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Research Article

On a Noor-Waseem-type method for solving nonlinear equations

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Abstract

A generalized method due to Noor and Waseem is studied for solving nonlinear equations in Banach space. The Noor-Waseem method is of order three. But, the convergence of this method was shown assuming that the fourth derivative, not on the method, exists. This constraint is limiting its applicability. Moreover, neither computable error bounds nor results about the uniqueness of the solution were given. We address all these problems using only the first derivative which only appears on the method. Hence, we extend the applicability of the method under consideration. Our techniques can be used to obtain the convergence of other similar higher order methods using assumptions only on the first derivative of the operator involved.

Keywords: Noor-Waseem method; Banach space; convergence order; Taylor expansion; iterative method.

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1. Introduction

A plethora of problems from diverse disciplines are solved using high order iterative methods [1–13]. But there are some common problems with the applications of these methods. We demonstrate how to overcome these problems. In particular, Noor and Waseem [9] considered the following third convergence iterative method:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ x_{k+1} &= x_k - 4A_k^{-1}F(x_k), \end{aligned} \tag{1}$$

for solving the nonlinear equation

$$F(x) = 0, \tag{2}$$

where

$$A_k = 3F'\left(\frac{2x_k + y_k}{3}\right) + F'(y_k).$$

Here, $F : \Omega \subset X \longrightarrow Y$ is an operator acting between Banach spaces X and Y with $\Omega \neq \emptyset$. Throughout the article, we take $U(x_0, \rho) = \{x \in X : \|x - x_0\| < \rho\}$ and $U[x_0, \rho] = \{x \in X : \|x - x_0\| \leq \rho\}$ for some $\rho > 0$.

Our convergence analysis is not based on Taylor expansion (unlike earlier studies [7–13]), so we do not need assumptions on higher than one order derivatives of the operator involved. For example: Let $X = Y = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define f on Ω by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then, we have $f(1) = 0$ and $f'''(t) = 6 \log t^2 + 60t^2 - 24t + 22$. Obviously $f'''(t)$ is not bounded on Ω. So, the convergence of the Noor-Waseem method is not guaranteed by the analysis in [9]. The same problem exists with other methods [1–8, 10–13] although the method may converge. Other problems have already been reported in the abstract of this study. That is why it is imporatant to use assumptions only on the derivatives appearing on the method.

The paper contains local convergence analysis in Section 2 and the numerical examples are given in Section 3.

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2. Ball convergence

The convergence is based on some functions and parameters. Let $B = [0, \infty)$. Assume

(i) Function $\psi_0(t) - 1$ has a minimal zero $R_0 \in M - \{0\}$, where $\psi_0 : B \longrightarrow B$ is continuous and nondecreasing. Set $B_0 = [0, R_0)$.

(ii) Function $\varphi_1(t) - 1$ has a minimal zero $r_1 \in B_0 - \{0\}$, where $\psi : B_0 \longrightarrow B$ is continuous, nondecresing and

$$\varphi_1(t) = \frac{\int_0^1 \psi(\theta t) d\theta}{1 - \psi_0(t)}.$$

(iii) Function $q(t) - 1$ has a minimal zero $R_1 \in B - \{0\}$, where

$$q(t) = \frac{1}{4}\left(3\psi_0\left(\frac{(2+\varphi_1(t))t}{3}\right) + \psi_0(t)\right).$$

Set $R = \min\{R_0, R_1\}$, and $B_1 = [0, R)$.

(iv) Function $\varphi_2(t) - 1 = 0$ has a minimal zero $r_2 \in B_1 - \{0\}$, where

$$\varphi_2(t) = \varphi_1(t) + \frac{p(t)\int_0^1 \psi_1(\theta t) d\theta}{4(1-\psi_0(t))(1-q(t))},$$

where $\psi_1 : B_1 \longrightarrow B$ is continuous, nondecreasing and

$$p(t) = 3\left(3\psi_0\left(\frac{(2+\varphi_1(t))t}{3}\right) + \psi_0(t)\right) + \psi_0(\varphi_1(t)t) + \psi_0(t).$$

The parameter r given by

$$r = \min\{r_i\}, \quad i = 1, 2, \tag{3}$$

shall be shown to be a radius of convergence for method (1). Let $B_2 = [0, r)$. Then, it follows from (3) that for all $t \in B_2$

$$0 \leq \psi_0(t) < 1 \tag{4}$$
$$0 \leq q(t) < 1 \tag{5}$$
$$0 \leq p(t) \tag{6}$$

and

$$0 \leq \varphi_i(t) < 1. \tag{7}$$

Assume:

(C1) There exists $x^* \in \Omega$ which is a simple solution of (2).

