $L \supseteq K$, ν induces an injective map (denoted with the same symbol) $\nu : Y(L) \longrightarrow \mathbb{P}^r(L)$. The elements of $\mathbb{P}^r(L)$ are the equivalence classes (up to a non-zero multiplicative constant) of tensors of format $(n_1 + 1) \times \cdots \times (n_k + 1)$ with coefficients in L. Let $\pi_i : Y \longrightarrow \mathbb{P}^{n_i}_K$ be the projection of Y onto its i-th factor. Set $Y_i := \prod_{j \neq i} \mathbb{P}^{n_j}_K$ and let $\eta_i : Y \longrightarrow Y_i$ denote the projection. Thus for any $p = (p_1, \dots, p_k) \in Y$, $\pi_i(p) = p_i$ is the i-th components of p, while $\eta_i(p) = (p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_k)$ deletes the i-th component of p. These formulas show that π_i and η_i are defined over K and that for any field $L \supseteq K$ they induces surjections (denoted with the same symbols) $\pi_i : Y(L) \longrightarrow \mathbb{P}^{n_i}(L)$ and $\eta_i : Y(L) \longrightarrow Y_i(L)$. Since in our definition the decomposition of Y into k factors $\mathbb{P}^{n_i}_K$ is defined over K, the Segre variety $\nu(Y_K)$ has k rulings by projective subspaces. For any field L such that $K \subseteq L \subseteq \overline{K}$ set $X(L) := \nu(Y(L)) \subseteq \mathbb{P}^r(L)$.

We fix $q \in \mathbb{P}^r(K)$ (unless otherwise stated) and call $Y'(\overline{K}) \subseteq Y(\overline{K})$ the minimal multiprojective space such that $q \in \langle \nu(Y'(\overline{K})) \rangle_{\overline{K}}$. We often says that $X'(\overline{K}) := \nu(Y'(\overline{K}))$ is the *concise Segre of* q. By Autarky (see [8, Proposition 3.1.3.1]) $r_{X'(\overline{K})}(q) = r_{X(\overline{K})}(q)$ and $\mathcal{S}(X'(\overline{K}), q) = \mathcal{S}(X(\overline{K}), q)$ (the proof of [8, Proposition 3.1.3.1] only requires that we can make limits and the Zariski topology is sufficient over an algebraically closed field).

For any $q \in \mathbb{P}^r(\overline{K})$ we say that a finite set $S \subset \mathbb{P}^r(\overline{K})$ irredundantly spans q if $q \in \langle S \rangle_{\overline{K}}$ and $q \notin \langle S' \rangle_{\overline{K}}$ for any $S' \subsetneq S$.

Take any $q \in \mathbb{P}^r(\overline{K})$. We saw that, after fixing the coordinates, we get a field K_q and it is natural to study the notions of ranks and solutions for q with respect to the field K_q. Of course, if $q \in \mathbb{P}^r(K)$, then $K_q = K$ and so, as we have seen, the two notions may be different.

Fix positive integers k and n_i, $1 \leq i \leq k$. Call $\mathcal{E}_{X(\overline{K})}$ the set of all $q \in \mathbb{P}^r(\overline{K})$ which are concise for $X(\overline{K})$, i.e. there is no multiprojective space $Y' \subsetneq Y(\overline{K})$ with Y' defined over $\overline{K}$ and $q \in \langle \nu(Y') \rangle_{\overline{K}}$. By concision over any algebraically closed field for each $q \in \mathcal{E}_{X(\overline{K})}$ every $A \in \mathcal{S}(Y(\overline{K}), q)$ spans $Y(\overline{K})$ (the proof of [8, Proposition 3.1.3.1] only requires that we can make limits and the Zariski topology is sufficient over an algebraically closed field).

Definition 3.1. *Let $A \subset Y(\overline{K}) = \mathbb{P}^{n_1}(\overline{K}) \times \cdots \times \mathbb{P}^{n_k}(\overline{K})$, $n_i > 0$ for all i, be a finite set, $A \neq \emptyset$. Fix $o = (o_1, \dots, o_k) \in A$, an integer $i \in \{1, \dots, k\}$, a line $L_i(\overline{K}) \subseteq \mathbb{P}^{n_i}(\overline{K})$ such that $o_i \in L_i$ and two points $u_i, v_i \in L_i(\overline{K}) \setminus \{o_i\}$, $u_i \neq v_i$. Write $u = (a_1, \dots, a_k)$ and $v = (b_1, \dots, b_k)$ with $a_j = b_j = o_j$ for all $j \neq i$, $a_i = u_i$ and $b_i = v_i$. Set $A' := (A \setminus \{o\}) \cup \{u, v\}$. The set A' is said to be obtained from A making an elementary increasing with respect to the i-th factor.*

