Research Article

Cubic binomial Fibonacci sums

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Abstract

As there is a paucity of cubic Fibonacci and Lucas identities in the existing literature, this paper is devoted to evaluating some cubic binomial Fibonacci and Lucas sums.

Keywords: Fibonacci number; Lucas number; summation identity; binomial coefficient; cubic Fibonacci identity.

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1. Introduction

The Fibonacci numbers, $F_j$, and the Lucas numbers, $L_j$, are defined, for $j \in \mathbb{Z}$, through the following recurrence relations:

$$F_j = F_{j-1} + F_{j-2}, \quad F_0 = 0, \quad F_1 = 1;$$

and

$$L_j = L_{j-1} + L_{j-2}, \quad L_0 = 2, \quad L_1 = 1;$$

with

$$F_{-j} = (-1)^{j-1}F_j, \quad L_{-j} = (-1)^{j}L_j.$$  

Details about the Fibonacci and Lucas numbers can be found in the excellent books written by Koshy [2] and Vajda [6].

Throughout this paper, we denote the golden ratio, $(1 + \sqrt{5})/2$, by $\alpha$ and write

$$\beta = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\alpha},$$

so that $\alpha \beta = -1$ and $\alpha + \beta = 1$. Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers in terms of $\alpha$ and $\beta$ are given as

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \quad j \in \mathbb{Z}.$$  

Nagy et al. [4] noted that there is a dearth of cubic Fibonacci and Lucas identities in the existing literature. Some cubic Fibonacci identities with binomial coefficients were derived recently by Kronenburg [3]. The main goal of the present paper is to evaluate the following sums:

$$\sum_{k=0}^{n} \binom{n}{k} F_{3k+s}, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{3k+s}, \quad \sum_{k=0}^{n} 2^k \binom{n}{k} F_{3k+s},$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{3k+s}, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k F_{3k+s}, \quad \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{3k+s},$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} F_{2k+s}, \quad \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} F_{2k+s},$$

and the corresponding series involving Lucas numbers, for any non-negative integer $n$ and for any integer $s$.

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2. Prerequisite identities for the main results

In this section, some results that are needed for proving the main results (presented in Section 3) are given.

**Lemma 2.2.** For a real or complex number \( z \), assume that a given well-behaved function \( h(z) \) have, in its domain, the representation \( h(z) = \sum_{k=0}^{c_2} g(k)z^f(k) \) where \( f(k) \) and \( g(k) \) are given real sequences and \( c_1, c_2 \in [-\infty, \infty] \). Let \( j \) be an integer. Then,

\[
\sum_{k=c_1}^{c_2} g(k)z^f(k)F_{f(k)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) h \left( \beta^i \alpha^{(m-i)j} z \right),
\]

\[
\sum_{k=c_1}^{c_2} g(k)z^f(k)L_{f(k)}^m = \sum_{i=0}^{m} \left( \binom{m}{i} \right) h \left( \beta^i \alpha^{(m-i)j} z \right).
\]

**Proof.** We have

\[
\sum_{k=c_1}^{c_2} g(k)z^f(k)F_{f(k)}^m = \sum_{k=c_1}^{c_2} g(k)z^f(k) \left( \alpha^{f(k)} - \beta^{f(k)} \right)^m
\]

\[
= \frac{1}{(\sqrt{5})^m} \sum_{k=c_1}^{c_2} g(k)z^f(k) \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) \beta^i \alpha^{(m-i)j} f(k)
\]

\[
= \frac{1}{(\sqrt{5})^m} \sum_{k=c_1}^{c_2} g(k) \left( \beta^i \alpha^{(m-i)j} z \right)^{f(k)}
\]

\[
= \frac{1}{(\sqrt{5})^m} \sum_{k=c_1}^{c_2} (-1)^i \left( \binom{m}{i} \right) h \left( \beta^i \alpha^{(m-i)j} z \right).
\]

The proof of (L) is similar.

