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Research Article

Regularity and decay of solutions for the 3D Kuramoto-Sivashinsky equation posed on smooth domains and parallelepipeds

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Abstract

Initial-boundary value problems for the 3D Kuramoto-Sivashinsky equation posed on smooth domains and parallelepipeds are considered. The existence and uniqueness of global regular and strong solutions as well as their exponential decay are established.

Keywords: Kuramoto-Sivashinsky equation; global solutions; decay in bounded and unbounded domains.

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1. Introduction

This work is concerned with the existence, uniqueness, regularity and exponential decay rates of solutions to initial-boundary value problems for the three-dimensional Kuramoto-Sivashinsky (KS) equation:

$$\phi_t + \Delta^2\phi + \Delta\phi + \frac{1}{2}|\nabla\phi|^2 = 0. \tag{1}$$

Here Δ and ∇ are the Laplacian and the gradient in $\mathbb{R}^3$. In [11], Kuramoto studied the turbulent phase waves and Sivashinsky in [21] obtained an asymptotic equation which simulated the evolution of a disturbed plane flame front; see also [8]. Mathematical results on initial and initial-boundary value problems for the one-dimensional KS equation are presented in [3, 5, 7, 13, 15, 18, 23, 25] (see references cited there for more information). In [4, 5], the initial value problem for multi-dimensional KS type equations has been considered. Two-dimensional periodic problems for the KS equation and its modifications posed on rectangles were studied in [2, 16–18, 20, 25], where some results on the existence of weak solutions and nonlinear stability have been established. In [14], three-dimensional Kuramoto-Sivashinsky-Zakharov-Kuznetsov equation has been studied.

For three dimensions, (x, y, z), Equation (1) can be rewritten in the form of the following system:

$$u_t + \Delta^2 u + \Delta u + uu_x + vv_x + ww_x = 0, \tag{2}$$

$$v_t + \Delta^2 v + \Delta v + uu_y + vv_y + ww_y = 0, \tag{3}$$

$$w_t + \Delta^2 w + \Delta w + uu_z + vv_z + ww_z = 0, \tag{4}$$

$$u_y = v_x,\ u_z = w_x,\ v_z = w_y, \tag{5}$$

where $u = \phi_x,\ v = \phi_y,\ w = \phi_z$. Let Ω be the minimal parallelepiped containing a given smooth domain $\bar{D}$:

$$\Omega = \{(x, y, z) \in R^3; x \in (0, L_1), y \in (0, L_2), z \in (0, L_3)\}.$$

The first essential problem that arises while one studies either (1) or (2)–(5), is destabilizing effects of $\Delta u, \Delta v, \Delta w$; they may be damped by dissipative terms $\Delta^2 u, \Delta^2 v, \Delta^2 w$ provided that D has some specific properties. Admissible domains called "thin domains" are appeared here naturally, where some L_i are sufficiently small while others L_j may be arbitrarily large $i, j = 1, 2, 3;\ i \neq j$, (see [2, 10, 20]).

The second essential problem is the presence of semi-linear terms in (2)–(4) which are interconnected. This does not allow to obtain the first estimate independent of u, v, w and leads to a connection between L_1, L_2, L_3 and $u(0), v(0), w(0)$. Taking into account these arguments, we study initial-boundary value problems for (2)–(5) posed on smooth domains D,

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where the existence and uniqueness of global regular solutions as well as their exponential decay of the $H^2(D)$-norm are established. Moreover, we obtained a "smoothing effect" for solutions with respect to initial data. On the other hand, we also study initial-boundary value problems for (2)–(5) posed on unbounded parallelepipeds and establish the existence and uniqueness of global strong solutions as well as their exponential decay of the $L^2(D)$-norm.

This paper has the following structure. Section 1 is the introduction. Section 2 consists of notations and auxiliary facts. In Section 3, formulation of an initial-boundary value problem for (2)–(5) in a smooth bounded domain D is given. The existence and uniqueness of global regular solutions, exponential decay of the $H^2(D)$-norm and a "smoothing effect" have been established. In Section 4, an initial-boundary value problem for (2)–(5) posed on a horizontal parallelepiped D_x is formulated. The existence, uniqueness and exponential decay of the $L^2(D_x)$-norm for global strong solutions are also established in Section 4. (Similar results can be proved for the 3D Kuramoto-Sivashinsky system posed on parallelepiped parallel to axes $0Y$ and $0Z$.) Section 5 gives concluding remarks.

2. Notations and auxiliary facts

Let Ω be a sufficiently smooth domain in $\mathbb{R}^3$ satisfying the cone condition (see [1]) and $x = (x_1, x_2, x_3) \in \Omega$. We use the standard notations of Sobolev spaces $W^{k,p}$, L^p and H^k for functions and the following notations for the norms [1, 6] for scalar functions $f(x,t)$:

$$\|f\|^2 = \int_\Omega |f|^2 d\Omega, \qquad \|f\|^p_{L^p(\Omega)} = \int_\Omega |f|^p \, d\Omega, \qquad \|f\|^p_{W^{k,p}(\Omega)} = \sum_{0\le\alpha\le k} \|D^\alpha f\|^p_{L^p(\Omega)}, \qquad \|f\|_{H^k(\Omega)} = \|f\|_{W^{k,2}(\Omega)}.$$

When $p = 2$, $W^{k,p}(\Omega) = H^k(\Omega)$ is a Hilbert space with the scalar product

$$((u,v))_{H^k(\Omega)} = \sum_{|j|\le k} (D^j u, D^j v), \ \|u\|_{L^\infty(\Omega)} = ess\ sup_\Omega |u(x)|.$$

We use the notation $H^k_0(\Omega)$ to represent the closure of $C^\infty_0(\Omega)$, the set of all C^∞ functions with compact support in Ω, with respect to the norm of $H^k(\Omega)$.

Lemma 2.1 (Steklov's Inequality [22]). *If $v \in H^1_0(0,L)$ then*

$$\frac{\pi^2}{L^2}\|v\|^2(t) \le \|v_x\|^2(t).$$

Lemma 2.2 (See [12], Theorem 7.1; [24], Lemma 3.5.). *If $v \in H^1_0(\Omega)$ and $n = 3$, then*

$$\|v\|_{L^4(\Omega)} \le 2^{1/2}\|v\|^{1/4}\|\nabla v\|^{3/4}.$$

Lemma 2.3 (Differential form of the Gronwall Inequality). *Let $I = [t_0, t_1]$. Suppose that the functions $a, b : I \to \mathbb{R}$ are integrable and the function $a(t)$ is of any sign. Let $u : I \to \mathbb{R}$ be a differentiable function satisfying*

$$u_t(t) \le a(t)u(t) + b(t), \text{ for } t \in I \text{ and } \ u(t_0) = u_0, \tag{6}$$

then

$$u(t) \le u_0 e^{\int_{t_0}^t a(t)\,dt} + \int_{t_0}^t e^{\int_{t_0}^s a(r)\,dr} b(s)\,ds.$$

Proof. Multiply (6) by the integrating factor $e^{\int_{t_0}^s a(r)\,dr}$ and integrate from t_0 to t. □

The next Lemmas will be used in estimates.

Lemma 2.4 (See [1, 9, 19]). *Let Ω be a sufficiently smooth bounded domain in $\mathbb{R}^3$ satisfying the cone condition. If $v \in H^2(\Omega) \cap H^1_0(\Omega)$, then*

$$\|v\|_{L^4(\Omega)} \le C_n\Big(\|v\|^{5/8}_{L^2(\Omega)}\|\Delta v\|^{3/8}_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\Big).$$

If $v \in H^2_0(\Omega)$, then

$$\|v\|_{L^4(\Omega)} \le C_n\|v\|^{5/8}_{L^2(\Omega)}\|\Delta v\|^{3/8}_{L^2(\Omega)}.$$

Lemma 2.5 (See [1, 9, 19]). *Let Ω be a sufficiently smooth bounded domain in $\mathbb{R}^3$ satisfying the cone condition and $v \in H^4(\Omega) \cap H^1_0(\Omega)$, then*

$$\sup_\Omega |v(x)| \le C_s\Big(\|\Delta^2 v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\Big)$$

Lemma 2.6. *Let $f(t)$ be a continuous positive function such that*

$$f'(t) + (\alpha - kf^n(t))f(t) \leq 0, \ t > 0, \ n \in \mathbb{N}, \tag{7}$$

$$\alpha - kf^n(0) > 0, \ k > 0.$$

Then $f(t) < f(0)$ for all $t > 0$.

