

Review Article

Constrained extension of linear operators, approximation, and the moment problem

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Abstract

Some Hahn-Banach-extension results for linear operators are reviewed. Several direct applications of these extension results to the existence of a solution for Markov moment problems are pointed out. A variant of the Mazur-Orlicz theorem in concrete spaces is also proved. Moreover, polynomial-approximation results on unbounded subsets and their applications to the existence and uniqueness of the solutions for the full Markov moment problem on \mathbb{R}^n and on \mathbb{R}_+^n are reviewed. The multi-dimensional case is considered and the one-dimensional case follows as a consequence. Finally, the truncated moment problem is briefly discussed.

Keywords: extension of linear operators; sandwich constraints; full Markov moment problem; existence of a solution; polynomial approximation; unbounded subsets; uniqueness.

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1. Introduction

Studying and proving extension results for linear functionals and operators are usually motivated by (i) proving existence of solutions for some moment problems (see [1,2,5,6,11,13,14,21–26,28,30,31]), (ii) sub-differentiability of convex functions and convex operators (see [17,18,20]), (iii) decomposition of linear operators acting on ordered vector spaces (the codomain being an order complete vector lattice) as a difference of two positive linear operators, the first one being dominated by a given convex operator (see [17,19,20]), (iv) the Krein-Milman theorem and some other problems. The Krein-Milman theorem has applications also to representation theorems, convex optimization and some other fields of analysis (see [4,5,18]). Many results on the moment problem, not necessarily or explicitly using extension of linear functional and operators, have been published in [3,4,8–10,12,15,16,28,32,33]. Old results of the present author published in [19–21] have found several applications specified in [2,17,22,23,25]. Also, several polynomial-approximation results on unbounded subsets reviewed in the recent paper [24] have found applications [25,26]. Density theorems in a different framework were published in [27]. In present paper, the following notations are used: $\mathbb{N} = \{0, 1, 2, \dots\}$ is the set of all nonnegative integers, \mathcal{P} is the real vector space of all polynomial functions with real coefficients. If $F \subseteq \mathbb{R}^n$ is a closed unbounded subset, then denote by $C_c(F)$ the vector space of all real-valued continuous compactly supported functions defined on F . Take $\mathbb{R}_+ = [0, +\infty)$. Basic notions and results in analysis related to this paper can be found in the monographs [1,4,7,13,18,29,31].

The remaining part of this paper is organized as follows. Section 2 is devoted to general extension results for linear operators. A variant of the Mazur-Orlicz theorem is also considered in Section 2. In Section 3, some direct applications of the results of Section 2 are specified. Section 4 is concerned with a polynomial approximation on unbounded subsets and its applications to the existence and uniqueness of the solution for some classical full Markov moment problems. Section 5 is devoted to some comments and remarks on the truncated moment problem. Section 6 concludes the paper.

2. On constrained extensions for linear operators

One of the earlier results on extension of linear operators refers to the existence of a positive linear extension for a positive linear operator defined on a majorizing subspace of an ordered vector space. The codomain space is an order complete vector space. Here, the constraint is the positivity of the extension on the positive cone of the domain space. Let X_1 be an ordered vector space whose positive cone $X_{1,+}$ is generating ($X_1 = X_{1,+} - X_{1,+}$). Recall that in such an ordered vector space X_1 , a vector subspace S is said to be a majorizing subspace if for any $x \in X_1$ there exists $s \in S$ such that $x \leq s$. Following

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is a significant example of a majorizing subspace. Let $F \subseteq \mathbb{R}^n$ be a closed unbounded subset and $1 \leq \alpha < +\infty$. Let ν be a positive regular Borel measure on F , with finite moments of all orders. Define $X := L_\nu^\alpha(F)$ and let X_1 be the vector subspace of all functions $f \in X$ for which there exists a polynomial p such that $|f| \leq p$ on F . Then the subspace $S := \mathcal{P}$ of all polynomial functions on F is a majorizing subspace of X_1 . The space X_1 contains $C_c(F)$ (the subspace of all continuous compactly supported real functions on F), as well as the subspace \mathcal{P} ($p \in \mathcal{P} \implies |p| = \sqrt{1 \cdot p^2} \leq ((1 + p^2)/2) \in \mathcal{P}$). The subspace X_1 is dense in X , since it contains $C_c(F)$, which is dense in $L_\nu^\alpha(F) = X$.

Theorem 2.1 (see [14], Theorem 1.2.1). *Let X_1 be an ordered vector space whose positive cone is generating, $X_0 \subset X_1$ a majorizing vector subspace, Y an order complete vector space, $T_0 : X_0 \rightarrow Y$ a positive linear operator. Then T_0 admits a positive linear extension $T : X_1 \rightarrow Y$.*

A much more general form of Theorem 2.1 was proved in [19] (see also [20] for details and completions). In the next theorem, X is a real vector space, Y an order-complete vector lattice, $A, B \subseteq X$ convex subsets, $Q : A \rightarrow Y$ a concave operator, $P : B \rightarrow Y$ a convex operator, $M \subset X$ a vector subspace, $T_0 : M \rightarrow Y$ a linear operator. All vector spaces and linear operators are considered over the real field.