Since \( \beta^i \alpha^{m-i} = (-1)^i \alpha^{m-2i} \), identities (F) and (L) can also be written as

\[
\sum_{k=c_1}^{c_2} g(k)z^f(k)F_{f(k)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) h \left( (-1)^i \alpha^{(m-2i)j} z \right),
\]

\[
\sum_{k=c_1}^{c_2} g(k)z^f(k)L_{f(k)}^m = \sum_{i=0}^{m} \left( \binom{m}{i} \right) h \left( (-1)^i \alpha^{(m-2i)j} z \right).
\]

**Lemma 2.2.** For the non-negative integers \( m \) and \( n \), the integers \( j, r \) and \( s \), and the real or complex numbers \( x \) and \( z \), the following identities hold:

\[
\sum_{k=0}^{n} \left( \binom{n}{k} \right) x^{n-k} z^k F_{(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) \alpha^{(m-2i)js} \left( x + (-1)^i \alpha^{(m-2i)j} z \right)^n,
\]

\[
\sum_{k=0}^{n} \left( \binom{n}{k} \right) x^{n-k} z^k L_{(rk+s)}^m = \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) \alpha^{(m-2i)js} \left( x + (-1)^i \alpha^{(m-2i)j} z \right)^n.
\]

**Proof.** Consider the binomial identity

\[
h(z) = \sum_{k=0}^{n} g(k)z^f(k) = z^s(x + z^r)^n,
\]

where

\[
f(k) = rk + s, \quad g(k) = \left( \binom{n}{k} \right) x^{n-k}.
\]

Thus,

\[
b \left( (-1)^i \alpha^{(m-2i)j} z \right) = (-1)^i \alpha^{(m-2i)js} z^s \left( x + (-1)^i \alpha^{(m-2i)j} z \right)^n.
\]

Use of (1) and (2) in identity (F'), with \( c_1 = 0, c_2 = n \), gives

\[
\sum_{k=0}^{n} \left( \binom{n}{k} \right) x^{n-k} z^k F_{(rk+s)}^m = \frac{1}{(\sqrt{5})^m} \sum_{i=0}^{m} (-1)^i \left( \binom{m}{i} \right) \alpha^{(m-2i)js} \left( x + (-1)^i \alpha^{(m-2i)j} z \right)^n,
\]

from which identity (BF) follows when we write \( z^{1/r} \) for \( z \). To prove (BL), use (1) and (2) in identity (L').

\[\square\]
It is sometimes convenient to use the $(\alpha \text{ vs } \beta)$ versions of identities (BF) and (BL):

\[
\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k F^n_{m(k)} = \frac{1}{(\sqrt{5})^n} \sum_{i=0}^{m} (-1)^i \binom{m}{i} \beta^{ij} \alpha^{(m-i)j} \left( x + \beta^{ij} \alpha^{(m-i)j} z \right)^n,
\]

\[
\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k L^n_{m(k)} = \sum_{i=0}^{m} \binom{m}{i} \beta^{ij} \alpha^{(m-i)j} \left( x + \beta^{ij} \alpha^{(m-i)j} z \right)^n.
\]

**Lemma 2.3** (Hoggatt et al. [1]). For the integers $p$ and $q$, the following identities hold:

\[
L_{p+q} - L_p \alpha^q = \alpha^q F_q \sqrt{5},
\]

\[
L_{p+q} - L_p \beta^q = \alpha^q F_q \sqrt{5},
\]

\[
F_{p+q} - F_p \alpha^q = \beta^p F_q,
\]

\[
F_{p+q} - F_p \beta^q = \alpha^p F_q.
\]

**Lemma 2.4.** Let $a$, $b$, $c$ and $d$ be rational numbers and $\lambda$ be an irrational number. Then,

\[a + \lambda b = c + \lambda d \iff a = c, \ b = d.\]

