Research Article

Uniqueness of the tensor decomposition for tensors with small ranks over a field

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Abstract

We study the uniqueness of a decomposition of a tensor over a field as a sum of rank 1 tensors, when the tensor has low rank, up to 3. We put this in a more general framework (X-rank) and study two different definitions of decompositions over a given (not algebraically closed) field.

Keywords: Segre variety; tensor decomposition; perfect field.

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1. Introduction

Let K be a field. Let \( \overline{K} \) be the algebraic closure of K. Unless otherwise stated we assume that K is a perfect field. We just mention that this assumption is satisfied if either K is a finite field or if \( \text{char}(K) = 0 \). Let \( X \subset \mathbb{P}^r \) be a geometrically integral subvariety defined over K and such that \( X(\overline{K}) \) is non-degenerate, i.e. no hyperplane of \( \mathbb{P}^r(\overline{K}) \) contains \( X(\overline{K}) \). Recall that for any \( q \in \mathbb{P}^r(\overline{K}) \) the \((X(\overline{K}), q)\)-rank \( r_{X(\overline{K}), q} \) of \( q \) is the minimal cardinality of a finite set \( A \subset X(\overline{K}) \) such that \( q \in \langle A \rangle_{\overline{K}} \), where \( \langle A \rangle_{\overline{K}} \) denotes the linear span over \( \overline{K} \). The solution set \( S(X(\overline{K}), q) \) of \( q \) with respect to \( X(\overline{K}) \) is the set of all finite sets \( A \subset X(\overline{K}) \) such that \( \#A = r_{X(\overline{K}), q} \) and \( q \in \langle A \rangle_{\overline{K}} \). This definition implies \( S(X(\overline{K}), q) \neq \emptyset \). If \( \#S(X(\overline{K}), q) = 1 \) we say that \( q \) satisfies uniqueness or that it has uniqueness with respect to \( X(\overline{K}) \). Now assume \( q \in \mathbb{P}^r(K) \). There are at least two very different ways to define the K-rank of \( q \) and each of these two ways gives a different definition of solution set. These definitions may give different ranks (Example 2.1) or the same rank, but different solution sets (Example 2.2).

Definition 1.1. Let \( r_{X(K), q} \) be the minimal cardinality of a set \( A \subset X(K) \) such that \( A \) spans \( q \) with the convention \( r_{X(K), q} = +\infty \) if there is no such set \( A \) exists, i.e. the set \( X(K) \) is contained in a hyperplane not containing \( q \).

Definition 1.2. The \((X, K)\)-rank \( r_{X, K}(q) \) of \( q \) is the minimal cardinality of a finite set \( A \subset X(\overline{K}) \) defined over \( K \) and whose linear span contains \( q \) (we do not require that all points of \( A \) are defined over \( q \)).

If \( r_{X(K), q} < +\infty \) let \( S(X(K), q) \) denote the set of all \( A \subset X(K) \) spanning \( q \) and with \( \#A = r_{X(K), q} \). The integer \( r_{X(K), q} \) is often called the \((X(K))-rank \) of \( q \).

Call \( S(X, K, q) \) the solution set of \( q \) for Definition 1.2, i.e., let \( S(X, K, q) \) denote the set of all \( A \subset X(\overline{K}) \) defined over \( K \) such that \( \#A = r_{X, K}(q) \) and \( A \) spans \( q \).

In the next two theorems \( X(\overline{K}) \subset \mathbb{P}^r(\overline{K}) \) is a Segre variety defined over \( K \). In their statements \( X(\overline{K}) \) and \( X(K) \) are the images by the Segre embedding \( \nu \) of a multiprojective space

\[
Y_K = \mathbb{P}^{n_1}_K \times \cdots \times \mathbb{P}^{n_k}_K
\]

and conciseness over \( \overline{K} \) means that there is no proper multiprojective space \( Y'(\overline{K}) \subset Y(\overline{K}) \) such that \( q \in \langle \nu(Y'(\overline{K})) \rangle_{\overline{K}} \).

Theorem 1.1. Assume \( \#K \geq 5 \). Take

\[
X(\overline{K}) \cong \mathbb{P}^{n_1}(\overline{K}) \times \cdots \times \mathbb{P}^{n_k}(\overline{K})
\]

with \( n_i > 0 \) for all \( i \) and assume that this decomposition is defined over \( K \). Fix \( q \in \mathbb{P}^r(K) \) such that \( r_{X(K), q} = 2 \) (respectively \( r_{X, K}(q) = 2 \)) and \( X(\overline{K}) \) is concise for \( q \). Then \( \#S(X(K), q) > 1 \) (respectively \( \#S(X, K, q) > 1 \)) if and only if \( k = 2 \) and \( n_1 = n_2 = 1 \). Moreover, the solution sets are infinite in each of these cases if \( k \) is infinite.

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The next result only uses Definition 1.1.

**Theorem 1.2.** Assume $K$ infinite and $\text{char}(K) \neq 2, 3$. Fix $q \in \mathbb{P}^r(K)$ which is concise over $\overline{K}$, i.e. there is no Segre variety $X'(\overline{K}) \subseteq X(\overline{K})$ such that $q \in \langle X'(\overline{K}) \rangle_{\overline{K}}$. Assume $r_{X(\overline{K})}(q) = 3$. We have $\#S(X(\overline{K}), q) > 1$ if and only if $q$ and $Y_K = \mathbb{P}^n_1 \times \cdots \times \mathbb{P}^n_k$ are in one of the following 6 cases (up to a permutation of the factors of $Y$):

1. $k = 2$ and $n + 1 = n_2 = 1$;
2. $k = 3$, $n_1 = n_2 = n_3 = 1$ and $q$ is contained in the tangential variety of $X(\overline{K})$.
3. $k = 4$, $n_1 = n_2 = n_3 = n_4 = 1$;
4. $k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$;
5. $k = 3$, $n_1 \in \{1, 2\}$, $n_2 \in \{1, 2\}$, $n_4 = 1$ for all $i > 2$ and $q$ is represented by a tensor which is the sum of a rank 1 tensor and a rank 2 tensor equivalent to a $2 \times 2$ matrix.
6. $r_{X(\overline{K})}(q) = 2$, $\#S(X(\overline{K}), q) = 1$ and $r_{X(\overline{K})}(q) = 3$.