(C2) $\|F'(x^*)^{-1}(F'(w) - F'(x^*))\| \leq \psi_0(\|w - x^*\|)$ for all $w \in \Omega$. Let $\Omega_0 = U(x^*, R_0) \cap \Omega$.

(C3) $\|F'(x^*)^{-1}(F'(w) - F'(z))\| \leq \psi(\|w - z\|)$,
$\|F'(x^*)^{-1}F'(v)\| \leq \psi_1(\|v - x^*\|)$ for all $w, v \in \Omega_0$.

(C4) $U[x^*, r] \subset \Omega$.

We now can prove the main local convergence result for method (1) using conditions (C).

Theorem 2.1. *Assume conditions (C) hold. Then, sequence* $\{x_n\}$ *generated by method (1) with starting point* $x_0 \subset U(x^*, r) - \{x^*\}$*, is well defined in* $U(x^*, r)$*, remains in* $U(x^*, r)$ *for all* $n = 0, 1, 2, \ldots$ *and converges to* x^*.

Proof. Mathematical induction is used to show

$$\|y_k - x^*\| \leq \varphi_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r \tag{8}$$

and

$$\|x_{k+1} - x^*\| \leq \varphi_2(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_n - x^*\| < r. \tag{9}$$

Let $z \in U(x^*, r)$. Then, using (C2) one gets in turn that

$$\|F'(x^*)^{-1}(F'(z) - F'(x^*))\| \leq \psi_0(\|z - x^*\|) \leq \psi_0(r) < 1.$$

Hence, the inequality

$$\|F'(z)^{-1}F'(x^*)\| \leq \frac{1}{1 - \psi_0(\|v - x^*\|)} \tag{10}$$

follows from the Banach lemma on invertible operators [10]. By the first substep of method (1) and (10) for $z = x_0$, one has

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= F'(x_0)^{-1} \int_0^1 [F'(x_0) - F'(x^* + \theta(x_0 - x^*))d\theta](x_0 - x^*). \end{aligned} \tag{11}$$

By (C2), (10) (for $z = x_0$) and (11), we obtain

$$\begin{aligned} \|y_0 - x^*\| &\leq \frac{\int_0^1 \psi(\theta\|x_0 - x^*\|)d\theta}{1 - \varphi_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \\ &\leq \varphi_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r. \end{aligned} \tag{12}$$

That is (8) holds for $k = 0$ and $y_0 \in U(x^*, r)$. Next, it is shown $A_0^{-1} \in L(Y, X)$. By (5) and (12), we can write

$$\begin{aligned} \|(4F'(x^*))^{-1}(A_0 - F'(x^*))\| &\leq \frac{1}{4}\left[3\|F'(x^*)^{-1}\left(F'\left(\frac{2x_0 + y_0}{3}\right) - F'(x^*)\right)\|\right. \\ &\quad \left. + \|F'(x^*)^{-1}(F'(y_0) - F'(x^*))\|\right] \\ &\leq \frac{1}{4}\left(3\psi_0\left(\frac{2\|x_n - x^*\| + \|y_0 - x^*\|}{3}\right) + \psi_0(\|y_0 - x^*\|)\right) \\ &\leq \frac{1}{4}\left(3\psi_0\left(\frac{(2 + \varphi_1(\|x_n - x^*\|))\|x_0 - x^*\|}{3}\right)\right. \\ &\quad \left. + \psi_0(\varphi_1(\|x_0 - x^*\|)\|x_0 - x^*\|)\right. \\ &= q(\|x_0 - x^*\|) < 1. \end{aligned} \tag{13}$$

Hence, we obtain

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{4(1 - p(\|x_0 - x^*\|))}. \tag{14}$$

