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Research Article

On irregular and antiregular domination in graphs

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Abstract

A set S of vertices in a connected graph G of diameter d is an irregular dominating set if it is possible to assign distinct labels from the set $\{1, 2, \ldots, d\}$ to the vertices of S in such a way that for every vertex v of G, there exists a vertex u of S such that the distance from v to u is the label of u. If exactly two vertices of S are permitted to have the same label, then S is an antiregular dominating set. Several classes of graphs are investigated to determine whether they have irregular or antiregular dominating sets, with the primary emphasis on trees. For graphs possessing these sets, the minimum size of such a set is also studied.

Keywords: distance; vertex orbits; domination; irregular domination; antiregular domination.

2020 Mathematics Subject Classification: 05C05, 05C12, 05C15, 05C69.

1. Introduction

In recent decades, domination in graphs has become a popular area of study. While this area evidently began with the work of Berge [2] in 1958 and Ore [13] in 1962, domination did not become an active area of research until 1977 with the appearance of a survey paper by Cockayne and Hedetniemi [8]. Since then, a large number of variations of domination have surfaced and provided numerous applications to different areas of science and real-life problems (see [10, 11]). For a vertex v in a nontrivial connected graph G, let $N(v)$ denote the *neighborhood* of v and $N[v] = \{v\} \cup N(v)$ the *closed neighborhood* of v. A vertex v in a graph G is said to *dominate* a vertex u if either $u = v$ or $uv \in E(G)$. That is, a vertex v dominates the vertices in its closed neighborhood $N[v]$. A set S of vertices in G is a *dominating set* of G if every vertex of G is dominated by at least one vertex in S. The minimum number of vertices in a dominating set of G is the *domination number* $\gamma(G)$ of G.

Of the many variations of domination that have been introduced, probably the most common and most studied is total domination, introduced by Cockayne, Dawes and Hedetniemi [7]. In total domination, a vertex u dominates a vertex v in a graph G if uv is an edge of G and so a vertex does not dominate itself. It is this manner of domination that we use here, that is, in this paper domination is total domination. A set S of vertices in a graph G is a *total dominating set* of G if for every vertex v of G, there is a vertex $u \in S$ such that u dominates v. The minimum cardinality of a total dominating set of G is the *total domination number* $\gamma_t(G)$ of G. A graph G has a total domination number if and only if G has no isolated vertices. The book by Henning and Yeo [12] deals exclusively with total domination in graphs. Here we only consider nontrivial connected graphs.

Total domination and some other types of domination can be described with the aid of distance in graphs. We denote the *distance* (the length of a shortest path) between two vertices u and v in a graph G by $d(u, v)$. The greatest distance from a vertex v to a vertex of G is its *eccentricity*, denoted by $e(v)$. The minimum eccentricity among the vertices of G is the *radius* $\text{rad}(G)$ of G and the maximum eccentricity is the *diameter* $\text{diam}(G)$. Therefore, the diameter of G is the maximum distance between any two vertices of G. In total domination, a vertex u dominates a vertex v if $d(u, v) = 1$. For a total dominating set S in a nontrivial connected graph G, one can think of assigning each vertex of S the label 1 and assigning no label to the vertices of G not in S. Thus, if $u \in S$, then u is labeled 1, indicating that u dominates all vertices of G whose distance from u is 1. Thus, every vertex of G has distance 1 from a vertex of S.

In [9], a generalization of (total) domination was introduced called *orbital domination*. For a positive integer r and a vertex v in a connected graph G, the *r-orbit* $O_r(v)$ of v is $O_r(v) = \{u \in V(G) : d(u, v) = r\}$. A set $S = \{u_1, u_2, \ldots, u_k\}$ of vertices in a nontrivial connected graph G is an *orbital dominating set* of G if each vertex $u_i \in S$ can be labeled with a

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positive integer r_i, where $r_i \leq e(u_i)$, such that $\bigcup_{i=1}^{k} O_{r_i}(u_i) = V(G)$. Thus, if S is an orbital dominating set of G, then for every vertex v of G, there exists a vertex u_i in S such that $d(u_i, v) = r_i$. Here, u_i is said to *dominate* v. The minimum cardinality of an orbital dominating set is denoted by $\gamma_o(G)$, called the *orbital domination number* of G. This concept has been studied further in [5].

If all labels of an orbital dominating set S are the same positive integer r, then S is an r*-regular orbital dominating set*. It was shown in [9] that a nontrivial connected graph G has an r-regular orbital dominating set if and only if $1 \leq r \leq \mathrm{rad}(G)$. If $r = 1$, then S is a *total dominating set*. At the other extreme, if no two vertices of an orbital dominating set S have the same label, then S is an *irregular orbital dominating set* or, more simply, an *irregular dominating set*. This concept was introduced and studied in [4] and studied further in [3, 6]. While every nontrivial connected graph has an orbital dominating set (indeed, a total dominating set), not every graph has an irregular dominating set.

Proposition 1.1. [6] *No connected vertex-transitive graph has an irregular dominating set.*

If G is a graph possessing an irregular dominating set, then the minimum cardinality of such a set in G is the *irregular domination number* $\tilde{\gamma}(G)$ of G. Thus, $\gamma_o(G) \leq \tilde{\gamma}(G)$ for every graph G possessing an irregular dominating set. If S is an orbital dominating set where exactly two vertices of S have the same label, then S is an *antiregular orbital dominating set* or, more simply, an *antiregular dominating set*. If a graph has an irregular dominating set, then it also has an antiregular dominating set, but not conversely. If a graph G possesses an antiregular dominating set, then the minimum cardinality of such a set in G is the *antiregular domination number* $\gamma_A(G)$ of G.

The primary goal here concerns investigating graphs that possess irregular or antiregular dominating sets and determining the irregular domination and antiregular domination numbers of these graphs. These concepts are members of classes of concepts which are opposite in a sense to those that deal with regularity in graphs. Many of these irregularity topics are discussed in [1].

While there are well-known classes of graphs that do not possess irregular dominating sets, including vertex-transitive graphs as mentioned in Proposition 1.1, such sets exist in nearly all trees.

Theorem 1.1. [4] *A nontrivial tree T has an irregular dominating set if and only if T is neither a star nor a path of order* 2 *or* 6.

2. Graphs of small diameter

First, we investigate the irregular and antiregular domination numbers of graphs of small diameter. If $S = \{v_1, v_2, \ldots, v_k\}$ is an orbital dominating set in a graph G and f is the corresponding labeling, then $(f(v_1), f(v_2), \ldots, f(v_k))$ is called an *orbital sequence* of G. An orbital sequence of length $\gamma_o(G)$ is a *minimum orbital sequence*. It is easy to see that $1, 1$ is the only orbital sequence of length 2 in any nontrivial connected graph. Thus, we have the following observation.