Proof. Obviously, $f'(0) + (\alpha - kf^n(0))f^n(0) \leq 0$. Since f is continuous, there exists $T > 0$ such that $f(t) < f(0)$ for every $t \in [0, T)$. Suppose that $f(0) = f(T)$. Integrating (7), we find

$$f(T) + \int_0^T (\alpha - kf^n(t))f(t)\,dt \leq f(0).$$

Since

$$\int_0^T (\alpha - kf^n(t))f(t)\,dt > 0,$$

one gets $f(T) < f(0)$. This contradicts the equation $f(T) = f(0)$. Therefore, $f(t) < f(0)$ for all $t > 0$. □

3. KS equation posed on smooth domains

Let Ω be the minimal parallelepiped containing a given bounded smooth domain $\bar{D}$:

$$\Omega = \{(x, y, z) \in R^3; x \in (0, L_1), y \in (0, L_2), z \in (0, L_3)\}.$$

Differentiating (1) with respect to x, y, z and denoting $u = \phi_x, \ v = \phi_y, w = \phi_z$, we come in $Q_t = D \times (0, t)$ to the following initial-boundary value problem:

$$u_t + \Delta^2 u + \Delta u + uu_x + vv_x + ww_x = 0, \tag{8}$$

$$v_t + \Delta^2 v + \Delta v + uu_y + vv_y + ww_y = 0, \tag{9}$$

$$w_t + \Delta^2 w + \Delta w + uu_z + vv_z + ww_z = 0, \tag{10}$$

$$u_y = v_x, \ u_z = w_x, \ v_z = w_y, \tag{11}$$

$$u|_{\partial D} = v|_{\partial D} = w|_{\partial D} = \Delta u|_{\partial D} = \Delta v|_{\partial D} = \Delta w|_{\partial D} = 0, \ t > 0, \tag{12}$$

$$u(x, y, z, 0) = u_0(x, y, z), \ \ v(x, y, z, 0) = v_0(x, y, z),$$

$$w(x, y, z, 0) = w_0(x, y, z) \text{ in } D. \tag{13}$$

Lemma 3.1. *Let $f \in H^4(D) \cap H_0^1(D)$ and $\Delta f|_{\partial D} = 0$. Then*

$$a\|f\|^2 \leq \|\nabla f\|^2, \ \ a^2\|f\|^2 \leq \|\Delta f\|^2, \ \ a\|\nabla f\|^2 \leq \|\Delta f\|^2, \ \ a^2\|\Delta f\|^2 \leq \|\Delta^2 f\|^2, \ \ \textit{where } a = \sum_{i=1}^{3} \frac{\pi^2}{L_i}.$$

Proof. Since

$$\|\nabla f\|^2 = \|f_x\|^2 + \|f_y\|^2 + \|f_z\|^2,$$

define

$$\tilde{f}(x, t) = \begin{cases} f(x, t) \ \text{ if } x \in D; \\ 0 \ \text{ if } x \in \Omega/D. \end{cases}$$

Making use of Steklov's inequalities for $\tilde{f}(x, t)$ and taking into account that $\|\nabla f\| = \|\nabla \tilde{f}\|$, we get

$$\|\nabla f\|^2 \geq a\|f\|^2, \ \text{ where } a = \sum_{i=1}^{3} \frac{\pi^2}{L_i^2} = \frac{\pi^2(L_1^2L_2^2 + L_1^2L_3^2 + L_2^2L_3^2)}{L_1^2L_2^2L_3^2}.$$

On the other hand,

$$a\|f\|^2 \leq \|\nabla f\|^2 = -\int_D f\Delta f dxdy \leq \|\Delta f\|\|f\|.$$

This implies

$$a\|f\| \leq \|\Delta f\| \ \text{ and } \ a^2\|f\|^2 \leq \|\Delta f\|^2.$$

Consequently, $a\|\nabla f\|^2 \le \|\Delta f\|^2$. Similarly,

$$\|\Delta f\|^2 = \int_D f\Delta^2 f dxdy \le \|\Delta^2 f\|\|f\| \le \frac{1}{a}\|\Delta^2 f\|\|\Delta f\|.$$

Hence, $a\|\Delta f\| \le \|\Delta^2 f\|$. Proof of Lemma 3.1 is now complete. □

Remark 3.1. *Assertions of Lemma 3.1 are true if the function f is replaced respectively by u, v, w.*

Definition 3.1. *A triplet (u, v, w) of functions*

$$u, v, w \in L^\infty(R^+; H^2(D) \cap H_0^1(D)) \cap L^2(R^+; H^4(D)),$$
$$u_t, v_t, w_t \in L^2(R^+; L^2(D))$$

satisfying a.e. in D (8)–(11) and (12) as well as (13) is a regular solution to (8)–(13).

Theorem 3.1 (Special basis)**.** *Let*

$$\theta = 1 - \frac{1}{a} = 1 - \frac{L_1^2 L_2^2 L_3^2}{\pi^2(L_1^2 L_2^2 + L_1^2 L_3^2 + L_2^2 L_3^2)} > 0. \tag{14}$$

Given

$$u_0, v_0, w_0 \in H^2(D) \cap H_0^1(D)$$

such that

$$\theta - \frac{C_1}{\theta}\Big(\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2\Big) > 0, \tag{15}$$

where $C_1 = 4C_n^4\Big(1 + \frac{1}{a^2}\Big)$. Then there exists a unique triplet (u, v, w) of functions:

$$u, v, w \in L^\infty(R^+; H^2(D) \cap H_0^1(D)) \cap L^2(R^+; H^4(D)),$$
$$u_t, v_t, w_t \in L^2(R^+; L^2(D))$$

which is a regular solution to (8)–(13). Moreover,

$$\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t) \le \Big[\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2\Big] \exp\left\{-\frac{\theta a^2 t}{2}\right\}.$$

If in addition,

$$\theta - \frac{12}{a\theta}\left(1 + \frac{1}{a^4}\right) C_s^2 \Big(\|\Delta u_0\|^2 + \|\Delta v_0\|^2 + \|\Delta w_0\|^2\Big) > 0, \tag{16}$$

then

$$\|\Delta u\|^2(t) + \|\Delta v\|^2(t) + \|\Delta w\|^2(t) \le \Big(\|\Delta u_0\|^2 + \|\Delta v_0\|^2 + \|\Delta w_0\|^2\Big) \exp\{-a^2\theta/2t\}.$$

Remark 3.2. *In Theorem 3.1, there are two types of restrictions: the first one is pure geometrical,*

$$1 - \frac{1}{a} = 1 - \frac{L_1^2 L_2^2 L_3^2}{\pi^2\big(L_1^2 L_2^2 + L_1^2 L_3^2 + L_2^2 L_3^2\big)} > 0$$

which is needed to eliminate destabilizing effects of the terms $\Delta u, \Delta v, \Delta w$ in (8)–(10):

$$\|\Delta u\|^2 - \|\nabla u\|^2, \ \|\Delta v\|^2 - \|\nabla v\|^2, \ \|\Delta w\|^2 - \|\nabla w\|^2.$$

It is clear that

$$\lim_{L_1, L_2, L_3 \to 0} \frac{L_1^2 L_2^2 L_3^2}{\pi^2\big(L_1^2 L_2^2 + L_1^2 L_3^2 + L_2^2 L_3^2\big)} = 0,$$

hence to achieve the first restriction (14), it is possible either to decrease L_2, L_3 allowing L_1 to grow, transforming as a limit domain D into a horizontal unbounded thin domain or to decrease L_1, L_3 allowing L_2 to grow, or to decrease L_1, L_2 allowing L_3 to grow transforming respectively as a limit the domain D into unbounded thin domain oriented along the axis OZ. There is a number of various possible combinatios, for example, L_3 is small while L_1, L_2 may grow etc.
A situation with condition (15) is more complicated: if initial data are not small, then it is possible either to decrease L_1, L_2, L_3 to fulfill this condition or for fixed L_1, L_2, L_3 to decrease initial data $\|u_0\|, \|v_0\|, \|w_0\|$.
The same approach is valid to fulfill the additional restriction (16)

$$\theta - \frac{12}{a\theta}(1 + \frac{1}{a^4})C_s^2\Big(\|\Delta u_0\|^2 + \|\Delta v_0\|^2 + \|\Delta w_0\|^2\Big) > 0$$

which guarantees decay of the $H^2(D)$-norm.

Proof. It is possible to construct Galerkin's approximations to (8)–(13) by the following way: Let $w_j(x,y,z)$ be eigenfunctions of the problem:

$$\Delta^2 w_j + \lambda_j w_j = 0 \text{ in } D; \;\; w_j|_{\partial D} = \Delta w_j|_{\partial D} = 0, \; j = 1, 2, \dots.$$

Define

$$u^N(x,y,z,t) = \sum_{i=1}^N g_i^u(t) w_i(x,y,z), \;\; v^N(x,y,z,t) = \sum_{i=1}^N g_i^v(t) w_i(x,y,z), \;\; w^N(x,y,z,t) = \sum_{i=1}^N g_i^w(t) w_i(x,y,z).$$

Unknown functions $g_i^u(t), \;\; g_i^v(t), \;\; g^w(t)$ satisfy the following initial problems:

$$\frac{d}{dt}(u^N, w_j)(t) + (\Delta^2 u^N, w_j)(t) + (\Delta u^N, w_j)(t) + (u^N u_x^N, w_j)(t) + (v^N v_x^N, w_j)(t) + (w^N w_x^N, w_j)(t) = 0,$$

$$\frac{d}{dt}(v^N, w_j)(t) + (\Delta^2 v^N, w_j)(t) + (\Delta v^N, w_j)(t) + (u^N u_y^N, w_j)(t) + (v^N v_y^N, w_j)(t) + (w^N w_y^N, w_j)(t) = 0,$$

$$\frac{d}{dt}(w^N, w_j)(t) + (\Delta^2 w^N, w_j)(t) + (\Delta w^N, w_j)(t) + (u^N u_z^N, w_j)(t) + (v^N v_z^N, w_j)(t) + (w^N w_z^N, w_j)(t) = 0,$$

$$g_j^u(0) = g_{0j}^u, \;\; g_j^v(0) = g_{0j}^v,, \;\; g_j^w(0) = g_{0j}^w, \;\; j = 1, 2, \dots.$$

The estimates that follow may be established on Gallerkin's approximations (see [5, 7]), but it is more explicitly to prove them on smooth solutions of (8)–(13).