So, iterate x_1 exists by the second substep of method (1) for $k = 0$. By (C3) the following estimate is obtained

$$\begin{aligned} \|F'(x^*)^{-1}(A_0 - 4F'(x_n))\| &\leq \left(3\psi_0\left(\left\|\frac{2x_0 + y_0}{3} - x^*\right\|\right) + \psi_0(\|x_0 - x^*\|)\right) \\ &\quad + \psi_0(\|y_0 - x^*\|) + \psi_0(\|x_0 - x^*\|) \\ &\leq 3\left(\psi_0\left(\frac{(2 + \varphi_1(\|x_0 - x^*\|))\|x_0 - x^*\|}{3}\right)\right) + \psi_0(\|x_0 - x^*\|) \\ &\quad + \psi_0(\varphi(\|x_0 - x^*\|)\|x_0 - x^*\|) + \psi_0(\|x_0 - x^*\|) \qquad (15) \\ &= p\,\|x_0 - x^*\|. \qquad (16) \end{aligned}$$

Then, by the second substep of method (1) one has

$$x_1 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + F'(x_0)^{-1}(A_0 - 4F'(x_0))A_0^{-1}F(x_0). \tag{17}$$

Using (C3), (7), (10) (for $z = x_0$), (14), (15) and (17)

$$\begin{aligned} \|x_1 - x^*\| &\leq \varphi_1(\|x_0 - x^*\|)\|x_0 - x^*\| + p(\|x_0 - x^*\|) \int_0^1 \varphi_1(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\| \\ &\leq \varphi_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x_*\| < r. \end{aligned} \tag{18}$$

So, (9) holds for $k = 0$ and $x_1 \in U(x^*, r)$. Replace, x_0, y_0, x_1 by x_k, y_k, x_{k+1} in the previous stimulations to complete the induction for (8) and (9). Then, from the estimation

$$\|x_{i+1} - x^*\| \leq c\|x_i - x^*\| < r, \tag{19}$$

where $c = \varphi_2(\|x_0 - x^*\|) \in [0, 1)$, we conclude $\lim_{k\longrightarrow\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

□

The uniqueness of the solution is presented in the next proposition.

Proposition 2.1. *Assume:*

(a) *There exists a simple solution $x^* \in \Omega$ of equation $F(x) = 0$.*

(b) *There exists $\bar{r} \geq r$ such that*

$$\int_0^1 \psi_0(\theta\bar{r})d\theta < 1. \tag{20}$$

Let $\Omega_1 = \Omega \cap U[x^*, \bar{r}]$. Then, x^* is the only solution of equation $F(x) = 0$ in the region Ω_1.

Proof. Let $\lambda \in \Omega_1$ be a solution of equation $F(x) = 0$. Set

$$M = \int_0^1 F'(x^* + t(\lambda - x^*))dt.$$

Then, by (C2) and (20), we have in turn that

$$\begin{aligned}\|F'(x^*)^{-1}(M - F'(x^*))\| &\leq \int_0^1 \psi_0(\theta\|\lambda - x^*\|)d\theta \\ &\leq \int_0^1 \psi_0(\theta\bar{r})d\theta < 1.\end{aligned}$$

So, $\lambda = x^*$ follows from the invertibility of M and the identity $0 = F(\lambda) - F(x^*) = M(\lambda - x^*)$.

□

3. Numerical examples

We compute the radius of convergence in this section.

Example 3.1. *Take $\psi_0(t) = \psi(t) = 96.6629073t$ and $\psi_1(t) = 2$. Then, we have*

$$r_1 = 0.0069 \text{ and } r_2 = 0.0054 = r.$$

Example 3.2. *Let $X = Y = \mathbb{R}^3, D = B[0, 1], x_* = (0, 0, 0)^T$. Define the function F on D for $w = (x, y, z)^T$ by*

$$F(w) = \left(e^x - 1, \, \frac{e-1}{2}y^2 + y, \, z\right)^T.$$

Then, we get

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and hence for

$$\psi_0(t) = (e-1)t, \quad \psi(t) = e^{\frac{1}{e-1}}t \quad \text{and} \quad \psi_1(t) = e^{\frac{1}{e-1}},$$

we have

$$r_1 = 0.3827 = r \text{ and } r_2 = 0.6092.$$

4. Basins of attractions

The basins of attraction [8], also referred as Fatou sets, of an iterative method are the collection of all initial points from which the iterative method converges to a solution of a given equation. The complement of Fatou set is known as Julia set. The following test problems which are systems of polynomials in two variables are considered and the basins of attraction associated to each root of the corresponding system are displayed in Figure 1.

