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Research Article

A note on the symmetric division deg coindex of graphs

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Abstract

The symmetric division deg coindex $\overline{SDD}(G)$ of a simple connected graph G is defined as the sum of the terms $\frac{d_G(u)^2+d_G(v)^2}{d_G(u)d_G(v)}$ over all pairs of distinct non-adjacent vertices of G, where $d_G(u)$ denotes the degree of a vertex u of G. In this paper, upper bounds on the symmetric division deg coindex of edge corona product of two graphs and Mycielskian of a graph are presented. Also, it is proved that the symmetric division deg coindex of the double graph of a connected graph G with n vertices can be written in terms of the symmetric division deg coindex of G and n.

Keywords: symmetric division deg coindex; edge corona graph; Mycielskian of a graph; double graph.

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1. Introduction

A topological index of a graph is a parameter that does not depend on the labeling or pictorial representation of the graph. In theoretical chemistry, molecular structure descriptors (also called topological indices) are used for modeling physicochemical, pharmacologic, toxicologic, biological and other properties of chemical compounds. Several types of such indices exist, especially the ones that depend on the vertex and edge distances.

Topological indices have found applications in modeling several physicochemical properties in QSAR (Quantitative structure-activity relationship) and QSPR (quantitative structure-property relationships) studies [4, 9]. Many particular types of topological indices are defined using the structure of the underlying molecular graph, such as the Wiener index [11], first Zagreb index [2] and Balaban index [3]. Vukičević and Gašperov [10] observed that most of these indices are defined via the sum of individual bond contributions. Among the 148 discrete Adriatic indices studied in [10], whose predictive properties were evaluated against the benchmark datasets of the International Academy of Mathematical Chemistry, 20 indices were selected as significant predictors of physicochemical properties. One of these useful discrete Adriatic indices is the symmetric division deg index SDD which is defined as

$$SDD(G) = \sum_{xy\in E(G)} \left(\frac{d_G(x)}{d_G(y)} + \frac{d_G(y)}{d_G(x)}\right).$$

Among all the existing topological indices, SDD index has the best correlating ability for predicting the total surface area of polychlorobiphenys [10].

The first Zagreb index [2] and its coindex of a connected graph G are defined as

$$M_1(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$$

and

$$\overline{M}_1(G) = \sum_{uv\notin E(G)} (d_G(u) + d_G(v)),$$

respectively. The harmonic index and its coindex are, respectively, defined as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u) + d_G(v)}$$

and

$$\overline{H}(G) = \sum_{uv\notin E(G)} \frac{2}{d_G(u) + d_G(v)}.$$

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Motivated by the studies on the coindices, like the first Zagreb coindex and harmonic coindex, the symmetric division deg coindex is proposed here:

$$\overline{SDD}(G) = \sum_{uv \notin E(G)} \frac{d_G(u)^2 + d_G(v)^2}{d_G(u)d_G(v)}.$$

In this paper, upper bounds on the symmetric division deg coindex of edge corona product of two graphs and Mycielskian of a graph are given. Also, it is proved that the symmetric division deg coindex of the double graph of a connected graph G with n vertices can be written in terms of the symmetric division deg coindex of G and n.

2. Mathematical properties of $\overline{SDD}$

A modification of the second Zagreb index was proposed by Nikolić et al. [7] in 2003. The modified second Zagreb index and its coindex of G are, respectively, defined as

$$M_2^*(G) = \sum_{uv \in E(G)} \frac{1}{d_G(u)d_G(v)}$$

and

$$\overline{M}_2^*(G) = \sum_{uv \notin E(G)} \frac{1}{d_G(u)d_G(v)}.$$

The inverse degree index of G is defined as

$$ID(G) = \sum_{u \in V(G)} \frac{1}{d_G(u)}.$$

The redefined first and second Zagreb coindices of G are, respectively, defined as

$$\overline{RZ}_1(G) = \sum_{uv \notin E(G)} \frac{d_G(u) + d_G(v)}{d_G(u)d_G(v)}$$

and

$$\overline{RZ}_2(G) = \sum_{uv \notin E(G)} \frac{d_G(u)^2 + d_G(v)^2}{d_G(u) + d_G(v)}.$$

In this section, we present upper bounds on the symmetric division deg coindex of edge corona product graph, Mycielskian of a graph and double graphs.