**Lemma 2.5.** For the integers $p$ and $q$,

\[1 + (-1)^p \alpha^{2q} = \begin{cases} (-1)^p \alpha^q F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^p \alpha^q L_q, & \text{if } p \text{ and } q \text{ have the same parity;} \end{cases}\]

and

\[1 - (-1)^p \alpha^{2q} = \begin{cases} (-1)^p \alpha^{-q} L_q, & \text{if } p \text{ and } q \text{ have different parity;} \\ (-1)^p \alpha^{-q} F_q \sqrt{5}, & \text{if } p \text{ and } q \text{ have the same parity.} \end{cases}\]

**Proof:** We have

\[
(-1)^{p+q} + (-1)^p \alpha^{2q} = \alpha^{p+q} \beta^{p+q} + \alpha^{p+2q} \beta^p = \alpha^{p+q} \beta^p (\alpha^q + \beta^q) = (-1)^p \alpha^q L_q. \tag{3}
\]

Similarly,

\[
(-1)^{p+q} - (-1)^p \alpha^{2q} = (-1)^{p-1} \alpha^q F_q \sqrt{5}. \tag{4}
\]

Corresponding to (3) and (4), we have

\[
(-1)^{p+q} + (-1)^p \beta^{2q} = (-1)^p \beta^q L_q \tag{5}
\]

and

\[
(-1)^{p+q} - (-1)^p \beta^{2q} = (-1)^{p-1} \beta^q F_q \sqrt{5}. \tag{6}
\]

Identities (3), (4), (5) and (6) imply

\[
(-1)^p + \alpha^{2q} = \alpha^q L_q,
\]

\[
(-1)^q - \alpha^{2q} = -\alpha^q F_q \sqrt{5},
\]

\[
(-1)^p + \beta^{2q} = \beta^q L_q,
\]

\[
(-1)^q - \beta^{2q} = \beta^q F_q \sqrt{5}.
\]

**Lemma 2.6** (Hoggatt et al [1]). For $p$ and $q$ integers,

\[
L_{p+q} - L_p \alpha^q = -\beta^p F_q \sqrt{5},
\]

\[
L_{p+q} - L_p \beta^q = \alpha^p F_q \sqrt{5},
\]

\[
F_{p+q} - F_p \alpha^q = \beta^p F_q,
\]

\[
F_{p+q} - F_p \beta^q = \alpha^p F_q.
\]
Lemma 2.7. The following identities hold:

\[ 1 - \alpha = \beta, \quad 1 - \beta = \alpha, \quad 1 + \alpha^3 = 2\alpha^2, \quad 1 + \beta^3 = 2\beta^2, \]

(7)

\[ 1 + \alpha = \alpha^2, \quad 1 + \beta = \beta^2, \quad 1 - \alpha^3 = -2\alpha, \quad 1 - \beta^3 = -2\beta, \]

(8)

\[ 1 - 2\alpha = -\sqrt{5}, \quad 1 - 2\beta = \sqrt{5}, \quad 1 + 2\alpha^3 = \alpha^3\sqrt{5}, \quad 1 + 2\beta^3 = -\beta^3\sqrt{5}, \]

(9)

\[ 2 + \alpha = \alpha\sqrt{5}, \quad 2 + \beta = -\beta\sqrt{5}, \quad 2 - \alpha^3 = -\sqrt{5}, \quad 2 - \beta^3 = \sqrt{5}, \]

(10)

\[ 1 + 3\alpha = \alpha^2\sqrt{5}, \quad 1 + 3\beta = -\beta^2\sqrt{5}, \quad 1 - 3\alpha^3 = -2\alpha^2\sqrt{5}, \quad 1 - 3\beta^3 = 2\beta^2\sqrt{5}, \]

(11)

\[ 3 - \alpha = -\beta\sqrt{5}, \quad 3 - \beta = \alpha\sqrt{5}, \quad 3 + \alpha^3 = 2\alpha^2\sqrt{5}, \quad 3 + \beta^3 = -2\beta\sqrt{5}. \]

(12)

Proof. Each identity is obtained by making appropriate substitutions for \( p \) and \( q \) in the identities given in Lemma 2.6.