Remark 3.1. *Take A, o, u, v and A' as in Definition 3.1. Obviously $\#A' = \#A + 1$. Take a field $K \subseteq K' \subseteq \overline{K}$ and assume that the finite set $A \setminus \{o\}$ is defined over K' and $\{o_i, u_i, v_i\} \subset \mathbb{P}^{n_i}(K')$. Then A' is defined over K'. If each point of A is defined over K' and $\{u_i, v_i\} \subset \mathbb{P}^{n_i}(K')$, then each point of A' is defined over K'.*

Remark 3.2. *Take A, o, u, v and A' as in Definition 3.1 with $\#A > 1$ and such that at least one point $a \in A \setminus \{o\}$ is not defined over K. Then for no choice of u_i, v_i all points of A' are defined over K.*

Remark 3.3. *Take $S \subset Y(\overline{K})$ such that $e(S) > 0$ and $\#S \leq 3$. Since ν is an embedding, $\#S = 3$ and $e(S) = 1$, i.e. $L := \langle \nu(S) \rangle_{\overline{K}}$ is a line. Since L contains 3 points of $\nu(Y(\overline{K}))$ and any Segre variety is cut out by quadrics, $L \subseteq Y(\overline{K})$. The structure of linear subspaces of $Y(\overline{K})$ shows that there is $i \in \{1, \dots, k\}$ such that $\#\pi_h(S) = 1$ for all $h \neq i$, while $\pi_i(S)$ are 3 collinear points.*

4. Segre varieties: lemmas and quoted results

We use the following result (see [2, Proposition 5.3]) (alternatively, the reader may just use [1, Theorem 1.1] and do a little work).

Proposition 4.1. *Fix $q \in \mathbb{P}^r(\overline{K})$ such that $r_{X(\overline{K})}(q) = 2$ and take a multiprojective space $Y'(\overline{K}) \subseteq Y(\overline{K})$ concise for q. Fix any $A \subset Y(\overline{K})$ such that $\nu(A) \in \mathcal{S}(X(\overline{K}), q)$. Fix $B \subset Y(\overline{K})$ such that $\#B = 3$ and $\nu(B)$ irredundantly spans q. and call $Y'(\overline{K}) \subseteq Y(\overline{K})$ is the minimal multiprojective space containing B. Then $Y'(\overline{K}) \cong (\mathbb{P}^1(\overline{K}))^s$ for some $s \geq 2$, $A \subset Y'(\overline{K})$ and one of the following cases occurs:*

(1). *$A \cap B \neq \emptyset$, B is obtained from A making and elementary increasing as in Definition 3.1 and either $Y'(\overline{K}) = Y(\overline{K})$ or $Y(\overline{K}) \cong \mathbb{P}^2(\overline{K}) \times (\mathbb{P}^1(\overline{K}))^{s-1}$ or $Y(\overline{K}) \cong (\mathbb{P}^1(\overline{K}))^{s+1}$;*

(2). *$A \cap B = \emptyset$; in this case either $Y(\overline{K}) \cong \mathbb{P}^2(\overline{K}) \times \mathbb{P}^1(\overline{K})$ or $Y(\overline{K}) \cong \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$ or $Y(\overline{K}) \cong \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$.*

For Proposition 4.2 the reader is referred to Section 3 for our conventions concerning Segre varieties. For instance (case $k = 2$ and $n_1 = n_2 = 1$) over many fields, e.g. the real field or a finite field, there are smooth quadric surfaces of $\mathbb{P}^3(K)$ with no ruling defined over K, but with K-points.

Proposition 4.2. *Take $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = 2$ and $\#\mathcal{S}(X(\overline{K}), q) > 1$. Then $r_{X,K}(q) = r_{X(K)}(q) = 2$, $\#\mathcal{S}(X, K, q) \geq \#\mathcal{S}(X(K), q) > 1$, $\mathcal{S}(X(K), q)$ is infinite if K is infinite.*

Proof. Fix any $A \subset Y(\overline{K})$ such that $\nu(A) \in \mathcal{S}(\nu(Y(\overline{K})), q)$. Let $Y'(\overline{K}) \subseteq Y(\overline{K})$ the minimal multiprojective space containing A. By Autarky $Y'(\overline{K})$ is the minimal multiprojective space such that $q \in \langle \nu(Y'(\overline{K})) \rangle_{\overline{K}}$. Concision also implies that all elements of $\mathcal{S}(X(\overline{K}), q)$ are contained in $\nu(Y'(\overline{K}))$. By [3, Proposition 2.3] $Y'(\overline{K}) \cong \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$. The structure of Segre varieties shows that that the isomorphism of $Y'(\overline{K})$ with $\mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$ is defined over K (we use that $\#\mathbb{P}^1(K)$ is infinite if K is infinite and that $\#\mathbb{P}^1(K) = \#K + 1 \geq 3$ if K is finite). Since $Y'(\overline{K})$ has only 2 factors, the tensor q is basically a matrix. The classification of rank 2 matrices over K gives the thesis. □