Theorem 2.2 (see [19], Theorem 1). *Assume that $T_0(x) \geq Q(x) \forall x \in M \cap A$ and $T_0(x) \leq P(x) \forall x \in M \cap B$. The following statements are equivalent:*

(a). *There exists a linear extension $T : X \rightarrow Y$ of the operator T_0 such that*

$$T|_A \geq Q \quad \text{and} \quad T|_B \leq P.$$

(b). *There exists a convex operator $P_1 : A \rightarrow Y$ and a concave operator $Q_1 : B \rightarrow Y$ such that for all*

$$(\rho, t, \lambda, a_1, a, b_1, b, v) \in [0, 1]^2 \times (0, \infty) \times A^2 \times B^2 \times M,$$

the following implication holds:

$$(1 - t)a_1 - tb_1 = v + \lambda((1 - \rho)a - \rho b) \implies (1 - t)P_1(a_1) - tQ_1(b_1) \geq T_0(v) + \lambda((1 - \rho)Q(a) - \rho P(b)),$$

where the convex subsets A, B are arbitrary, with no restriction on the existence of relative interior points or on their position with respect to the subspace M .

Theorem 2.3 (see [19], Theorem 2). *Let X be an ordered vector space, Y an order complete vector space, $M \subset X$ a vector subspace, $T_1 : M \rightarrow Y$ a linear operator, $P : X \rightarrow Y$ a convex operator. The following statements are equivalent:*

(a). *there exists a positive linear extension $T : X \rightarrow Y$ of T_1 such that $T \leq P$ on E ;*

(b). *it holds that $T_1(h) \leq P(x)$ for all $(h, x) \in M \times X$ such that $h \leq x$.*

One observes that in the very particular case $E_+ = \{0\}$, when the order relation on E is the equality then from Theorem 2.3 one obtains Hahn-Banach extension theorem for linear operators dominated by convex operators. Unlike Theorems 2.2 and 2.3, which provide necessary and sufficient conditions for the existence of an extension, the next result gives only sufficient conditions.

Theorem 2.4 (see [20]). *Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $C \subset X$ be a convex subset with the following properties:*

(a). *There exists a neighborhood D of the origin such that $(S + D) \cap C = \emptyset$ (that is, by definition, C and S are distanced).*

(b). *C is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j \in J} \subset \mathcal{L}(S, Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{T_j\}_{j \in J} \subset \mathcal{L}(X, Y)$ such that

$$T_j(s) = f_j(s), \quad s \in S, \quad T_j(\cdot) \geq \tilde{y}, \quad \cdot \in C, \quad j \in J.$$

Moreover, if D is a convex balanced neighbourhood of the origin such that

$$f_j(D \cap S) \subset [-u_0, u_0], \quad (S + D) \cap C = \emptyset$$

and if $\acute{a} > 0$ is such that $P_D(a) \leq \acute{a} \forall a \in C$ and $\acute{a}_1 > 0$ is large enough such that $\tilde{y} \leq \acute{a}_1 u_0$, then the following relations hold:

$$T_j(x) \leq (1 + \acute{a} + \acute{a}_1) P_D(x), \quad x \in X, \quad j \in J,$$

where P_D denotes the gauge (Minkowski functional) attached to D .

Theorem 2.5 (see [21], Theorem 5). *Let X be a preordered vector space and Y be an order complete vector space. Let $\{x_j\}_{j \in J}$ and $\{y_j\}_{j \in J}$ be the families of elements in X and Y , respectively, and let $P : X \rightarrow Y$ be a sublinear operator. The following statements are equivalent:*

(a). *There exists a linear positive operator $T : X \rightarrow Y$ such that*

$$T(x_j) \geq y_j, j \in J, T(x) \leq P(x), x \in X.$$

(b). *For any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}_+$, the following implication holds*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \implies \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

3. Applications to the Markov moment problem and Mazur-Orlicz theorem

We start this section with two applications to the abstract moment problem. The first one is another way of writing of Theorem 2.3.

Theorem 3.1 (see [21], Theorem 1). *Let X be a preordered vector space, Y an order complete vector lattice, $P : X \rightarrow Y$ a convex operator, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families. The following statements are equivalent:*

(a). *There exists a linear positive operator $T : X \rightarrow Y$ such that*

$$T(x_j) = y_j, j \in J, T(x) \leq P(x), x \in X.$$

(b). *For any finite subset $J_0 \subseteq J$, and any $\{\lambda_j; j \in J_0\} \subseteq \mathbb{R}$, it holds that*

$$\sum_{j \in J_0} \lambda_j x_j \leq x \in X \implies \sum_{j \in J_0} \lambda_j y_j \leq P(x).$$

In addition, if we assume that P is isotone ($u \leq v \implies P(u) \leq P(v)$), the assertions (a) and (b) are equivalent to (c), where:

(c). *For any finite subset $J_0 \subset J$ and any $\{\lambda_j\}_{j \in J_0} \subset \mathbb{R}$, the following inequality holds:*

$$\sum_{j \in J_0} \lambda_j y_j \leq P \left(\sum_{j \in J_0} \lambda_j x_j \right).$$

Theorem 3.2 (see [21], Theorem 4). *Let X be an ordered vector space and Y be an order complete vector lattice. Also, let $\{\varphi_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ be two given arbitrary families and $T_1, T_2 \in L(X, Y)$ be two linear operators. The following statements are equivalent:*

(a). *There is a linear operator $T \in L(X, Y)$ such that*

$$T_1(x) \leq T(x) \leq T_2(x) \quad \forall x \in X_+, T(\varphi_j) = y_j \quad \forall j \in J.$$

(b). *For any finite subset $J_0 \subset J$ and any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds*

$$\left(\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+ \right) \implies \sum_{j \in J_0} \lambda_j y_j \leq T_2(\psi_2) - T_1(\psi_1).$$

If X is a vector lattice, then each of the assertions (a) and (b) is equivalent to the following assertion:

