Electronic Journal of Mathematics
www.shahindp.com/locate/ejm

Electron. J. Math. **2** (2021) 44–51
DOI: 10.47443/ejm.2021.0025

Research Article

Cubic binomial Fibonacci sums

Kunle Adegoke[1,*], Adenike Olatinwo[1], Sourangshu Ghosh[2]

[1]*Department of Physics and Engineering Physics, Obafemi Awolowo University, Ile-Ife, Nigeria*
[2]*Department of Civil Engineering, Indian Institute of Technology, Kharagpur, India*

(Received: 22 November 2021. Received in revised form: 23 December 2021. Accepted: 27 December 2021. Published online: 30 December 2021.)

Abstract

As there is a paucity of cubic Fibonacci and Lucas identities in the existing literature, this paper is devoted to evaluating some cubic binomial Fibonacci and Lucas sums.

Keywords: Fibonacci number; Lucas number; summation identity; binomial coefficient; cubic Fibonacci identity.

2020 Mathematics Subject Classification: 11B39, 11B37.

1. Introduction

The Fibonacci numbers, F_j, and the Lucas numbers, L_j, are defined, for $j \in \mathbb{Z}$, through the following recurrence relations:

$$F_j = F_{j-1} + F_{j-2}, (j \geq 2), \quad F_0 = 0, F_1 = 1;$$

and

$$L_j = L_{j-1} + L_{j-2}, (j \geq 2), \quad L_0 = 2, L_1 = 1;$$

with

$$F_{-j} = (-1)^{j-1} F_j, \quad L_{-j} = (-1)^j L_j.$$

Details about the Fibonacci and Lucas numbers can be found in the excellent books written by Koshy [2] and Vajda [6].

Throughout this paper, we denote the golden ratio, $(1+\sqrt{5})/2$, by α and write

$$\beta = \frac{1-\sqrt{5}}{2} = -\frac{1}{\alpha},$$

so that $\alpha\beta = -1$ and $\alpha + \beta = 1$. Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers in terms of α and β are given as

$$F_j = \frac{\alpha^j - \beta^j}{\alpha - \beta}, \quad L_j = \alpha^j + \beta^j, \quad j \in \mathbb{Z}.$$

Nagy et al. [4] noted that there is a dearth of cubic Fibonacci and Lucas identities in the existing literature. Some cubic Fibonacci identities with binomial coefficients were derived recently by Kronenburg [3]. The main goal of the present paper is to evaluate the following sums:

$$\sum_{k=0}^{n} \binom{n}{k} F_{k+s}^3, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} F_{k+s}^3, \quad \sum_{k=0}^{n} 2^k \binom{n}{k} F_{k+s}^3,$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{k+s}^3, \quad \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k F_{k+s}^3, \quad \sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{k+s}^3,$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} F_{2k+s}^3, \quad \sum_{k=1}^{\lceil n/2 \rceil} \binom{n}{2k-1} F_{2k+s}^3,$$

and the corresponding series involving Lucas numbers, for any non-negative integer n and for any integer s.

*Corresponding author (adegoke00@gmail.com).

Shahin

2. Prerequisite identities for the main results

In this section, some results that are needed for proving the main results (presented in Section 3) are given.

Lemma 2.1. *For a real or complex number z, assume that a given well-behaved function $h(z)$ have, in its domain, the representation $h(z)=\sum_{k=c_1}^{c_2} g(k)z^{f(k)}$ where $f(k)$ and $g(k)$ are given real sequences and $c_1, c_2 \in [-\infty,\infty]$. Let j be an integer. Then,*

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}F_{jf(k)}^m=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^i\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right), \tag{F}$$

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}L_{jf(k)}^m=\sum_{i=0}^m\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right). \tag{L}$$