The edge corona product $G \bullet H$ of G and H is defined as the graph obtained by taking one copy of G and $|E(G)|$ copies of H, and then joining two end vertices of the i^{th} edge of G to every vertex in the i^{th} copy of H (see [1,3,8] for more details).

Lemma 2.1. [6] *Let f be a convex function on the interval I and $x_1, x_2, \ldots, x_n \in I$. Then*

$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \ldots f(x_n)}{n},$$

with equality if and only if $x_1 = x_2 = \ldots = x_n$.

Theorem 2.1. *Let G_1 and G_2 be two graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\overline{SDD}(G_1 \bullet G_2) \leq \overline{SDD}(G_1) + m_1\Big(\overline{SDD}(G_2) + \overline{RZ_1}(G_2) + 8\overline{M}_2^*(G_2) + \overline{RZ_2}(G_2) + \overline{H}(G_2)\Big) + \frac{2n_1(n_2+m_2)}{(n_2+1)}(ID(G_1) - 1)$$
$$+ \Big((n_2+1)(2n_1m_1 - M_1(G_1)) + 2m_1(m_1-1)(m_2+1)\Big)ID(G_2) + 2m_1(m_2^2 - m_2 - 2n_2).$$

Proof. Let x_{ij} be the jth vertex in the ith copy of H, $i \in \{1, 2, \ldots, m_1\}$, $j \in \{1, 2, \ldots, n_2\}$, and let y_k be the kth vertex in G_1, $k \in \{1, 2, \ldots, n_1\}$. Also let x_j be the jth vertex in G_2. By the definition of edge corona of G_1 and G_2, for each vertex x_{ij}, we have $d_{G_1 \bullet G_2}(x_{ij}) = d_{G_2}(x_j) + 2$, and for every vertex y_k in G_1, $d_{G_1 \bullet G_2}(y_k) = d_{G_1}(y_k)n_2 + d_{G_1}(y_k) = (n_2+1)d_{G_1}(y_k)$.

Now, we consider the following four cases of nonadjacent vertex pairs in $G_1 \bullet G_2$.

Case 1. The nonadjacent vertex pairs $\{x_{ij}, x_{ih}\}$, $1 \leq i \leq m_1$, $1 \leq j < h \leq n_2$, are considered and it is assumed that $x_jx_h \notin E(G_2)$.

$$S_1 = \sum_{i=1}^{m_1} \sum_{x_{ij}x_{ih} \notin E(G_1 \bullet G_2)} \frac{d_{G_1 \bullet G_2}(x_{ij})^2 + d_{G_1 \bullet G_2}(x_{ih})^2}{d_{G_1 \bullet G_2}(x_{ij})d_{G_1 \bullet G_2}(x_{ih})}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \frac{(d_{G_2}(x_j)+2)^2 + (d_{G_2}(x_h)+2)^2}{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)} \\
&= \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \frac{(d_{G_2}(x_j)^2 + d_{G_2}(x_h)^2) + 4(d_{G_2}(x_h)+d_{G_2}(x_j)) + 8}{d_{G_2}(x_j)d_{G_2}(x_h) + 2(d_{G_2}(x_j)+d_{G_2}(x_h)) + 4}.
\end{aligned}
$$

By Lemma 2.1, we have

$$
\frac{1}{d_{G_2}(x_j)d_{G_2}(x_h) + 2(d_{G_2}(x_j)+d_{G_2}(x_h)) + 4} \le \frac{1}{4d_{G_2}(x_j)d_{G_2}(x_h)} + \frac{1}{8(d_{G_2}(x_j)+d_{G_2}(x_h)) + 16}
$$

with equality if and only if $d_{G_2}(x_j)d_{G_2}(x_h) = 8(d_{G_2}(x_j)+d_{G_2}(x_h)) + 16$. Thus,

$$
\begin{aligned}
S_1 \le\ & \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \Bigg(\frac{(d_{G_2}(x_j)^2 + d_{G_2}(x_h)^2) + 4(d_{G_2}(x_h)+d_{G_2}(x_j)) + 8}{4d_{G_2}(x_j)d_{G_2}(x_h)} \\
&+ \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \frac{(d_{G_2}(x_j)^2 + d_{G_2}(x_h)^2) + 4(d_{G_2}(x_h)+d_{G_2}(x_j)) + 8}{8(d_{G_2}(x_j)+d_{G_2}(x_h)) + 16} \Bigg).
\end{aligned}
$$