\[ \square \]

3. Cubic binomial Fibonacci identities

Lemma 3.1. For a non-negative integer \( n \), integers \( j \), \( r \) and \( s \), and real or complex numbers \( x \) and \( z \), the following identities hold:

\[ 5\sqrt{5}\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k F_{j(rk+s)}^3 = \alpha^{3j} (x + \alpha^{3j} z)^n - \beta^{3j} (x + \beta^{3j} z)^n \]

\[ - (-1)^j 3\alpha^{3j} (x + (-1)^j \alpha^{2j} z)^n + (-1)^j 3\beta^{3j} (x + (-1)^j \beta^{2j} z)^n, \]

(F1)

(14)

\[ 5\sqrt{5}\sum_{k=0}^{n} \binom{n}{k} x^{n-k} z^k L_{j(rk+s)}^3 = \alpha^{3j} (x + \alpha^{3j} z)^n + \beta^{3j} (x + \beta^{3j} z)^n \]

\[ + (-1)^j 3\alpha^{3j} (x + (-1)^j \alpha^{2j} z)^n + (-1)^j 3\beta^{3j} (x + (-1)^j \beta^{2j} z)^n. \]

(L1)

Proof. Set \( m = 3 \) in identities (BF′) and (BL′).

\[ \square \]

Theorem 3.1. For a non-negative integer \( n \) and for any integer \( s \), the following identities hold:

\[ \sum_{k=0}^{n} \binom{n}{k} F_{k+s}^3 = \frac{1}{5}(2^n F_{2n+3s} + 3 F_{n+s}), \]

(13)

\[ \sum_{k=0}^{n} \binom{n}{k} L_{k+s}^3 = 2^n L_{2n+3s} + 3 L_{n+s}. \]

(14)

Proof. By setting \( x = 1, z = 1, j = 1, r = 1 \) in (F1) and utilizing identity (7), we obtain

\[ 5\sqrt{5}\sum_{k=0}^{n} \binom{n}{k} F_{k+s}^3 = 2^n (\alpha^{3s+2n} - \beta^{3s+2n}) + 3(\alpha^{n-s} - \beta^{n-s}); \]

and hence identity (13). To prove identity (14), use these \((x, z, j, \ldots)\) values in (L1).

\[ \square \]

A special case of (13), when \( s = 0 \), was obtained by Stanica [5].

Theorem 3.2. For a non-negative integer \( n \) and for any integer \( s \),

\[ \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_{k+s}^3 = \frac{1}{5}((-1)^n 2^n F_{n+s} - (-1)^s 3 F_{2n+s}), \]

(15)

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{k+s}^3 = (-1)^n 2^n L_{n+3s} + (-1)^s 3 L_{2n+s}, \]

(16)

Proof. To prove identity (15), set \( x = 1, z = -1, j = 1, r = 1 \) in (F1), noting the identities in (8), to get

\[ 5\sqrt{5}\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{k+s}^3 = (-1)^n 2^n (\alpha^{n+3s} - \beta^{n+3s}) - 3(-1)^s (\alpha^{2n+s} - \beta^{2n+s}), \]

from which the identity follows. The proof of (16) is similar. Use these values in (L1).

\[ \square \]
Stanica [5] also found the special case of identity (15) when \( s = 0 \).

**Theorem 3.3.** For a non-negative integer \( n \) and for any integer \( s \),
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{k+s}^3 = \begin{cases} 
5^{n/2-1}(F_{3n+3s} - (-1)^s3F_s), & \text{if } n \text{ is even;} \\
5^{(n-3)/2}(L_{3n+3s} + (-1)^s3L_s), & \text{if } n \text{ is odd;}
\end{cases}
\tag{17}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} 2^k F_{k+s}^3 = \begin{cases} 
5^{n/2}(L_{3n+3s} + (-1)^s3L_s), & \text{if } n \text{ is even;} \\
5^{(n+1)/2}(F_{3n+3s} - (-1)^s3F_s), & \text{if } n \text{ is odd;}
\end{cases}
\tag{18}
\]