Proof. We have

$$\begin{aligned}\sum_{k=c_1}^{c_2} g(k)z^{f(k)}F_{jf(k)}^m&=\sum_{k=c_1}^{c_2} g(k)z^{f(k)}\frac{\left(\alpha^{jf(k)}-\beta^{jf(k)}\right)^m}{(\sqrt{5})^m}\\&=\frac{1}{(\sqrt{5})^m}\sum_{k=c_1}^{c_2} g(k)z^{f(k)}\sum_{i=0}^m(-1)^i\binom{m}{i}\beta^{ijf(k)}\alpha^{(m-i)jf(k)}\\&=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^i\binom{m}{i}\sum_{k=c_1}^{c_2} g(k)\left(\beta^{ij}\alpha^{(m-i)j}z\right)^{f(k)}\\&=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^i\binom{m}{i}h\left(\beta^{ij}\alpha^{(m-i)j}z\right).\end{aligned}$$

The proof of (L) is similar. □

Since $\beta^i\alpha^{m-i}=(-1)^i\alpha^{m-2i}$, identities (F) and (L) can also be written as

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}F_{jf(k)}^m=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^i\binom{m}{i}h\left((-1)^{ij}\alpha^{(m-2i)j}z\right), \tag{F$'$}$$

$$\sum_{k=c_1}^{c_2} g(k)z^{f(k)}L_{jf(k)}^m=\sum_{i=0}^m\binom{m}{i}h\left((-1)^{ij}\alpha^{(m-2i)j}z\right). \tag{L$'$}$$

Lemma 2.2. *For the non-negative integers m and n, the integers j, r and s, and the real or complex numbers x and z, the following identities hold:*

$$\sum_{k=0}^n\binom{n}{k}x^{n-k}z^kF_{j(rk+s)}^m=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^{i(js+1)}\binom{m}{i}\alpha^{(m-2i)js}\left(x+(-1)^{ijr}\alpha^{(m-2i)jr}z\right)^n, \tag{BF}$$

$$\sum_{k=0}^n\binom{n}{k}x^{n-k}z^kL_{j(rk+s)}^m=\sum_{i=0}^m(-1)^{ijs}\binom{m}{i}\alpha^{(m-2i)js}\left(x+(-1)^{ijr}\alpha^{(m-2i)jr}z\right)^n. \tag{BL}$$

Proof. Consider the binomial identity

$$h(z)=\sum_{k=0}^n g(k)z^{f(k)}=z^s(x+z^r)^n,$$

where

$$f(k)=rk+s,\quad g(k)=\binom{n}{k}x^{n-k}. \tag{1}$$

Thus,

$$h\left((-1)^{ij}\alpha^{(m-2i)j}z\right)=(-1)^{ijs}\alpha^{(m-2i)js}z^s(x+(-1)^{ijr}\alpha^{(m-2i)jr}z^r)^n. \tag{2}$$

Use of (1) and (2) in identity (F$'$), with $c_1=0$, $c_2=n$, gives

$$\sum_{k=0}^n\binom{n}{k}x^{n-k}z^{rk}F_{j(rk+s)}^m=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^m(-1)^{i(js+1)}\binom{m}{i}\alpha^{(m-2i)js}\left(x+(-1)^{ijr}\alpha^{(m-2i)jr}z^r\right)^n,$$

from which identity (BF) follows when we write $z^{1/r}$ for z. To prove (BL), use (1) and (2) in identity (L$'$). □

It is sometimes convenient to use the (α vs β) versions of identities (BF) and (BL):

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kF_{j(rk+s)}^m=\frac{1}{(\sqrt{5})^m}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\beta^{ijs}\alpha^{(m-i)js}\left(x+\beta^{ijr}\alpha^{(m-i)jr}z\right)^n, \tag{BF$'$}$$

$$\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kL_{j(rk+s)}^m=\sum_{i=0}^{m}\binom{m}{i}\beta^{ijs}\alpha^{(m-i)js}\left(x+\beta^{ijr}\alpha^{(m-i)jr}z\right)^n. \tag{BL$'$}$$