Lemma 4.1. *Let $C \subset \mathbb{P}^3_K$ be a rational normal curve defined over K. Fix $q \in \mathbb{P}^3_K$ with $r_{C(K)}(q) = 3$. If K is infinite, then $\mathcal{S}(C(K), q)$ is infinite. If K is finite and $r_{X(\overline{K})}(q) = 2$, then $\#\mathcal{S}(C(K), q) > 1$ if and only if $\#K \geq 5$.*

Proof. Since $C(\overline{K})$ is a rational normal curve of $\mathbb{P}^3(\overline{K})$, no 3 of its points are collinear. By assumption $q \notin C(K)$ and q is not contained in any secant line of $C(K)$. Hence the linear projection $\ell_q : \mathbb{P}^3_K \setminus \{q\} \to \mathbb{P}^2_K$ induces an injective map $\ell : C(K) \to \mathbb{P}^2(K)$ and $\ell(C(\overline{K}))$ is a degree 3 integral and rational plane curve with arithmetic genus 1 and hence with a unique singular point, o, which is either an ordinary node or an ordinary cusp. In all cases $o \in \mathbb{P}^2(K)$. The point o is an ordinary node if and only if $r_{X(\overline{K})}(q) = 2$ and this occurs if and only if $o \notin \ell(C(K))$. The assumption $r_{C(K)}(q) = 3$ is equivalent to assuming that $\ell(C(K))$ has 3 collinear points. Call L a line (necessarily defined over K) such that $\#(\ell(C(K)) \cap L) = 3$. Since C is a rational normal curve over K, $C(K)$ is infinite if K is infinite and $\#C(K) = \#K + 1$ if K is finite.

Claim 1. $\ell(C(K)) \cap L \subsetneq \ell(C(K))$, unless $K = \mathbb{F}_2$. If either $K \neq \mathbb{F}_3$ or o is an ordinary node, there is $a \in \ell(C(K)) \setminus \ell(C(K)) \cap L$ such that $a \neq o$.

Proof of Claim 1. $\#\ell(C(K))\cap L=3$, $\ell(C(K))$ is infinite if K is infinite, while $\#\ell(C(K))=\#K+1$ if K is finite.

(a). Assume that K is infinite. The set $W(\overline{K})$ of all lines $R\subset\mathbb{P}^2(\overline{K})$ such that $o\notin R$ and $\#(R\cap\ell(C(\overline{K})))=3$ is a non-empty open Zariski open subset of the dual projective space $\mathbb{P}^2(\overline{K})^\vee$. The set W is defined over K. Since $L\in W(K)$, $W(K)\neq\emptyset$. Since K is infinite, $W(K)$ is Zariski dense in $W(\overline{K})$. Thus $\ell(C(K))$ has infinitely many trisecant lines.

(b). Assume $r_{X(\overline{K})}(q)=2$ and K finite. Thus o is an ordinary node and $o\notin\ell(C(K))$. Set $x:=\#K$. The set $\ell(C(K))$ has cardinality $x+1$. By Bezout's theorem each line through o contains at most another point of $\ell(C(K))$, for any $a\in\ell(C(K))$ the tangent line to $\ell(C(\overline{K}))$ does not contain o and no line is quadrisecant to $\ell(C(K))$. Thus each line though 2 points of $\ell(C(K))$ either is tangent to $\ell(C(\overline{K}))$ at one of these 2 points or meets a third point of $\ell(C(\overline{K}))$. There are $\binom{x+1}{2}$ subsets of $\ell(C(\overline{K}))$ with cardinality 2 and at most $x+1$ of them are on a secant line of $\ell(C(\overline{K}))$. The union of three collinear points has 3 subsets with cardinality 2. Thus if $\binom{x+1}{2}-x-1>6$, i.e. if $x\geq 5$, then $\ell(C(K))$ has at least 2 trisecant lines. Now assume $x=4$ and that $\ell(C(K))$ has 2 trisecant lines, say R and D. Set $\{a\}:=L\cap D$. Since $\#\ell(C(K))=5$, $a\in\ell(C(K))$ and $\ell(C(K))\subset R\cup D$. Bezout gives that $T_a\ell(C(\overline{K}))\notin\{R,D\}$. Since $T_a\ell(C(\overline{K}))$ is defined over K, it meets $\ell(C(K))$ at a point not in $R\cup D$, a contradiction. The exclusion of the cases $x=2,3$ is easier, because $\ell(C(K))$ has no quadrisecant line. □

Proposition 4.3. *Take $q\in\mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q)=2$ and $\#\mathcal{S}(X(\overline{K}),q)=1$. Set $\{A\}:=\mathcal{S}(X(\overline{K}),q)$. Then:*

(1). $r_{X,K}(q)=2$ *and* $\{A\}=\mathcal{S}(X,K,q)$.