Note that $8(d_{G_2}(x_j)+d_{G_2}(x_h)) + 16 \ge d_{G_2}(x_j)+d_{G_2}(x_h)$. This implies

$$
\frac{1}{8(d_{G_2}(x_j)+d_{G_2}(x_h)) + 16} \le \frac{1}{d_{G_2}(x_j)+d_{G_2}(x_h)}.
$$

Therefore,

$$
\begin{aligned}
S_1 \le\ & \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \Bigg(\frac{(d_{G_2}(x_j)^2 + d_{G_2}(x_h)^2) + 4(d_{G_2}(x_h)+d_{G_2}(x_j)) + 8}{d_{G_2}(x_j)d_{G_2}(x_h)} \\
&+ \sum_{i=1}^{m_1} \sum_{x_j x_h \notin E(G_2)} \frac{(d_{G_2}(x_j)^2 + d_{G_2}(x_h)^2) + 4(d_{G_2}(x_h)+d_{G_2}(x_j)) + 8}{d_{G_2}(x_j)+d_{G_2}(x_h)} \Bigg) \\
=\ & m_1\Big(\overline{SDD}(G_2) + \overline{RZ_1}(G_2) + 8\overline{M}_2^*(G_2) + \overline{RZ_2}(G_2) + \overline{H}(G_2) + 4\overline{m_2}\Big).
\end{aligned}
$$

Case 2. The nonadjacent vertex pairs $\{y_k, y_s\}$, $1 \le k < s \le n_1$, are considered and it is assumed that $y_k y_s \notin E(G_1)$. Thus,

$$
\begin{aligned}
S_2 &= \sum_{y_k y_s \notin E(G_1 \bullet G_2)} \frac{d_{G_1\bullet G_2}(y_k)^2 + d_{G_1\bullet G_2}(y_s)^2}{d_{G_1\bullet G_2}(y_k)d_{G_1\bullet G_2}(y_s)} \\
&= \sum_{y_k y_s \notin E(G_1)} \frac{(n_2+1)^2 d_{G_1}(y_k)^2 + (n_2+1)^2 d_{G_1}(y_s)^2}{(n_2+1)^2 d_{G_1}(y_k)d_{G_1}(y_s)} \\
&= \sum_{y_k y_s \notin E(G_1)} \frac{d_{G_1}(y_k)^2 + d_{G_1}(y_s)^2}{d_{G_1}(y_k)d_{G_1}(y_s)} \\
&= \overline{SDD}(G_1).
\end{aligned}
$$

Case 3. The nonadjacent vertex pairs $\{x_{ij}, y_k\}$, $1 \le i \le m_1$, $1 \le j \le n_2$, $1 \le k \le n_1$, are considered and it is assumed that the ith edge e_i, $1 \le i \le m_1$, in G_1 does not pass through y_k. Note that each vertex y_k is adjacent to all vertices of $d_{G_1}(y_k)$ copies of G_2, that is, each y_k is not adjacent to any vertex of $m_1 - d_{G_1}(y_k)$ copies of G_2. Hence,