Lemma 2.3 (Hoggatt et al. [1])**.** *For the integers p and q, the following identities hold:*

$$L_{p+q}-L_p\alpha^q=-\beta^pF_q\sqrt{5},$$

$$L_{p+q}-L_p\beta^q=\alpha^pF_q\sqrt{5},$$

$$F_{p+q}-F_p\alpha^q=\beta^pF_q,$$

$$F_{p+q}-F_p\beta^q=\alpha^pF_q.$$

Lemma 2.4. *Let a, b, c and d be rational numbers and λ be an irrational number. Then,*

$$a+\lambda b=c+\lambda d \iff a=c,\quad b=d\,.$$

Lemma 2.5. *For the integers p and q,*

$$1+(-1)^p\alpha^{2q}=\begin{cases}(-1)^p\alpha^qF_q\sqrt{5}, & \text{if } p \text{ and } q \text{ have different parity;}\\ (-1)^p\alpha^qL_q, & \text{if } p \text{ and } q \text{ have the same parity;}\end{cases}$$

and

$$1-(-1)^p\alpha^{2q}=\begin{cases}(-1)^{p-1}\alpha^qL_q, & \text{if } p \text{ and } q \text{ have different parity;}\\ (-1)^{p-1}\alpha^qF_q\sqrt{5}, & \text{if } p \text{ and } q \text{ have the same parity.}\end{cases}$$

Proof. We have

$$\begin{aligned}(-1)^{p+q}+(-1)^p\alpha^{2q}&=\alpha^{p+q}\beta^{p+q}+\alpha^{p+2q}\beta^p\\&=\alpha^{p+q}\beta^p(\alpha^q+\beta^q)\\&=(-1)^p\alpha^qL_q.\end{aligned} \tag{3}$$

Similarly,

$$(-1)^{p+q}-(-1)^p\alpha^{2q}=(-1)^{p-1}\alpha^qF_q\sqrt{5}. \tag{4}$$

□

Corresponding to (3) and (4), we have

$$(-1)^{p+q}+(-1)^p\beta^{2q}=(-1)^p\beta^qL_q \tag{5}$$

and

$$(-1)^{p+q}-(-1)^p\beta^{2q}=(-1)^p\beta^qF_q\sqrt{5}. \tag{6}$$

Identities (3), (4), (5) and (6) imply

$$(-1)^q+\alpha^{2q}=\alpha^qL_q,$$

$$(-1)^q-\alpha^{2q}=-\alpha^qF_q\sqrt{5},$$

$$(-1)^q+\beta^{2q}=\beta^qL_q,$$

$$(-1)^q-\beta^{2q}=\beta^qF_q\sqrt{5}.$$

Lemma 2.6 (Hoggatt et al [1])**.** *For p and q integers,*

$$L_{p+q}-L_p\alpha^q=-\beta^pF_q\sqrt{5},$$

$$L_{p+q}-L_p\beta^q=\alpha^pF_q\sqrt{5},$$

$$F_{p+q}-F_p\alpha^q=\beta^pF_q,$$

$$F_{p+q}-F_p\beta^q=\alpha^pF_q.$$

Lemma 2.7. *The following identies hold:*

$$1-\alpha=\beta,\quad 1-\beta=\alpha,\quad 1+\alpha^3=2\alpha^2,\quad 1+\beta^3=2\beta^2, \tag{7}$$

$$1+\alpha=\alpha^2,\quad 1+\beta=\beta^2,\quad 1-\alpha^3=-2\alpha,\quad 1-\beta^3=-2\beta, \tag{8}$$

$$1-2\alpha=-\sqrt5,\quad 1-2\beta=\sqrt5,\quad 1+2\alpha^3=\alpha^3\sqrt5,\quad 1+2\beta^3=-\beta^3\sqrt5, \tag{9}$$

$$2+\alpha=\alpha\sqrt5,\quad 2+\beta=-\beta\sqrt5,\quad 2-\alpha^3=-\sqrt5,\quad 2-\beta^3=\sqrt5, \tag{10}$$

$$1+3\alpha=\alpha^2\sqrt5,\quad 1+3\beta=-\beta^2\sqrt5,\quad 1-3\alpha^3=-2\alpha^2\sqrt5,\quad 1-3\beta^3=2\beta^2\sqrt5, \tag{11}$$

$$3-\alpha=-\beta\sqrt5,\quad 3-\beta=\alpha\sqrt5,\quad 3+\alpha^3=2\alpha\sqrt5,\quad 3+\beta^3=-2\beta\sqrt5. \tag{12}$$