Observation 2.1. *Let G be a nontrivial connected graph. Then $\gamma_o(G) = 2$ if and only if $\gamma_A(G) = 2$ and so $\gamma_A(G) = 2$ if and only if $\gamma_t(G) = 2$. Consequently, if $\gamma_A(G) = 2$, then $\mathrm{diam}(G) \leq 3$.*

Chartrand, Henning, and Schultz [5] obtained the following two results concerning minimum orbital sequences of length 3.

Theorem 2.1. [5] *The only minimum orbital sequences (r_1, r_2, r_3) where $r_1 \leq r_2 \leq r_3$ of length* 3 *in connected graphs are* $(1, 1, 1)$, $(1, 1, 2)$, $(1, 1, 3)$, $(1, 1, 4)$, $(1, 2, 2)$, $(1, 2, 3)$, $(2, 2, 2)$, $(2, 2, 3)$, $(2, 3, 3)$, *and* $(3, 3, 3)$.

Theorem 2.2. [5] *The only minimum orbital sequences (r_1, r_2, r_3), $r_1 \leq r_2 \leq r_3$, of length* 3 *in trees are* $(1, 1, 1)$, $(1, 1, 3)$, $(1, 1, 4)$, *and* $(1, 2, 3)$.

If $S = \{v_1, v_2, \ldots, v_k\}$ is an irregular dominating set in a graph G and f is the corresponding labeling, then $(f(v_1), f(v_2), \ldots, f($ is called an *irregular dominating sequence* or, more simply, an *irregular sequence* of G. An irregular sequence of length $\tilde{\gamma}(G)$ is a *minimum irregular sequence*. An *antiregular sequence* and a *minimum antiregular sequence* of a graph are defined similarly. The following are consequences of Theorems 2.1 and 2.2.

Corollary 2.1. *If s is an irregular sequence of length* 3 *in a connected graph, then the three terms of s are* $1, 2, 3$.

Corollary 2.2. *If $s = (r_1, r_2, r_3)$ is a minimum antiregular sequence of length* 3 *in a connected graph G, where $r_1 \leq r_2 \leq r_3$, then s is one of* $(1, 1, 2)$, $(1, 1, 3)$, $(1, 1, 4)$, $(1, 2, 2)$, $(2, 2, 3)$, *and* $(2, 3, 3)$. *Furthermore, if G is a tree, then s is either* $(1, 1, 3)$ *or* $(1, 1, 4)$.

If G is a connected graph of diameter 2 with $\gamma_A(G) = 3$, then there is no restriction on the structure of the subgraph of G induced by an antiregular dominating set of size 3.

Observation 2.2. *For every graph $F \in \{\overline{K}_3, K_2 + K_1, P_3, K_3\}$ of order 3, there exists a connected graph G of diameter 2 with $\gamma_A(G) = 3$ for which G has an antiregular dominating set S such that $G[S] \cong F$.*

Observation 2.2 can be illustrated with a single graph. The graph G of Figure 1 has diameter 2 and $\gamma_A(G) = 3$ and the subgraphs of G induced by the four minimum antiregular dominating sets shown in that figure are the four non-isomorphic graphs of order 3. The following result characterizes those connected bipartite graphs having irregular domination number 3.

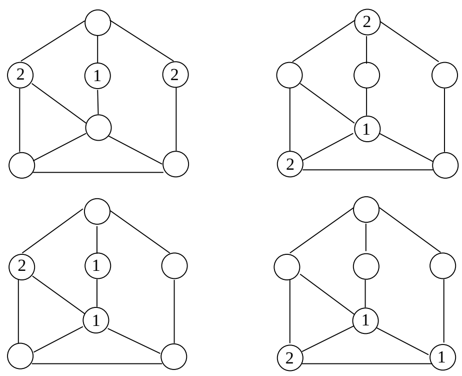


Figure 1: Four minimum antiregular dominating sets in a graph G.

Theorem 2.3. *Let G be a connected bipartite graph. Then $\tilde{\gamma}(G) = 3$ if and only if (i) some vertex in a partite set U of G is adjacent to every vertex in the other partite set W and (ii) there exists a vertex $w \in W$ that is not adjacent to a vertex $u \in U$ but $N(w) \subseteq O_2(u)$.*

Proof. First, assume that there is $v \in U$ that is adjacent to every vertex in W and there is $w \in W$ that is not adjacent to some vertex $u \in U$ and $N(w) \subseteq O_2(u)$. In particular, $v \in N(w)$ and $d(u, w) = 3$. Define a labeling f of G by assigning the label 1 to v, the label 2 to u, and the label 3 to w. Then each vertex of W is dominated by v, each vertex of $N(w) \subseteq U$ is dominated by u, and every vertex in $U - N(w)$ is dominated by w. Thus, f is an irregular dominating labeling of G and so $\tilde{\gamma}(G) = 3$.

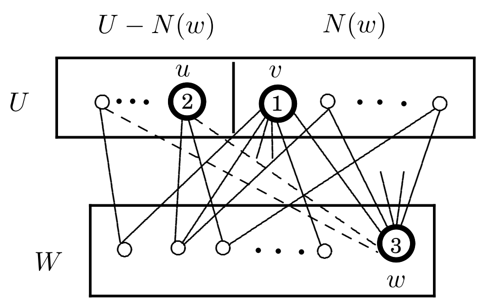


Figure 2: A step in the proof of Theorem 2.3.

We now establish the converse. Let G be a connected bipartite graph with $\tilde{\gamma}(G) = 3$. Since $\tilde{\gamma}(G) = 3$, it follows by Corollary 2.1 that there is an irregular dominating labeling g of G that assigns $1, 2, 3$ to three vertices of G. If x is a labeled vertex and $g(x) = r$, then (i) $O_r(x) \subseteq U$ or $O_r(x) \subseteq W$ and (ii) x and $O_r(x)$ belong to the same partite set if and only if r is even. Since there are exactly three labeled vertices of G, some labeled vertex must dominate all vertices in one partite set of G and the other two labeled vertices dominate all vertices in the other partite set of G. We may assume that v is a labeled vertex that dominates all vertices of W. Since v does not dominate itself, $v \in U$. Because every neighbor of v belongs to W and the distance from v to these vertices is 1, the vertex v must be labeled 1, implying that v is adjacent to every vertex of W. At this stage, every vertex of W is dominated, in fact by v, and no vertex of U is dominated. In particular, v is not dominated.

Because every vertex of W is adjacent to v and the label 1 has already been assigned, no vertex of W can dominate v. Therefore, v can only be dominated by a vertex of U. Since the distance from a vertex of $U - \{v\}$ to v is 2, some vertex of $U - \{v\}$ must be labeled 2. Let $u \in U - \{v\}$ such that $g(u) = 2$. Thus, u dominates all vertices of U at distance 2 from u, including v. At this stage, no vertex in the set $U - O_2(u)$ is dominated. In particular, u is not dominated. By Corollary 2.1, there must be a vertex $w \in W$ that is labeled 3 and dominates the remaining vertices of U, namely all vertices in $U - O_2(u)$. Hence, $d(u,w) = 3$ and w is not adjacent to u. Since a vertex in $N(w) \subseteq U$ cannot be dominated by v or by w, it follows that each vertex in $N(w)$ is dominated by u and so $N(w) \subseteq O_2(u)$. Consequently, G has the desired property. □

Suppose that G is a connected bipartite graph with $\tilde{\gamma}(G) = 3$. By Theorem 2.3, there is a vertex v in one partite set U of G that is adjacent to all vertices in the other partite set W of G. If $x, y \in U$, where $x, y \neq v$, then $d(v,x) = d(v,y) = 2$ and so $d(x,y) = 2$ or $d(x,y) = 4$. The following corollary is therefore a consequence of Theorem 2.3.