Moreover, $S(X(\overline{K}), q)$ is infinite in all these cases.

See Example 5.6 for case (6) of the list (of course, it does not occur for all $K$: it does not occur if $K = \mathbb{C}$). Case (6) does not occur for the $(X, K)$-rank by Lemma 2.1.

The first 5 items of the list are as the ones of [3, Theorem 7.1], except that case 4 covers two cases (case 4 and 5 of [3, Theorem 7.1]), because the integers $k$ and $n_i$ are the same and the thesis in both cases is that $S(X(\overline{K}), q)$ is infinite (see Example 5.5 for an explanation of the geometry involved). The last one is handled in End of Proof of Theorem 1.2 with a quotation to Proposition 4.2 proved in section 4.

A key tool for the proof of Theorem 1.1 is [3, Proposition 2.3]. A key tool for the proof of Theorem 1.2 is [3, Theorem 7.1], which is also listed in the introduction of [3]. To use [3, Proposition 2.3] it will be sufficient to quote it at a key point. The use of [3, Theorem 7.1] is more complicated, because as any reader of [3] can see it says that a concise tensor $q \in \mathbb{P}(\overline{K})$ such that $r_{X(\overline{K})}(q) = 3$ has $\#S(X(\overline{K}), q) > 1$ if and only if $q$ is as in 6 listed classes, with some of the classes described with the parameters of the concise Segre of $q$, the integer $\dim S(X(\overline{K}), q)$ (which is always $> 0$) and, sometimes, the additional words: see Example so and so for a description of $q$ and $S(X(\overline{K}), q)$. In each case we will give all the details needed for our proofs over $K$ (Examples 5.1, 5.2, 5.3, 5.4). Then in the end of proof of Theorem 1.2 we will connect the dots and explain the use of [3, Theorem 7.1] in the other cases, too.

### 2. Arbitrary $X$

In this section we only assume that $X \subseteq \mathbb{P}^r$ is a geometrically integral and defined over $K$ and that $X(\overline{K})$ is non-degenerate. For any $q = (a_0 : \cdots : a_r) \in \mathbb{P}^r(K)$ let $K_q$ be the subfield of $\overline{K}$ generated by $K$ and all fractions $a_i/a_j$ with $a_j \neq 0$. Note that for all $t \in \overline{K} \setminus \{0\}$ and $(a_0 : \cdots : a_r)$ and $(ta_0 : \cdots : ta_r)$ give the same ratios with non-zero denominators. The field $K_q$ is invariant for the action of $GL(r + 1, K)$ and it is often called the field generated by $q$ and $K$. Since $\overline{K}$ is algebraic over $K$, the field $K_q$ is a finite extension of $q$.

Let $A \subseteq \mathbb{P}^r(K)$ be a finite set. Let $K'_A \subseteq \overline{K}$ be the subfield generated by $\cup q \in A K_q$. The field $K'_A$ will be called the subfield of $\overline{K}$ generated by the points of $A$. Since $K$ is a perfect field, there is a finite extension $K'_1$ of $K'_A$ such that the extension $K_1/K$ is Galois, say with Galois group $G$. Set $H := \{g \in G \mid g(A) = A\}$ and $K_A := K'_1^H$ (the fixed field). The field $K_A$ is called the Galois subfield of $\overline{K}$ generated by $A$. If $K_A = K$ we say that $A$ is defined over $K$. Fix any $q \in \mathbb{P}(K)$. The $(X, K)$-rank $r_{X,K}(q)$ of $q$ is the minimal cardinality of a finite set $A \subseteq \mathbb{P}(\overline{K})$ defined over $K$ and spanning $q$. We always have $r_{X,K}(q) < +\infty$. Obviously

$$r_{X(\overline{K})}(q) = r_{X,K}(q)$$

Recall that $S(X, K, q)$ denotes the set of all finite sets $S \subseteq Y(\overline{K})$ such that $S$ is defined over $K$ (but we are not assuming that all points of $S$ are defined over $q$), $\#S = r_{X,K}(q)$ and $q \in \langle \nu(S) \rangle_{\overline{K}}$.

For any field $L \supseteq K$ and any finite set $S \subseteq X(L)$ let $\langle S \rangle_L$ denote the linear span of $S$ in $\mathbb{P}(L)$. For any $q \in \mathbb{P}(K)$ and any $S \subseteq S(X(\overline{K}), q)$ there is a finite extension $L$ of $K$ such that $q \in S(X(L), q)$. The field $L$ depends on $S$. If $S(X(\overline{K}), q)$ is infinite there should not be, in general, a finite extension $L$ of $K$ such that $S(X(\overline{K}), q) = S(X(L), q)$ (but it may exist, e.g. for $K = \mathbb{R}$, we may take as $L$ the field $\mathbb{C}$).

**Example 2.1.** Take $K = \mathbb{R}$ and $\overline{K} = \mathbb{C} = \mathbb{R}(i)$. Let $C \subseteq \mathbb{P}^2$ be a smooth curve defined over $\mathbb{R}$ and with $C(\mathbb{R}) \neq \emptyset$. All $q \in \mathbb{P}(\mathbb{R}) \setminus C(\mathbb{R})$ have $r_{X(\overline{C})}(q) = 2$, but there are many example in [4, §3] of pairs $(X, q)$ with $r_{C(\mathbb{R})}(q) = 3$. 