(2). *If* $r_{X(K)}(q)=2$*, then* $\{A\}=\mathcal{S}(X(K),q)$.

(3). *Assume* $r_{X(K)}(q)=3$*. Then* $\#\mathcal{S}(X(K),q)>1$ *if and only if either K is infinite or* $\#K\geq 5$.

Proof. Since $\#\mathcal{S}(X(\overline{K}),q)=1$, $k>2$. Parts (1) and (2) follow from Lemma 2.1. Next, assume $r_{X(K)}(q)=3$ and take $B\in\mathcal{S}(X(K),q)$. Since $r_{X(K)}(q)\neq 2$, not all points of A are defined over K. Since $\#A=2$, no point of A is defined over K. Fix a finite extension $L\supset K$ such that each point of A is defined over L, say $A=\{a,b\}\subset Y(L)$. Since K is a perfect field and each subgroup with index 2 of a finite group is normal, we may take as L a degree 2 Galois extension of K. Call g the non-zero element of the Galois group G of the extension L/K. Note that $g(a)=b$ and $g(b)=a$. Remark 3.2 implies that B is not an elementary increasing of A. Since $k>2$, Proposition 4.1 implies that the minimal multiprojective space W over $\overline{K}$ containing B is isomorphic to $(\mathbb{P}^1(\overline{K}))^3$. We also have $A\cap B=\emptyset$ and $A\cup B$ is contained in the image of a morphism $f:\mathbb{P}^1(\overline{K})\longrightarrow(\mathbb{P}^1(\overline{K}))^3$ whose composition with the projections π_i are isomorphisms (see [1, Lemma 5.8]. Thus $\pi_{i|B}$ is injective for all i. Write $B=\{u,v,z\}$, $u_i=\pi_i(u)$, $v_i=\pi_i(v)$ and $z_i=\pi_i(z)$. Note that $\{u_i,v_i,z_i\}\subset\mathbb{P}^1(K)$. Fix 3 distinct points $0,1,\infty\in\mathbb{P}^1(K)$. Let $f_i:\mathbb{P}^1_K\longrightarrow\mathbb{P}^1_K$ denote the only isomorphism such that $f_i(0)=u_i$, $f_i(1)=v_i$ and $f_i(\infty)=z_i$. The isomorphism f_i is defined over K and $f=(f_1,f_2,f_3)$ is an embedding $f:\mathbb{P}^1_K\longrightarrow(\mathbb{P}^1_K)^3$ such that $\{u,v,z\}\subset f(\mathbb{P}^1(K))$. The curve $\nu(f((\mathbb{P}^1)^3))$ is a degree 3 rational normal curve. Each point of $\langle\nu(f((\mathbb{P}^1)^3))\rangle_K$ is in more than one way irredundantly spanned by 3 points of $\langle\nu(f((\mathbb{P}^1)^3))\rangle_K$ if $\#K\leq 5$. (Lemma 4.1). Thus in this case $\#\mathcal{S}(X(K),q)>1$, unless $\#K<5$. The cases with $\#K<5$ are excluded by Lemma 4.1. □

Lemma 4.2. *Let $C\subset\mathbb{P}^4$ be a degree 4 rational normal curve defined over K. Assume the existence of $q\in\mathbb{P}^4(K)$ such that $r_{C(\overline{K})}(q)=r_{C(K)}(q)=3$. Assume* $\mathrm{char}(K)\neq 2,3$ *and K infinite. Then $\mathcal{S}(C(K),q)$ is infinite and Zariski dense in $\mathbb{P}^1(\overline{K})$.*

Proof. Take $B\subset C(K)$ with $B\in\mathcal{S}(C(K),q)$. Since $\mathrm{char}(K)\neq 2,3$ and K infinite we may apply Sylvester's theorem (see [7, pp. 36–39]) and get that the set $\mathcal{S}(C(\overline{K}),q)$ is a non-empty open subset of a one-dimensional projective space. The algebraic variety $\mathcal{S}(C(\overline{K}),q)$ is defined over K and it contains $B\subset C(K)$. Thus $\mathcal{S}(C(K),q)$ is infinite and Zariski dense in $\mathbb{P}^1(\overline{K})$. □

5. Segre varieties: the main proofs

Let $\tau(\overline{K})\subseteq\mathbb{P}^r(\overline{K})$ denote the tangential variety of $X(\overline{K})$. We recall the following lemma, part (a) being well-known (e.g. [5, Table 1, $n=3$]) and part (b) be being proved (but not stated) in arbitrary characteristic in [3].