Corollary 2.3. *If G is a connected bipartite graph with $\tilde{\gamma}(G) = 3$, then* $\operatorname{diam}(G) \in \{3,4\}$.

Figure 3 shows two graphs G with $\tilde{\gamma}(G) = 3$, one of which has diameter 3 while the other has diameter 4.

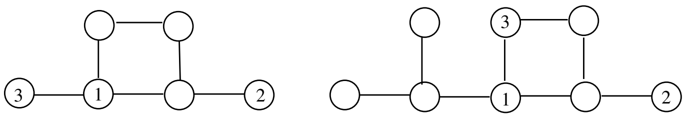


Figure 3: Two graphs G with $\tilde{\gamma}(G) = 3$.

By Observation 2.1, if G is a nontrivial connected graph, then $\gamma_A(G) = 2$ if and only if $\gamma_t(G) = 2$. For example, if $G = F \,\square\, K_2$, where F is a graph with $\gamma(F) = 1$, or $G = F \vee H$ (the join of two graphs F and H), then $\gamma_t(G) = 2$ and so $\gamma_A(G) = 2$. Every complete bipartite graph has total domination number 2 and so has an antiregular dominating set. The only connected bipartite graphs of diameter 2 and girth 4 are the complete bipartite graphs $K_{r,s}$ where $r, s \geq 2$, all of which have an antiregular dominating set. Thus, if there exists a connected graph G of diameter 2 other than C_5 that does not have an antiregular dominating set, then it must have girth 3 or 5 or it is a non-bitartite graph of girth 4. The famous Petersen graph and the 7-regular Hoffman-Singleton graph both have diameter 2 and girth 5 and each has antiregular dominating sets. In fact, for both graphs, any three neighbors of any vertex that are labeled $1, 2, 2$ is a minimum antiregular dominating set. This is shown for the Petersen graph in Figure 4. It is not known if there exists a 57-regular graph of diameter 2 and girth 5 but if such a graph exists, then it too possesses a minimum antiregular dominating set of the same type. We have the following conjecture.

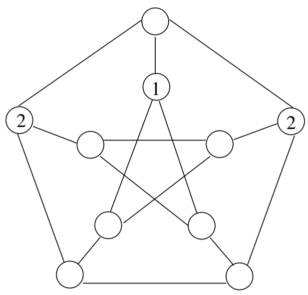


Figure 4: An antiregular dominating labeling of the Petersen graph.

Conjecture 2.1. *If G is a graph of diameter 2 and $G \neq C_5$, then G has an antiregular dominating set.*

If a graph of diameter 2 has a vertex of sufficiently small degree or sufficiently large degree, then it must have an antiregular dominating set. The following result will be useful in establishing this fact.

Lemma 2.1. *Let G be a graph of diameter 2. If G contains two adjacent vertices u and v such that either (1) every vertex different from u or v is a neighbor of u or v or (2) every neighbor of u different from v is a neighbor of v or every neighbor of v different from u is a neighbor of u, then G has an antiregular dominating set.*

Proof. First, suppose that (1) occurs. Then $\{u,v\}$ is a total dominating set of G and so $\gamma_t(G) = \gamma_A(G) = 2$ by Observation 2.1. Next, suppose that (1) does not occur but (2) occurs, say every neighbor of u different from v is a neighbor of v. Since (1) does not occur, there is a neighbor w of v that is not neighbor of u. Then the labeling $f : \{u,v,w\} \to \{1,2\}$ defined by $f(u) = 2$ and $f(v) = f(w) = 1$ is an antiregular dominating labeling of G. □

Theorem 2.4. *Let G be a graph of order $n \geq 4$ and diameter 2 such that $G \neq C_5$. If $\delta(G) \leq 2$ or $\Delta(G) \geq \max\{3, n-4\}$, then G has an antiregular dominating set.*

Proof. First, suppose that $\delta(G) \leq 2$. Since $G \neq C_5$, there is $v \in V(G)$ such that $\deg v = \delta(G)$ and v is adjacent to a vertex u of degree 3 or more. If $\deg v = 1$, then every vertex different from u is a neighbor of u and so $\{u, v\}$ is a total dominating set of G. Thus, $\gamma_A(G) = 2$ by Observation 2.1. If $\deg v = 2$, let $N(v) = \{u, w\}$. Since $\mathrm{diam}(G) = 2$, every vertex different from u or w is a neighbor of u or w. If $uw \in E(G)$, then G has an antiregular dominating set by Lemma 2.1. Thus, we may assume that $uw \notin E(G)$. First, suppose that there is a vertex x_1 different from v that is adjacent to both u and w. Since $\deg u \geq 3$, there is a neighbor x_2 of u different from x_1 such that $d(x_2, v) = 2$. Then the labeling $f : \{v, x_1, x_2\} \to \{1, 2\}$ defined by $f(x_1) = 1$ and $f(v) = f(x_2) = 2$ is an antiregular dominating labeling of G. Next, suppose that there is no vertex different from v that is adjacent to both u and w. Since $\deg u \geq 3$, there are two distinct neighbors x_1 and x_2 of u different from v. Then the labeling $f : \{v, x_1, x_2\} \to \{1, 2\}$ defined by $f(x_1) = 1$ and $f(v) = f(x_2) = 2$ is an antiregular dominating labeling of G.

Next, suppose that $\Delta(G) \geq \max\{3, n-4\}$. Let $uv \in E(G)$ such that $\deg u = \Delta(G)$ and $\deg v = \max\{\deg w : w \in N(u)\}$. By Lemma 2.1, we may assume that there is $x \in N(u) - \{v\}$ such that $x \notin N(v)$ and there is $y \in N(v) - \{u\}$ such that $y \notin N(u)$. Let X be the set of neighbors of u distinct from v that are not neighbors of v and let Y be the set of neighbors of v distinct from u that are not neighbors of u. Thus, $x \in X$ and $y \in Y$. Also, let $Z = N(u) \cap N(v)$ and let $S = V(G) - [N(u) \cup N(v)]$. By Lemma 2.1, we may assume that S is not empty. Thus, X, Y and S are not empty, while Z may be empty. Hence, $\Delta(G) = \deg u = n - 1 - (|Y| + |S|) \leq n - 3$. Since $\mathrm{diam}(G) = 2$, every vertex of S is adjacent to at least one vertex in $X \cup Z \cup Y$ and $d(u, s) = d(v, s) = 2$ for each $s \in S$. See Figure 5.