Example 2.2. Take $K = \mathbb{R}$ and $\overline{K} = C = \mathbb{R}(i)$. Let $C \subset \mathbb{P}^2$ be a real smooth conic with $C(\mathbb{R}) \neq \emptyset$. Up to a real change of variables we may take $C = \{x^2 + y^2 - z^2 = 0\}$, where $x, y, z$ are homogeneous coordinates. Fix $q \in \mathbb{P}^2(\mathbb{C}) \setminus C(\mathbb{C})$. Since $q \notin C(\mathbb{C})$, $r_{X,C}(q) = 2$. There are 2 tangent lines of $C(\mathbb{R})$ passing through $q$. Call $\alpha_1, \alpha_2$ the points of $C(\mathbb{R})$ whose tangent lines contain $q$. For any real line $L(\mathbb{R}) \subset \mathbb{P}^2(\mathbb{R})$ containing $q$ and not intersecting $\{\alpha_1, \alpha_2\}$ the set $L(\mathbb{R}) \cap C(\mathbb{R})$ is formed by two distinct points of $C(\mathbb{R})$ and the set $L(\mathbb{R}) \cap C(\mathbb{C})$ is the complex conjugation. Thus $r_{X,R}(q) = 2$ and $S(X,R,q)$ is a real $\mathbb{P}^1(\mathbb{R})$ (the real pencil of all lines through $q$) minus 2 points. Thus topologically $S(X,R,q)$ is the union of 2 disjoint circles. In the same way $r_{X,C}(q) = 2$ and that $S(C,C,q)$ is a complex $\mathbb{P}^1(\mathbb{C})$ minus 2 points. Let $\ell_q : \mathbb{P}^2(\mathbb{C}) \setminus \{q\} \to \mathbb{P}^1(\mathbb{C})$ denote the linear projection from $q$. Since $q \notin C(\mathbb{C})$, $\ell := \ell_{q,C}(C) : C(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a degree 2 surjection. Since $q \in \mathbb{P}^2(\mathbb{R})$, $\ell_q$ and $\ell$ are defined over $\mathbb{R}$. The set $C(\mathbb{R})$ is a circle, while $C(\mathbb{R}) \setminus \{\alpha_1, \alpha_2\}$ is the union of two disjoint intervals. To see that $S(C,C,q) \subset S(C,\mathbb{R},q)$ for some $q$ it is sufficient to dehomogenize the equation of $C$ and take as $q$ a point outside the circle $\{x^2 + y^2 = 1\} \subset \mathbb{R}^2$, i.e. to take $q = (x : y : 1)$ with $x^2 + y^2 > 1$.

Lemma 2.1. Fix $q \in \mathbb{P}^r(K)$ such that $\#S(X,K) = 1$. Then:

1. $r_{X,K} = r_{X,K}(q)$ and $S(X,K,q) = S(X,K,q)$.

2. If $r_{X,K} = r_{X,K}(q)$, then $S(X,K,q) = S(X,K,q)$.

Proof. Write $S(X,K,q) = \{A\}$ for some $A \in X(K)$. Since $K$ is perfect, there is a finite Galois extension $K'$ of $K$ such that each point of $A$ is defined over $K'$. Call $G$ the Galois group of the extension $K'/K$. Fix $g \in G$. Since $q \in \mathbb{P}^r(K) \cap (A)_K$ and $g$ is the identity map, $q \in (g(A))_K$. Thus $g(A) = A$. Thus $A \in S(X,K,q)$ and $r_{X,K}(q) = \#A$. Now assume $r_{X,K} = r_{X,K}(q)$. Since $S(X,K,q) \neq \emptyset$, $S(X,K,q) = \{A\}$.

Example 2.1 shows that in part (2) of Lemma 2.1 the assumption “$r_{X,K} = r_{X,K}(q)$” is not always satisfied.

3. Segre varieties: notation and preliminaries

Remember that $K$ is a perfect field. We call $\mathbb{P}^n_K$ an $n$-dimensional projective space defined over $K$. Note that we impose in the definition of $\mathbb{P}^n_K$ that the degree 1 line bundle is defined over $K$. For all fields $L \supseteq K$ we denote the set of all $L$-points of $\mathbb{P}^n_K$. Fix positive integers $k$ and $n_i$, $1 \leq i \leq k$ and set $Y_K := \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$ (or just $Y$ since $K$ is fixed). We impose that $Y_K$ splits over $K$ as a product of $k$ projective spaces, each of them defined over $K$. For any multiprojective space $Y$ let $\nu$ denote its Segre embedding. Thus if $Y = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, $\nu$ is an embedding $\nu : Y \to \mathbb{P}^r$, $r = -1 + \sum_{i=1}^k(n_i + 1)$, defined over $K$. For instance, for $k = 2$ and $n_1 = n_2 = 1$ the scheme $\nu(Y(K)) \subset \mathbb{P}^3$ is projectively equivalent to the smooth hyperbolic quadric surface. For many $K$ there are non-hyperbolic smooth quadric surfaces. The non-hyperbolic smooth quadric surfaces are not counterexamples to many of the statement of this paper, because our assumptions prevent such objects as subjects of the theorems. See [6] for a description of the Segre varieties over a finite field. For any field $L \supseteq K$, $\nu$ induces an injective map (denoted with the same symbol) $\nu : Y(L) \to \mathbb{P}^r(L)$. The elements of $\mathbb{P}^r(L)$ are the equivalence classes (up to a non-zero multiplicative constant) of tensors of format $(n_1 + 1) \times \cdots \times (n_k + 1)$ with coefficients in $L$. Let $\pi_i : Y \to \mathbb{P}^{n_i}_K$ be the projection of $Y$ onto its $i$-th factor. Set $Y_i := \prod_{j \neq i} \mathbb{P}^{n_j}_K$ and let $\eta_i : Y_i \to Y_i$ denote the projection. Thus for any $p = (p_1, \ldots, p_k) \in Y$, $\pi_i(p) = p_i$ is the $i$-th component of $p$, while $\eta_i(p) = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_k)$ deletes the $i$-th component of $p$. These formulas show that $\pi_i$ and $\eta_i$ are defined over $K$ and that for any field $L \supseteq K$ they induces surjections (denoted with the same symbols) $\pi_i : Y(L) \to \mathbb{P}^{n_i}(L)$ and $\eta_i : Y(L) \to Y_i(L)$. Since in our definition the decomposition of $Y$ into $k$ factors $\mathbb{P}^{n_i}_K$ is defined over $K$, the Segre variety $\nu(Y_K)$ has $k$ rulings by projective subspaces. For any field $L$ such that $K \subseteq L \subseteq K$ set $X(L) := \nu(Y(L)) \subseteq \mathbb{P}^r(L)$.