**Proof.** The proof of (17) proceeds with the choice \( j = 1, r = 1, x = 1, z = 2 \) in (F1), employing the set of identities (9), giving
\[
5\sqrt{5} \sum_{k=0}^{n} 2^k \binom{n}{k} F_{k+s}^3 = (\sqrt{5})^n(\alpha^{3n+3s} - (-1)^n\beta^{3n+3s}) - 3(-1)^{n+s}(\sqrt{5})^n(\alpha^s - (-1)^n\beta^s),
\]
from which the identity follows in accordance with the parity of \( n \). The proof of (18) is similar. Use these \((x, z, j, \ldots)\) values in (L1).

**Theorem 3.4.** For a non-negative integer \( n \) and for any integer \( s \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{k+s}^3 = \begin{cases} 
5^{n/2-1}((-1)^s-13F_{n+s} + F_{3s}), & \text{if } n \text{ is even;} \\
5^{(n-3)/2}((-1)^s-13L_{n+s} - L_{3s}), & \text{if } n \text{ is odd;}
\end{cases}
\tag{19}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} L_{k+s}^3 = \begin{cases} 
5^{n/2}((-1)^s3L_{n+s} + L_{3s}), & \text{if } n \text{ is even;} \\
5^{(n+1)/2}((-1)^s3F_{n+s} - F_{3s}), & \text{if } n \text{ is odd.}
\end{cases}
\tag{20}
\]

**Proof.** The coice \( x = 2, z = -1, j = 1, z = 1 \) in (F1), noting the set of identities (10) gives
\[
5\sqrt{5} \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{k+s}^3 = (\sqrt{5})^n((-1)^n(\alpha^{3s} - (-1)^n\beta^{3s}) - (\sqrt{5})^n((-1)^s3(\alpha^{s} - (-1)^n\beta^{s});
\]
from which we get (19). The proof of (20) is similar.

**Theorem 3.5.** For a non-negative integer \( n \) and for any integer \( s \),
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} F_{k+s}^3 = \begin{cases} 
5^{n/2-1}(2^n F_{2n+3s} - (-1)^s3F_{2n+s}), & \text{if } n \text{ is even;} \\
-5^{(n-3)/2}(2^n L_{2n+3s} + (-1)^s3L_{2n+s}), & \text{if } n \text{ is odd;}
\end{cases}
\tag{21}
\]
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} L_{k+s}^3 = \begin{cases} 
5^{n/2}(2^n L_{2n+3s} + (-1)^s3L_{2n+s}), & \text{if } n \text{ is even;} \\
-5^{(n+1)/2}(2^n F_{2n+3s} - (-1)^s3F_{2n+s}), & \text{if } n \text{ is odd.}
\end{cases}
\tag{22}
\]

**Proof.** Choose \( x = 1, z = -3, j = 1, r = 1 \) in (F1). This gives, with the use of the identities in (11),
\[
5\sqrt{5} \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k} F_{k+s}^3 = (\sqrt{5})^n((-1)^n2^n(\alpha^{2n+3s} - (-1)^n\beta^{2n+3s}) - (\sqrt{5})^n((-1)^s3(\alpha^{n+s} - (-1)^n\beta^{n+s});
\]
Identity (21) now follows. The proof of (22) is similar.