Example 4.1. *The system*

$$\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases}$$

has the solution set $\{(-1,-1),(0,0),(1,1)\}$.

Example 4.2. *The system*

$$\begin{cases} 3x^2y - y^3 = 0 \\ x^3 - 3xy^2 - 1 = 0 \end{cases}$$

has the solution set $\left\{\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right),\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right),(1,0)\right\}$.

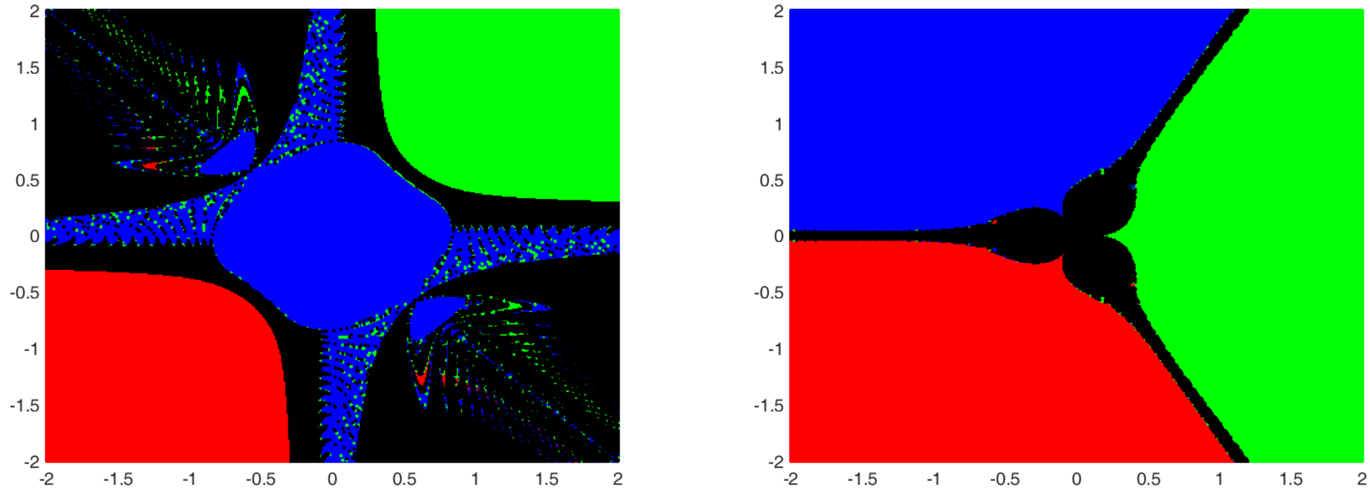

Figure 1: Dynamical plane of Method (1) with basins of attraction for Examples 4.1 and 4.2.

We apply method (1) to obtain the basins of attraction for each test problem and compare the results. For generating the basin of attraction associated to each root of a given system of nonlinear equations, we consider the rectangular region $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : -2 \le x \le 2, -2 \le y \le 2\}$ which contains all the roots of test problems. We consider an equidistant grid of 401×401 points in $\mathcal{R}$ and we choose these points as the initial guess x_0, for the method (1). A fixed tolerance 10^{-8} and a maximum of 50 iterations are used for all the cases. A color is being assigned to each attracting basin corresponding to each root. If we do not obtain the desired tolerance with the fixed iterations, we do not continue and we decide that the iterative method starting at x_0 does not converge to any of the roots and assign black color to those points. In this way, we distinguish the basins of attraction by their respective colors for distinct roots of each method.

Figure 1 demonstrates the basin of attraction corresponding to each root of the method (1). The Julia set (black region), which contains all the initial points from which the iterative method does not converge to any of the roots, can easily be observed in the figure.

The figure presented in this work is generated by a 4-core 64 bit Windows machine with Intel Core i7-3770 processor using MATLAB programming language.

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