We fix $q \in \mathbb{P}^r(K)$ (unless otherwise stated) and call $Y' \subseteq Y(K)$ the minimal multiprojective space such that $q \in \langle \nu(Y'(K)) \rangle$. We often says that $X'(K) := \nu(Y'(K))$ is the concise Segre of $q$. By Autarky (see [8, Proposition 3.1.3.1]) $r_{X'(K)}(q) = r_{X,K}(q)$ and $S(X',K,q) = S(X,K,q)$ (the proof of [8, Proposition 3.1.3.1] only requires that we can make limits and the Zariski topology is sufficient over an algebraically closed field).

For any $q \in \mathbb{P}^r(K)$ we say that a finite set $S \subset \mathbb{P}^r(K)$ irredundantly spans $q$ if $q \in \langle S \rangle$ and $q \notin \langle S' \rangle$ for any $S' \subset S$.

Take any $q \in \mathbb{P}^r(K)$. We saw that, after fixing the coordinates, we get a field $K_q$ and it is natural to study the notions of ranks and solutions for $q$ with respect to the field $K_q$. Of course, if $q \in \mathbb{P}^r(K)$, then $K_q = K$ and so, as we have seen, the two notions may be different.

Fix positive integers $k$ and $n_i$, $1 \leq i \leq k$. Call $E_{X,K}$ the set of all $q \in \mathbb{P}^r(K)$ which are concise for $X(K)$, i.e. there is no multiprojective space $Y' \subset Y(K)$ with $Y'$ defined over $K$ and $q \in \langle \nu(Y') \rangle$. By concision over any algebraically closed field for each $q \in E_{X,K}$, every $A \in S(Y(K),q)$ spans $Y(K)$ (the proof of [8, Proposition 3.1.3.1] only requires that we can make limits and the Zariski topology is sufficient over an algebraically closed field).
Definition 3.1. Let $A \subseteq Y(K) = \mathbb{P}^{n_1}(K) \times \cdots \times \mathbb{P}^{n_k}(K)$, $n_i > 0$ for all $i$, be a finite set, $A \neq \emptyset$. Fix $o = (o_1, \ldots, o_k) \in A$, an integer $i \in \{1, \ldots, k\}$, a line $L_i(K) \subseteq \mathbb{P}^{n_i}(K)$ such that $o_i \in L_i$, and two points $u_i, v_i \in L_i(K) \setminus \{o_i\}$, $u_i \neq v_i$. Write $u = (a_1, \ldots, a_k)$ and $v = (b_1, \ldots, b_k)$ with $a_j = b_j = o_j$ for all $j \neq i$, $a_i = u_i$ and $b_i = v_i$. Set $A' := (A \setminus \{o\}) \cup \{u, v\}$. The set $A'$ is said to be obtained from $A$ making an elementary increasing with respect to the $i$-th factor.

Remark 3.1. Take $A, o, u, v$ and $A'$ as in Definition 3.1. Obviously $\#A' = \#A + 1$. Take a field $K \subseteq K' \subseteq K$ and assume that the finite set $A \setminus \{o\}$ is defined over $K'$ and $\{o_i, u_i, v_i\} \subseteq \mathbb{P}^{n_i}(K')$. Then $A'$ is defined over $K'$. If each point of $A$ is defined over $K'$ and $\{u_i, v_i\} \subseteq \mathbb{P}^{n_i}(K')$, then each point of $A'$ is defined over $K'$.

Remark 3.2. Take $A, o, u, v$ and $A'$ as in Definition 3.1 with $\#A > 1$ and such that at least one point $a \in A \setminus \{o\}$ is not defined over $K$. Then for no choice of $u_i, v_i$ all points of $A'$ are defined over $K$.

Remark 3.3. Take $S \subseteq Y(K)$ such that $\nu(S) > 0$ and $\#S \leq 3$. Since $\nu$ is an embedding, $\#S = 3$ and $\nu(S) = 1$, i.e. $L := \langle \nu(S) \rangle_K$ is a line. Since $L$ contains 3 points of $\nu(Y(K))$ and any Segre variety is cut out by quadrics, $L \subseteq Y(K)$. The structure of linear subspaces of $Y(K)$ shows that there is $i \in \{1, \ldots, k\}$ such that $\#\pi_i(S) = 1$ for all $h \neq i$, while $\pi_i(S)$ are 3 collinear points.

4. Segre varieties: lemmas and quoted results

We use the following result (see [2, Proposition 5.3]) (alternatively, the reader may just use [1, Theorem 1.1] and do a little work).

Proposition 4.1. Fix $q \in \mathbb{P}^r(K)$ such that $r_{X(K)}(q) = 2$ and take a multiprojective space $Y'(K) \subseteq Y(K)$ concise for $q$. Fix any $A \subseteq Y(K)$ such that $\nu(A) \in S(X(K), q)$. Fix $B \subseteq Y(K)$ such that $\#B = 3$ and $\nu(B)$ irredundantly spans $q$, and call $Y'(K) \subseteq Y(K)$ the minimal multiprojective space containing $B$. Then $Y'(K) \cong (\mathbb{P}^{1}(K))^s$ for some $s \geq 2$, $A \subseteq Y'(K)$ and one of the following cases occurs:

1. $A \cap B \neq \emptyset$, $B$ is obtained from $A$ making and elementary increasing as in Definition 3.1 and either $Y'(K) = Y(K)$ or $Y'(K) \cong \mathbb{P}^2(K) \times (\mathbb{P}^{1}(K))^s-1$ or $Y'(K) \cong (\mathbb{P}^{1}(K))^s+1$;

2. $A \cap B = \emptyset$; in this case either $Y'(K) \cong \mathbb{P}^2(K) \times \mathbb{P}^{1}(K)$ or $Y'(K) \cong \mathbb{P}^1(K) \times \mathbb{P}^{1}(K)$ or $Y'(K) \cong \mathbb{P}^1(K) \times \mathbb{P}^{1}(K) \times \mathbb{P}^{1}(K)$.