**Theorem 3.6.** For a non-negative integer \( n \) and for any integer \( s \),
\[
\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{k+s}^3 = \begin{cases} 
5^{n/2-1}(2^n F_{n+3s} + 3F_{n-s}), & \text{if } n \text{ is even;} \\
5^{(n-3)/2}(2^n L_{n+3s} + 3L_{n-s}), & \text{if } n \text{ is odd;}
\end{cases}
\tag{23}
\]
\[
\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} L_{k+s}^3 = \begin{cases} 
5^{n/2}(2^n L_{n+3s} + 3L_{n-s}), & \text{if } n \text{ is even;} \\
5^{(n+1)/2}(2^n F_{n+3s} + 3F_{n-s}), & \text{if } n \text{ is odd.}
\end{cases}
\tag{24}
\]

**Proof.** Set \( x = 3, z = 1, j = 1 = r \) in (F1) and use the set of identities in (12) to obtain
\[
5\sqrt{5} \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{k+s}^3 = (\sqrt{5})^n2^n(\alpha^{n+3s} - (-1)^n\beta^{n+3s}) + (\sqrt{5})^n3(\alpha^{n-s} - (-1)^n\beta^{n-s});
\]
from which (23) follows. The proof of (24) is similar. Use the same \((x, z, \ldots)\) values in (L1).
Lemma 3.2. For a non-negative integer \( n \), integers \( j \), \( r \) and \( s \) and real or complex \( z \),

\[
5\sqrt{5} \sum_{k=0}^{[n/2]} 2 \left( \frac{n}{2k} \right) z^{2k} F^3_{j(2k+s)} = \alpha^{3j}(1 + \alpha^{3r} z)^n + \alpha^{3j}(1 - \alpha^{3r} z)^n - \beta^{3j}(1 + \beta^{3r} z)^n - \beta^{3j}(1 - \beta^{3r} z)^n
\]

\[
= (-1)^j \alpha^{3j}3(1 + (-1)^j \alpha^{3r} z)^n - (-1)^j \alpha^{3j}3(1 - (-1)^j \alpha^{3r} z)^n
\]

\[
+ (-1)^j \beta^{3j}3(1 + (-1)^j \beta^{3r} z)^n + (-1)^j \beta^{3j}3(1 - (-1)^j \beta^{3r} z)^n.
\]

\[
2 \sum_{k=0}^{[n/2]} \left( \frac{n}{2k} \right) z^{2k} L^3_{j(2k+s)} = \alpha^{3j}(1 + \alpha^{3r} z)^n + \alpha^{3j}(1 - \alpha^{3r} z)^n + \beta^{3j}(1 + \beta^{3r} z)^n + \beta^{3j}(1 - \beta^{3r} z)^n
\]

\[
= (-1)^j \alpha^{3j}3(1 + (-1)^j \alpha^{3r} z)^n + (-1)^j \alpha^{3j}3(1 - (-1)^j \alpha^{3r} z)^n
\]

\[
+ (-1)^j \beta^{3j}3(1 + (-1)^j \beta^{3r} z)^n + (-1)^j \beta^{3j}3(1 - (-1)^j \beta^{3r} z)^n.
\]

\[
5\sqrt{5} \sum_{k=1}^{[n/2]} 2 \left( \frac{n}{2k-1} \right) z^{2k-1} F^3_{j(2k+s)}
\]

\[
= \alpha^{3j(r+s)}(1 + \alpha^{3r+s} z)^n - \alpha^{3j(r+s)}(1 - \alpha^{3r+s} z)^n - \beta^{3j(r+s)}(1 + \beta^{3r+s} z)^n - \beta^{3j(r+s)}(1 - \beta^{3r+s} z)^n
\]

\[
= (-1)^{j(r+s)} \alpha^{3j(r+s)}3(1 + (-1)^{j(r+s)} \alpha^{3r} z)^n + (-1)^{j(r+s)} \alpha^{3j(r+s)}3(1 - (-1)^{j(r+s)} \alpha^{3r} z)^n
\]

\[
+ (-1)^{j(r+s)} \beta^{3j(r+s)}3(1 + (-1)^{j(r+s)} \beta^{3r} z)^n + (-1)^{j(r+s)} \beta^{3j(r+s)}3(1 - (-1)^{j(r+s)} \beta^{3r} z)^n.
\]