Lemma 5.1. *Take $Y:=(\mathbb{P}^1_K)^3$ and hence $r=7$.*

(a). *Over $\overline{K}$, $\sigma_2(X(\overline{K}))=\mathbb{P}^7(\overline{K})$, $\mathcal{E}_X$ has 2 orbits, O_2 and O_3, for the action of* $\mathrm{Aut}(\mathbb{P}^1(\overline{K}))^3$ *with $O_2=\tau(X(\overline{K}))\cap\mathcal{E}_X$ and $O_3=\mathbb{P}^7(\overline{K})\setminus\tau(X(\overline{K}))$.*

(b). *Each $q\in O_3$ has $r_{X(\overline{K})}(q)=2$ and $\#\mathcal{S}(Y(\overline{K},q)=1$.*

(c). *Each $q\in\tau(X(\overline{K}))\cap\mathcal{E}_X$ has $r_{X(\overline{K})}(q)=3$ and* $\dim\mathcal{S}(X(\overline{K}),q)>0$.

Lemma 5.2. *Take Y, O_1, O_2 and O_3 as in Lemma 5.1. Fix $q \in \mathbb{P}^7(K) \cap \mathcal{E}_X$.*

(a). *If $q \in O_3$, then $r_{X,K}(q) = 2$ and $\#\mathcal{S}(X,K,q) = 1$. If $r_{X(K)}(q) = 2$, then $\#\mathcal{S}(X(K),q) = 1$. If $r_{X(K)}(q) = 3$, then $\#\mathcal{S}(X(K),q) > 1$ if and only if $\#K \ge 5$ and $\mathcal{S}(X(K),q)$ is infinite if K is infinite.*

(b). *If $q \in O_2$, then $r_{X(K)}(q) = r_{X,K}(q) = 3$, $\#\mathcal{S}(X(K),q) > 1$ and $\mathcal{S}(X(K),q)$ is infinite if K is infinite.*

Proof. Fix $q \in O_3$. Part (a) for (X,K) follows from Lemma 2.1 and part (a) of Lemma 5.1. If $r_{X(K)}(q) = 2$, then $\#\mathcal{S}(X(K),q) = 1$ (Lemma 2.1). Now assume $r_{X(K)}(q) = 3$. Take A such that $\{\nu(A)\} = \mathcal{S}(X(\overline{K}),q)$. By assumption A is defined over K, but not all points of A are defined over K. By assumption there is $B \subset Y(K)$ such that $\#B = 3$ and $q \in \langle \nu(B)\rangle_K$. By Remark 3.2, Proposition 4.1 and the assumption $q \in \mathcal{E}_X$, $A \cap B = \emptyset$. A key part of the proof of Proposition 4.1 was [1, Theorem 1.1 and Lemma 5.8] which gives the existence of a curve $C \subset (\mathbb{P}^1(\overline{K})^3$ such that $A \cup B \subset C$ and $\pi_{i|C}$ is an isomorphism for $i = 1,2,3$. Hence $\pi_{i|B}$ is injective. Moreover, [1] also studies all such curves (called of tridegree $(1,1,1)$ in [1]). A key point is that C is uniquely determined in a constructive way from B and hence it is uniquely determined by B. In this way we see that C is defined over K. Since $q \in \langle \nu(B)\rangle_K$, $q \in \langle \nu(C)\rangle_K$. The curve $\nu(C)$ is a degree 3 rational normal curve in its linear span. Use Lemma 4.1 and the last part of the proof of Proposition 4.3.