(c). *$T_1(w) \leq T_2(w)$ for all $w \in X_+$ and for any finite subset $J_0 \subset J$ and for all $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have*

$$\sum_{j \in J_0} \lambda_j y_j \leq T_2 \left(\left(\sum_{j \in J_0} \lambda_j \varphi_j \right)^+ \right) - T_1 \left(\left(\sum_{j \in J_0} \lambda_j \varphi_j \right)^- \right).$$

The next results provide applications of some of the preceding theorems to concrete spaces. We recall that a sequence $(u_n)_{n \geq 0}$ in an ordered vector space Y is called nonnegative with respect to an interval $\mathbb{I} \subseteq \mathbb{R}$ if for any $n \in \mathbb{N}$, and any coefficients $\lambda_j, j = 0, 1, \dots, n$, we have

$$\lambda_0 + \lambda_1 t + \dots + \lambda_n t^n = 0 \quad \forall t \in \mathbb{I} \implies \lambda_0 u_0 + \lambda_1 u_1 + \dots + \lambda_n u_n \in Y_+.$$

Theorem 3.3 (see [22], Theorem 31). *Let*

$$0 < b \in \mathbb{R}, X = C([0, b]), \varphi_j(t) = t^j, j \in \mathbb{N}, j \geq 1, t \in [0, b],$$

$$\{\psi_k\}_{k=0} \subset X, \|\psi_k\| \leq 1, \psi_0 = \mathbf{1}, \psi_k(0) = 1, k \in \mathbb{N}, k \geq 1.$$

Let Y be an order complete vector lattice with strong order unit u_0 and $(y_n)_{n \geq 1}$ a sequence in Y such that (u_0, y_1, y_2, \dots) is nonnegative with respect to $[0, b]$. Then for any $\alpha_1 \in (0, +\infty)$, there exists a linear operator $T \in L(X, Y)$ such that

$$T(\varphi_j) = y_j, j \in \mathbb{N}, j \geq 1, T(\psi_k) \geq \alpha_1 u_0, k \in \mathbb{N}, T(\varphi) \leq (2 + \alpha_1) \|\varphi\| u_0, \varphi \in X.$$

If we additionally assume that $\alpha_1 \geq 1$, and Y is a Banach lattice whose closed unit ball is $[-u_0, u_0]$, then $\|T\| \leq 2 + \alpha_1$, T is continuous and positive.

Proof. One applies Theorem 2.4, where S stands for $\text{Span}\{\varphi_j; j = 1\}$ and C stands for the convex hull of the set $\{\psi_k; k \in \mathbb{N}\}$. We have $\|s - a\| \geq |s(0) - a(0)| = |0 - 1| = 1$ for all $s \in S$ and all $a \in C$, so that $d(S, C) = 1$. This can be written as $(S + B(0, 1)) \cap C = \emptyset$, so that V stands for $B(0, 1)$ and $\|\cdot\|_\infty$ stands for P_V in Theorem 2.4. On the other hand, $\|\psi_k\| = 1, k \in \mathbb{N}$ implies that $\|a\| \leq 1$ for all $a \in C$, so that the number 1 stands for α in Theorem 2.4. We also take in Theorem 2.4 $\tilde{y} = \alpha_1 u_0$. Define

$$f\left(\sum_{j=1}^n \lambda_j \varphi_j\right) = \sum_{j=1}^n \lambda_j y_j, n \in \mathbb{N}, n \geq 1, \lambda_j \in \mathbb{R}, j = 1, \dots, n.$$

Let $\sum_{j=1}^n \lambda_j \varphi_j \in S \cap B(0, 1)$. Using the hypothesis on the sequence which is nonnegative on $[0, b]$, one has:

$$\begin{aligned} \sup \left\{ \left| \sum_{j=1}^n \lambda_j t^j \right| ; t \in [0, b] \right\} \leq 1 &\implies \\ \sum_{j=1}^n \lambda_j t^j + 1 \geq 0, 1 - \sum_{j=1}^n \lambda_j t^j \geq 0, \forall t \in [0, b] &\implies \\ \sum_{j=1}^n \lambda_j y_j + u_0 \geq 0, u_0 - \sum_{j=1}^n \lambda_j y_j \geq 0 &\implies \\ f\left(\sum_{j=1}^n \lambda_j \varphi_j\right) = \sum_{j=1}^n \lambda_j y_j \in [-u_0, u_0] &\implies \\ f(S \cap B(0, 1)) \subseteq [-u_0, u_0]. \end{aligned}$$

The conclusion is that f satisfies the conditions of Theorem 2.4. According to this theorem, there exists a linear extension T of f such that

$$T|_C \geq \alpha_1 u_0, T(\varphi) \leq (1 + 1 + \alpha_1) \|\varphi\| u_0 = (2 + \alpha_1) \|\varphi\| u_0, \varphi \in X.$$