For Proposition 4.2 the reader is referred to Section 3 for our conventions concerning Segre varieties. For instance (case $k = 2$ and $n_1 = n_2 = 1$) over many fields, e.g. the real field or a finite field, there are smooth quadric surfaces of $\mathbb{P}^3(K)$ with no ruling defined over $K$, but with $K$-points.

Proposition 4.2. Take $q \in \mathbb{P}^r(K)$ such that $r_{X(K)}(q) = 2$ and $\#S(X(K), q) > 1$. Then $r_{X,K}(q) = r_{X(K)}(q) = 2$, $\#S(X, K, q) \geq \#S(X(K), q) > 1$, $S(X(K), q)$ is infinite if $K$ is infinite.

Proof. Fix any $A \subseteq Y(K)$ such that $\nu(A) \in S(\nu(Y(K)), q)$. Let $Y'(K) \subseteq Y(K)$ the minimal multiprojective space containing $A$. By Autarky $Y'(K)$ is the minimal multiprojective space such that $q \in (\nu(Y'(K)))_K$. Concision also implies that all elements of $S(X(K), q)$ are contained in $\nu(Y'(K))$. By [3, Proposition 2.3] $Y'(K) \cong \mathbb{P}^2(K) \times \mathbb{P}^{1}(K)$. The structure of Segre varieties shows that that the isomorphism of $Y'(K)$ with $\mathbb{P}^2(K) \times \mathbb{P}^{1}(K)$ is defined over $K$ (we use that $\#\mathbb{P}^1(K)$ is infinite if $K$ is infinite and that $\#\mathbb{P}^1(K) = \#K + 1 \geq 3$ if $K$ is finite). Since $Y'(K)$ has only 2 factors, the tensor $q$ is basically a matrix. The classification of rank 2 matrices over $K$ gives the thesis.

Lemma 4.1. Let $C \subseteq \mathbb{P}^3_K$ be a rational normal curve defined over $K$. Fix $q \in \mathbb{P}^3_K$ with $\nu(C(K)) = 3$. If $K$ is infinite, then $S(C(K), q)$ is infinite. If $K$ is finite and $r_{X(K)}(q) = 2$, then $\#S(C(K), q) > 1$ if and only if $\#K \geq 5$.

Proof. Since $C(K)$ is a rational normal curve of $\mathbb{P}^3(K)$, no 3 of its points are collinear. By assumption $q \notin C(K)$ and $q$ is not contained in any secant line of $C(K)$. Hence the linear projection $\ell_q : \mathbb{P}^3_K \setminus \{q\} \to \mathbb{P}^2_K$ induces an injective map $\ell : C(K) \to \mathbb{P}^2(K)$ and $\ell_C(K)$ is a degree 3 integral and rational plane curve with arithmetic genus 1 and hence with a unique singular point, $o$, which is either an ordinary node or an ordinary cusp. In all cases $o \in \mathbb{P}^2(K)$. The point $o$ is an ordinary node if and only if $r_{X(K)}(q) = 2$ and this occurs if and only if $o \notin \ell(C(K))$. The assumption $r_{C(K)}(q) = 3$ is equivalent to assuming that $\ell(C(K))$ has 3 collinear points. Call $L$ a line (necessarily defined over $K$) such that $\#(\ell(C(K)) \cap L) = 3$. Since $C$ is a rational normal curve over $K$, $C(K)$ is infinite if $K$ is infinite and $\#C(K) = \#K + 1$ if $K$ is finite.

Claim 1. $\ell(C(K)) \cap L \subseteq \ell(C(K))$, unless $K = \mathbb{F}_2$. If either $K \neq \mathbb{F}_3$ or $o$ is an ordinary node, there is $a \in \ell(C(K)) \setminus \ell(C(K)) \cap L$ such that $a \neq o$. 

Proof of Claim 1. Let $\ell (C(K)) \cap L = 3$, $\ell (C(K))$ is infinite if $K$ is infinite, while $\# \ell (C(K)) = \# K + 1$ if $K$ is finite.

(a). Assume that $K$ is infinite. The set $W(K)$ of all lines $R \subset \mathbb{P}^2(K)$ such that $0 \not\in R$ and $\# (R \cap \ell (C(K))) = 3$ is a non-empty open Zariski open subset of the dual projective space $\mathbb{P}^2(K)$. The set $W$ is defined over $K$. Since $L \in W(K)$, $W(K)$ is infinite if $K$ is infinite, $W(K)$ is Zariski dense in $W(K)$. Thus $\ell (C(K))$ has infinitely many trisecant lines.

(b). Assume $r_X(\mathbb{P}^1(K)) = 2$ and $K$ finite. Thus $\sigma$ is an ordinary node and $0 \not\in \ell (C(K))$. Set $x := \# K$. The set $\ell (C(K))$ has cardinality $x + 1$. By Bezout’s theorem each line through $o$ contains at most another point of $\ell (C(K))$, for any $a \in \ell (C(K))$ the tangent line to $\ell (C(K))$ does not contain $a$ and no line is trisecant to $\ell (C(K))$. Thus each line through $2$ points of $\ell (C(K))$ either is tangent to $\ell (C(K))$ at one of these two points or meets a third point of $\ell (C(K))$. There are $\binom{r + 1}{3}$ subsets of $\ell (C(K))$ with cardinality $2$ and at most $x + 1$ of them are on a secant line of $\ell (C(K))$. The union of three collinear points has $3$ subsets with cardinality $2$. Thus if $\binom{r + 1}{3} - x - 1 > 6$, i.e. if $x \geq 5$, then $\ell (C(K))$ has at least $2$ trisecant lines. Now assume $x = 4$ and that $\ell (C(K))$ has $2$ trisecant lines, say $R$ and $D$. Set $\{a\} := L \cap 2 \ell (C(K))$ and $\ell (C(K)) \cap R \cup D$. Bezout’s theorem implies that $\ell (C(K)) \not\in \{R, D\}$. Since $\ell (C(K))$ is defined over $K$, it meets $\ell (C(K))$ at a point not in $R \cup D$, a contradiction. The exclusion of the cases $x = 2, 3$ is easier, because $\ell (C(K))$ has no quadrisecant line. 