Now we prove part (b). Since $r_{X(\overline{K})}(q) = 3$, we have $r_{X(K)}(q) \ge r_{X,K}(q) \ge 3$. Since $q \in \tau(X(\overline{K})) \setminus X(\overline{K})$, there is a degree 2 connected zero-dimensional scheme $v \subset Y(\overline{K})$ such that $q \in \langle \nu(v)\rangle_{\overline{K}}$. Since $q \in \mathcal{E}_X \cap \mathbb{P}^7(K)$, it is easy to check as in the proof of [3, Proposition 2.3] that v is unique. Hence v is defined over K. Thus $\{o\}_{\mathrm{red}}$ is defined over K (here we use that K is perfect). Write $o = (o_1, o_2, o_3)$ with $o_i \in \mathbb{P}^1(K)$. Let $v_i \subset \mathbb{P}^1(\overline{K})$ connected degree scheme with o_i as its reduction. Set $T := \eta_1^{-1}(o_1) \cup \eta_2^{-1}(o_2) \cup \eta_3^{-1}(o_3)$. The set $\nu(T(\overline{K}))$ is the union of 3 lines through $\nu(o)$, each of them defined over K and $\dim\langle \nu(T(\overline{K}))\rangle_{\overline{K}} = 3$. Fix $e_3 \in \mathbb{P}^1(K) \setminus \{o_3\}$ (it exists for any K). Let $\ell : \langle \nu(T(\overline{K}))\rangle_{\overline{K}} \setminus \{\nu(e_3)\} \to \mathbb{P}^2(\overline{K})$ denote the linear projection from $\nu(e_3)$. Since $\nu(e_3)$ and $\nu(T)$ are defined over K, ℓ is defined over K. Since $q \in \mathcal{E}_X$, $q \ne \nu(o_3)$ and hence $\ell(q)$ is a well-defined point of $\mathbb{P}^2(K)$. The plane $\mathbb{P}^2(\overline{K})$ is spanned by the reducible conic $D := \nu(\eta_2^{-1}(o_2) \cup \eta_3^{-1}(o_3))$. Since $\ell(q) \notin D$, we easily see the existence of more that one (and infinitely many if K is infinite) $S \subset D(K)$ such that $\ell(q) \in \langle S\rangle$. Thus $r_{X(K)}(q) \le 3$. Hence $r_{X(K)}(q) \ge r_{X,K}(q) \ge 3$. We also proved that $\#\mathcal{S}(X(K),q) > 1$ and $\mathcal{S}(X(K),q)$ is infinite if K is infinite. □

Lemma 5.3. *Take $Y := (\mathbb{P}^1_K)^4$ and hence $r = 15$. We have $\dim \sigma_3(X(\overline{K})) = 14$, $\dim \mathcal{S}(Y(\overline{K}) \ge 1$ for all q with $r_{X(\overline{K})}(q)) = 3$ and $\dim \mathcal{S}(Y(\overline{K}),q) = 1$ for a general $q \in \sigma_3(X(\overline{K}))$. Take $q \in \mathbb{P}^{15}(K)$ such that $r_{X(\overline{K})}(q)) = r_{X(K)}(q)) = 3$. If K is infinite, then $\mathcal{S}(X(K),q)$ is infinite.*

Proof. The part over $\overline{K}$ is well-known and written down at least in characteristic 0 in all lists of defective Segre varieties. In arbitrary characteristic it is essentially proved in the following way, which we also need for later proofs over K. Fix $B \subset Y(K)$ such that $\#B = 3$ and $\#\pi_i(B) = 3$ for all i. Write $B = \{u,v,z\}$. Let $f_i : \mathbb{P}^1_K \to \mathbb{P}^1_K$ be the only isomorphism such that $f_i(0) = \pi_i(u)$, $f_i(1) = \pi_i(v)$ and $f_i(\infty) = \pi_i(z)$. The morphism $f = (f_1,f_2,f_3,f_4) : \mathbb{P}^1(\overline{K}) \to \mathbb{P}^1(\overline{K})^4$ is defined over K and $B \subseteq f(\mathbb{P}^1(K))$. Since $\nu(f(\mathbb{P}^1(\overline{K})))$ is a degree 4 rational normal curve in its linear span, we may apply Lemma 4.2.

By assumption $r_{X(K)}(q) = 3$. Fix $B \subset Y(K)$ such that $\nu(B) \in \mathcal{S}(X(K),q)$. We just proved the case "$\#\pi_i(B) = 3$ for all i ". Thus we may assume $\#\pi_i(B) \le 2$, for at least one index i. Since $X(\overline{K})$ is concise for q, $\#\pi_i(B) = 2$. If this is true for at least two indices i, then we are in case described in Example 5.3 below. Thus we may assume that $\#\pi_i(B) = 2$ for exactly one index i, say $i = 4$. Call $F \subset B$ the set with $\#F = 2$ and $\#\pi_4(F) = 1$. Set $\{a\} := \pi_4(B)$ and $H := \pi_4^{-1}(a)$. H is a multiprojective space isomorphic to $(\mathbb{P}^1_K)^3$ and $H = (\mathbb{P}^1_K)^3 \times \{a\}$ is embedded in Y by the inclusion $\{a\} \hookrightarrow \mathbb{P}^1_K$. Since $\nu(B)$ irredundantly spans q, there is a unique $q' \in \langle \nu(F)\rangle_K$ such that $q \in \langle \{q', \nu(o)\}\rangle$. Now we vary $b \in (\mathbb{P}^1_K)^4$. Take any $o' \in (\mathbb{P}^1(K))^4$. The line $\langle \{q, \nu(b)\}\rangle_K$ meets $\langle \nu(H)\rangle_K$ at a unique point q_b. We take $F_b \in \mathcal{S}(H(K), q_b)$ and set $B_b := F_b \cup \{b\}$. We need to justify the existence of F_b, i.e. that $r_{H(K)}(q_b) = 2$. For any field $L \supseteq K$ all sets $A \subset H(L)$ such that $\#\pi_i(A) = 2$ for all $i = 1,2,3$, form a unique orbit for the action of $\mathrm{Aut}(\mathbb{P}^1(L))^3$. Taking their linear spans we cover all $q'' \in \langle \nu(Y(L)\rangle_L$ with $r_{H(L)}(q'') = 2$. Since K is infinite, we get as q_b a Zariski dense subset of the projective space $\langle \nu(H(\overline{K}))\rangle_{\overline{K}}$. □