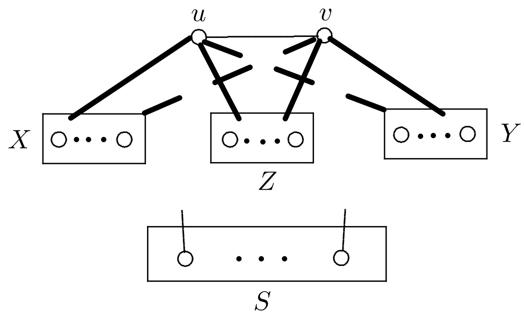


Figure 5: A step in the proof of Theorem 2.4.

(1) If there is $w \in X \cup Z$ that is adjacent to every vertex of Y or there is $w \in Z \cup Y$ that is adjacent to every vertex of X, say the latter, then the labeling $f : \{u, v, w\} \to \{1, 2\}$ defined by $f(u) = 2$ and $f(v) = f(w) = 1$ is an antiregular dominating labeling of G.

(2) If there is $w \in X \cup Z \cup Y$ such that w is not adjacent to any vertex of S, then the labeling $f : \{u, v, w\} \to \{1, 2\}$ defined by $f(u) = f(v) = 1$ and $f(w) = 2$ is an antiregular dominating labeling of G.

By (1) and (2), we may assume that G satisfies the following two conditions:

(i) For every $w \in X \cup Z$, there is $y \in Y$ such that $wy \notin E(G)$ and for every $w \in Z \cup Y$, there is $x \in X$ such that $wx \notin E(G)$.

(ii) Every vertex in $X \cup Z \cup Y$ is adjacent to some vertex of S.

Since $\Delta(G) = \deg u = n - 1 - (|Y| + |S|) \geq n - 4$ and Y and S are nonempty sets, it follows that $|Y| + |S| \leq 3$ and so either $|Y| = 1$ or $|S| = 1$. First, suppose that $|Y| = 1$. Let $Y = \{y\}$ and $|S| \in \{1, 2\}$. By (i), the vertex y is not adjacent to any vertex $x \in X$ and so $d(x, y) = 2$ for each $x \in X$. If $\deg v \geq 3$, then let $w \in N(v) - \{u, y\}$ and the labeling $f : \{u, w, y\} \to \{1, 2\}$ defined by $f(w) = 1$ and $f(u) = f(y) = 2$ is an antiregular dominating labeling of G. If $\deg v = 2$, then $Z = \emptyset$, $|X| \geq 2$, and $\deg x = 2$ for each $x \geq X$ by (ii). Let $x_1, x_2 \in X$. The labeling $f : \{v, x_1, x_2\} \to \{1, 2\}$ defined by $f(x_1) = 1$ and $f(v) = f(x_2) = 2$ is an antiregular dominating labeling of G. Next, suppose that $|S| = 1$ and $|Y| = 2$. Let $S = \{s\}$. By (ii), the vertex s is adjacent to every vertex in $X \cup Z \cup Y$. If $Z \neq \emptyset$, then $N(s) \cup N(z) = V(G)$ for each vertex $z \in Z$ and so $\gamma_t(G) = \gamma_A(G) = 2$. Hence, we may assume that $Z = \emptyset$ and so $|X| \geq 2$. Let $x_1, x_2 \in X$. Then the labeling $f : \{s, x_1, x_2\} \to \{1, 2\}$ defined by $f(s) = f(x_1) = 1$ and $f(x_2) = 2$ is an antiregular dominating labeling of G. □

Conjecture 2.2. *If G is a graph of order $n \geq 9$ and diameter 2 such that $\Delta(G) = n - 5$, then G has an antiregular dominating set.*

By Theorem 2.4, if there should exist a graph G of order $n \geq 8$ and diameter 2 that has no antiregular dominating labeling, then $3 \leq \deg v \leq n - 5$ for every vertex v of G. In the case where $n = 8$, there are two 3-regular graphs of order 8 and diameter 2 (shown in Figure 6), each of which has an antiregular dominating labeling (also shown in Figure 6). Hence, if there is a graph of diameter 2 (different from the cycle of order 5) with no antiregular dominating labeling, then it has order 9 or more.

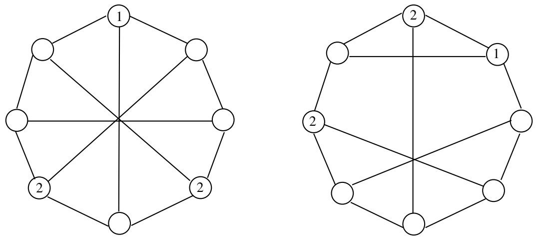


Figure 6: Two 3-regular graphs of order 8 and diameter 2.

3. Antiregular domination in paths

We saw in Theorem 1.1 that none of P_2, P_3 and P_6 has an irregular dominating labeling, while all other paths do. The irregular domination numbers of all paths P_n of order n with $4 \le n \le 26$ and $n \ne 6$ have been determined in [3], namely

$$\tilde{\gamma}(P_n) = \begin{cases} n-1 & \text{if } n = 4, 5 \\ \frac{n+2}{2} & \text{if } n = 10 \\ \frac{n+3}{2} & \text{if } n \text{ is odd and } 7 \le n \le 25 \\ \frac{n+4}{2} & \text{if } n \text{ is even, } 8 \le n \le 26, \text{ and } n \ne 10. \end{cases}$$

We now turn our attention to antiregular domination numbers of paths. All nontrivial paths possess an antiregular dominating labeling. The path P_6, which has no irregular dominating labeling, has an antiregular dominating labeling in which any number in the set $[5]$ is repeated. This is illustrated in Figure 7.

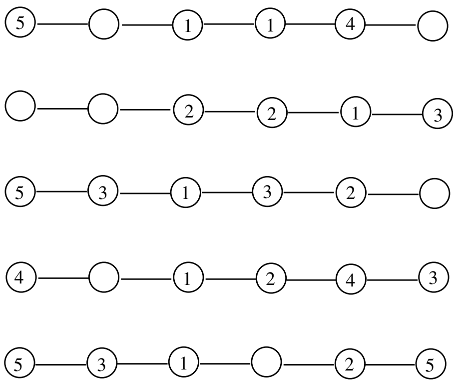


Figure 7: Antiregular dominating labelings of P_6.

In order to establish a lower bound for antiregular domination numbers of paths, we first present an observation concerning bipartite graphs.

Observation 3.1. *Let G be a nontrivial connected bipartite graph with partite sets U and W and let f be an irregular dominating labeling of G. If x is a labeled vertex and $f(x) = r$, then (i) $O_r(x) \subseteq U$ or $O_r(x) \subseteq W$ and (ii) x and $O_r(x)$ belong to the same partite set if and only if r is even. Consequently, if a labeled vertex dominates two vertices u and v, then $d(u, v)$ is even.*

Theorem 3.1. *For each integer $n \ge 5$,*

$$\gamma_A(P_n) \ge \lceil (n+1)/2 \rceil = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n+2}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $P_n = (u_0, u_1, \ldots, u_{n-1})$ be a path of order $n \ge 5$. Assume, to the contrary, that there is either an odd integer $n \ge 5$ such that $\gamma_A(P_n) < \frac{n+1}{2}$ or an even integer $n \ge 6$ such that $\gamma_A(P_n) < \frac{n+2}{2}$. We consider these two cases.