$$
\begin{aligned}
S_3 &= \sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k)) \sum_{j=1}^{n_2} \frac{(d_{G_2}(x_j)+2)^2 + (n_2+1)^2 d_{G_1}(y_k)^2}{(d_{G_2}(x_j)+2)(n_2+1)d_{G_1}(y_k)} \\
&\le \sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k)) \sum_{j=1}^{n_2} \left(\frac{(d_{G_2}(x_j)+2)}{(n_2+1)d_{G_1}(y_k)} + \frac{(n_2+1)d_{G_1}(y_k)}{d_{G_2}(x_j)} \right) \\
&= \sum_{k=1}^{n_1}(n_1 - d_{G_1}(y_k)) \left(\frac{2(n_2+m_2)}{(n_2+1)d_{G_1}(y_k)} + (n_2+1)d_{G_1}(y_k) ID(G_2) \right) \\
&= \frac{2n_1(n_2+m_2)}{(n_2+1)} ID(G_1) + 2n_1 m_1 (n_2+1) ID(G_2) - \frac{2n_1(n_2+m_2)}{(n_2+1)} - (n_2+1)M_1(G_1)ID(G_2)
\end{aligned}
$$

$$= \frac{2n_1(n_2+m_2)}{(n_2+1)}(ID(G_1)-1)+(n_2+1)(2n_1m_1-M_1(G_1))ID(G_2).$$

Case 4. The nonadjacent vertex pairs $\{x_{ij}, x_{\ell h}\}$, $1 \le i < \ell \le m_1$, $1 \le j, h \le n_2$, are considered.

$$\begin{aligned} S_4 &= \sum_{x_{ij}x_{\ell h}\notin E(G_1\bullet G_2)} \frac{d_{G_1\bullet G_2}(x_{ij})^2 + d_{G_1\bullet G_2}(x_{\ell h})^2}{d_{G_1\bullet G_2}(x_{ij})d_{G_1\bullet G_2}(x_{\ell h})} \\ &= \frac{m_1(m_1-1)}{2}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2} \frac{(d_{G_2}(x_j)+2)^2+(d_{G_2}(x_h)+2)^2}{(d_{G_2}(x_j)+2)(d_{G_2}(x_h)+2)} \\ &\le \frac{m_1(m_1-1)}{2}\sum_{j=1}^{n_2}\sum_{h=1}^{n_2}\left(\frac{(d_{G_2}(x_j)+2)}{d_{G_2}(x_h)}+\frac{(d_{G_2}(x_h)+2)}{d_{G_2}(x_j)}\right) \\ &= 2m_1(m_1-1)(m_2+1)ID(G_2). \end{aligned}$$

From the above four cases of nonadjacent vertex pairs, one obtains the desired result. □

The Mycielskian $\mu(G)$ (see [5]) of G contains G itself as an isomorphic subgraph, together with $n+1$ additional vertices: a vertex u_i corresponding to each vertex v_i of G, and another vertex w. Each vertex u_i is connected by an edge to w, so that these vertices form a subgraph in the form of a star $K_{1,n}$. The minimum and maximum vertex degrees of G, respectively, are denoted by $\delta(G)$ and $\Delta(G)$.

By the definition of the Mycielskian of a graph G, for each edge v_iv_j of G, the Mycielskian of G includes two edges, u_iv_j and v_iu_j. Now, we find an upper bound for symmetric division deg coindex of the Mycielskian of a graph.

Theorem 2.2. *Let G be a graph on n vertices and m edges. Then*

$$\overline{SDD}(\mu(G)) \le \left(\frac{n(n-1)-2(m-1)}{2}\right)\overline{SDD}(G) + \left(\frac{n(n-1)-2m}{2}\right)(2\overline{RZ}_1(G)+\overline{M}_2^*(G)+\overline{RZ}_2(G)+\overline{H}(G)+n(n-1)-m)$$
$$+m\Big(SDD(G)+M_2^*(G)+RZ_2(G)+H(G)+2(m+n)\Big)+(n^2+2n+2)ID(G)+(3n+8m)+\frac{3\Delta^2+2\Delta+1}{\delta^2}.$$

Proof. Let $V(G)=\{v_1,\dots,v_n\}$ and let $V(\mu(G))=\{v_1,\dots,v_n,u_1,\dots,u_n,w\}$. By the structure of the Mycielskian of G, if $v_iv_j \notin E(G)$, then $v_iu_j \notin E(G)$, and $v_ju_i \notin E(G)$. By the definition of $\mu(G)$, for each $i \in \{1,2,\dots,n\}$, we have $d_{\mu(G)}(v_i) = 2d_G(v_i)$, $d_{\mu(G)}(u_i) = d_G(v_i)+1$ and $d_{\mu(G)}(w) = n$. Now, we consider the following cases of nonadjacent vertex pairs in $\mu(G)$.