Estimate I: $u \in L^\infty(\mathbb{R}^+; H) \cap L^2(\mathbb{R}^+; H^2(D) \cap H_0^1(D))$.
Multiply (8) by $2u$, (9) by $2v$ and (10) by $2w$ to obtain

$$\frac{d}{dt}\|u\|^2(t) + 2\|\Delta u\|^2(t) - 2\|\nabla u\|^2(t) + 2(uu_x, u)(t) + 2(vv_x, u)(t) + 2(ww_x, u)(t) = 0, \tag{17}$$

$$\frac{d}{dt}\|v\|^2(t) + 2\|\Delta v\|^2(t) - 2\|\nabla v\|^2(t) + 2(uu_y, v)(t) + 2(vv_y, v)(t) + 2(ww_y, v)(t) = 0, \tag{18}$$

$$\frac{d}{dt}\|w\|^2(t) + 2\|\Delta w\|^2(t) - 2\|\nabla w\|^2(t) + 2(uu_z, w)(t) + 2(vv_z, w)(t) + 2(ww_z, w)(t) = 0. \tag{19}$$

Making use of (14) and the equalities $(u^2, u_x)(t) = (v^2, v_y)(t) = (w^2, w_z)(t) = 0$, we get

$$\frac{d}{dt}\|u\|^2(t) + \theta\|\Delta u\|^2(t) + \theta\|\Delta u\|^2(t) - (v^2, u_x)(t) - (w^2, u_x)(t) \le 0, \tag{20}$$

$$\frac{d}{dt}\|v\|^2(t) + \theta\|\Delta v\|^2(t) + \theta\|\Delta v\|^2(t) - (u^2, v_y)(t) - (w^2, v_y)(t) \le 0, \tag{21}$$

$$\frac{d}{dt}\|w\|^2(t) + \theta\|\Delta w\|^2(t) + \theta\|\Delta w\|^2(t) - (u^2, w_z)(t) - (v^2, w_z)(t) \le 0. \tag{22}$$

Making use of Lemmas 2.4, 3.1, and the Young inequality, we estimate

$$\begin{aligned}
I_1 = (v^2, u_x) &\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{1}{2\epsilon}\|v\|^4_{L^4(D)} \\
&\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{1}{2\epsilon}C_n^4\Big(\|v\|^{5/8}_{L^2(D)}\|\Delta v\|^{3/8}_{L^2(D)} + \|v\|_{L^2(D)}\Big)^4 \\
&\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{2}{\epsilon}C_n^4\Big(\|v\|^{5/2}_{L^2(D)}\|\Delta v\|^{3/2}_{L^2(D)} + \|v\|^4_{L^2(D)}\Big) \\
&\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{4}{\epsilon}C_n^4\Big(\|v\|^2_{L^2(D)}\|\Delta v\|^2_{L^2(D)} + \|v\|^4_{L^2(D)}\Big) \\
&\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{4}{\epsilon}C_n^4\Big(\|v\|^2_{L^2(D)}\|\Delta v\|^2_{L^2(D)} + \frac{\|v\|^2_{L^2(D)}}{a^2}\|\Delta v\|^2\Big) \\
&\le \frac{\epsilon}{2}\|u_x\|^2 + \frac{4}{\epsilon}C_n^4\Big(1 + \frac{1}{a^2}\Big)\|v\|^2_{L^2(D)}\|\Delta v\|^2_{L^2(D)} \\
&\le \frac{\epsilon}{2a}\|\Delta u\|^2 + \frac{1}{\epsilon}C_1\|v\|^2_{L^2(D)}\|\Delta v\|^2_{L^2(D)},
\end{aligned}$$

where $C_1 = 4C_n^4\Big(1+\frac{1}{a^2}\Big)$. Similarly,

$$I_2 = (u_x, w^2) \le \frac{\epsilon}{2a}\|\Delta u\|^2 + \frac{1}{\epsilon}C_1\|w\|^2_{L^2(D)}\|\Delta w\|^2_{L^2(D)},$$

$$I_3 = (u^2, v_y) \le \frac{\epsilon}{2a}\|\Delta v\|^2 + \frac{1}{\epsilon}C_1\|u\|^2_{L^2(D)}\|\Delta u\|^2_{L^2(D)},$$

$$I_4 = (w^2, v_y) \le \frac{\epsilon}{2a}\|\Delta v\|^2 + \frac{1}{\epsilon}C_1\|w\|^2_{L^2(D)}\|\Delta w\|^2_{L^2(D)},$$

$$I_5 = (u^2, w_z) \le \frac{\epsilon}{2a}\|\Delta w\|^2 + \frac{1}{\epsilon}C_1\|u\|^2_{L^2(D)}\|\Delta u\|^2_{L^2(D)},$$

$$I_6 = (v^2, w_z) \le \frac{\epsilon}{2a}\|\Delta w\|^2 + \frac{1}{\epsilon}C_1\|v\|^2_{L^2(D)}\|\Delta v\|^2_{L^2(D)}.$$

Substituting $I_1 - I_6$ into (20)–(22), taking $2\epsilon = 1$, making use of Lemma 3.1 and summing the results, we find

$$\frac{d}{dt}\Big(\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t)\Big) + \frac{\theta}{2}\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)$$
$$+\Big[\theta - \frac{C_1}{\theta}\Big(\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t)\Big)\Big]\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big) \le 0. \tag{23}$$

By condition (15) of Theorem 3.1,

$$\theta - \frac{C_1}{\theta}\Big(\|u_0\|^2+\|v_0\|^2+\|w_0\|^2\Big) > 0.$$

Making use of positivity of the second term in (23) and Lemma 2.6, we obtain

$$\theta - \frac{C_1}{\theta}\Big(\|u\|^2(t)+\|v|^2(t)+\|w\|^2(t)\Big) > 0 \text{ for a.e. } t>0.$$

Then (23) becomes

$$\frac{d}{dt}\Big(\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t)\Big) + \frac{\theta}{2}\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big) \le 0. \tag{24}$$

This implies

$$\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t) + \frac{\theta}{2}\int_0^t\Big(\|\Delta u\|^2(\tau)+\|\Delta v\|^2(\tau)+\|\Delta w\|^2(\tau)\Big)d\tau \le \|u_0\|^2+\|v_0\|^2+\|w_0\|^2.$$

On the other hand, by Lemma 3.1, (24) is reduced to the form

$$\frac{d}{dt}\Big(\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t)\Big) + \frac{a^2\theta}{2}\Big(\| u\|^2(t)+\|v\|^2(t)+\|w\|^2(t)\Big) \le 0.$$

This implies

$$\|u\|^2(t)+\|v\|^2(t)+\|w\|^2(t) \le \Big[\|u_0\|^2+\|v_0\|^2+\|w_0\|^2\Big]e^{-\frac{a^2\theta t}{2}}.$$

Estimate II: $u \in L^\infty(\mathbb{R}^+; H^2(D)\cap H^1_0(D)) \cap L^2(\mathbb{R}^+; H^4(D)\cap H^1_0(D))$.
Multiply (8) by $2\Delta^2 u$, (9) by $2\Delta^2 v$ and (10) by $2\Delta^2 w$ to obtain

$$\frac{d}{dt}\|\Delta u\|^2(t) + 2\|\Delta^2 u\|^2(t) + 2\|\Delta^2 u\|(t)\|\Delta u\|(t) + 2(uu_x, \Delta^2 u)(t) + 2(vv_x, \Delta^2 u)(t) + 2(ww_x, \Delta^2 u)(t) = 0, \tag{25}$$

$$\frac{d}{dt}\|\Delta v\|^2(t) + 2\|\Delta^2 v\|^2(t) + 2\|\Delta^2 v\|(t)\|\Delta v\|(t) + 2(uu_y, \Delta^2 v)(t) + 2(vv_y, \Delta^2 v)(t) + 2(ww_y, \Delta^2 v)(t) = 0, \tag{26}$$

$$\frac{d}{dt}\|\Delta w\|^2(t) + 2\|\Delta^2 w\|^2(t) + 2\|\Delta^2 w\|(t)\|\Delta w\|(t) + 2(uu_z, \Delta^2 w)(t) + 2(vv_z, \Delta^2 w)(t) + 2(ww_z, \Delta^2 w)(t) = 0. \tag{27}$$