Case 1. *n is odd.* Then $n = 2k + 1$ for some integer $k \ge 2$ and $\lceil \frac{n+1}{2} \rceil = \frac{n+1}{2} = k + 1$. Thus, there is a minimum antiregular dominating labeling f of P_{2k+1} using at most k labels from the set $[2k]$. If $f(v) \in [k]$, then v dominates one or two vertices of P_{2k+1} (according to the location of v). If $f(v) \in \{k+1, k+2, \ldots, 2k\} = [k+1, 2k]$, then v dominates at most one vertex of P_{2k+1}. Thus, these k labeled vertices can dominate at most $2k$ vertices of P_{2k+1}, which is a contradiction.

Case 2. *n is even*. Let $n = 2k+2$ for some integer $k \geq 2$. Then $\lceil \frac{n+1}{2} \rceil = \frac{n+2}{2} = k+2$. Thus, there is a minimum antiregular dominating labeling f of P_{2k+2} using at most $k+1$ labels from the set $[2k+1]$. If $f(v) \in [k]$, then v dominates one or two vertices of P_{2k+2}. If $f(v) \in [k+1, 2k+1]$, then v dominates at most one vertex of P_{2k+2}. This implies that (i) $f(v) \in [k]$ for every labeled vertex v of P_{2k+2}, (ii) there is $a \in [k]$ such that exactly two vertices of P_{2k+2} are labeled a, (iii) for each $b \in [k] - \{a\}$, there is exactly one vertex of P_{2k+2} labeled b, and (iv) no vertex of P_{2k+2} is dominated by more than one vertex and every vertex dominates two vertices. Furthermore, any subset of ℓ vertices labeled from elements of the set $[k]$ must dominate 2ℓ vertices of P_{2k+2}. If $f(v) = k$ and v dominates two vertices of P_{2k+2}, then $v \in \{u_k, u_{k+1}\}$. By symmetry, we may assume that $f(u_k) = k$ and u_k dominates u_0 and u_{2k}. Since a vertex labeled $k-1$ must dominate exactly two vertices in $V(P_{2k+2}) - \{u_0, u_{2k}\}$, this forces $f(u_{k+2}) = k-1$ and u_{k+2} dominates u_3 and u_{2k+1}.

If $k = 2$ (and $n = 6$), then the 3rd labeled vertex cannot dominate u_1 and u_2, a contradiction. If $k = 3$ (and $n = 8$), then the vertex labeled 1 cannot dominate two vertices in $\{u_1, u_2, u_4, u_5\}$ not already dominated, a contradiction. Hence, $k \geq 4$. This forces $f(u_{k-1}) = k-2$ and u_{k-1} dominates u_1 and u_{2k-3} and $f(u_{k+1}) = k-3$ and u_{k+1} dominates u_4 and u_{2k-2}. If $k = 4$ (and $n = 10$), then the 5th labeled vertex cannot dominate the remaining two undominated vertices u_2 and u_7, a contradiction. If $k = 5$ (and $n = 12$), then the vertex labeled 1 cannot dominate two of the four undominated vertices u_2, u_5, u_6, u_9, a contradiction. Hence, $k \geq 6$. Then either $f(u_{k+3}) = k-4$ or $f(u_{k-2}) = k-4$. If $k = 6$ (and $n = 12$), then the vertex labeled 1 cannot dominate the two undominated vertices, a contradiction. Thus, we may assume that $k \geq 7$. If $f(u_{k+3}) = k-4$, then u_{k+3} dominates u_7 and u_{2k-1}. Since the vertex labeled $k-5$ must dominate two undominated vertices, this forces $f(u_{k-3}) = k-5$ and u_{k-3} dominates u_2 and u_{2k-8}. If $f(u_{k-2}) = k-4$, then u_{k-2} dominates u_2 and u_{2k-6}. Since the vertex labeled $k-5$ must dominate two undominated vertices, this forces $f(u_{k+4}) = k-5$ and u_{k+4} dominates u_9 and u_{2k-1}. In either case, if $i \in \{0, 1, 2, 3, 4\} \cup \{2k-3, 2k-2, 2k-1, 2k, 2k+1\}$, then u_i is already dominated. Hence, there is no unlabeled vertex that can be labeled $k-6$ and dominates two vertices not already dominated, which is impossible. □

While equality in Theorem 3.1 holds for $n = 5, 6, 7, 8, 10, 12$ (as a minimum antiregular dominating labeling of P_n is given in Figure 8 for each $n \in \{5, 6, 7, 8, 10, 12\}$), strict inequality holds for $n = 9$ and $n = 11$.

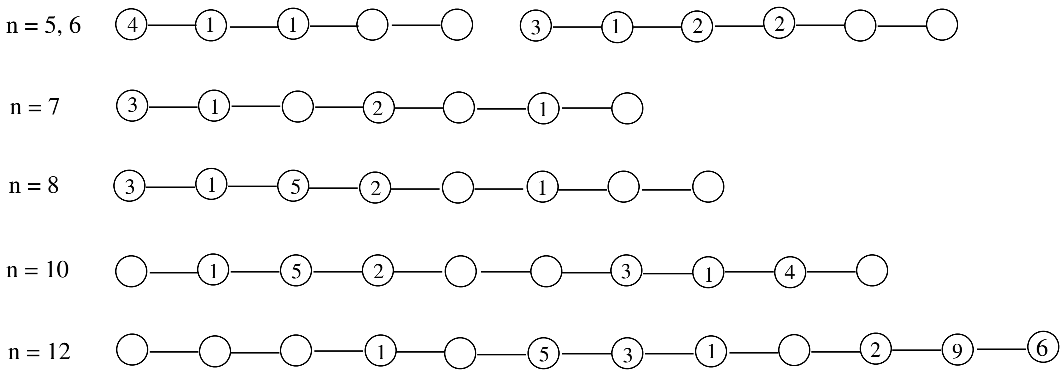


Figure 8: Antiregular dominating labelings of P_n for $n = 5, 6, 7, 8, 10, 12$.

Proposition 3.1. $\gamma_A(P_9) = 6$ *and* $\gamma_A(P_{11}) = 7$.

Proof. The antiregular dominating labelings of P_9 and P_{11} in Figure 9 shows that $\gamma_A(P_9) \leq 6$ and $\gamma_A(P_{11}) \leq 7$. By Theorem 3.1, $\gamma_A(P_9) \geq 5$ and $\gamma_A(P_{11}) \geq 6$. It remains to show that $\gamma_A(P_9) \neq 5$ and $\gamma_A(P_{11}) \neq 6$. We will only verify $\gamma_A(P_9) \neq 5$ since the argument for $\gamma_A(P_{11}) \neq 6$ is similar.