Proof. In the identities

\[
h_1(z) = 2 \sum_{k=0}^{[n/2]} \left( \frac{n}{2k} \right) z^{2rk+s} = z^s(1 + z^r)^n + z^s(1 - z^r)^n,
\]

\[
h_2(z) = 2 \sum_{k=1}^{[n/2]} \left( \frac{n}{2k-1} \right) z^{2rk+s} = z^{r+s}(1 + z^r)^n - z^{r+s}(1 - z^r)^n,
\]

identify

\[
g(k) = 2 \left( \frac{n}{2k} \right), \quad f(k) = 2rk + s, \quad c_1 = 0, \quad c_2 = [n/2], \quad h(z) = z^s(1 + z^r)^n + z^s(1 - z^r)^n,
\]

and use these in (F) and (L) to obtain (F2) and (L2).

Similarly, use of

\[
g(k) = 2 \left( \frac{n}{2k-1} \right), \quad f(k) = 2rk + s, \quad c_1 = 1, \quad c_2 = [n/2], \quad h(z) = z^s(1 + z^r)^n - z^s(1 - z^r)^n,
\]

in (F) and (L) gives (F3) and (L3).

Theorem 3.7. For a non-negative integer \( n \) and for any integer \( s \),

\[
10 \sum_{k=0}^{[n/2]} \left( \frac{n}{2k} \right) F^3_{2k+s} = 2^n(F_{2n+3s} - (-1)^n F_{n+3s}) - 3(-1)^s(F_{2n+s} - (-1)^s F_{n-s}),
\]

\[
2 \sum_{k=0}^{[n/2]} \left( \frac{n}{2k} \right) L^3_{2k+s} = 2^n(L_{2n+3s} - (-1)^n L_{n+3s}) + 3(-1)^s(L_{2n+s} - (-1)^s L_{n-s}).
\]
Proof. The choice of $z = 1 = j = r$ in (F2) gives

$$10\sqrt{5} \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} F_{2k+s}^3 = 2^n(\alpha^{2n+3s} - \beta^{2n+3s}) + (-1)^n2^n(\alpha^{n+3s} - \beta^{n+3s}) + 3(-1)^1(\beta^n - \alpha^n) - 3(-1)^n(\alpha^{2n+s} - \beta^{2n+s});$$

from which identity (25) follows. The proof of (26) is similar; use $z = 1 = j = r$ in (L2).

Corollary 3.1. For a non-negative integer $n$ and for any integer $s$,

$$10 \sum_{k=0}^{n} \frac{2n}{2k} F_{2k+s}^3 = \begin{cases} 4^n L_n F_{3n+3s} - (-1)^1 3 F_{n+s} L_{3n} & \text{if } n \text{ is even}; \\ 4^n F_n L_{3n+3s} - (-1)^1 3 L_{n+s} F_{3n} & \text{if } n \text{ is odd}; \end{cases}$$

$$2 \sum_{k=0}^{n} \frac{2n}{2k} L_{2k+s}^3 = \begin{cases} 4^n L_n L_{3n+3s} + (-1)^1 3 L_{n+s} L_{3n} & \text{if } n \text{ is even}; \\ 5(4^n F_n F_{3n+3s} + (-1)^1 3 F_{n+s} F_{3n}) & \text{if } n \text{ is odd}. \end{cases} \tag{27}$$

Proof. Write $2n$ for $n$ in each of the identities (25) and (26). Simplification is achieved by the use of the following well-known Fibonacci identities which are valid for any two integers $u$ and $v$ having the same parity:

$$F_u + (-1)^{(u-v)/2} F_v = L_{(u-v)/2} F_{(u+v)/2}, \tag{28}$$

$$F_u - (-1)^{(u-v)/2} F_v = L_{(u-v)/2} L_{(u+v)/2}, \tag{29}$$

$$L_u + (-1)^{(u-v)/2} L_v = L_{(u-v)/2} L_{(u+v)/2}, \tag{30}$$

$$L_u - (-1)^{(u-v)/2} L_v = 5F_{(u-v)/2} F_{(u+v)/2}. \tag{31}$$

Corollary 3.2. For a non-negative integer $n$,

$$10 \sum_{k=0}^{n} \frac{2n-1}{2k} F_{2k}^3 = \begin{cases} (2^{2n-1} - 3) F_{2n-1} L_{n-1} L_n & \text{if } n \text{ is even}; \\ (2^{2n-1} - 3) 5F_{2n-1} F_{n-1} F_n & \text{if } n \text{ is odd}; \end{cases} \tag{32}$$