Case 1. The nonadjacent vertex pairs $\{v_i, v_j\}$ in $\mu(G)$ are considered.

$$C_1 = \sum_{v_iv_j\notin E(\mu(G))}\frac{d_{\mu(G)}(v_i)^2+d_{\mu(G)}(v_j)^2}{d_{\mu(G)}(v_i)d_{\mu(G)}(v_j)} = \sum_{v_iv_j\notin E(G)}\frac{4d_G(v_i)^2+4d_G(v_j)^2}{4d_G(v_i)d_G(v_j)} = \overline{SDD}(G).$$

Case 2. The nonadjacent vertex pairs $\{u_i, u_j\}$ in $\mu(G)$ are considered.

Case 2.1. $u_iu_j \notin E(\mu(G))$ and $v_iv_j \notin E(G)$.

$$\begin{aligned} C_2' &= \sum_{u_iu_j\notin E(\mu(G))}\frac{d_{\mu(G)}(u_i)^2+d_{\mu(G)}(u_j)^2}{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)} \\ &= \sum_{v_iv_j\notin E(G)}\frac{(d_G(v_i)+1)^2+(d_G(v_j)+1)^2}{(d_G(v_i)+1)(d_G(v_j)+1)} \\ &= \sum_{v_iv_j\notin E(G)}\frac{(d_G(v_i)^2+d_G(v_j)^2)+2(d_G(v_i)+d_G(v_j))+2}{d_G(v_i)d_G(v_j)+(d_G(v_i)+d_G(v_j))+1}. \end{aligned}$$

By Lemma 2.1, one obtains

$$\begin{aligned} C_2' &\le \sum_{v_iv_j\notin E(G)}\frac{(d_G(v_i)^2+d_G(v_j)^2)+2(d_G(v_i)+d_G(v_j))+2}{4d_G(v_i)d_G(v_j)} \\ &\quad+\sum_{v_iv_j\notin E(G)}\frac{(d_G(v_i)^2+d_G(v_j)^2)+2(d_G(v_i)+d_G(v_j))+2}{4(d_G(v_i)+d_G(v_j))+1)} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)^2 + d_G(v_j)^2) + 2(d_G(v_i) + d_G(v_j)) + 2}{d_G(v_i)d_G(v_j)} \\
&\quad + \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)^2 + d_G(v_j)^2) + 2(d_G(v_i) + d_G(v_j)) + 2}{d_G(v_i) + d_G(v_j)} \\
&= \overline{SDD}(G) + 2\overline{RZ}_1(G) + \overline{M}_2^*(G) + \overline{RZ}_2(G) + \overline{H}(G) + n(n-1) - m.
\end{aligned}
$$

Case 2.2. $u_iu_j \notin E(\mu(G))$ and $v_iv_j \in E(G)$.

$$
\begin{aligned}
C_2'' &= \sum_{u_iu_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)^2 + d_{\mu(G)}(u_j)^2}{d_{\mu(G)}(u_i)d_{\mu(G)}(u_j)} \\
&= \sum_{v_iv_j \in E(G)} \frac{(d_G(v_i)+1)^2 + (d_G(v_j)+1)^2}{(d_G(v_i)+1)(d_G(v_j)+1)} \\
&\leq SDD(G) + M_2^*(G) + RZ_2(G) + H(G) + 2(m+n).
\end{aligned}
$$

If $u_iu_j \notin E(\mu(G))$, then there are m edges $v_iv_j \in E(G)$ and $\frac{n(n-1)}{2} - m$ nonadjacent vertex pairs $\{v_i, v_j\}$ in G as well as $\mu(G)$. By Cases 2.1 and 2.2, we have the contribution of nonadjacent vertex pair of Case 2 given as

$$
\begin{aligned}
C_2 &= \Big(\frac{n(n-1)}{2} - m\Big)C_2' + mC_2'' \\
&\leq \Big(\frac{n(n-1)}{2} - m\Big)\Big(\overline{SDD}(G) + 2\overline{RZ}_1(G) + \overline{M}_2^*(G) + \overline{RZ}_2(G) + \overline{H}(G) + n(n-1) - m\Big) \\
&\quad + m\Big(SDD(G) + M_2^*(G) + RZ_2(G) + H(G) + 2(m+n)\Big).
\end{aligned}
$$