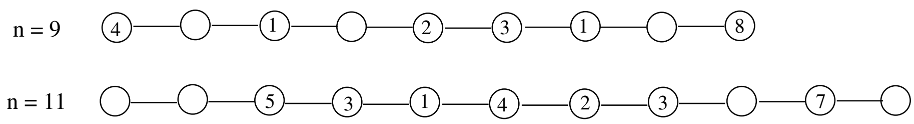
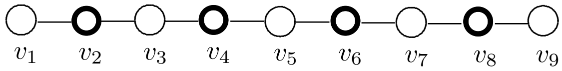


Figure 9: Antiregular dominating labelings of P_9 and P_{11}.

Let $P_9 = (v_1, v_2, \ldots, v_9)$. Assume, to the contrary, that there is an antiregular dominating labeling f of P_9 in which five vertices are labeled. Since a labeled vertex can dominate at most two vertices of P, there must be four labeled vertices that dominate eight vertices of P. By Observation 3.1, if a labeled vertex dominates two vertices v_i and v_j of P, then

$d(v_i, v_j) = |i-j|$ is even and so i and j are of the same parity. Let X be the set of four labeled vertices of P that dominate eight vertices of P. Then the vertices of X must dominate all the vertices v_2, v_4, v_6, v_8 and four of the five vertices v_1, v_3, v_5, v_7, v_9. Furthermore, each vertex of X must dominate one pair of vertices in $\{v_2, v_4, v_6, v_8\}$ or one pair of vertices in $\{v_1, v_3, v_5, v_7, v_9\}$.



The two vertices in X that dominate v_2, v_4, v_6, v_8 produce a partition of $\{v_2, v_4, v_6, v_8\}$ into two 2-element subsets. There are three possible such partitions:

$$\text{(a) } \{v_2, v_4\}, \{v_6, v_8\}, \text{ (b) } \{v_2, v_6\}, \{v_4, v_8\}, \text{ and (c) } \{v_2, v_8\}, \{v_4, v_6\}.$$

If (a) occurs, then $f(v_3) = f(v_7) = 1$; if (b) occurs, then $f(v_4) = f(v_6) = 2$; and if (c) occurs, then $f(v_5) = 3$ and $f(v_5) = 1$, which is impossible. Thus, only (a) and (b) are possible and so either $f(v_3) = f(v_7) = 1$ or $f(v_4) = f(v_6) = 2$.

The two vertices of X that dominate four vertices in $\{v_1, v_3, v_5, v_7, v_9\}$ produce a partition of these four vertices into two 2-element subsets. By symmetry, there are nine distinct possible partitions:

$$\begin{gathered}(1)\ \{v_1, v_3\}, \{v_5, v_7\},\ (2)\ \{v_1, v_3\}, \{v_5, v_9\},\ (3)\ \{v_1, v_5\}, \{v_7, v_9\},\\ (4)\ \{v_1, v_5\}, \{v_3, v_7\},\ (5)\ \{v_1, v_5\}, \{v_3, v_9\},\ (6)\ \{v_1, v_7\}, \{v_3, v_5\},\\ (7)\ \{v_1, v_7\}, \{v_3, v_9\},\ (8)\ \{v_1, v_9\}, \{v_3, v_5\},\ (9)\ \{v_1, v_9\}, \{v_3, v_7\}.\end{gathered}$$

- ⋆ If (1) occurs, then $f(v_2) = f(v_6) = 1$, which is impossible since no label can be used three or more times and no two labels can be duplicated.
- ⋆ If (2) occurs, then $f(v_2) = 1$ and $f(v_7) = 2$, which is impossible because no label can be used three times.
- ⋆ If (3) occurs, then $f(v_3) = 2$ and $f(v_8) = 1$, which is impossible because no label can be used three times.
- ⋆ If (4) occurs, then $f(v_3) = f(v_5) = 2$, which is impossible because no label can be used three or more times and no two labels can be duplicated.
- ⋆ If (5) occurs, then $f(v_3) = 2$ and $f(v_6) = 3$. This forces that (a) occurs in which v_3 is already labeled 1, a contradiction.
- ⋆ If (6) occurs, then $f(v_4) = 3$ and $f(v_4) = 1$, which is impossible.
- ⋆ If (7) occurs, then $f(v_4) = f(v_6) = 3$, which is impossible because two labels cannot be duplicated.
- ⋆ If (8) occurs, then $f(v_5) = 4$ and $f(v_4) = 1$. This forces that (b) occurs in which v_4 is already labeled 2, a contradiction.
- ⋆ If (9) occurs, then $f(v_5) = 4$ and $f(v_5) = 2$, which is impossible.

Consequently, $\gamma_A(P_9) \neq 5$ and so $\gamma_A(P_9) = 6$. □

By Proposition 3.1, if $n = 9, 11$, then $\gamma_A(P_n) = \tilde{\gamma}(P_n) = \frac{n+3}{2}$. This gives rise to the following question.

Problem 3.1. *Does $\gamma_A(P_n) = \tilde{\gamma}(P_n) = \frac{n+3}{2}$ for all odd integers $n \geq 13$?*

While each nontrivial path different from P_2, P_3, and P_6 has an irregular dominating labeling, it follows from Proposition 1.1 that no cycle C_n of order $n \geq 3$ has an irregular dominating labeling. Thus, we have the following question: *Which cycles have an antiregular dominating labeling?* Not only does the path of size 5 fail to have an irregular dominating labeling, the cycle of size 5 fails to have an antiregular dominating labeling.

Example 3.1. *The 5-cycle C_5 does not have an antiregular dominating labeling.*

Proof. Assume, to the contrary, that $C_5 = (v_1, v_2, v_3, v_4, v_5, v_1)$ has an antiregular dominating labeling whose corresponding antiregular dominating labeling is f. Since at least one vertex of C_5 must be labeled 2, we may assume that $f(v_1) = 2$. Thus, v_1 dominates v_3 and v_4. Because v_1 cannot dominate itself, it follows that v_1 is dominated by a vertex labeled 1 or 2. First, suppose that v_1 is dominated by a vertex labeled 1, say $f(v_2) = 1$. However then, the remaining two undominated vertices v_2 and v_5 cannot be dominated by the 3rd labeled vertex (which is not v_1), producing a contradiction. Next, suppose that v_1 is dominated by a vertex labeled 2, say $f(v_3) = 2$. The remaining undominated vertex v_2 must be dominated by a vertex labeled 1. Since $f(v_1) = f(v_3) = 2$, this is impossible. □

It can be shown that for all integers n with $6 \leq n \leq 30$, the n-cycle C_n has an antiregular dominating labeling. Such a labeling is shown in Figure 10 for $n = 29$ and $n = 30$. In fact, we have the following conjecture.