For an arbitrary $\epsilon > 0$, making use of (14) and Lemmas 2.5, 3.1, we can write

$$\frac{d}{dt}\|\Delta u\|^2(t) + (2\theta - 3\epsilon)\|\Delta^2 u\|^2(t) \le \frac{1}{\epsilon}\Big[\sup_D u^2(x,y,z,t)\|\nabla u\|^2(t) + \sup_D v^2(x,y,z,t)\|\nabla v\|^2(t) + \sup_D w^2(x,y,z,t)\|\nabla w\|^2(t)\Big]$$
$$\le \frac{1}{\epsilon}\Big[C_s^2\Big(\|\Delta^2 u\|(t)+\|u\|(t)\Big)^2\|\nabla u\|^2(t) + C_s^2\Big(\|\Delta^2 v\|(t)+\|v\|(t)\Big)^2$$
$$\times\|\nabla v\|^2(t) + C_s^2\Big(\|\Delta^2 w\|(t)+\|w\|(t)\Big)^2\|\nabla w\|^2(t)\Big]$$

$$\leq \frac{1}{\epsilon}\Big[2C_s^2\Big(\|\Delta^2 u\|^2(t)\|\nabla u\|^2(t)+\|\Delta^2 v\|^2(t)\|\nabla v\|^2(t)$$
$$+\|\Delta^2 w\|^2(t)\|\nabla w\|^2(t)\Big)+2C_s^2\Big(\|u\|^2(t)\|\nabla u\|^2(t)$$
$$+\|v\|^2(t)\|\nabla v\|^2(t)+\|w\|^2(t)\|\nabla w\|^2(t)\Big)\Big]. \tag{28}$$

In a similar way, we obtain

$$\frac{d}{dt}\|\Delta v\|^2(t)+(2\theta-3\epsilon)\|\Delta^2 v\|^2(t)\leq \frac{1}{\epsilon}\Big[2C_s^2\Big(\|\Delta^2 u\|^2(t)\|\nabla u\|^2(t)+\|\Delta^2 v\|^2(t)\|\nabla v\|^2(t)$$
$$+\|\Delta^2 w\|^2(t)\|\nabla w\|^2(t)\Big)+2C_s^2\Big(\|u\|^2(t)\|\nabla u\|^2(t)$$
$$+\|v\|^2(t)\|\nabla v\|^2(t)+\|w\|^2(t)\|\nabla w\|^2(t)\Big)\Big], \tag{29}$$

$$\frac{d}{dt}\|\Delta w\|^2(t)+(2\theta-3\epsilon)\|\Delta^2 w\|^2(t)\leq \frac{1}{\epsilon}\Big[2C_s^2\Big(\|\Delta^2 u\|^2(t)\|\nabla u\|^2(t)+\|\Delta^2 v\|^2(t)\|\nabla v\|^2(t)$$
$$+\|\Delta^2 w\|^2(t)\|\nabla w\|^2(t)\Big)+2C_s^2\Big(\|u\|^2(t)\|\nabla u\|^2(t)$$
$$+\|v\|^2(t)\|\nabla v\|^2(t)+\|w\|^2(t)\|\nabla w\|^2(t)\Big)\Big]. \tag{30}$$

Taking $\epsilon=\frac{\theta}{6}$ and making use of Lemma 3.1, we rewrite (28)–(30) in the form:

$$\frac{d}{dt}\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)+\frac{\theta}{2}\Big(\|\Delta^2 u\|(t)+\|\Delta^2 v\|^2(t)+\|\Delta^2 w\|^2(t)\Big)+\theta\Big(\|\Delta^2 u\|(t)+\|\Delta^2 v\|^2(t)+\|\Delta^2 w\|^2(t)\Big)$$
$$-\frac{12}{a\theta}C_s^2\Big[\|\Delta^2 u\|^2(t)\|\Delta u\|^2(t)+\|\Delta^2 v\|^2(t)\|\Delta v\|^2(t)+\|\Delta^2 w\|^2(t)\|\Delta w\|^2(t)$$
$$+\frac{1}{a^2}\Big(\|u\|^2(t)\|\Delta^2 u\|^2(t)+\|v\|^2(t)\|\Delta^2 v\|^2(t)+\|w\|^2(t)\|\Delta^2 w\|^2(t)\Big)\Big]\leq 0.$$

Since, by Lemma 3.1, $a^2\|f\|^2\leq\|\Delta f\|^2$; we can rewrite it in a more convenient form:

$$\frac{d}{dt}\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)+\frac{\theta}{2}\Big(\|\Delta^2 u\|(t)+\|\Delta^2 v\|^2(t)+\|\Delta^2 w\|^2(t)\Big)$$
$$+\Big[\theta-\frac{12}{a\theta}(1+\frac{1}{a^4})C_s^2\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)\Big]\Big[\Delta^2 u\|^2(t)+\|\Delta^2 v\|^2(t)+\|\Delta^2 v\|^2(t)\Big]\leq 0. \tag{31}$$

Condition (16) and Lemma 2.6 guarantee that

$$\theta-\frac{12}{a\theta}\left(1+\frac{1}{a^4}\right)C_s^2\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)>0,\ t>0.$$

Hence, (31) can be rewritten as

$$\frac{d}{dt}\Big(\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)+\frac{a^2\theta}{2}\Big(\|\Delta u\|(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\Big)\leq 0. \tag{32}$$

By integrating, we get

$$\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)\leq\Big(\|\Delta u_0\|^2+\|\Delta v_0\|^2+\|\Delta w_0\|^2\Big)e^{(-a^2\theta/2t)}. \tag{33}$$

Returning to (32), we find

$$\|\Delta u\|^2(t)+\|\Delta v\|^2(t)+\|\Delta w\|^2(t)+\int_0^t\Big(\|\Delta^2 u\|^2(\tau)+\|\Delta^2 v\|^2(\tau)+\|\Delta^2 w\|^2(\tau)\Big)d\tau\leq C\Big(\|\Delta u_0\|^2+\|\Delta v_0\|^2+\|\Delta w_0\|^2\Big). \tag{34}$$

Finally, directly from (8)–(10), we obtain $u_t,v_t,w_t\in L^2(\mathbb{R}^+;L^2(D))$.
This completes the proof of the existence part of Theorem 3.1.

Lemma 3.2. *A regular solution of Theorem 3.1 is uniquely defined.*

Proof. Let u_1,v_1,w_1 and u_2,v_2,w_2 be two distinct solutions to (8)–(13). Denoting $p=u_1-u_2,\ q=v_1-v_2, r=w_1-w_2$, we come to the following system:

$$p_t+\Delta^2 p+\Delta p+\frac{1}{2}\Big(u_1^2-u_2^2\Big)_x+\frac{1}{2}\big(v_1^2-v_2^2\big)_x+\Big(w_1^2-w_2^2\Big)_x=0, \tag{35}$$

$$q_t + \Delta^2 q + \Delta q + \frac{1}{2}\Big(u_1^2 - u_2^2\Big)_y + \frac{1}{2}\big(v_1^2 - v_2^2\big)_y + \Big(w_1^2 - w_2^2\Big)_y = 0, \tag{36}$$

$$r_t + \Delta^2 r + \Delta r + \frac{1}{2}\Big(u_1^2 - u_2^2\Big)_z + \frac{1}{2}\big(v_1^2 - v_2^2\big)_z + \Big(w_1^2 - w_2^2\Big)_z = 0; \tag{37}$$

$$p_y = q_x,\ p_z = r_x,\ q_z = r_y, \tag{38}$$

$$p|_{\partial D} = q|_{\partial D} = r|_{\partial D} = \Delta p|_{\partial D} = \Delta q|_{\partial D} = \Delta r|_{\partial D} = 0,\ t > 0, \tag{39}$$

$$p(x,y,z,0) = q(x,y,z,0) = r(x,y,z,0) = 0 \text{ in } D. \tag{40}$$

Multiply (35) by $2p$, (36) by $2q$ and (37) by $2r$ to obtain

$$\frac{d}{dt}\|p\|^2(t) + 2\|\Delta p\|^2(t) - 2\|\nabla p\|^2(t) - (\{u_1+u_2\}p, p_x)(t) - (\{v_1+v_2\}q, p_x)(t) - (\{w_1+w_2\}r, p_x)(t) = 0, \tag{41}$$

$$\frac{d}{dt}\|q\|^2(t) + 2\|\Delta q\|^2(t) - 2\|\nabla q\|^2(t) - (\{u_1+u_2\}p, q_y)(t) - (\{v_1+v_2\}q, q_y)(t) - (\{w_1+w_2\}r, q_y)(t) = 0, \tag{42}$$

$$\frac{d}{dt}\|r\|^2(t) + 2\|\Delta r\|^2(t) - 2\|\nabla r\|^2(t) - (\{u_1+u_2\}p, r_z)(t) - (\{v_1+v_2\}q, r_z)(t) - (\{w_1+w_2\}r, r_z)(t) = 0. \tag{43}$$

We estimate

$$\begin{aligned} I_1 = (\{u_1+u_2\}p, p_x) &\le \frac{\epsilon}{2}\|p_x\|^2 + \frac{1}{\epsilon}(\{u_1^2+u_2^2\}, p^2) \\ &\le \frac{\epsilon}{2a}\|\Delta p\|^2 + \frac{1}{\epsilon}\sup_D(u_1^2(x,y,z,t) + u_2^2(x,y,z,t))\|p\|^2 \\ &\le \frac{\epsilon}{2a}\|\Delta p\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 u_1\|^2 + \|\Delta^2 u_2\|^2 + \|u_1\|^2(t) + \|u_2\|^2(t)\}\|p\|^2. \end{aligned}$$