Proof. Each identity is obtained by making appropriate substitutions for p and q in the identities given in Lemma 2.6. □

3. Cubic binomial Fibonacci identities

Lemma 3.1. *For a non-negative integer n, integers j, r and s, and real or complex numbers x and z, the following identities hold:*

$$\begin{aligned}5\sqrt5\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kF^3_{j(rk+s)}&=\alpha^{3js}(x+\alpha^{3jr}z)^n-\beta^{3js}(x+\beta^{3jr}z)^n\\&\quad-(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\&\quad+(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n,\end{aligned} \tag{F1}$$

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}z^kL^3_{j(rk+s)}&=\alpha^{3js}(x+\alpha^{3jr}z)^n+\beta^{3js}(x+\beta^{3jr}z)^n\\&\quad+(-1)^{js}3\alpha^{js}(x+(-1)^{jr}\alpha^{jr}z)^n\\&\quad+(-1)^{js}3\beta^{js}(x+(-1)^{jr}\beta^{jr}z)^n.\end{aligned} \tag{L1}$$

Proof. Set $m=3$ in identities (BF$'$) and (BL$'$). □

Theorem 3.1. *For a non-negative integer n and for any integer s, the following identities hold:*

$$\sum_{k=0}^{n}\binom{n}{k}F^3_{k+s}=\frac15(2^nF_{2n+3s}+3F_{n-s}), \tag{13}$$

$$\sum_{k=0}^{n}\binom{n}{k}L^3_{k+s}=2^nL_{2n+3s}+3L_{n-s}\,. \tag{14}$$

Proof. By setting $x=1$, $z=1$, $j=1$, $r=1$ in (F1) and utilizing identity (7), we obtain

$$5\sqrt5\sum_{k=0}^{n}\binom{n}{k}F^3_{k+s}=2^n(\alpha^{3s+2n}-\beta^{3s+2n})+3(\alpha^{n-s}-\beta^{n-s});$$

and hence identity (13). To prove identity (14), use these $(x,z,j,\ldots)$ values in (L1). □

A special case of (13), when $s=0$, was obtained by Stanica [5].

Theorem 3.2. *For a non-negative integer n and for any integer s,*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^kF^3_{k+s}=\frac15((-1)^n2^nF_{n+3s}-(-1)^s3F_{2n+s}), \tag{15}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}L^3_{k+s}=(-1)^n2^nL_{n+3s}+(-1)^s3L_{2n+s}, \tag{16}$$

Proof. To prove identity (15), set $x=1$, $z=-1$, $j=1$, $r=1$ in (F1), noting the identities in (8), to get

$$5\sqrt5\sum_{k=0}^{n}(-1)^k\binom{n}{k}F^3_{k+s}=(-1)^n2^n(\alpha^{n+3s}-\beta^{n+3s})-3(-1)^s(\alpha^{2n+s}-\beta^{2n+s}),$$

from which the identity follows. The proof of (16) is similar. Use these values in (L1). □

Stanica [5] also found the special case of identity (15) when $s = 0$.

Theorem 3.3. *For a non-negative integer n and for any integer s,*

$$\sum_{k=0}^{n} \binom{n}{k} 2^k F_{k+s}^3 = \begin{cases} 5^{n/2-1}(F_{3n+3s} - (-1)^s 3F_s), & \text{if } n \text{ is even;} \\ 5^{(n-3)/2}(L_{3n+3s} + (-1)^s 3L_s) & \text{if } n \text{ is odd,} \end{cases} \tag{17}$$

$$\sum_{k=0}^{n} \binom{n}{k} 2^k L_{k+s}^3 = \begin{cases} 5^{n/2}(L_{3n+3s} + (-1)^s 3L_s), & \text{if } n \text{ is even;} \\ 5^{(n+1)/2}(F_{3n+3s} - (-1)^s 3F_s) & \text{if } n \text{ is odd.} \end{cases} \tag{18}$$