Conjecture 3.1. *For each integer $n \geq 6$, the n-cycle C_n has an antiregular dominating labeling.*

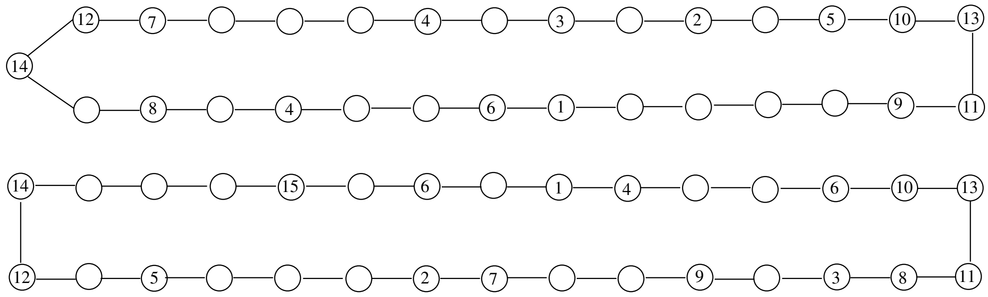


Figure 10: Antiregular dominating labelings of C_{29} and C_{30}.

4. Antiregular domination in trees

In Section 3, we discussed antiregular domination in paths. We now turn our attention to trees in general. For an integer $d \geq 3$, let $\tilde{T}_d$ denote the unique tree (up to isomorphism) of largest order having diameter d with the property that there is a longest path P in $\tilde{T}_d$ such that each interior vertex of P has degree 3 and all other vertices of $\tilde{T}_d$ have degree 1 or 2. Figure 11 shows the trees $\tilde{T}_d$ of diameter $d = 3, 4, 5, 6$.

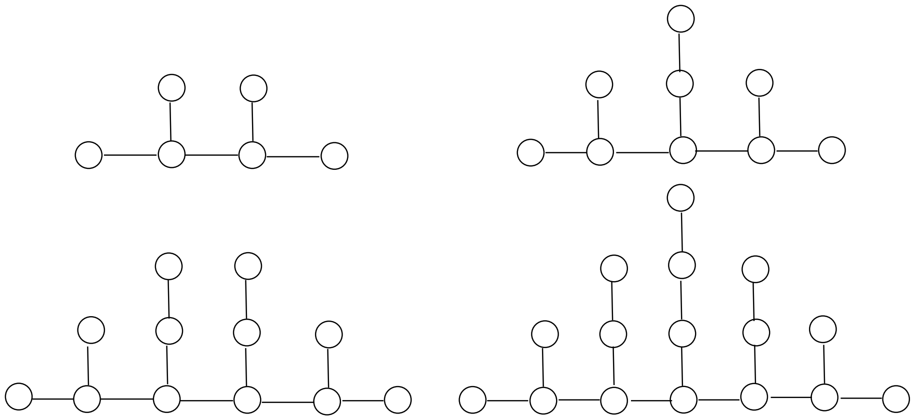


Figure 11: The trees $\tilde{T}_d$ of diameter $d = 3, 4, 5, 6$.

Lemma 4.1. *Let T be a tree of diameter d and let P be a longest path in the tree $\tilde{T}_d$ of diameter $d \geq 3$ each of whose interior vertices has degree 3 in $\tilde{T}_d$. If f is an antiregular dominating labeling of $\tilde{T}_d$ such that each labeled vertex of $\tilde{T}_d$ belongs to P, then f gives rise to an antiregular dominating labeling f' of T such that f and f' have the same set of labels.*

Proof. Let $P = (u_0, u_1, u_2, \ldots, u_d)$ be a longest path in $\tilde{T}_d$, where u_i is adjacent to the vertex w_i not on P and B_i is the branch (path) at u_i containing u_iw_i for $1 \leq i \leq d-1$. Let f be an antiregular dominating labeling of $\tilde{T}_d$ such that each labeled vertex of $\tilde{T}_d$ belongs to P. Next, let T be a tree of diameter $d \geq 3$ and let $P' = (u'_0, u'_1, u'_2, \ldots, u'_d)$ be a longest path in T. Denote the set of all labeled vertices of $\tilde{T}_d$ by X. Thus, $X \subseteq V(P)$. Let $X' = \{x' : x \in X\}$. Then $X' \subseteq V(P')$. Define a labeling $f' : X' \to [d]$ of T by $f'(x') = f(x)$ for each $x' \in X'$ and all vertices in $V(T) - X'$ are not labeled. We show that f' is an antiregular dominating labeling of T. Let $v' \in V(T)$. We show that v' is dominated by a vertex of X'.

- ⋆ If $v' \in V(P')$, then $v' = u'_i$ for some integer i with $0 \leq i \leq d$. Since f is an antiregular dominating labeling of $\tilde{T}_d$, it follows that u_i is dominated by a vertex $x \in X$ in $\tilde{T}_d$. Since $d_{\tilde{T}_d}(u_i, x) = d_T(u'_i, x')$ and $f(x) = f'(x')$, it follows that u'_i is dominated by $x' \in X'$.
- ⋆ If $v' \notin V(P')$, then v' belongs to a branch B'_i at u'_i for some integer i with $1 \leq i \leq d-1$ such that $V(B'_i) \cap V(P') = \{u'_i\}$. Suppose that $d_T(v', u'_i) = j \geq 1$. Let B_i be the branch at u_i in $\tilde{T}_d$ containing u_iw_i and let $v \in V(B_i)$ such that $d_{\tilde{T}_d}(v, u_i) = j$. Then v is dominated by a vertex $x \in X$ in $\tilde{T}_d$. Since $d_{\tilde{T}_d}(u_i, x) = d_T(u'_i, x')$, it follows that $d_{\tilde{T}_d}(v, x) = j + d_{\tilde{T}_d}(u'_i, x') = j + d_T(u_i, x) = d_T(v', x')$. Furthermore, $f(x) = f'(x')$. Hence, v' is dominated by a vertex $x' \in X'$ in T.

Therefore, f' is an antiregular dominating labeling of T and f and f' have the same set of labels. □

Theorem 4.1. *If T is a tree of diameter $d \geq 3$, then $\gamma_A(T) \leq \gamma_A(P_{d+1})$.*

Proof. By Lemma 4.1, it suffices to show that $\tilde{T}_d$ has an antiregular dominating labeling for every integer $d \geq 3$ such that every labeled vertex of $\tilde{T}_d$ belongs to a longest path P in $\tilde{T}_d$ in which every interior vertex of P has degree 3 in $\tilde{T}_d$. We proceed by induction to verify the following statement.

For each integer $d \geq 3$, there is an antiregular dominating labeling f of $\tilde{T}_d$ such that (1) all labeled vertices of $\tilde{T}_d$ belong to a longest path P, each of whose interior vertices has degree 3, and (2) at least one end-vertex of P is not labeled by f.

Figure 12 shows an antiregular dominating labeling with the desired properties for the tree $\tilde{T}_3$. Thus, the statement holds for $d = 3$. Suppose that the statement holds for the tree $\tilde{T}_d$ for some integer $d \geq 3$. We show that the statement also holds for the tree $\tilde{T}_{d+1}$.