Similarly,

$$I_2 = (\{v_1+v_2\}q, p_x) \le \frac{\epsilon}{2a}\|\Delta p\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 v_1\|^2 + \|\Delta^2 v_2\|^2 + \|v_1\|^2(t) + \|v_2\|^2(t)\}\|q\|^2,$$

$$I_3 = (\{w_1+w_2\}r, p_x) \le \frac{\epsilon}{2a}\|\Delta p\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 w_1\|^2 + \|\Delta^2 w_2\|^2 + \|w_1\|^2(t) + \|w_2\|^2(t)\}\|r\|^2,$$

$$I_4 = (\{u_1+u_2\}p, q_y) \le \frac{\epsilon}{2a}\|\Delta q\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 u_1\|^2 + \|\Delta^2 u_2\|^2 + \|u_1\|^2(t) + \|u_2\|^2(t)\}\|p\|^2,$$

$$I_5 = (\{v_1+v_2\}q, q_y) \le \frac{\epsilon}{2a}\|\Delta q\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 v_1\|^2 + \|\Delta^2 v_2\|^2 + \|v_1\|^2(t) + \|v_2\|^2(t)\}\|q\|^2,$$

$$I_6 = (\{w_1+w_2\}r, q_y) \le \frac{\epsilon}{2a}\|\Delta q\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 w_1\|^2 + \|\Delta^2 w_2\|^2 + \|w_1\|^2(t) + \|w_2\|^2(t)\}\|r\|^2,$$

$$I_7 = (\{u_1+u_2\}p, r_z) \le \frac{\epsilon}{2a}\|\Delta r\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 u_1\|^2 + \|\Delta^2 u_2\|^2 + \|u_1\|^2(t) + \|u_2\|^2(t)\}\|p\|^2,$$

$$I_8 = (\{v_1+v_2\}q, r_z) \le \frac{\epsilon}{2a}\|\Delta r\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 v_1\|^2 + \|\Delta^2 v_2\|^2 + \|v_1\|^2(t) + \|v_2\|^2(t)\}\|q\|^2,$$

$$I_9 = (\{w_1+w_2\}r, r_z) \le \frac{\epsilon}{2a}\|\Delta r\|^2 + \frac{2C_s^2}{\epsilon}\{\|\Delta^2 w_1\|^2 + \|\Delta^2 w_2\|^2 + \|w_1\|^2(t) + \|w_2\|^2(t)\}\|r\|^2,$$

Substituting $I_1 - I_9$ into (41)–(43), we get

$$\begin{aligned} \frac{d}{dt}\|p\|^2(t) + (2 - \frac{1}{a} - \frac{3\epsilon}{a})\|\Delta p\|^2(t) \le 2\frac{C_s^2}{\epsilon}\Big[&\Big(\|\Delta^2 u_1\|^2(t) + \|\Delta^2 u_2\|^2(t) + \|u_1\|^2(t) + \|u_2\|^2(t)\Big)\|p\|^2(t) \\ &+ \Big(\|\Delta^2 v_1\|^2(t) + \|\Delta^2 v_2\|^2(t) + \|v_1\|^2(t) + \|v_2\|^2(t)\Big)\|q\|^2(t) \\ &+ \Big(\|\Delta^2 w_1\|^2(t) + \|\Delta^2 w_2\|^2(t) + \|w_1\|^2(t) + \|w_2\|^2(t)\Big)\|r\|^2(t)\Big], \end{aligned} \tag{44}$$

$$\frac{d}{dt}\|q\|^2(t) + (2 - \frac{1}{a} - \frac{3\epsilon}{a})\|\Delta q\|^2(t) \le 2\frac{C_s^2}{\epsilon}\Big[\Big(\|\Delta^2 u_1\|^2(t) + \|\Delta^2 u_2\|^2(t) + \|u_1\|^2(t) + \|u_2\|^2(t)\Big)\|p\|^2(t)$$

$$+ \Big(\|\Delta^2 v_1\|^2(t) + \|\Delta^2 v_2\|^2(t) + \|v_1\|^2(t) + \|v_2\|^2(t)\Big)\|q\|^2(t)$$
$$+ \Big(\|\Delta^2 w_1\|^2(t) + \|\Delta^2 w_2\|^2(t) + \|w_1\|^2(t) + \|w_2\|^2(t)\Big)\|r\|^2(t)\Big], \tag{45}$$

$$\frac{d}{dt}\|r\|^2(t) + (2 - \frac{1}{a} - \frac{3\epsilon}{a})\|\Delta r\|^2(t) \leq 2\frac{C_s^2}{\epsilon}\Big[\Big(\|\Delta^2 u_1\|^2(t) + \|\Delta^2 u_2\|^2(t) + \|u_1\|^2(t) + \|u_2\|^2(t)\Big)\|p\|^2(t)$$
$$+ \Big(\|\Delta^2 v_1\|^2(t) + \|\Delta^2 v_2\|^2(t) + \|v_1\|^2(t) + \|v_2\|^2(t)\Big)\|q\|^2(t)$$
$$+ \Big(\|\Delta^2 w_1\|^2(t) + \|\Delta^2 w_2\|^2(t) + \|w_1\|^2(t) + \|w_2\|^2(t)\Big)\|r\|^2(t)\Big]. \tag{46}$$

By taking $\epsilon = \frac{1}{3}$, we transform (44)–(46) as follows:

$$\frac{d}{dt}(\|p\|^2(t) + \|q\|^2(t) + \|r\|^2(t)) \leq C\Big(\|\Delta^2 u_1\|^2(t) + \|\Delta^2 u_2\|^2(t) + \|\Delta^2 v_1\|^2(t) + \|\Delta^2 v_2\|^2(t)$$
$$+ \|\Delta^2 w_1\|^2(t) + \|\Delta^2 w_2\|^2(t) + \|u_1\|^2(t) + \|u_2\|^2(t) + \|v_1\|^2(t)$$
$$+ \|v_2\|^2(t) + \|w_1\|^2(t) + \|w_2\|^2(t)\Big)\{\|p\|^2(t) + \|q\|^2(t) + \|r\|^2(t)\}.$$

Since by (34) and Lemma 3.1,

$$\|\Delta^2 u_i\|^2(t), \|\Delta^2 v_i\|^2(t), \|\Delta^2 w_i\|^2(t), \in L^1(\mathbb{R}^+)$$

and

$$\|u_i\|(t), \|v_i\|^2(t), \|w_i\|^2(t) \in L^1(\mathbb{R}^+),\ i = 1, 2,$$

hence by (40) and Lemma 2.3,

$$\|p\|^2(t) + \|q\|^2(t) + \|r\|(t) \equiv 0 \ \text{ for all } \ t > 0.$$

Thus,

$$u_1(x, y, z, t) \equiv u_2(x, y, z, t); \quad v_1(x, y, z, t) \equiv v_2(x, y, z, t); \quad w_1(x, y, z, t) \equiv w_2(x, y, z, t).$$

This completes the proof of Lemma 3.2, and consequently, of Theorem 3.1.

□

4. KS system posed on unbounded parallelepipeds

Define

$$D_x = \{(x, y, z) \in \mathbb{R}^3;\ x \in (0, +\infty),\ y \in (0, L_2,),\ z \in (0, L_3)\},\ Q_{xt} = (0, t) \times D_x,\ \|f\|^2 = \|f\|^2_{L^2(D_x)}.$$

Assertions of Lemma 3.1 are true also for D_x provided that

$$a, \theta \text{ are replaced by } a_x = \lim_{L_1 \to +\infty} a = \sum_{i=2}^{3} \frac{\pi^2}{L_i^2},\ \theta_x = 1 - \frac{1}{a_x}.$$

In Q_{xt}, consider the following initial-boundary value problem:

$$u_t + \Delta^2 u + \Delta u + uu_x + vv_x + ww_x = 0, \tag{47}$$

$$v_t + \Delta^2 v + \Delta v + uu_y + vv_y + ww_y = 0, \tag{48}$$

$$w_t + \Delta^2 w + \Delta w + uu_z + vv_z + ww_z = 0, \tag{49}$$

$$u_y = v_x,\ u_z = w_x,\ v_z = w_y, \tag{50}$$

$$u|_{\partial D_x} = v|_{\partial D_x} = w|_{\partial D_x} = \frac{\partial}{\partial N} u|_{\partial D_x} = \frac{\partial}{\partial N} v|_{\partial D_x} \tag{51}$$

$$= \frac{\partial}{\partial N} w|_{\partial D_x} = 0,\ t > 0, \tag{52}$$

$$u(x, y, z, 0) = u_0(x, y, z), \quad v(x, y, z, 0) = v_0(x, y, z),$$

$$w(x, y, z, 0) = w_0(x, y, z) \text{ in } D_x. \tag{53}$$

where $\frac{\partial}{\partial N}$ is an exterior normal derivative on ∂D_x.