Next, we prove the last assertion of the statement. For this, additionally assume that $\alpha_1 \geq 1$ and Y is a Banach lattice. From what is already proved, replacing φ by $-\varphi$, we infer that $|T(\varphi)| \leq (2 + \alpha_1) \|\varphi\| u_0$, which further yields:

$$\|T(\varphi)\| \leq (2 + \alpha_1) \|\varphi\|, \varphi \in X,$$

since the norm on Y is solid and $\|u_0\| = 1$. Hence T is continuous and $\|T\| \leq 2 + \alpha_1$. Thanks to its continuity and to Weierstrass approximation theorem, to prove the positivity of T , it is sufficient to show that $T(p) \geq 0$ for all polynomials p , with $p(t) \geq 0$ for all $t \in [0, b]$. Indeed, let $p(t)$ be defined by $p(t) = \lambda_0 + \sum_{j=1}^n \lambda_j t^j \geq 0, t \in [0, b]$. According to our hypothesis, it holds that

$$\begin{aligned} \lambda_0 u_0 + \sum_{j=1}^n \lambda_j y_j \geq 0 &\implies \sum_{j=1}^n \lambda_j y_j \geq -\lambda_0 u_0 \implies \\ T\left(\lambda_0 \mathbf{1} + \sum_{j=1}^n \lambda_j \varphi_j\right) &= \lambda_0 T(\psi_0) + \sum_{j=1}^n \lambda_j y_j \geq \lambda_0 \alpha_1 u_0 - \lambda_0 u_0 = \lambda_0 (\alpha_1 - 1) u_0 \geq 0. \end{aligned}$$

We have used the fact that $\lambda_0 = p(0) = 0$. The proof is now complete. □

The preceding result leads to the next one. We need the following basic example of a commutative algebra of self-adjoint operators, which is also an order complete vector lattice. Let H be a Hilbert space and A be a self-adjoint operator acting on H . Let $Y = Y(A)$ be the space of self-adjoint operators acting on H (see pp.303–305 in [7]). Namely, if $\mathcal{A} = \mathcal{A}(H)$ is the space of all self-adjoint operators acting on H then the natural order relation on \mathcal{A} is, by definition,

$$V \leq W \iff \langle Vh, h \rangle \leq \langle Wh, h \rangle \quad \forall h \in H.$$

With respect to this order relation, $\mathcal{A}(H)$ is an ordered Banach space, which is not a lattice. Moreover, the multiplication operation on $\mathcal{A}(H)$ is not commutative. Therefore, using the following notations, we define a suitable subspace $Y(A)$ of $\mathcal{A}(H)$:

$$Y_1(A) = \{V \in \mathcal{A}; AV = VA\}, Y(A) = \{W \in Y_1(A); WV = VW, \forall V \in Y_1(A)\}, \tag{1}$$

Then $Y(A)$ is the codomain space that we are interested in for the next two results. This space is an order complete Banach lattice (see [7]) and is a commutative real algebra (this last assertion is obvious).

Theorem 3.4 (see [22], Corollary 32). *Let H be a Hilbert space, $A \in \mathcal{A}(H)$ a positive self-adjoint operator acting on H , and $Y = Y(A)$ the order complete Banach lattice defined by (1). Take $b \in \mathbb{R}_+$ such that the spectrum $\sigma(A) = [0, b]$ and $\alpha_1 \geq 1$. Then there exists a nondecreasing mapping $\sigma : [0, b] \rightarrow Y$ such that*

$$\int_0^b t^j d\sigma = A^j, \quad j \in \mathbb{N}, \quad j \geq 1, \quad \int_0^b e^{-kt} d\sigma \geq \alpha_1 I, \quad k \in \mathbb{N}, \quad \int_0^b \varphi(t) d\sigma \leq (2 + \alpha_1) \|\varphi\| I, \quad \varphi \in C([0, b]).$$

In particular, for such a mapping σ , the following inequality holds

$$\alpha_1 + 1 \leq \|\sigma(b) - \sigma(0)\| + \|\exp(-kU)\|, \quad k \in \mathbb{N}.$$

Proof. We apply Theorem 3.3 to

$$Y = Y(A), \quad u_0 = I \in Y, \quad y_j = A^j, \quad j \in \{1, 2, \dots\}, \quad \psi_k(t) = \exp(-kt), \quad k \in \mathbb{N}, \quad t \in [0, b], \quad \varphi_j(t) = t^j, \quad j \in \mathbb{N}, \quad j \geq 1, \quad t \in [0, b].$$

Here, $I : H \rightarrow H$ is the identity operator. The sequence $(I, A, A^2, \dots, A^n, \dots)$ is nonnegative with respect to the interval $[0, b]$. Indeed, if $p(t) = \lambda_0 + \lambda_1 t + \dots + \lambda_n t^n \geq 0 \quad \forall t \in [0, b] = \sigma(A)$, then we obtain:

$$\lambda_0 + \lambda_1 A + \dots + \lambda_n A^n = \int_{\sigma(A)} p(t) dE_A \geq 0,$$

since the spectral measure dE_A is positive. Thus, all the requirements of the statement of Theorem 3.3 are satisfied. According to this theorem, there exists a linear positive (bounded) operator $T : C([0, b]) \rightarrow Y$, with the properties specified in the conclusion of Theorem 3.3. According to [7] (see p.272), there is a monotone increasing mapping $\sigma : [0, b] \rightarrow Y$ such that

$$T(\varphi) = \int_0^b \varphi(t) d\sigma, \quad \varphi \in C([0, b]).$$

This leads to:

$$\int_0^b t^j d\sigma = T(\varphi_j) = A^j, \quad j \in \mathbb{N}, \quad j \geq 1, \quad \int_0^b \exp(-kt) d\sigma = T(\psi_k) \geq \alpha_1 I, \quad k \in \mathbb{N},$$

$$\int_0^b \varphi(t) d\sigma = T(\varphi) \leq (2 + \alpha_1) \|\varphi\| I, \quad \varphi \in C([0, b]).$$