Proof. The proof of (17) proceeds with the choice $j = 1$, $r = 1$, $x = 1$, $z = 2$ in (F1), employing the set of identities (9), giving

$$5\sqrt{5}\sum_{k=0}^{n} 2^k \binom{n}{k} F_{k+s}^3 = (\sqrt{5})^n(\alpha^{3n+3s} - (-1)^n\beta^{3n+3s}) - 3(-1)^{n+s}(\sqrt{5})^n(\alpha^s - (-1)^n\beta^s),$$

from which the identity follows in accordance with the parity of n. The proof of (18) is similar. Use these $(x, z, j, \ldots)$ values in (L1). □

Theorem 3.4. *For a non-negative integer n and for any integer s,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{k+s}^3 = \begin{cases} 5^{n/2-1}((-1)^{s-1} 3F_{n+s} + F_{3s}), & \text{if } n \text{ is even;} \\ 5^{(n-3)/2}((-1)^{s-1} 3L_{n+s} - L_{3s}), & \text{if } n \text{ is odd;} \end{cases} \tag{19}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} L_{k+s}^3 = \begin{cases} 5^{n/2}((-1)^s 3L_{n+s} + L_{3s}), & \text{if } n \text{ is even;} \\ 5^{(n+1)/2}((-1)^s 3F_{n+s} - F_{3s}), & \text{if } n \text{ is odd.} \end{cases} \tag{20}$$

Proof. The coice $x = 2$, $z = -1$, $j = 1$, $z = 1$ in (F1), noting the set of identities (10) gives

$$5\sqrt{5}\sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^{n-k} F_{k+s}^3 = (\sqrt{5})^n(-1)^n(\alpha^{3s} - (-1)^n\beta^{3s}) - (\sqrt{5})^n(-1)^s 3(\alpha^{n+s} - (-1)^n\beta^{n+s});$$

from which we get (19). The proof of (20) is similar. □

Theorem 3.5. *For a non-negative integer n and for any integer s,*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k F_{k+s}^3 = \begin{cases} 5^{n/2-1}(2^n F_{2n+3s} - (-1)^s 3F_{2n+s}), & \text{if } n \text{ is even;} \\ -5^{(n-3)/2}(2^n L_{2n+3s} + (-1)^s 3L_{2n+s}), & \text{if } n \text{ is odd;} \end{cases} \tag{21}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k L_{k+s}^3 = \begin{cases} 5^{n/2}(2^n L_{2n+3s} + (-1)^s 3L_{2n+s}), & \text{if } n \text{ is even;} \\ -5^{(n+1)/2}(2^n F_{2n+3s} - (-1)^s 3F_{2n+s}), & \text{if } n \text{ is odd.} \end{cases} \tag{22}$$

Proof. Choose $x = 1$, $z = -3$, $j = 1$, $r = 1$ in (F1). This gives, with the use of the identities in (11),

$$5\sqrt{5}\sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^k F_{k+s}^3 = (\sqrt{5})^n(-1)^n 2^n(\alpha^{2n+3s} - (-1)^n\beta^{2n+3s}) - (\sqrt{5})^n(-1)^s 3(\alpha^{2n+s} - (-1)^n\beta^{2n+s}).$$

Identity (21) now follows. The proof of (22) is similar. □

Theorem 3.6. *For a non-negative integer n and for any integer s,*

$$\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{k+s}^3 = \begin{cases} 5^{n/2-1}(2^n F_{n+3s} + 3F_{n-s}), & \text{if } n \text{ is even;} \\ 5^{(n-3)/2}(2^n L_{n+3s} + 3L_{n-s}), & \text{if } n \text{ is odd;} \end{cases} \tag{23}$$

$$\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} L_{k+s}^3 = \begin{cases} 5^{n/2}(2^n L_{n+3s} + 3L_{n-s}), & \text{if } n \text{ is even;} \\ 5^{(n+1)/2}(2^n F_{n+3s} + 3F_{n-s}), & \text{if } n \text{ is odd.} \end{cases} \tag{24}$$