Definition 4.1. *A triplet*

$$u, v, w \in L^\infty(\mathbb{R}^+; H_0^2(D_x)), \quad \Delta^2 u, \Delta^2 v, \Delta^2 w \in L^\infty(\mathbb{R}^+; L^2(D_x)),$$
$$u_t, v_t, w_t \in L^\infty(\mathbb{R}^+; L^2(D_x)) \cap L^2(\mathbb{R}^+; H_0^2(D_x))$$

satisfying (51)–(53) *and the following identities:*

$$(u_t, \phi)(t) + (\Delta u, \Delta\phi)(t) + (\Delta u, \phi)(t) + (uu_x, \phi)(t) + (vv_x, \phi)(t) + (ww_x, \phi)(t) = 0, \ t > 0, \tag{54}$$

$$(v_t, \phi)(t) + (\Delta v, \Delta\phi)(t) + (\Delta v, \phi)(t) + (uu_y, \phi)(t) + (vv_y, \phi)(t) + (ww_y, \phi)(t) = 0, \ t > 0, \tag{55}$$

$$(w_t, \phi)(t) + (\Delta w, \Delta\phi)(t) + (\Delta w, \phi)(t) + (uu_z, \phi)(t) + (vv_z, \phi)(t) + (ww_z, \phi)(t) = 0, \ t > 0, \tag{56}$$

where $\phi(x, y)$ *is an arbitrary function from* $H_0^2(D_x)$*, is a strong solution to the problem* (47)–(53).

Theorem 4.1. *Let*

$$a_x = \lim_{L_1 \to +\infty} a = \sum_{i=2}^{3} \frac{\pi^2}{L_i^2}, \ \theta_x = 1 - \frac{1}{a_x} > 0. \tag{57}$$

Given $u_0, v_0, w_0 \in H^4(D_x) \cap H_0^2(D_x)$ *such that*

$$\theta_x > \frac{24}{\theta_x a_x^{3/2}} \Big(\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2 \Big). \tag{58}$$

Then the problem (47)–(52) *has a unique strong solution*

$$u, v, w \in L^\infty(\mathbb{R}^+; H_0^2(D_x)), \quad \Delta^2 u, \Delta^2 v, \Delta^2 w \in L^\infty(\mathbb{R}^+; L^2(D_x)),$$
$$u_t, v_t, w_t \in L^\infty(\mathbb{R}^+; L^2(D_x)) \cap L^2(\mathbb{R}^+; H_0^2(D_x)).$$

Moreover, u, v, w *satisfy the following inequalities:*

$$\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t) + \frac{\theta_x}{2} \int_0^t \Big(\|\Delta u\|^2(\tau) + \|\Delta v\|^2(\tau) + \|\Delta w\|^2(\tau) \Big) d\tau \le \|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2; \tag{59}$$

$$\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t) \le \Big[\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2 \Big] \exp\left\{ -\frac{a_x^2 \theta_x t}{2} \right\}; \tag{60}$$

$$\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t) \le \Big(\|u_t\|^2(0) + \|v_t\|^2(0) + \|w_t\|^2(0) \Big) \exp\left\{ -\frac{a_x^2 \theta_x t}{2} \right\}; \tag{61}$$

$$\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t) + \frac{\theta_x}{2} \int_0^t \Big(\|\Delta u_\tau\|^2(\tau) + \|\Delta v_\tau\|^2(\tau) + \|\Delta w_\tau\|^2(\tau) \Big) d\tau \le \|u_t\|^2(0) + \|v_t\|^2(0) + \|w_t\|^2(0). \tag{62}$$

Proof. Define the space $W = H^4(D_x) \cap H_0^2(D_x)$ and let $\{w_i(x, y, z), \ i \in \mathbb{N}\}$ be a countable dense set in W. We construct approximate solutions to (47)–(52) in the form

$$u^N(x, y, z, t) = \sum_{i=1}^N g_i^u(t) w_i(x, y, z); \quad v^N(x, y, z, t) = \sum_{i=1}^N g_i^v(t) w_i(x, y, z); \quad w^N(x, y, z, t) = \sum_{i=1}^N g_i^w(t) w_i(x, y, z).$$

The unknown functions $g_i^u(t), \ g_i^v(t), \ g_i^w(t)$ satisfy the following initial problems:

$$\frac{d}{dt}(u^N, w_j)(t) + (\Delta u^N, \Delta w_j)(t) - (\nabla u^N, \nabla w_j)(t) + (u^N u_x^N, w_j)(t) + (v^N v_x^N, w_j)(t) + (w^N w_x^N, w_j)(t) = 0, \tag{63}$$

$$\frac{d}{dt}(v^N, w_j)(t) + (\Delta v^N, \Delta w_j)(t) - (\nabla v^N, \nabla w_j)(t) + (u^N u_y^N, w_j)(t) + (v^N v_y^N, w_j)(t) + (w^N w_y^N, w_j)(t) = 0, \tag{64}$$

$$\frac{d}{dt}(w^N, w_j)(t) + (\Delta w^N, \Delta w_j)(t) - (\nabla w^N, \nabla w_j)(t) + (u^N u_z^N, w_j)(t) + (v^N v_z^N, w_j)(t) + (w^N w_z^N, w_j)(t) = 0, \tag{65}$$

$$g_j^u(0) = g_{0j}^u, \quad g_j^v(0) = g_{0j}^v, \quad g_j^w(0) = g_{0j}^w, \quad j = 1, 2, \ldots. \tag{66}$$

Estimate I. Dropping indices N, j, multiply (47) by $2u$, (48) by $2v$ and (49) by $2w$ to obtain

$$\frac{d}{dt}\|u\|^2(t) + 2\|\Delta u\|^2(t) - 2\|\nabla u\|^2(t) + 2(uu_x, u)(t) + 2(vv_x, u)(t) + 2(ww_x, u)(t) = 0, \tag{67}$$

$$\frac{d}{dt}\|v\|^2(t) + 2\|\Delta v\|^2(t) - 2\|\nabla v\|^2(t) + 2(uu_y, v)(t) + 2(vv_y, v)(t) + 2(ww_y, v)(t) = 0, \tag{68}$$

$$\frac{d}{dt}\|w\|^2(t) + 2\|\Delta w\|^2(t) - 2\|\nabla w\|^2(t) + 2(uu_z, w)(t) + 2(vv_z, w)(t) + 2(ww_z, w)(t) = 0. \tag{69}$$

Since

$$(u^2, u_x)(t) = (v^2, v_y)(t) = (w^2, w_z)(t) = 0,$$

we get

$$\frac{d}{dt}\|u\|^2(t) + \theta\|\Delta u\|^2(t) + \theta\|\Delta u\|^2(t) - (v^2, u_x)(t) - (w^2, u_x)(t) \le 0, \tag{70}$$

$$\frac{d}{dt}\|v\|^2(t) + \theta\|\Delta v\|^2(t) + \theta\|\Delta v\|^2(t) - (u^2, v_y)(t) - (w^2, v_y)(t) \le 0, \tag{71}$$

$$\frac{d}{dt}\|w\|^2(t) + \theta\|\Delta w\|^2(t) + \theta\|\Delta w\|^2(t) - (u^2, w_z)(t) - (v^2, w_z)(t) \le 0. \tag{72}$$

Making use of Lemmas 2.2 and 3.1, we estimate

$$\begin{aligned} I_1 = (v^2, u_x) &\le \|u_x\|_{L^4(D_x)}\|v\|_{L^4(D_x)}\|v\| \\ &\le 2\|v\|\|v\|^{1/4}\|\nabla v\|^{3/4}\|u_x\|^{1/4}\|\nabla u_x\|^{3/4} \\ &\le \frac{2}{a_x^{1/4}}\|v\|\|\nabla v\|\|\Delta u\| \le \epsilon\|\Delta u\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|v\|^2\|\Delta v\|^2. \end{aligned}$$

Similarly,

$$\begin{aligned} I_2 &= (w^2, u_x) \le \epsilon\|\Delta u\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|w\|^2\|\Delta w\|^2, \\ I_3 &= (u^2, v_y) \le \epsilon\|\Delta v\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|u\|^2\|\Delta u\|^2, \\ I_4 &= (w^2, v_y) \le \epsilon\|\Delta v\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|w\|^2\|\Delta w\|^2, \\ I_5 &= (u^2, w_z) \le \epsilon\|\Delta w\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|u\|^2\|\Delta u\|^2, \\ I_6 &= (v^2, w_z) \le \epsilon\|\Delta w\|^2 + \frac{1}{\epsilon a_x^{3/2}}\|v\|^2\|\Delta v\|^2. \end{aligned}$$

Substituting $I_1 - I_6$ into (70)–(72), choosing $4\epsilon = \theta_x$ and taking into account (53), we get

$$\begin{aligned} &\frac{d}{dt}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big) + \frac{\theta_x}{2}\Big(\|\Delta u\|^2(t) + \|\Delta v\|^2(t) + \|\Delta w\|^2(t)\Big) \\ &+ \left[\theta_x - \frac{4}{\theta_x a_x^{3/2}}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big)\right]\Big(\|\Delta u\|^2(t) + \|\Delta v\|^2(t) + \|\Delta w\|^2(t)\Big) < 0. \end{aligned} \tag{73}$$