Next, we prove the last inequality of the statement. From the previously mentioned inequality, for any $k \in \mathbb{N}$ we have

$$\alpha_1 I \leq \int_0^b e^{-kt} d\sigma = \int_0^b \left(1 + \sum_{m=1}^{\infty} \frac{(-k)^m}{m!} t^m \right) d\sigma = \sigma(b) - \sigma(0) + \sum_{m=1}^{\infty} \frac{(-k)^m}{m!} A^m = \sigma(b) - \sigma(0) - I + \exp(-kA) \implies$$

$$(\alpha_1 + 1) I \leq \sigma(b) - \sigma(0) + \exp(-kA).$$

Due to the fact that the norm on the Banach lattice Y is monotone (increasing) on the positive cone Y_+ , the preceding inequality implies

$$\alpha_1 + 1 \leq \|\sigma(b) - \sigma(0)\| + \|\exp(-kA)\|, \quad k \in \mathbb{N}.$$

The proof is now complete. □

In the particular case when $H = \mathbb{R}^n, n \geq 2$, and A is a symmetric $n \times n$ matrix with real entries, the space $Y(A)$ defined by (1) is isomorphic to a commutative algebra of the space $Sym(n, \mathbb{R})$ of all symmetric $n \times n$ matrices with real entries, and is also an order complete Banach lattice with respect to the trace of the usual order relation defined in $Sym(n, \mathbb{R})$ on $Y(A)$.

We go on with a Mazur-Orlicz type result in concrete spaces. Let H be an arbitrary Hilbert space, $A \in \mathcal{A}(H)$ be a positive self-adjoint operator, $Y = Y(A)$ be the space defined by (1). Let X be the space $C(\sigma(A))$, $\varphi_j(t) = t^j, j \in \mathbb{N}, t \in \sigma(A)$, and $(B_j)_{j \in \mathbb{N}}$ be a sequence in Y .

Theorem 3.5. *The following statements are equivalent:*

(a). *There exists a linear bounded positive operator $T \in B_+(X, Y)$ such that*

$$T(\varphi_j) \geq B_j, j \in \mathbb{N}, T(\varphi) \leq \int_{\sigma(U)} |\varphi| dE_U, \forall \varphi \in X, \|T\| \leq 1.$$

(b). $B_j \leq A^j, j \in \mathbb{N}$.

Proof. The implication (a) \Rightarrow (b) is obvious:

$$B_j \leq T(\varphi_j) \leq \int_{\sigma(A)} |\varphi_j| dE_A = \int_{\sigma(U)} t^j dE_A = A^j, j \in \mathbb{N};$$

we have used the positivity of the operator A which leads to $|\varphi_j| = \varphi_j$ on $\sigma(A) \subset \mathbb{R}_+$. For the converse, that is (b) \Rightarrow (a), one applies Theorem 2.5, where J stands for \mathbb{N} , x_j stands for φ_j and y_j stands for $B_j, j \in \mathbb{N}$. Let J_0 and $\{\lambda_j\}_{j \in J_0}$ be as mentioned in part (b) of Theorem 2.5. The following implications hold

$$\sum_{j \in J_0} \lambda_j \varphi_j \leq \varphi \in X \Rightarrow \sum_{j \in J_0} \lambda_j \int_{\sigma(A)} \varphi_j dE_A = \sum_{j \in J_0} \lambda_j A^j \leq \int_{\sigma(A)} \varphi dE_A \leq \int_{\sigma(A)} |\varphi| dE_A := P(\varphi).$$

The positivity of the spectral measure dE_A has been used. On the other hand, the hypothesis (b), the fact that the scalars λ_j are nonnegative and the preceding evaluation yield

$$\lambda_j B_j \leq \lambda_j A^j \quad \forall j \Rightarrow \sum_{j \in J_0} \lambda_j B_j \leq \sum_{j \in J_0} \lambda_j A^j \leq P(\varphi),$$

where $P(\varphi)$ is defined above. Thus, the implication (b) of Theorem 2.5 is satisfied. Application of the latter theorem leads to the existence of a “feasible solution” T having the property mentioned in part (a) of the present theorem. The last property is a consequence of the preceding one, using the fact that the norm on Y is solid. Namely,

$$\pm T(\varphi) = T(\pm\varphi) \leq \int_{\sigma(A)} |\varphi| dE_A := P(\varphi) \implies$$

$$\|T(\varphi)\| \leq P(\varphi) \implies \|T(\varphi)\| \leq \|P(\varphi)\| \leq \|\varphi\| \cdot \|I\| = \|\varphi\|, \varphi \in X.$$

Hence $\|T\| \leq 1$. This concludes the proof. □

For the Markov moment problem discussed in Theorem 3.2, we recall one of its applications (see [2] and [23]). A convex subset \mathcal{C} of the real vector space X is called finite-simplicial if for any finite dimensional compact $K \subseteq \mathcal{C}$, there is a finite dimensional simplex S such that $K \subseteq S \subseteq \mathcal{C}$. Information about finite-dimensional simplexes can be found in [18].

Theorem 3.6 (see [2], Corollary 3.5). *Let X be a vector space, $\mathcal{C} \subseteq X$ be a finite-simplicial set, $f : \mathcal{C} \rightarrow \mathbb{R}$ be a convex function, $g : \mathcal{C} \rightarrow \mathbb{R}$ be a concave function, such that $f \leq g$ on \mathcal{C} . There exists an affine function $h : \mathcal{C} \rightarrow \mathbb{R}$, satisfying the sandwich condition $f \leq h \leq g$ on \mathcal{C} .*

In Theorem 3.6, the novelty is that a finite-simplicial set $\mathcal{C} \subseteq X$ may be unbounded in any locally convex topology on \mathcal{C} . Next, we review the statement of a topological version of the preceding theorem.