Proof. Set $x = 3$, $z = 1$, $j = 1 = r$ in (F1) and use the set of identities in (12) to obtain

$$5\sqrt{5}\sum_{k=0}^{n} \binom{n}{k} 3^{n-k} F_{k+s}^3 = (\sqrt{5})^n 2^n(\alpha^{n+3s} - (-1)^n\beta^{n+3s}) + (\sqrt{5})^n 3(\alpha^{n-s} - (-1)^n\beta^{n-s});$$

from which (23) follows. The proof of (24) is similar. Use the same $(x, z, \ldots)$ values in (L1). □

Lemma 3.2. *For a non-negative integer n, integers j, r and s and real or complex z,*

$$\begin{aligned}5\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}2\binom{n}{2k}z^{2k}F^3_{j(2rk+s)} &= \alpha^{3js}(1+\alpha^{3jr}z)^n+\alpha^{3js}(1-\alpha^{3jr}z)^n-\beta^{3js}(1+\beta^{3jr}z)^n-\beta^{3js}(1-\beta^{3jr}z)^n\\ &\quad -(-1)^{js}\alpha^{js}3(1+(-1)^{jr}\alpha^{jr}z)^n-(-1)^{js}\alpha^{js}3(1-(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{js}\beta^{js}3(1+(-1)^{jr}\beta^{jr}z)^n+(-1)^{js}\beta^{js}3(1-(-1)^{jr}\beta^{jr}z)^n,\end{aligned} \tag{F2}$$

$$\begin{aligned}2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}z^{2k}L^3_{j(2rk+s)} &= \alpha^{3js}(1+\alpha^{3jr}z)^n+\alpha^{3js}(1-\alpha^{3jr}z)^n+\beta^{3js}(1+\beta^{3jr}z)^n+\beta^{3js}(1-\beta^{3jr}z)^n\\ &\quad +(-1)^{js}\alpha^{js}3(1+(-1)^{jr}\alpha^{jr}z)^n+(-1)^{js}\alpha^{js}3(1-(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{js}\beta^{js}3(1+(-1)^{jr}\beta^{jr}z)^n+(-1)^{js}\beta^{js}3(1-(-1)^{jr}\beta^{jr}z)^n,\end{aligned} \tag{L2}$$

$$\begin{aligned}&5\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}2\binom{n}{2k-1}z^{2k-1}F^3_{j(2rk+s)}\\ &\quad =\alpha^{3j(r+s)}(1+\alpha^{3jr}z)^n-\alpha^{3j(r+s)}(1-\alpha^{3jr}z)^n-\beta^{3j(r+s)}(1+\beta^{3jr}z)^n+\beta^{3j(r+s)}(1-\beta^{3jr}z)^n\\ &\quad -(-1)^{j(r+s)}\alpha^{j(r+s)}3(1+(-1)^{jr}\alpha^{jr}z)^n+(-1)^{j(r+s)}\alpha^{j(r+s)}3(1-(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{j(r+s)}\beta^{j(r+s)}3(1+(-1)^{jr}\beta^{jr}z)^n-(-1)^{j(r+s)}\beta^{j(r+s)}3(1-(-1)^{jr}\beta^{jr}z)^n,\end{aligned} \tag{F3}$$

$$\begin{aligned}&2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}z^{2k-1}L^3_{j(2rk+s)}\\ &\quad =\alpha^{3j(r+s)}(1+\alpha^{3jr}z)^n-\alpha^{3j(r+s)}(1-\alpha^{3jr}z)^n+\beta^{3j(r+s)}(1+\beta^{3jr}z)^n-\beta^{3j(r+s)}(1-\beta^{3jr}z)^n\\ &\quad +(-1)^{j(r+s)}\alpha^{j(r+s)}3(1+(-1)^{jr}\alpha^{jr}z)^n-(-1)^{j(r+s)}\alpha^{j(r+s)}3(1-(-1)^{jr}\alpha^{jr}z)^n\\ &\quad +(-1)^{j(r+s)}\beta^{j(r+s)}3(1+(-1)^{jr}\beta^{jr}z)^n-(-1)^{j(r+s)}\beta^{j(r+s)}3(1-(-1)^{jr}\beta^{jr}z)^n.\end{aligned} \tag{L3}$$