Case 3. The nonadjacent vertex pairs $\{u_i, v_i\}$ in $\mu(G)$ are considered for each $i = 1, 2, \dots, n$.

$$
\begin{aligned}
C_3 &= \sum_{i=1}^{n} \frac{d_{\mu(G)}(u_i)^2 + d_{\mu(G)}(v_i)^2}{d_{\mu(G)}(u_i)d_{\mu(G)}(v_i)} \\
&= \sum_{i=1}^{n} \frac{(d_G(v_i)+1)^2 + 4d_G(v_i)^2}{2(d_G(v_i)+1)(d_G(v_i))} \\
&\leq \sum_{i=1}^{n} \left(\frac{d_G(v_i)+1}{d_G(v_i)} + 2\right).
\end{aligned}
$$

Thus, $C_3 \leq ID(G) + 3n$.

Case 4. The nonadjacent vertex pairs $\{u_i, v_j\}$ in $\mu(G)$ are considered.

$$
\begin{aligned}
C_4 &= \sum_{u_iv_j \notin E(\mu(G))} \frac{d_{\mu(G)}(u_i)^2 + d_{\mu(G)}(v_j)^2}{d_{\mu(G)}(u_i)d_{\mu(G)}(v_j)} \\
&= \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)+1)^2 + 4d_G(v_j)^2}{2(d_G(v_i)+1)d_G(v_j)}.
\end{aligned}
$$

For any vertex $x \in V(G)$, one has $\delta(G) \leq d_G(x) \leq \Delta(G)$ and hence

$$
\begin{aligned}
C_4 &\leq \sum_{v_iv_j \notin E(G)} \frac{(d_G(v_i)+1)^2 + 4d_G(v_j)^2}{d_G(v_i)d_G(v_j)} \\
&\leq \overline{SDD}(G) + \frac{3\Delta^2 + 2\Delta + 1}{\delta^2}.
\end{aligned}
$$

Case 5. The nonadjacent vertex pairs $\{w, v_i\}$ in $\mu(G)$ are considered for each $i = 1, 2, \dots, n$.

$$
\begin{aligned}
C_5 &= \sum_{v_iw \notin E(\mu(G))} \frac{d_{\mu(G)}(v_i)^2 + d_{\mu(G)}(w)^2}{d_{\mu(G)}(v_i)d_{\mu(G)}(w)} \\
&= \sum_{v_i \in V(G)} \frac{4d_G(v_i)^2 + (n+1)^2}{2d_G(v_i)(n+1)}
\end{aligned}
$$

$$\begin{aligned} &\leq \sum_{v_i \in V(G)} \frac{4d_G(v_i)^2 + (n+1)^2}{d_G(v_i)} \\ &= (n+1)^2 ID(G) + 8m. \end{aligned}$$

From the above five cases of nonadjacent vertex pairs, one obtains the desired result. □

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. The vertices of the double graph G^* are given by the two sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Thus for each vertex $v_i \in V(G)$, there are two vertices x_i and y_i in $V(G^*)$. The double graph G^* includes the initial edge set of each copy of G, and for any edge $v_iv_j \in E(G)$, two more edges x_iy_j and x_jy_i are added. For a given vertex v in G, let

$$T_G(v) = \sum_{uv \notin E(G)} \frac{d_G(u)^2 + d_G(v)^2}{d_G(u)d_G(v)}.$$

Now, we find the exact value of the symmetric division deg coindex for the double graph of a given graph.