Theorem 3.7 (see [23], Theorem 4). *Let X be an ordered Banach space. Assume that the positive cone X_+ is finite-simplicial and there exists $x_0 \in X_+$ such that $X_+ - x_0$ contains a balanced and absorbing convex subset. Let*

$$f, -g : X_+ \rightarrow \mathbb{R}$$

be convex continuous functions such that $f \leq g$. Assume also that $f(0) = g(0) = 0$. Then there exists a continuous linear functional $L : X \rightarrow \mathbb{R}$ such that $f \leq L \leq g$ on X_+ .

4. Polynomial approximation on unbounded subsets and the classical Markov moment problem

We recall the classical formulation of the moment problem, under the terms of Stieltjes (see the basic book of Akhiezer [1] for details): find the repartition of the positive mass on the nonnegative semi-axis if the moments of arbitrary orders j ($j = 0, 1, 2, \dots$) are given. More precisely, in the Stieltjes moment problem, a sequence of real numbers $(y_j)_{j \geq 0}$ is given and one looks for a nondecreasing real function $\sigma(t)$ ($t \geq 0$), which satisfies the moment conditions:

$$\int_0^\infty t^j d\sigma = y_j, \quad (j = 0, 1, 2, \dots)$$

This is a one-dimensional moment problem on an unbounded interval. Namely, it is an interpolation problem with the constraint on the positivity of the measure $d\sigma$. The numbers $y_j, j \in \mathbb{N} = \{0, 1, 2, \dots\}$ are called the moments of the measure $d\sigma$. The present section concerns firstly the existence problem. However, the uniqueness is studied as well. The moment problem is an inverse problem: we are looking for an unknown measure, starting from its moments. The direct problem might be: being given the measure $d\sigma$, compute its moments $\int_0^\infty t^j d\sigma, j = 0, 1, 2, \dots$. The connection with the positive polynomials and extensions of linear positive functional and operators is quite clear. Namely, if one takes $\varphi_j(t) := t^j, j \in \mathbb{N}, t \in [0, +\infty)$, \mathcal{P} the vector space of polynomials with real coefficients and

$$T_0 : \mathcal{P} \rightarrow \mathbb{R}, T_0 \left(\sum_{j \in J_0} \alpha_j \varphi_j \right) := \sum_{j \in J_0} \alpha_j y_j, \tag{2}$$

where $J_0 \subset \mathbb{N}$ is a finite subset, then the moment conditions $T_0(\varphi_j) = y_j, j \in \mathbb{N}$ are clearly verified. It remains to check whether the linear form T_0 has nonnegative values at all nonnegative polynomials. If this condition is satisfied, due to Haviland theorem, there exists a positive regular Borel measure μ on $[0, +\infty)$ such that $\int_0^{+\infty} t^j d\mu = y_j, j \in \mathbb{N}$. In other words, the measure μ satisfies the condition

$$\int_0^{+\infty} p d\mu = T_0(p), \quad p \in \mathcal{P}.$$

The multidimensional moment problem is formulated similarly. Let $n \in \mathbb{N}, n \geq 2$, and $F \subseteq \mathbb{R}^n$ be a closed subset. One denotes

$$\varphi_j(t) = t^j = t_1^{j_1} \cdots t_n^{j_n}, \quad j = (j_1, \dots, j_n) \in \mathbb{N}^n, \quad t = (t_1, \dots, t_n) \in F, \quad \mathbb{R}_+ = [0, +\infty), \quad \mathcal{P}_+(F) = \{p \in \mathcal{P}; p(t) \geq 0 \quad \forall t \in F\}.$$

Being given a sequence $(y_j)_{j \in \mathbb{N}^n}$ of real numbers, the existence moment problem means to find a necessary and sufficient condition on the numbers $y_j, j \in \mathbb{N}^n$ for the existence of a positive regular Borel measure μ on F such that

$$\int_F t^j d\mu = y_j, \quad j \in \mathbb{N}^n. \tag{3}$$

Due to Haviland theorem [11] (which works in more than one dimension), the desired condition is

$$T_0(p) \geq 0 \quad \text{for all } p \in \mathcal{P}_+(F), \tag{4}$$

where T_0 is defined by (2) and $j \in \mathbb{N}^n$. It is well-known that there exists nonnegative polynomials on the entire space $\mathbb{R}^n, n \geq 2$, which are not sums of squares of polynomials (see [3,31]), unlike the case $n = 1$, where $p \in \mathcal{P}_+(\mathbb{R}) \iff p = p_1^2 + p_2^2$ for some $p_1, p_2 \in \mathbb{R}[t] = \mathcal{P}$. Also, for $p \in \mathcal{P}$, we have $p(t) \geq 0$ for all $t \in [0, +\infty)$ if and only if $p(t) = p_1^2(t) + tp_2^2(t) \quad \forall t \in [0, +\infty)$ for some $p_1, p_2 \in \mathcal{P}$. These well-known facts make possible writing condition (4) for $F = \mathbb{R}$ and for $F = \mathbb{R}_+$ in terms of quadratic forms. For $F = \mathbb{R}^n$, or $F = \mathbb{R}_+^n, n \geq 2$, this is no more possible. If there exists a measure μ such that (3) holds, we say that $(y_j)_{j \in \mathbb{N}^n}$ is a sequence of moments or a moment sequence, and μ is a representing measure for $(y_j)_{j \in \mathbb{N}^n}$. From now on, by a measure on F we mean a positive regular measure on F with finite moments of all orders. A sequence $(y_j)_{j \in \mathbb{N}^n}$ of numbers is called positive semi-definite if for any finite subset $J_0 \subset \mathbb{N}^n$ and for any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, we have

$$\sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \geq 0.$$

Clearly, any moment sequence is positive semi-definite. Indeed, if $(y_j)_{j \in \mathbb{N}^n}$ is a moment sequence, we can write

$$\sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} = \sum_{i,j \in J_0} \lambda_i \lambda_j \int_F t^{i+j} d\mu = \int_F \left(\sum_{j \in J_0} \lambda_j t^j \right)^2 d\mu \geq 0,$$