Proof. In the identities

$$h_1(z)=2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}z^{2rk+s}=z^s(1+z^r)^n+z^s(1-z^r)^n,$$

$$h_2(z)=2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}z^{2rk+s}=z^{r+s}(1+z^r)^n-z^{r+s}(1-z^r)^n,$$

identify

$$g(k)=2\binom{n}{2k},\quad f(k)=2rk+s,\quad c_1=0,\quad c_2=\lfloor n/2\rfloor,\quad h(z)=z^s(1+z^r)^n+z^s(1-z^r)^n,$$

and use these in (F) and (L) to obtain (F2) and (L2).
Similarly, use of

$$g(k)=2\binom{n}{2k-1},\quad f(k)=2rk+s,\quad c_1=1,\quad c_2=\lceil n/2\rceil,\quad h(z)=z^s(1+z^r)^n-z^s(1-z^r)^n,$$

in (F) and (L) gives (F3) and (L3). □

Theorem 3.7. *For a non-negative integer n and for any integer s,*

$$10\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F^3_{2k+s}=2^n(F_{2n+3s}+(-1)^nF_{n+3s})-3(-1)^s(F_{2n+s}-(-1)^sF_{n-s}), \tag{25}$$

$$2\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}L^3_{2k+s}=2^n(L_{2n+3s}+(-1)^nL_{n+3s})+3(-1)^s(L_{2n+s}+(-1)^sL_{n-s}). \tag{26}$$

Proof. The choice of $z = 1 = j = r$ in (F2) gives

$$10\sqrt{5}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}F_{2k+s}^3 = 2^n(\alpha^{2n+3s}-\beta^{2n+3s}) + (-1)^n 2^n(\alpha^{n+3s}-\beta^{n+3s}) + 3(-1)^s(\beta^s\alpha^n-\alpha^s\beta^n) - 3(-1)^s(\alpha^{2n+s}-\beta^{2n+s});$$

from which identity (25) follows. The proof of (26) is similar; use $z = 1 = j = r$ in (L2). □

Corollary 3.1. *For a non-negative integer n and for any integer s,*

$$10\sum_{k=0}^{n}\binom{2n}{2k}F_{2k+s}^3 = \begin{cases} 4^n L_n F_{3n+3s} - (-1)^s 3F_{n+s}L_{3n}, & \text{if } n \text{ is even};\\ 4^n F_n L_{3n+3s} - (-1)^s 3L_{n+s}F_{3n}, & \text{if } n \text{ is odd};\end{cases}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{2k+s}^3 = \begin{cases} 4^n L_n L_{3n+3s} + (-1)^s 3L_{n+s}L_{3n}, & \text{if } n \text{ is even};\\ 5(4^n F_n F_{3n+3s} + (-1)^s 3F_{n+s}F_{3n}), & \text{if } n \text{ is odd}.\end{cases} \tag{27}$$

Proof. Write $2n$ for n in each of the identities (25) and (26). Simplification is achieved by the use of the following well-known Fibonacci identities which are valid for any two integers u and v having the same parity:

$$F_u + (-1)^{(u-v)/2}F_v = L_{(u-v)/2}F_{(u+v)/2}, \tag{28}$$

$$F_u - (-1)^{(u-v)/2}F_v = F_{(u-v)/2}L_{(u+v)/2}, \tag{29}$$

$$L_u + (-1)^{(u-v)/2}L_v = L_{(u-v)/2}L_{(u+v)/2}, \tag{30}$$

$$L_u - (-1)^{(u-v)/2}L_v = 5F_{(u-v)/2}F_{(u+v)/2}. \tag{31}$$

□

Corollary 3.2. *For a non-negative integer n,*

$$10\sum_{k=0}^{n}\binom{2n-1}{2k}F_{2k}^3 = \begin{cases} (2^{2n-1}-3)F_{2n-1}L_{n-1}L_n, & \text{if } n \text{ is even};\\ (2^{2n-1}-3)5F_{2n-1}F_{n-1}F_n, & \text{if } n \text{ is odd};\end{cases} \tag{32}$$

$$2\sum_{k=0}^{n}\binom{2n}{2k}L_{2k}^3 = \begin{cases} (4^n+3)L_nL_{3n}, & \text{if } n \text{ is even};\\ (4^n+3)5F_nF_{3n}, & \text{if } n \text{ is odd}.\end{cases} \tag{33}$$