End of the proof of Theorem 1.1: Fix $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = 2$. The case with $\#\mathcal{S}(X(\overline{K}),q) > 1$ is true by Proposition 4.2. Assume $\#\mathcal{S}(X(\overline{K}),q) = 1$. Lemma 2.1 gives $r_{X,K}(2) = 2$, $\#\mathcal{S}(X,K,q) = 1$ and that $\#\mathcal{S}(X(K),q) = 1$ if $r_{X(K)}(q) = 2$. □

Example 5.1. Case 1 of [3, Theorem 7.1] is the case $k = 2$, i.e. when the tensor T whose equivalence class represent q is a rank 3 matrix. In this case $r_{X(K)}(q) = r_{X(\overline{K})}$, $\#\mathcal{S}(X(K),q) > 1$ for any field K and $\mathcal{S}(X(K),q)$ is infinite if K is infinite.

Example 5.2. Take $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = r_{X,K}(q) = 3$, $X(\overline{K})$ is concise for $X(\overline{K})$ and q is as in case 2 of [3, Theorem 7.1]. Thus $k = 3$, $n_1 = n_2 = n_3 = 1$ and $q \in \tau(X(\overline{K}))$. We described this case in Lemma 5.2.

Example 5.3. Take $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = r_{X,K)}(q) = 3$, $X(\overline{K})$ is concise for $(\overline{K})$ and q is as in case 6 of [3, Theorem 7.1]. Thus $k \geq 3$, $n_i = 1$ for all $i \geq 3$, $n_1 \in \{1,2\}$, $n_2 \in \{1,2\}$ and there is $E \in \mathcal{S}(X(\overline{K}), q)$ and $F \subset E$ with $\#F = 2$ and $\#\pi_i(F) = 1$ for all $i \geq 3$. Set $\{o\} := E \setminus F$. Since $\nu(E)$ irredundantly spans q, there is a unique $q' \in \langle \nu(F) \rangle_{\overline{K}}$ such that $q \in \langle \{q', \nu(o)\} \rangle_{\overline{K}}$. If $n_1 + n_2 + k \geq 4$, we also proved that each $E' \in \mathcal{S}(X(\overline{K}), q)$ contains o and $\#\pi(E' \setminus \{o\}) = 1$ for all $i \geq 3$. Under these stronger assumptions o is uniquely determined by q and hence it is defined over K. Thus q' is defined over k. By the definition of q' and the description of this case in [3] the minimal multiprojective space $Y'(\overline{K})$ such that $q' \in \langle \nu(Y'(\overline{K})) \rangle_{\overline{K}}$ is isomorphic to $\mathbb{P}^1(\overline{K})^2$, i.e. q' is represented by a tensor, which is equivalent to a 2×2 rank 2 matrix. Since q' is defined over K, the classification of matrices of rank > 1 gives $\#\mathcal{S}(X(K), q') > 1$ and that $\mathcal{S}(X(K), q')$ is infinite. Adding o we get $\#\mathcal{S}(X(K), q') > 1$ and that $\mathcal{S}(X(K), q)$ is infinite if K is infinite. Now assume $k = 3$ and $n_1 = n_2 = n_3 = 1$. Concision and the assumption "$r_{X(\overline{K})}(q) = 3$" give $q \in \tau(X(\overline{K})) \cap \mathcal{E}_X$ and we handled this case in Lemma 5.1.

Example 5.4. Take $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = r_{X,K)}(q) = 3$, $X(\overline{K})$ is concise for q, and q is as in case 3 of [3, Theorem 7.1]. Thus $k = 4$, $n_1 = n_2 = n_3 = n_4 = 1$. We described this case in Lemma 5.3.