Making use of (54) and Lemma 2.6, we obtain

$$\frac{d}{dt}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big) + \frac{\theta_x}{2}\Big(\|\Delta u\|^2(t) + \|\Delta v\|^2(t) + \|\Delta w\|^2(t)\Big) \le 0. \tag{74}$$

This implies

$$\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t) + \frac{\theta_x}{2}\int_0^t \Big(\|\Delta u\|^2(\tau) + \|\Delta v\|^2(\tau) + \|\Delta w\|^2(\tau)\Big)d\tau \le \|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2. \tag{75}$$

On the other hand, by Lemma 3.1, (71) can be rewritten as

$$\frac{d}{dt}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big) + \frac{a_x^2\theta_x}{2}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big) \le 0$$

that gives

$$\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t) \le \Big[\|u_0\|^2 + \|v_0\|^2 + \|w_0\|^2\Big]e^{(-\frac{a_x^2\theta_x t}{2})}. \tag{76}$$

Estimate II. Differentiate (47), (48), (49) with respect to t, then multiply the results respectively by $2u_t, 2v_t, 2w_t$ to get

$$\frac{d}{dt}\|u_t\|^2(t) + 2\|\Delta u_t\|^2(t) - 2\|\nabla u_t\|^2(t) = 2(uu_t, u_{xt})(t) + 2(vv_t, u_{xt})(t) + 2(ww_t, u_{xt})(t), \tag{77}$$

$$\frac{d}{dt}\|v_t\|^2(t) + 2\|\Delta v_t\|^2(t) - 2\|\nabla v_t\|^2(t) = 2(uu_t, v_{yt})(t) + 2(vv_t, v_{yt})(t) + 2(ww_t, v_{yt})(t), \tag{78}$$

$$\frac{d}{dt}\|w_t\|^2(t) + 2\|\Delta w_t\|^2(t) - 2\|\nabla v_t\|^2(t) = 2(uu_t, w_{zt})(t) + 2(vv_t, w_{zt})(t) + 2(ww_t, w_{zt})(t). \tag{79}$$

Making use of Lemmas 2.2 and 3.1, we estimate

$$\begin{aligned} I_1 &= 2(uu_t, u_{xt})(t) \le 2\|u\|(t)\|u_t\|_{L^4(D_X)}(t)\|\nabla u_t\|_{L^4(D_x)}(t) \\ &\le 4\|u\|(t)\|u_t\|^{1/4}(t)\|\nabla u_t\|^{3/4}(t)\|\nabla u_t\|^{1/4}(t)\|\Delta u_t\|^{3/4}(t) \\ &\le \frac{4}{a_x^{1/4}}\|u\|(t)\|\nabla u_t\|(t)\|\Delta u_t\|(t) \le \epsilon\|\Delta u_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|u\|^2(t)\|\Delta u_t\|^2(t), \end{aligned}$$

where ϵ is an arbitrary positive number. Similarly,

$$I_2 = 2(vv_t, u_{xt})(t) \le \epsilon\|\Delta u_t\|^2(t) + \frac{4}{a_x^{1/2}\epsilon}\|v\|^2(t)\|\Delta v_t\|^2(t),$$

$$I_3 = 2(ww_t, u_{xt})(t) \le \epsilon\|\Delta u_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|w\|^2(t)\|\Delta w_t\|^2(t),$$

$$I_4 = 2(uu_t, v_{yt})(t) \le \epsilon\|\Delta v_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|u\|^2(t)\|\Delta u_t\|^2(t),$$

$$I_5 = 2(vv_t, v_{yt})(t) \le \epsilon\|\Delta v_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|v\|^2(t)\|\Delta v_t\|^2(t),$$

$$I_6 = 2(ww_t, v_{yt})(t) \le \epsilon\|\Delta v_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|w\|^2(t)\|\Delta w_t\|^2(t),$$

$$I_7 = 2(uu_t, w_{zt})(t) \le \epsilon\|\Delta w_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|u\|^2(t)\|\Delta u_t\|^2(t),$$

$$I_8 = 2(vv_t, w_{zt})(t) \le \epsilon\|\Delta w_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|v\|^2(t)\|\Delta v_t\|^2(t),$$

$$I_9 = 2(ww_t, w_{zt})(t) \le \epsilon\|\Delta w_t\|^2(t) + \frac{4}{a_x^{3/2}\epsilon}\|w\|^2(t)\|\Delta w_t\|^2(t).$$

Taking into account that

$$\begin{aligned} 2\|\Delta u_t\|^2(t) - 2\|\nabla u_t\|^2(t) &> 2\theta_x\|\Delta u_t\|^2(t), \\ 2\|\Delta v_t\|^2(t) - 2\|\nabla v_t\|^2(t) &> 2\theta_x\|\Delta v_t\|^2(t), \\ 2\|\Delta w_t\|^2(t) - 2\|\nabla w_t\|^2(t) &> 2\theta_x\|\Delta w_t\|^2(t), \end{aligned}$$

and choosing $6\epsilon = \theta_x$, we get

$$\begin{aligned} &\frac{d}{dt}\Big(\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t)\Big) + \frac{\theta_x}{2}\Big(\|\Delta u_t\|^2(t) + \|\Delta v_t\|^2(t) + \|\Delta w_t\|^2(t)\Big) \\ &+ \left[\theta_x - \frac{24}{a_x^{3/2}\theta_x}\Big(\|u\|^2(t) + \|v\|^2(t) + \|w\|^2(t)\Big)\right]\Big(\|\Delta u_t\|^2(t) + \|\Delta v_t\|^2(t) + \|\Delta w_t\|^2(t)\Big) < 0. \end{aligned} \tag{80}$$

Taking into account (53), (54), we obtain

$$\frac{d}{dt}\Big(\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t)\Big) + \frac{\theta_x}{2}\Big(\|\Delta u_t\|^2(t) + \|\Delta v_t\|^2(t) + \|\Delta w_t\|^2(t)\Big) < 0. \tag{81}$$

By making use of Lemma 3.1, this can be rewritten as

$$\frac{d}{dt}\Big(\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t)\Big) + \frac{a_x^2\theta_x}{2}\Big(\|u_t\|^2(t) + \|v_t\|^2(t) + \|w_t\|^2(t)\Big) < 0. \tag{82}$$

Consequently,

$$\|u_t\|^2(t)+\|v_t\|^2(t)+\|w_t\|^2(t)\le\Big(\|u_t\|^2(0)+\|v_t\|^2(0)+\|w_t\|^2(0)\Big)\exp\Big(-\frac{a_x^2\theta_x t}{2}\Big). \tag{83}$$

Returning to (78), we find that

$$\|u_t\|^2(t)+\|v_t\|^2(t)+\|w_t\|^2(t)+\frac{\theta_x}{2}\int_0^t\Big(\|\Delta u_\tau\|^2(\tau)+\|\Delta v_\tau\|^2(\tau)+\|\Delta w_\tau\|^2(\tau)\Big)d\tau\le\|u_t\|^2(0)+\|v_t\|^2(0)+\|w_t\|^2(0). \tag{84}$$

Here,

$$\|u_t\|^2(0)\le C(\|u_0\|_W),\ \|v_t\|^2(0)\le C(\|v_0\|_W),\ \|w_t\|^2(0)\le C(\|w_0\|_W).$$

Jointly, (72) and (81) imply that

$$\begin{aligned}&u,v,w\in L^\infty(\mathbb{R}^+;H_0^2(D_x)),\\&u_t,v_t,w_t\in L^\infty(\mathbb{R}^+;L^2(D_x))\cap L^2(\mathbb{R}^+;H_0^2(D_x)).\end{aligned}\tag{85}$$

These inequalities guarantee the existence of strong solutions to (47)–(52), $u(x,y,z,t),v(x,y,z,t),w(x,y,z,t)$ satisfying (82) and the following identities:

$$(u_t,\phi)(t)+(\Delta u,\Delta\phi)(t)+(\Delta u,\phi)(t)+(uu_x,\phi)(t)+(vv_x,\phi)(t)+(ww_x,\phi)(t)=0,\ t>0, \tag{86}$$

$$(v_t,\phi)(t)+(\Delta v,\Delta\phi)(t)+(\Delta v,\phi)(t)+(uu_y,\phi)(t)+(vv_y,\phi)(t)+(ww_y,\phi)(t)=0,\ t>0, \tag{87}$$

$$(w_t,\phi)(t)+(\Delta w,\Delta\phi)(t)+(\Delta w,\phi)(t)+(uu_z,\phi)(t)+(vv_z,\phi)(t)+(ww_z,\phi)(t)=0,\ t>0, \tag{88}$$

where $\phi(x,y)$ is an arbitrary function from $H_0^2(D_x)$.
We can rewrite (83)–(85) in the form

$$\begin{aligned}(\Delta u,\Delta\phi)(t)&=-(u_t+\Delta u+uu_x+vv_x+ww_x,\phi)(t)\\(\Delta v,\Delta\phi)(t)&=-(v_t+\Delta v+uu_y+vv_y+ww_y,\phi)(t),\\(\Delta w,\Delta\phi)(t)&=-(w_t+\Delta w+uu_z+vv_z+ww_z,\phi)(t).\end{aligned}$$

It follows from here and (82) that

$$\Delta^2u,\ \Delta^2v,\ \Delta^2w\in L^\infty(\mathbb{R}^+;L^2(D_x)).$$

These jointly with (79) prove the existence part of Theorem 4.1.