Theorem 2.3. *Let G be a connected graph with n vertices. Then $\overline{SDD}(G^*) = 4\overline{SDD}(G) + 2n$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in G^*, of v_i for each $i \in \{1, 2, \ldots, n\}$. For any given vertex v_i in G and its clone vertices x_i and y_i, $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$ by the definition of the double graph. For $v_i, v_j \in V(G)$, if $v_iv_j \notin E(G)$, then $x_ix_j \notin E(G)$, $y_iy_j \notin E(G)$, $x_iy_j \notin E(G)$ and $y_ix_j \notin E(G)$. Hence, we only consider total contribution of the following three types of nonadjacent vertex pairs to calculate $\overline{SDD}(G)$.

Case 1. The nonadjacent vertex pairs $\{x_i, x_j\}$ and $\{y_i, y_j\}$ are considered, where $v_iv_j \notin E(G)$.

$$\begin{aligned} \sum_{y_iy_j \notin E(G^*)} \frac{d_{G^*}(y_i)^2 + d_{G^*}(y_j)^2}{d_{G^*}(y_i)d_{G^*}(y_j)} &= \sum_{x_ix_j \notin E(G^*)} \frac{d_{G^*}(x_i)^2 + d_{G^*}(x_j)^2}{d_{G^*}(x_i)d_{G^*}(x_j)} \\ &= \sum_{v_iv_j \notin E(G)} \frac{4d_G(v_i)^2 + 4d_Gv_j)^2}{4d_G(v_i)d_Gv_j)} \\ &= \overline{SDD}(G). \end{aligned}$$

Case 2. The nonadjacent vertex pairs $\{x_i, y_i\}$ are considered for each $i \in \{1, 2, \ldots, n\}$.

$$\sum_{i=1}^{n} \frac{d_{G^*}(x_i)^2 + d_{G^*}(y_i)^2}{d_{G^*}(x_i)d_{G^*}(y_i)} = \sum_{i=1}^{n} \frac{4d_G(v_i)^2 + 4d_G(v_i)^2}{4d_G(v_i)^2} = 2n.$$

Case 3. The nonadjacent vertex pairs $\{x_i, y_j\}$ and $\{y_i, x_j\}$ are considered, where $v_iv_j \notin E(G)$.
For each x_i, there exist $n-1-d_G(v_i)$ vertices in the set $\{y_1, y_2, \ldots, y_n\}$, among which every vertex together with x_i compose a nonadjacent vertex pairs of G^*. The total contribution of these $n-1-d_G(v_i)$ nonadjacent vertex pairs to calculate $\overline{SDD}(G^*)$ is

$$\sum_{x_iy_j \notin E(G^*)} \frac{d_{G^*}(x_i)^2 + d_{G^*}(y_j)^2}{d_{G^*}(x_i)d_{G^*}(y_j)} = \sum_{v_iv_j \notin E(G^*)} \frac{4d_G(v_i)^2 + 4d_G(v_j)^2}{4d_G(v_i)d_G(v_j)} = T_G(v_i).$$

Hence,

$$\sum_{i \neq j,\ x_iy_j \notin E(G^*)} \frac{d_{G^*}(x_i)2 + d_{G^*}(y_j)^2}{d_{G^*}(x_i)d_{G^*}(y_j)} = \sum_{i=1}^{n} T_G(v_i) = 2\overline{SDD}(G).$$

Therefore,

$$\begin{aligned} \overline{SDD}(G^*) &= \sum_{x_ix_j \notin E(G^*)} \frac{d_{G^*}(x_i)^2 + d_{G^*}(x_j)^2}{d_{G^*}(x_i)d_{G^*}(x_j)} + \sum_{y_iy_j \notin E(G^*)} \frac{d_{G^*}(y_i)^2 + d_{G^*}(y_j)^2}{d_{G^*}(y_i)d_{G^*}(y_j)} \\ &\quad + \sum_{i=1}^{n} \frac{d_{G^*}(x_i)2 + d_{G^*}(y_i)^2}{d_{G^*}(x_i)d_{G^*}(y_i)} + \sum_{i \neq j,\ x_iy_j \notin E(G^*)} \frac{d_{G^*}(x_i)^2 + d_{G^*}(y_j)^2}{d_{G^*}(x_i)d_{G^*}(y_j)} \\ &= 4\overline{SDD}(G) + 2n. \end{aligned}$$

□

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