Example 5.5. Take $q \in \mathbb{P}^r(K)$ such that $r_{X(\overline{K})}(q) = r_{X,K)}(q) = 3$, $X(\overline{K})$ is concise for $(\overline{K})$ and q is either as in case 4 or as in case 5 of [3, Theorem 7.1]. In both cases $k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$ and the 2 cases are distinguished in the following way. Fix any $E \subset Y(\overline{K})$ such that $\nu(E) \in \mathcal{S}(X(\overline{K}, q))$ and any $B \subset Y(K)$ such that $\nu(B) \in \mathcal{S}(X(K), q)$. Write $B = \{u, v, z\}$. Since $h^0(\mathcal{O}_{Y(\overline{K})}(0,1,1)) = 4$ and $\#E = 3$, $|\mathcal{I}_E(0,1,1)| \neq \emptyset$. It was proved in [3, Proposition 3.8] that $|\mathcal{I}_E(0,1,1)|$ is a singleton, say $|\mathcal{I}_E(0,1,1)| = \{G\}$. The two cases are distinguished as if G is irreducible (case 4) or G is reducible, say $G = G_1 \cup G_2$ with $G_1 \in |\mathcal{O}_{Y(\overline{K})}(0,1,0)|$ and $G_2 \in |\mathcal{O}_{Y(\overline{K})}(0,0,1)|$ (case 5). Each $A \subset Y(\overline{K})$ such that $\nu(A) \in \mathcal{S}(X(\overline{K}, q))$ is contained in G (see [3, Proposition 3.8]). In particular $B \subset G$ and G, being uniquely determined by B, is defined over K. The two cases are also distinguished according to the injectivity of both $\pi_{2|E}$ and $\pi_{2|E}$.

(a). Assume that G is irreducible and hence $\pi_{i|B}$ is injective for $i = 2, 3$. Thus for $i = 2, 3$ there is a unique isomorphism $f_i : \mathbb{P}^1_K \to \mathbb{P}^1_K$ defined over K such that $f_i(0) = \pi_i(u)$, $f_i(1) = \pi_1(v)$ and $f_i(\infty) = \pi_i(z)$. Since q is concise and $n_1 = 2$, $\pi_{1|B}$ is injective and $\pi_1(B)$ are 3 points of $\mathbb{P}^2(K)$ which are not collinear. Thus there is an embedding $f_1 : \mathbb{P}^1_K \to \mathbb{P}^2_K$ defined over K, with $f_1(0) = \pi_1(u)$, $f_1(1) = \pi_1(v)$, $f_1(\infty) = \pi_1(z)$ and $f_1(\mathbb{P}^1_K)$ a smooth conic containing $\pi_1(B)$. Set $f = (f_1, f_2, f_3) : \mathbb{P}^1_K \to Y_K$. The curve $\nu(\mathbb{P}^1_K)$ is a degree 4 rational normal curve in its linear span. We apply Lemma 4.2 to this rational normal curve.

(b). Assume $G = G_1 \cup G_2$. The Segre variety $X(\overline{K})$ is concise for q, $B \nsubseteq G_1$ and $B \nsubseteq G_2$. Exchanging if necessary the second and the third factor of Y_K we may assume $\#(B \cap G_1) = 2$ and $\#(B \cap G_2) = 1$. Set $F := B \cap G_1$, $\{o\} := B \setminus F$ and use Example 5.3.

End of the proof of Theorem 1.2: Fix $q \in \mathbb{P}^r(K)$.

(a). Assume $r_{X(\overline{K})}(q) = 2$ and $r_{X(K)}(q) = 3$. Use case 3 of Proposition 4.3.

(a1). Assume $r_{X(\overline{K})}(q) = 3$. Since $r_{X(K)}(q) = r_{X(\overline{K})}(q)$, Lemma 2.1 gives that $\#\mathcal{S}(X(K), q) = 1$ if $\#\mathcal{S}(X(\overline{K}), q) = 1$. Thus it is sufficient to check all 6 cases listed in [3, Theorem 7.1]. We did it in Examples 5.1, 5.2, 5.3, 5.4 and 5.5. □

Example 5.6. Assume that K has a degree 2 extension L. Call σ the generator of the Galois group the extension L/K. Fix $X(\overline{K})$ with $Y_K = (\mathbb{P}^1_K)^k$ for some $k > 2$ factors. Take $a_1, \dots, a_k \in K_1 \setminus K$. Using a_i and $\sigma(a_i)$ we may constructs a point $u \in Y(K_1) \setminus Y(K)$ such that $Y(\overline{K})$ is the minimal multiprojective space containing $\{u, \sigma(u)\}$. The line $\langle \{\nu(u), \nu(\sigma(u))\} \rangle_L$ is defined over K and hence it corresponds to a line R of $\mathbb{P}^r(K)$ containing no point of $\nu(Y(K))$. Fix $q \in R$. Since $\{u, \sigma(u)\} \in \mathcal{S}(\overline{L}), q)$ and $\#\mathcal{S}(X(\overline{K}), q) = 1$ (see [3, Proposition 2.3]), $r_{X,K}(q) = 2$ and $r_{X(K)}(q) > 2$. At least in some cases, e.g. $k = 3$ and $K = \mathbb{R}$, it is easy to find u and q such that $r_{X(K)}(q) = 3$.

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