Proposition 4.3. Take $q \in \mathbb{P}^r(K)$ such that $r_X(\mathbb{P}^1(K)) = 2$ and $\# S(X(\mathbb{P}^1(K)), q) = 1$. Set $\{A\} := S(X(\mathbb{P}^1(K)), q)$. Then:

1. $r_X(K) = 2$ and $\{A\} = S(X, K)$.

2. If $r_X(K) = 2$, then $\{A\} = S(X, K)$.

3. If $r_X(K) = 3$, then $\# S(X(K), q) > 1$ if and only if either $K$ is infinite or $K$ has at least one point.

Proof. Since $\# S(X(K), q) = 1$, $K$ is infinite if $\# S(X(K), q) = 2$, not all points of $A$ are defined over $K$. Since $\# A = 2$, not all points of $A$ are defined over $K$. Fix a finite extension $L \supset K$ such that each point of $A$ is defined over $L$, say $K = \{a, b\} \subseteq Y(L)$. Since $K$ is a perfect field and each subgroup with index $2$ of a finite group is normal, we may take as $L$ a degree $2$ Galois extension of $K$. Call $g$ the non-zero element of the Galois group $G$ of the extension $K$. Note that $q(a) = b$ and $q(b) = a$. Remark 3.2 implies that $B$ is an element of $\mathbb{P}^r(K)$, which is a non-empty open subset of a one-dimensional projective space if $\mathbb{P}^r(K)$ is $2$-dimensional. Now $\ell (C(K))$ is a degree $3$ rational normal curve. Each point of $\langle \nu (f(\langle \mathbb{P}^1(K) \rangle)^3) \rangle \cap K$ is more than one way irredundantly spanned by $3$ points of $\langle \nu (f(\langle \mathbb{P}^1(K) \rangle)^3) \rangle \cap K$ if $\# K \leq 5$. (Lemma 4.1). Thus in this case $\# S(X(K), q) > 1$, unless $\# K < 5$. The cases with $\# K < 5$ are excluded by Lemma 4.1.

Lemma 4.2. Let $C \subset \mathbb{P}^4$ be a degree $4$ rational normal curve defined over $K$. Assume the existence of $q \in \mathbb{P}^4(K)$ such that $r_C(\mathbb{P}^1(K)) = 3$. Assume $\# C(K) = 2$, and $K$ infinite. Then $S(C(K), q)$ is infinite and Zariski dense in $\mathbb{P}^1(K)$.

Proof. Take $B \subset C(K)$ with $B \subset S(C(K), q)$. Since $\# C(K) = 2$, and $K$ infinite we may apply Sylvester’s theorem (see [7, pp. 36–39]) and get that the set $S(C(K), q)$ is a non-empty open subset of a one-dimensional projective space. The algebraic variety $S(C(K), q)$ is defined over $K$ and it contains $B \subset C(K)$. Thus $S(C(K), q)$ is infinite and Zariski dense in $\mathbb{P}^1(K)$.

5. Segre varieties: the main proofs

Let $\tau(\mathbb{P}^1(K)) := (\mathbb{P}^1(K))^3$ and hence $r = 7$.

(a). Over $\mathbb{K}$, $\mathcal{O}_X(\mathbb{P}^1(K)) = \mathbb{P}^1(K)$, $\mathcal{E}_X$ has 2 orbits, $O_2$ and $O_3$, for the action of $\text{Aut}(\mathbb{P}^1(K))^3$ with $O_2 = \tau(\mathbb{P}^1(K)) \cap \mathcal{E}_X$ and $O_3 = \mathbb{P}^1(K) \setminus \tau(\mathbb{P}^1(K))$.

(b). Each $q \in O_3$ has $r_X(\mathbb{P}^1(K)) = 2$ and $\# S(\mathbb{P}^1(K), q) = 1$.

(c). Each $q \in \tau(\mathbb{P}^1(K)) \cap \mathcal{E}_X$ has $r_X(\mathbb{P}^1(K)) = 3$ and $\# S(\mathbb{P}^1(K), q) > 0$. 

Lemma 5.2. Take $Y$, $O_1$, $O_2$ and $O_3$ as in Lemma 5.1. Fix $q \in \mathbb{P}^7(K) \cap \mathcal{E}_X$.

(a). If $q \in O_3$, then $r_{X,K}(q) = 2$ and $\#S(X,K,q) = 1$. If $r_{X(K)}(q) = 2$, then $\#S(X,K,q) = 1$. If $r_{X(K)}(q) = 3$, then $\#S(X,K,q) > 1$ if and only if $K \geq 5$ and $S(X,K,q)$ is infinite if $K$ is infinite.

(b). If $q \in O_2$, then $r_{X,K}(q) = r_{X,K}(q) = 3$, $\#S(X,K,q) > 1$ and $S(X,K,q)$ is infinite if $K$ is infinite.

Proof: Fix $q \in O_3$. Part (a) for $(X,K)$ follows from Lemma 2.1 and part (a) of Lemma 5.1. If $r_{X(K)}(q) = 2$, then $\#S(X,K,q) = 1$ (Lemma 2.1). Now assume $r_{X,K}(q) = 3$. Take $A$ such that $\{v(A)\} = S(X,K,q)$. By assumption $A$ is defined over $K$, but not all points of $A$ are defined over $K$. By assumption there is $B \subset Y(K)$ such that $\#B = 3$ and $q \in \nu(B)$. By Remark 3.2, Proposition 4.1 and the assumption $q \in \mathcal{E}_X$, $A \cap B = \emptyset$. A key part of the proof of Proposition 4.1 was [1, Theorem 1.1 and Lemma 5.8] which gives the existence of a curve $C \subset \mathbb{P}^1(K)$ such that $A \cup B \subset C$ and $\nu(C)$ is an isomorphism for $i = 1, 2, 3$. Hence $\nu_{|B}$ is injective. Moreover, [1] also studies all such curves (called of tridegree $(1,1,1)$ in [1]). A key point is that $C$ is uniquely determined in a constructive way from $B$ and hence it is uniquely determined by $B$. In this way we see that $C$ is defined over $K$. Since $q \in \nu(B) \cap \mathcal{E}_X$, $\nu(C)_K$ is a degree 3 rational normal curve in its linear span. Use Lemma 4.1 and the last part of the proof of Proposition 4.3.