Proof. To prove (32), write $2n-1$ for n in (25) and set $s = 0$. To prove (33), set $s = 0$ in identity (27). □

Theorem 3.8. *For a non-negative integer n and for any integer s,*

$$10\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s}^3 = 2^n(F_{2n+3s+3} - (-1)^n F_{n+3s+3}) - (-1)^s 3(F_{2n+s+1} - (-1)^s F_{n-s-1}), \tag{34}$$

$$2\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}L_{2k+s}^3 = 2^n(L_{2n+3s+3} - (-1)^n L_{n+3s+3}) + (-1)^s 3(L_{2n+s+1} + (-1)^s L_{n-s-1}). \tag{35}$$

Proof. Set $z = 1 = j = r$ in identity (F3) to obtain

$$10\sqrt{5}\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}F_{2k+s}^3 = 2^n(\alpha^{2n+3s+3}-\beta^{2n+3s+3}) - (-1)^n 2^n(\alpha^{n+3s+3}-\beta^{n+3s+3}) + (-1)^{s+1}3(\alpha^{2n+s+1}-\beta^{2n+s+1}) + (-1)^{s+1}3(\alpha^n\beta^{s+1}-\alpha^{s+1}\beta^n);$$

from which identity (34) follows. The proof of (35) is similar. □

Corollary 3.3. *For a non-negative integer n and for any integer s, the following identities hold:*

$$10\sum_{k=1}^{n}\binom{2n}{2k-1}F_{2k+s}^3 = \begin{cases} 4^n F_n L_{3n+3s+3} - (-1)^s 3L_{n+s+1}F_{3n}, & \text{if } n \text{ is even};\\ 4^n L_n F_{3n+3s+3} - (-1)^s 3F_{n+s+1}L_{3n}, & \text{if } n \text{ is odd};\end{cases}$$

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}L_{2k+s}^3 = \begin{cases} 5(4^n F_n F_{3n+3s+3} + (-1)^s 3F_{n+s+1}F_{3n}), & \text{if } n \text{ is even};\\ 4^n L_n L_{3n+3s+3} + (-1)^s 3L_{n+s+1}L_{3n}, & \text{if } n \text{ is odd}.\end{cases}$$

Proof. Write $2n$ for n in each of the identities (34) and (35), and make use of identities (28) – (31). □

Corollary 3.4. *For a non-negative integer n, the following identities hold:*

$$10\sum_{k=1}^{n}\binom{2n-1}{2k-1}F_{2k-1}^3=\begin{cases}(2^{2n-1}+3)5F_{2n-1}F_{n-1}F_n, & \textit{if } n \textit{ is even};\\ (2^{2n-1}+3)F_{2n-1}L_{n-1}L_n, & \textit{if } n \textit{ is odd};\end{cases}$$

$$2\sum_{k=1}^{n}\binom{2n}{2k-1}L_{2k-1}^3=\begin{cases}(4^n-3)5F_nF_{3n}, & \textit{if } n \textit{ is even};\\ (4^n-3)L_nL_{3n}, & \textit{if } n \textit{ is odd}.\end{cases}$$

References

[1] V. E. Hoggatt, J. W. Phillips, H. T. Leonard, Twenty-four master identities, *Fibonacci Quart.* **9** (1971) 1–17.
[2] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.
[3] M. J. Kronenburg, Some weighted generalized Fibonacci number summation identities: Part 2, *arXiv*:2106.11838v2 [math.NT], Preprint, (2021).
[4] M. Nagy, S. R. Cowel, V. Beiu, Survey of cubic Fibonacci identities when cuboids carry weight, *arXiv*:1902.05944 [math.HO], Preprint, (2019).
[5] P. Stănică, Generating functions, weighted and non-weighted sums for powers of second order recurrence sequences, *Fibonacci Quart.* **41** (2003) 321–333.
[6] S. Vajda, *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*, Dover Press, New York, 2008.