$$2 \sum_{k=0}^{n} \frac{2n}{2k} L_{2k}^3 = \begin{cases} (4^n + 3) L_n L_{3n} & \text{if } n \text{ is even}; \\ (4^n + 3) 5F_n F_{3n} & \text{if } n \text{ is odd}. \end{cases} \tag{33}$$

Proof. To prove (32), write $2n-1$ for $n$ in (25) and set $s = 0$. To prove (33), set $s = 0$ in identity (27).

Theorem 3.8. For a non-negative integer $n$ and for any integer $s$,

$$10 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} F_{2k+s}^3 = 2^n(F_{2n+3s+3} - (-1)^n F_{n+3s+3}) - (-1)^n3(F_{2n+s+1} - F_{n-s-1}), \tag{34}$$

$$2 \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} L_{2k+s}^3 = 2^n(L_{2n+3s+3} - (-1)^n L_{n+3s+3}) + (-1)^n3(L_{2n+s+1} + (-1)^s L_{n-s-1}). \tag{35}$$

Proof. Set $z = 1 = j = r$ in identity (F3) to obtain

$$10\sqrt{5} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k-1} F_{2k+s}^3 = 2^n(\alpha^{2n+3s+3} - \beta^{2n+3s+3}) - (-1)^n2^n(\alpha^{n+3s+3} - \beta^{n+3s+3}) + (-1)^{r+1}3(\alpha^{2n+s+1} - \beta^{2n+s+1}) + (-1)^{s+1}3(\alpha^n \beta^{s+1} - \alpha^{s+1} \beta^n);$$

from which identity (34) follows. The proof of (35) is similar.

Corollary 3.3. For a non-negative integer $n$ and for any integer $s$, the following identities hold:

$$10 \sum_{k=1}^{n} \frac{2n}{2k-1} F_{2k+s}^3 = \begin{cases} 4^n F_n L_{3n+3s+3} - (-1)^n 3 L_{n+s+1} F_{3n} & \text{if } n \text{ is even}; \\ 4^n L_n F_{3n+3s+3} - (-1)^n 3 F_{n+s+1} L_{3n} & \text{if } n \text{ is odd}; \end{cases}$$

$$2 \sum_{k=1}^{n} \frac{2n}{2k-1} L_{2k+s}^3 = \begin{cases} 5(4^n F_n F_{3n+3s+3} + (-1)^n 3 F_{n+s+1} F_{3n}) & \text{if } n \text{ is even}; \\ 4^n L_n L_{3n+3s+3} + (-1)^n 3 L_{n+s+1} L_{3n} & \text{if } n \text{ is odd}. \end{cases}$$
Proof. Write $2n$ for $n$ in each of the identities (34) and (35), and make use of identities (28) – (31).

Corollary 3.4. For a non-negative integer $n$, the following identities hold:

$$2 \sum_{k=1}^{n} \left( \frac{2n}{2k-1} \right) F_{32k-1}^3 = \begin{cases} (2^{2n-1} + 3)5F_{2n-1}F_nF_{n-1} & \text{if $n$ is even;} \\ (2^{2n-1} + 3)F_{2n-1}L_{n-1}L_n & \text{if $n$ is odd;} \end{cases}$$

$$2 \sum_{k=1}^{n} \left( \frac{2n}{2k-1} \right) L_{32k-1}^3 = \begin{cases} (4^n - 3)5F_nF_{3n} & \text{if $n$ is even;} \\ (4^n - 3)L_nL_{3n} & \text{if $n$ is odd.} \end{cases}$$

References