Now we prove part (b). Since $r_{X(K)}(q) = 3$, we have $r_{X,K}(q) \geq r_{X,K}(q) \geq 3$. Since $q \in \tau(X(K)) \setminus X(K)$, there is a degree 2 connected zero-dimensional scheme $v \subset Y(K)$ such that $q \in \nu(v)$. Since $q \in X \cap \mathbb{P}^7(K)$, it is easy to check as in the proof of [3, Proposition 2.3] that $v$ is unique. Hence $v$ is defined over $K$. Thus $\nu_{v}$ fixed is defined over $K$ (here we use that $K$ is perfect). Write $v = (o_1, o_2, o_3)$ with $o_i \in \mathbb{P}^1(K)$. Let $v_1 \subset \mathbb{P}^1(K)$ connected degree scheme with $o_i$ as its reduction. Set $T := \eta_1^{-1}(v_1) \cup \eta_2^{-1}(o_2) \cup \eta_3^{-1}(o_3)$. The set $\nu(T) \subset \mathbb{P}^1(K)$ is the union of 3 lines through $\nu(o)$, each of them defined over $K$ and dim($\nu(T)$) $= 3$. Fix $e_3 \in \mathbb{P}^1(K) \setminus \{o_1\}$ (it exists for any $K$). Let $t$ : $\nu(T(K)) \setminus \{e_3\}$ $\to \mathbb{P}^2(K)$ denote the linear projection from $e_3$. Since $\nu(e_3)$ and $\nu(T)$ are defined over $K$, $t$ is defined over $K$. Since $q \in X, q \not\in \nu(o)$ and hence $\nu(T)$ is a well-defined point of $\mathbb{P}^2(K)$. The plane $\mathbb{P}^2(K)$ is spanned by the reducible conic $D := \nu(\eta_2^{-1}(o_2) \cup \eta_3^{-1}(o_3))$. Since $\nu(T)$ is a degree 3 rational normal curve in its linear span, we may apply Lemma 4.2.

Lemma 5.3. Take $Y := (\mathbb{P}_1)^4$ and hence $r = 15$. We have $\dim S_3(X(K)) = 14$, $\dim S(Y(K)) = 1$ for all $q$ with $r_{X,K}(q) = 3$ and $\dim S(Y(K)) = 1$ for a general $q \in S_3(X(K))$. Take $q \in \mathbb{P}^1(K)$ such that $r_{X,K}(q) = r_{X,K}(q) = 3$. If $K$ is infinite, then $S(X,K,q)$ is infinite.

Proof: The part over $K$ is well-known and written down at least in characteristic 0 in all lists of defective Segre varieties. In arbitrary characteristic it is essentially proved in the following way, which we also need for later proofs over $K$. Fix $B \subset Y(K)$ such that $\#B = 3$ and $\#_B(V) = 3$ for all $i$. Write $B = \{u, v, z\}$. Let $f_1 : \mathbb{P}_1 \to \mathbb{P}_1$ be the only isomorphism such that $f_i(0) = \pi_i(u)$, $f_i(1) = \pi_i(v)$ and $f_i(\infty) = \pi_i(z)$. The morphism $f_i = (f_i, f_2, f_3, f_4) : \mathbb{P}_1 \to \mathbb{P}_1$ is defined over $K$ and $B \subset f(\mathbb{P}_1)$. Since $\nu(f(\mathbb{P}_1))$ is a degree 4 rational normal curve in its linear span, we may apply Lemma 4.2.

By assumption $r_{X,K}(q) = 3$. Fix $B \subset Y(K)$ such that $\nu(B) \in S(X(K), q)$. We just proved the case $\#\pi_i(B) = 3$ for all $i$. Thus we may assume $\#\pi_i(B) \leq 2$, for at least one index i. Since $\nu(X(K))$ is concise for $q$, $\#\pi_i(B) = 2$. If this is true for at least two indices $i$, then we are in case described in Example 5.3 below. Thus we may assume that $\#\pi_i(B) = 2$ for exactly one index $i$, say $i = 4$. Call $F \subset B$ the set with $\#F = 2$ and $\#_F(4) = 1$. Set $\{a\} := \pi_i(B)$ and $H := \pi_i^{-1}(a)$. $H$ is a multiprojective space isomorphic to $\mathbb{P}_1^3$ and $H = \mathbb{P}^1_1 \times \{a\}$ is embedded in $Y$ by the inclusion $\{a\} \to \mathbb{P}^1_1$. Since $\nu(B)$ irredundantly spans $q$, there is a unique $q' \in \nu(F)_K$ such that $q \in \{q', \nu(o)\}$. Now we vary $b \in \mathbb{P}_1^3$. Take any $a' \in \mathbb{P}_1^3$. The line $\{q, \nu(b)\}_K$ meets $\nu(H)_K$ at a unique point $q_0$. We take $F_0 \in S(H(K), q_0)$ and set $R_0 := F_0 \cup \{b\}$. We need to justify the existence of $F_0$, i.e. that $r_{H(K)}(q_0) = 2$. For any field $L \supset K$ all sets $A \subset H(L)$ such that $\#_A(B) = 2$ for all $i = 1, 2, 3$, form a unique orbit for the action of $\text{Aut}(\mathbb{P}^1(L))^3$. Taking their linear spans we cover all $q' \in \nu(Y)_L$ with $r_{H(L)}(q') = 2$. Since $K$ is infinite, we get as $q_0$ a Zariski dense subset of the projective space $\nu(H)_K$.