For $n = 1$, the converse is true since any non-negative polynomial on \mathbb{R} is a sum of (two) squares of polynomials and a square of a polynomial can be written as

$$\left(\sum_{j \in J_0} \lambda_j \varphi_j \right)^2 = \sum_{i, j \in J_0} \lambda_i \lambda_j \varphi_{i+j};$$

then one applies Haviland theorem. For $n \geq 2$, there exist positive semi-definite sequences that are not moment sequences (see [4]). This is one of the main differences between the one-dimensional and multidimensional moment problems. More difficulties appear when one studies the uniqueness of the solution. The present section solves a part of these problems by means of polynomial approximation on Cartesian products of closed unbounded intervals. If in (3), the $y_j, j \in \mathbb{N}^n$, are elements of an ordered vector space Y then the solution is a positive linear operator $T : X \rightarrow Y$, where X is usually ordered Banach space (or even a Banach lattice) of real functions on F , which contains \mathcal{P} and the space $C_0(F)$ of all continuous compactly supported real functions defined on F . For example, if ν is a positive regular Borel measure on F with finite moments of all orders then we can take $X = L^1_\nu(F)$. Additionally, assuming that ν is a moment determinate measure, we recall in Lemma 4.1 that $\mathcal{P}_+(F)$ is dense in $(L^1_\nu(F))_+$. In particular, \mathcal{P} is dense in $X = L^1_\nu(F)$. If Y is an order complete Banach lattice and $T : X = L^1_\nu(F) \rightarrow Y$ is linear and positive, satisfying the interpolation moment conditions

$$T(\varphi_j) = y_j, \quad j \in \mathbb{N}^n, \tag{5}$$

then T is called a solution of the the moment problem defined by (5). The order completeness of Y is necessary for extending T_0 from a suitable chosen subspace of X to the entire space X , preserving positivity property. Then the solution T is unique, since any positive linear operator acting between Banach lattices is continuous (this is also true for T acting on ordered Banach spaces (see [17])), and \mathcal{P} is dense in X . We recall that, by definition, the measure ν is moment determinate (M -determinate) if it is uniquely determinate by its moments $\int_F t^j d\nu, j \in \mathbb{N}^n$. In what follows, we prove and apply polynomial approximation on unbounded subsets in order to characterize as simple as possible the existence and uniqueness of the solution $T : L^1_\nu(F) \rightarrow Y$ which satisfies the moment conditions (5), is positive (linear) and is dominated by a continuous convex operator $P : X \rightarrow Y$ on $X = L^1_\nu(F)$. In most cases, P is a continuous sublinear operator from X to Y . Another interesting sandwich condition on the solution T is $T_1 \leq T \leq T_2$ on X_+ , where $T_1 \leq T_2$ on X_+ . Here T_1 and T_2 are given bounded linear operators from X to Y . In this framework, we can control the norm of the solution T as well. We start by a key approximation lemma.

Lemma 4.1 (see [24], Lemma 3). *Let $F \subseteq \mathbb{R}^n$ be an unbounded closed subset and let ν be an M -determinate measure on F (with finite moments of all natural orders). Then, for any $x \in C_c(F), x(t) \geq 0, \forall t \in F$, there exists a sequence $(p_m)_m, p_m \geq x, m \in \mathbb{N}, p_m \rightarrow x$ in $L^1_\nu(F)$. In particular, it holds that*

$$\lim_m \int_F p_m(t) d\nu = \int_F x(t) d\nu,$$

the cone \mathcal{P}_+ of nonnegative polynomials is dense in $(L^1_\nu(F))_+$, and \mathcal{P} is dense in $L^1_\nu(F)$.

As an application of Lemma 4.1, we derive the next result.

Theorem 4.1 (see [24], Theorem 2). *Let F be a closed unbounded subset of \mathbb{R}^n, Y be an order complete Banach lattice, $(y_j)_{j \in \mathbb{N}^n}$ be a given sequence in Y, ν be a positive regular M -determinate Borel measure on F with finite moments of all orders. Let $T_2 \in B(L^1_\nu(F), Y)$ be a linear positive bounded operator from $L^1_\nu(F)$ to Y . The following statements are equivalent*

- (a). *There exists a unique linear operator $\tilde{T} \in B(L^1_\nu(F), Y)$ such that $\tilde{T}(\varphi_j) = y_j, j \in \mathbb{N}^n, \tilde{T}$ is between $\mathbf{0}$ and T_2 on the positive cone of $L^1_\nu(F)$, and $\|\tilde{T}\| \leq \|T_2\|$.*
- (b). *For any finite subset $J_0 \subset \mathbb{N}^n$, and for any $\{a_j\}_{j \in J_0} \subset \mathbb{R}$, we have*

$$\sum_{j \in J_0} a_j \varphi_j \geq 0 \text{ on } F \implies 0 \leq \sum_{j \in J_0} a_j y_j \leq \sum_{j \in J_0} a_j T_2(\varphi_j).$$

Lemma 4.2. *If $\nu = \nu_1 \times \dots \times \nu_n$ is a product of n M -determinate measures on \mathbb{R} , then any non-negative continuous compactly supported function f in $X = L^1_\nu(\mathbb{R}^n)$ can be approximated with sums of products*

$$p_1 \otimes \dots \otimes p_n, (p_1 \otimes \dots \otimes p_n)(t_1, \dots, t_n) := p_1(t_1) \dots p_n(t_n),$$

where p_j are nonnegative polynomial on the entire real line, $j = 1, \dots, n$, such that any such sum of special polynomials dominates f on \mathbb{R}^n .