Lemma 4.1. *A strong solution of Theorem 4.1 is uniquely defined.*

Proof. Let u_1,v_1,w_1 and u_2,v_2,w_2 be two distinct solutions to (47)–(52). Denoting $p=u_1-u_2,\ q=v_1-v_2,\ r=w_1-w_2$ and acting as by the proof of Lemma 3.2, we come to the following system:

$$\frac{d}{dt}\|p\|^2(t)+2\|\Delta p\|^2(t)-2\|\nabla p\|^2(t)-(\{u_1+u_2\}p,p_x)(t)-(\{v_1+v_2\}q,p_x)(t)-(\{w_1+w_2\}r,p_x)(t)=0, \tag{89}$$

$$\frac{d}{dt}\|q\|^2(t)+2\|\Delta q\|^2(t)-2\|\nabla q\|^2(t)-(\{u_1+u_2\}p,q_y)(t)-(\{v_1+v_2\}q,q_y)(t)-(\{w_1+w_2\}r,q_y)(t)=0, \tag{90}$$

$$\frac{d}{dt}\|r\|^2(t)+2\|\Delta r\|^2(t)-2\|\nabla r\|^2(t)-(\{u_1+u_2\}p,r_z)(t)-(\{v_1+v_2\}q,r_z)(t)-(\{w_1+w_2\}r,r_z)(t)=0; \tag{91}$$

$$p_y=q_x,\ p_z=r_x,\ q_z=rw_y, \tag{92}$$

$$p|_{\partial D}=q|_{\partial D}=r|_{\partial D}=\frac{\partial}{\partial N}p|_{\partial D}=\frac{\partial}{\partial N}q|_{\partial D}=\frac{\partial}{\partial N}r|_{\partial D}=0,\ t>0, \tag{93}$$

$$p(x,y,z,0)=q(x,y,z,0)=r(x,y,z,0)=0. \tag{94}$$

We estimate

$$\begin{aligned}I_1=(\{u_1+u_2\}p,p_x)&\le\|p_x\|_{L^4(D_x)}\|p\|\|u_1+u_2\|_{L^4(D_x)}\\&\le4\|p_x\|^{1/4}\|\nabla p_x\|^{3/4}\|p\|\|u_1+u_2\|^{1/4}\|\nabla(u_1+u_2)\|^{3/4}\end{aligned}$$

$$\leq \frac{4}{a_x^{1/4}}\|p\|\|\Delta p\|\|\nabla(u_1+u_2)\| \leq \frac{4}{a_x^{1/4}}\|p\|\|\Delta p\|(\|\nabla u_1\|+\|\nabla u_2\|)$$
$$\leq \epsilon\|\Delta p\|^2 + \frac{C}{\epsilon}\Big(\|\nabla u_1\|^2+\|\nabla u_2\|^2\Big)\|p\|^2.$$

Similarly,

$$I_2 = (\{v_1+v_2\}q, p_x) \leq \epsilon\|\Delta p\|^2 + \frac{C}{\epsilon}\Big(\|\nabla v_1\|^2+\|\nabla v_2\|^2\Big)\|q\|^2,$$
$$I_3 = (\{w_1+w_2\}r, p_x) \leq \epsilon\|\Delta p\|^2 + \frac{C}{\epsilon}\Big(\|\nabla w_1\|^2+\|\nabla w_2\|^2\Big)\|r\|^2,$$
$$I_4 = (\{u_1+u_2\}p, q_y) \leq \epsilon\|\Delta q\|^2 + \frac{C}{\epsilon}\Big(\|\nabla u_1\|^2+\|\nabla u_2\|^2\Big)\|p\|^2,$$
$$I_5 = (\{v_1+v_2\}q, q_y) \leq \epsilon\|\Delta q\|^2 + \frac{C}{\epsilon}\Big(\|\nabla v_1\|^2+\|\nabla v_2\|^2\Big)\|q\|^2,$$
$$I_6 = (\{w_1+w_2\}r, q_y) \leq \epsilon\|\Delta q\|^2 + \frac{C}{\epsilon}\Big(\|\nabla w_1\|^2+\|\nabla w_2\|^2\Big)\|r|^2,$$
$$I_7 = (\{u_1+u_2\}p, r_z) \leq \epsilon\|\Delta r\|^2 + \frac{C}{\epsilon}\Big(\|\nabla u_1\|^2+\|\nabla u_2\|^2\Big)\|p\|^2,$$
$$I_8 = (\{v_1+v_2\}q, r_z) \leq \epsilon\|\Delta r\|^2 + \frac{C}{\epsilon}\Big(\|\nabla v_1\|^2+\|\nabla v_2\|^2\Big)\|q\|^2,$$
$$I_9 = (\{w_1+w_2\}r, r_z) \leq \epsilon\|\Delta r\|^2 + \frac{C}{\epsilon}\Big(\|\nabla w_1\|^2+\|\nabla w_2\|^2\Big)\|r\|^2.$$

Substituting $I_1 - I_9$ into (86)–(88) and taking $6\epsilon = \theta_x$, we come to the following inequality:

$$\frac{d}{dt}\Big(\|p\|^2(t)+\|q\|^2(t)+\|r\|^2(t)\Big) \leq C\Big(\|\nabla u_1\|^2(t)+\|\nabla u_2\|^2(t)+\|\nabla v_1\|^2(t)+\|\nabla v_2\|^2(t)+\|\nabla w_1\|^2(t)+\|\nabla w_2\|^2(t)\Big)$$
$$\times\{\|p\|^2(t)+\|q\|^2(t)+\|r\|^2(t)\}.$$

Making use of (82) and Lemma 3.1, we get

$$\|\nabla u_1\|^2(t)+\|\nabla u_2\|^2(t)+\|\nabla v_1\|^2(t)+\|\nabla v_2\|^2(t)+\|\nabla w_1\|^2(t)+\|\nabla w_2\|^2(t) \in L^1(\mathbb{R}^+).$$

Also, by Lemma 2.3,

$$\|p\|^2(t)+\|q\|^2(t)+\|r\|(t) \equiv 0 \ \text{ for all } \ t>0.$$

Hence,

$$u_1(x,y,z,t) \equiv u_2(x,y,z,t); \quad v_1(x,y,z,t) \equiv v_2(x,y,z,t); \quad w_2(x,y,z,t) \equiv w_2(x,y,z,t).$$

This completes the proof of Lemma 4.1 and consequently of Theorem 4.1. □

Results similar to the ones presented in Theorem 4.1, can be established for the Kuramoto-Sivashinsky system (47)–(50) posed on unbounded parallelepipeds D_y and D_z, where

$$D_y = \{(x,y,z)\in\mathbb{R}^3;\ x\in(0,L_1),\ y\in(0,+\infty),\ z\in(0,L_3)\},$$
$$D_z = \{(x,y,z)\in\mathbb{R}^3;\ x\in(0,L_1),\ y\in(0,L_2,),\ z\in(0,+\infty)\}.$$

Assertions of Lemma 3.1 are true also for D_y provided that

$$a,\theta \text{ are replaced by } a_y = \lim_{L_2\to+\infty} a = \sum_{i=1,3}\frac{\pi^2}{L_i^2},\ \theta_y = 1-\frac{1}{a_y}$$

and for D_z provided that

$$a,\theta \text{ are replaced by } a_z = \lim_{L_3\to+\infty} a = \sum_{i=2}^{3}\frac{\pi^2}{L_i^2},\ \theta_z = 1-\frac{1}{a_z}.$$

5. Conclusions

This work is concerned with the formulation and solvability of initial-boundary value problems for the three-dimensional Kuramoto-Sivashinsky system (8)–(11) posed on smooth bounded domains and on unbounded parallelepipeds parallel to the principal axes $0X, 0Y, 0Z$. Theorem 3.1 contains results on the existence and uniqueness of global regular solutions as well as exponential decay of the $H^2(D)$-norm, where D is a smooth bounded 3D domain. We define a set of admissible domains, where destabilizing effects of terms $\Delta u, \Delta v, \Delta w$ are damped by dissipativity of $\Delta^2 u, \Delta^2 v, \Delta^2 w$ due to condition (14). This set contains "thin domains" (see [10, 17]), where some dimensions of D are small while others may be arbitrary large. The limiting cases of "thin domains" are unbounded parallelepipeds and they are presented in Section 4, where the existence and uniqueness of global strong solutions as well as exponential decay of the L^2-norms are also established. Since the initial-boundary value problems studied in this paper do not admit the a priori estimate independent of t, u, v, w, in order to prove the existence of global regular solutions, we put conditions (15) connecting geometrical properties of D with initial data u_0, v_0, w_0. Additionally, condition (16) guarantees exponential decay of the $H^2(D)$-norm. Moreover, Theorem 3.1 provides a "smoothing effect": initial data $u_0, v_0, w_0 \in H^2(D) \cap H^1_0(D)$ imply that $u, v, w \in L^2(\mathbb{R}^; H^4(D))$.

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