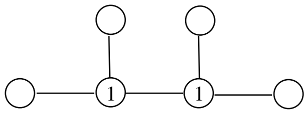


Figure 12: An antiregular dominating labeling for $\tilde{T}_3$.

Let $T = \tilde{T}_{d+1}$ and let $P = (u_0, u_1, \ldots, u_{d+1}) \cong P_{d+2}$ be a longest path (of diameter $d + 1$) in T such that each interior vertex of P has degree 3 in T. For $1 \leq i \leq d$, let B_i be the branch (path) of T at u_i such that $V(B_i) \cap V(P) = \{u_i\}$. Let $P' = (u_0, u_1, \ldots, u_d) = P - \{u_{d+1}\} \cong P_{d+1}$ and let $T' \cong \tilde{T}_d$ be the subtree of T with longest path P' such that each interior vertex of P' has degree 3 in T'. By the induction hypothesis, there is an antiregular dominating labeling f' of T' using elements from the set $[d]$ such that (1) all labeled vertices of T' belong to P' and (2) at least one end-vertex of P' is not labeled by f'. *We may assume that u_0 is not labeled by f'.* We now extend the labeling f' of the subtree T' to a labeling f of T by assigning the label $d + 1$ to u_0. This is illustrated in Figure 13 for $d = 3, 4, 5, 6$, where $\tilde{T}_d$ is obtained from $\tilde{T}_{d+1}$ by deleting those vertices and edges of $\tilde{T}_{d+1}$ indicated in bold and the labeling f' of $\tilde{T}_d$ of Figure 13 is extended to a labeling f of $\tilde{T}_{d+1}$ by defining $f(u_0) = d + 1$. Since f' is an antiregular dominating labeling, so is f. □

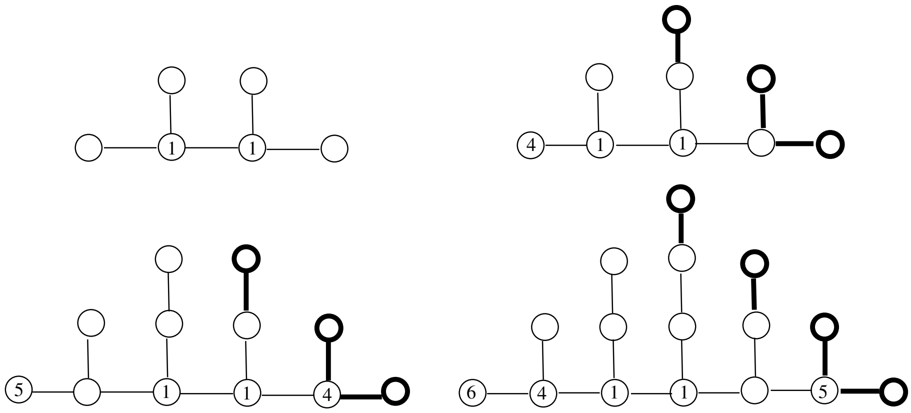


Figure 13: Antiregular dominating labelings of $\tilde{T}_d$ of diameter $d = 3, 4, 5, 6$.

The following result gives an upper bound for the diameter of a tree having a given antiregular domination number.

Theorem 4.2. *If T is a tree with $\gamma_A(T) = k$, then*

$$\mathrm{diam}(T) \leq \begin{cases} 2k - 1 & \text{if } k \text{ is even} \\ 2k - 2 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We consider two cases, according to whether k is even or k is odd.

Case 1. *k is even.* Since each labeled vertex dominates at most two vertices on a path of order $\mathrm{diam}(T) + 1$, it follows that $\mathrm{diam}(T) + 1 \leq 2k$.

Case 2. *k is odd.* Assume, to the contrary, that there is a tree with $\gamma_A(T) = k$ for some odd integer $k \geq 3$ such that $\mathrm{diam}(T) \geq 2k - 1$. Then T contains a path $P = (u_1, u_2, \ldots, u_{2k})$ of order $2k$. Let $U_1 = \{u_1, u_3, \ldots, u_{2k-1}\}$ and $U_2 = \{u_2, u_4, \ldots, u_{2k}\}$. Thus, $|U_1| = |U_2| = k$. Since $\gamma_A(T) = k$, there is an antiregular dominating labeling of T using exactly $k - 1$ distinct labels from the set $[2k - 1]$ and one of these label is used exactly twice. Necessarily, each of these k labeled vertices of T must dominate two vertices of P and the set of pairs of vertices of P dominated by a labeled vertex of T must result in a partition of $V(P)$. If $\{u_i, u_j\}$ is a pair of vertices dominated by a labeled vertex of T, then $d(u_i, u_j) = |i - j|$ must be even. Therefore, either $\{u_i, u_j\} \subseteq U_1$ or $\{u_i, u_j\} \subseteq U_2$. However, since both U_1 and U_2 consist of an odd number of vertices, such a labeling is impossible. □

As a consequence of Theorem 4.2, we then have the following result.

Corollary 4.1. *If T is a nontrivial tree, then*

$$\gamma_A(T) \geq \begin{cases} \dfrac{\operatorname{diam}(T)+1}{2} & \text{if } \gamma_A(T) \text{ is even} \\ \dfrac{\operatorname{diam}(T)+2}{2} & \text{if } \gamma_A(T) \text{ is odd.} \end{cases}$$

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References

[1] A. Ali, G. Chartrand, P. Zhang, *Irregularity in Graphs*, Springer, New York, 2021.
[2] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris* **247** (1958) 258–259.
[3] P. Broe, G. Chartrand, P. Zhang, Irregular domination in trees, *Electron. J. Math.* **1** (2021) 89–100.
[4] P. Broe, G. Chartrand, P. Zhang, Irregular orbital domination in graphs, *Int. J. Comput. Math. Comput. Syst. Theory*, In press,
[5] G. Chartrand, M. A. Henning, K. Schultz, On orbital domination numbers of graphs, *J. Combin. Math. Combin. Comput.* **37** (2001) 3–26.
[6] G. Chartrand, P. Zhang, A chessboard problem and irregular domination, Preprint.
[7] E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi, Total domination in graphs, *Networks* **10** (1977) 211–219.
[8] E. J. Cockayne, S. T. Hedetniemi, Towards a theory of domination in graphs, *Networks* **7** (1977) 247–261.
[9] L. Hayes, K. Schultz, J. Yates, Universal domination sequences of graphs, *Util. Math.* **54** (1998) 193–209.
[10] T. W. Haynes, S. T. Hedetniemi, P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1998.
[11] T. W. Haynes, S. T. Hedetniemi, P. J. Slater (Eds.), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.
[12] M. A. Henning, A. Yeo, *Total Domination in Graphs*, Springer, New York, 2013.
[13] O. Ore, *Theory of Graphs*, American Mathematical Society, Providence, 1962.