End of the proof of Theorem 1.1: Fix $q \in \mathbb{P}^1(K)$ such that $r_{X(K)}(q) = 2$. The case with $\#S(X(K), q) > 1$ true by Proposition 4.2. Assume $\#S(X(K), q) = 1$. Lemma 2.1 gives $r_{X,K}(q) = 2$, $\#S(X,K,q) = 1$ and that $\#S(X,K,q) = 1$ if $r_{X,K}(q) = 2$.

Example 5.1. Case 1 of [3, Theorem 7.1] is the case $k = 2$, i.e. when the tensor $T$ whose equivalence class represent $q$ is a rank 3 matrix. In this case $r_{X(K)}(q) = r_{X,K}(q) = 3$, $\#S(X(K), q)$ is concise for $X(K)$ and $q$ is as case 2 of [3, Theorem 7.1]. Thus $K = 3, n_1 = n_2 = n_3 = 1$ and $q \in \tau(X(K))$. We described this case in Lemma 5.2.
Example 5.3. Take $q \in P^r(K)$ such that $r_{X(K)}(q) = r_{X(K)}(q) = 3$, $X(K)$ is concise for $K$ and $q$ is in case 6 of [3, Theorem 7.1]. Thus $k \geq 3$, $n_1 = 1$ for all $i \geq 3$, $n_1 \in \{1, 2\}$, $n_2 \in \{1, 2\}$ and there is $E \in S(X(K), q)$ and $F \subseteq E$ with $\#F = 2$ and $\#\pi(F) = 1$ for all $i \geq 3$. Set $\{a\} := E \setminus F$. Since $\nu(E)$ irredundantly spans $q$, there is a unique $q' \in \{\nu(E)\}$ such that $q \in \{\nu(q'), \nu(o)\}$. If $n_1 + n_2 + k \geq 4$, we also proved that each $E' \in S(X(K), q)$ contains $o$ and $\#\pi(E' \setminus \{a\}) = 1$ for all $i \geq 3$. Under these stronger assumptions $o$ is uniquely determined by $q$ and hence it is defined over $K$. Thus $q'$ is defined over $K$. By the definition of $q'$ and the problem $X(K)$ is the minimal multiprojective space $\nu(X(K))$ such that $\nu \subset \{\nu(Y(K))\}$ is isomorphic to $P^1(K)^2$, i.e. $q'$ is represented by a tensor, which is equivalent to a $2 \times 2$ rank 2 matrix. Since $q'$ is defined over $K$, the classification of matrices of rank $1$ is as case 1. Thus $S(X(K), q') > 1$ and that $S(X(K), q')$ is finite. Now assume $k = n_1 = n_2 = 3$. Concision and the assumption $r_{X(K)}(q) = 3$ give $q \in \tau(X(K)) \cap E_K$ and we handled this case in Lemma 5.1.

Example 5.4. Take $q \in P^r(K)$ such that $r_{X(K)}(q) = r_{X(K)}(q) = 3$, $X(K)$ is concise for $q$, and $q$ is as in case 3 of [3, Theorem 7.1]. Thus $k = 4$, $n_1 = n_2 = n_3 = 4$. We described this case in Lemma 5.3.

Example 5.5. Take $q \in P^r(K)$ such that $r_{X(K)}(q) = r_{X(K)}(q) = 3$, $X(K)$ is concise for $K$ and $q$ is either as in case 4 or as in case 5 of [3, Theorem 7.1]. In both cases $k = 3$, $n_1 = 2$, $n_2 = n_3 = 1$ and the 2 cases are distinguished in the following way. Fix any $E \subset Y(K)$ such that $\nu(E) \in S(X(K), q)$ and any $B \subset Y(K)$ such that $\nu(B) \in S(X(K), q)$. Write $B = \{u, v, z\}$. Since $k^0(C_{X(K)}(0, 1, 1)) = 4$ and $\#E = 3$, $|E(0, 1, 1)| \neq 0$. It was proved in [3, Proposition 3.8] that $|E(0, 1, 1)|$ is a singleton, say $|E(0, 1, 1)| = \{G\}$. The two cases are distinguished as if $G$ is irreducible (case 4) or $G$ is reducible, say $G = G_1 \cup G_2$ with $G_1 \subseteq C_{X(K)}(0, 1, 0)$ and $G_2 \subseteq C_{X(K)}(0, 1, 0)$ (case 5). Each $A \subset Y(K)$ such that $\nu(A) \in S(X(K), q)$ is contained in $G$ (see [3, Proposition 3.8]). In particular $B \subset G$ and $G$, being uniquely determined, is defined over $K$. The two cases are also distinguished according to the injectivity of both $\pi_{2|E}$ and $\pi_{2|E}$.

(a). Assume that $G$ is irreducible and hence $\pi_{i|B}$ is injective for $i = 2, 3$. Thus $r_{X(K)}(q) = 2$ and $r_{X(K)}(q) = 3$. Use case 3 of Proposition 4.3.

Example 5.6. Assume that $K$ has a degree 2 extension $L$. Call $\sigma$ the generator of the Galois group the extension $L/K$. Fix $X(K)$ with $Y_K = (P^1_K)^k$ for some $k > 2$ factors. Take $a_1, \ldots, a_k \in K_1 \setminus K$. Using $a_i$ and $\sigma(a_i)$ we may construct a points $p \in Y(K_1) \setminus Y(K)$ such that $Y(K)$ is the minimal multiprojective space containing $\{u, \sigma(u)\}$. The line $\langle\nu(u), \nu(\sigma(u))\rangle$ is defined over $K$ and hence it corresponds to a line $R$ of $P^r(K)$ containing no point of $\nu(Y(K))$. Fix $R$. Since $\nu(\sigma(u)) \in S(T, q)$ and $\#S(X(K), q) = 1$ (see [3, Proposition 2.3]), $r_{X(K)}(q) = 2$ and $r_{X(K)}(q) > 2$. At least in some cases, e.g. $k = 3$ and $K = \mathbb{R}$, it is easy to find $u$ and $q$ such that $r_{X(K)}(q) = 3$.

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