To prove Lemma 4.2, one can use approximating Bernstein polynomial of n variables. Then, one applies Lemma 4.1 in each separate variable, for $n = 1$, $F = \mathbb{R}$, and the Fubini theorem.

The next result follows from Lemma 4.2, also using the explicit form of nonnegative polynomials on the entire real axis as a sum of (two) squares of polynomials with real coefficients.

Corollary 4.1 (see [24], Theorem 6). *Let $X = L^1_\nu(\mathbb{R}^n)$, where $\nu = \nu_1 \times \dots \times \nu_n$ is a product of n M -determinate positive regular Borel measure on \mathbb{R} with finite moments of all orders, Y be an order complete Banach lattice, $(y_j)_{j \in \mathbb{N}^n}$ be a sequence in Y , and $T_2 \in B_+(X, Y)$. The following statements are equivalent:*

- (a). *There exists a unique (bounded) linear operator $T \in B(X, Y)$ such that $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, T is between zero and T_2 on the positive cone of X , $\|T\| \leq \|T_2\|$;*
- (b). *For any finite subset $J_0 \subset \mathbb{N}^n$ and for any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, one has*

$$\sum_{j \in J_0} \lambda_j \varphi_j(t) \geq 0 \quad \forall t \in \mathbb{R}^n \implies \sum_{j \in J_0} \lambda_j y_j \in Y_+,$$

and for any finite subsets $J_k \subset \mathbb{N}$, $k = 1, \dots, n$, and for any $\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, $k = 1, \dots, n$, the following inequality holds

$$\sum_{i_1, j_1 \in J_1} \left(\dots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} y_{i_1+j_1, \dots, i_n+j_n} \right) \dots \right) \leq \sum_{i_1, j_1 \in J_1} \left(\dots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} T_2(\varphi_{i_1+j_1, \dots, i_n+j_n}) \right) \dots \right).$$

Theorem 4.2 (see [24], Theorem 7). *Let ν be as in Lemma 4.2, $X = L^1_\nu(\mathbb{R}^n)$, and Y be a Banach lattice. Assume that T is a linear bounded operator from X to Y . The following statements are equivalent:*

- (a). *T is positive on the positive cone of X .*
- (b). *For any finite subsets $J_k \subset \mathbb{N}$, $k = 1, \dots, n$, and for any $\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, $k = 1, \dots, n$, the following inequality holds*

$$0 \leq \sum_{i_1, j_1 \in J_1} \left(\dots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} T(x_{i_1+j_1, \dots, i_n+j_n}) \right) \dots \right).$$

Proof. Note that (b) says that T is positive on the convex cone generated by special positive polynomials $p_1 \otimes \dots \otimes p_n$, each factor of any term in the sum being non-negative on the whole real axis. Consequently, (a) \implies (b) is clear. In order to prove the converse, observe that any non-negative element of X can be approximated by non-negative continuous compactly supported functions. Such functions can be approximated by the sums of tensor products of positive polynomials in each separate variable, the latter being sums of squares. The conclusion is that any nonnegative function from X can be approximated in $X = L^1_\nu(\mathbb{R}^n)$ by the sums of tensor products of squares of polynomials in each separate variable. We know that on such special polynomials, T admits values in Y_+ , according to the condition (b). Now, the desired conclusion is a consequence of the continuity of T , also using the fact that the positive cone of Y is closed. This concludes the proof. □

Lemma 4.3. *If $\nu = \nu_1 \times \dots \times \nu_n$ is a product of n M -determinate positive regular Borel measures on $\mathbb{R}_+ = [0, +\infty)$ with finite moments of all natural orders then any nonnegative continuous compactly supported function f in $X = L^1_\nu(\mathbb{R}_+^n)$ can be approximated by means of sums of tensor products $p_1 \otimes \dots \otimes p_n$, where p_j is a nonnegative polynomial on the real nonnegative semi-axis, $j = 1, \dots, n$, such that all these sums of special nonnegative polynomials dominate f on \mathbb{R}_+^n .*

Corollary 4.2 (see [24], Theorem 5). *Let X be as in Lemma 4.3, $(y_j)_{j \in \mathbb{N}^n}$ be a sequence in Y , where Y is an order complete Banach lattice. Let $T_2 \in B_+(X, Y)$ be a positive (bounded) linear operator. The following statements are equivalent:*

- (a). *There exists a unique (bounded) linear operator $T \in B(X, Y)$ such that $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, T is between zero and T_2 on the positive cone of X , $\|T\| \leq \|T_2\|$.*
- (b). *For any finite subset $J_0 \subset \mathbb{N}^n$ and for any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, one has*

$$\sum_{j \in J_0} \lambda_j \varphi_j(t) \geq 0 \quad \forall t \in \mathbb{R}_+^n \implies \sum_{j \in J_0} \lambda_j y_j \in Y_+;$$

and for any finite subsets $J_k \subset \mathbb{N}$, $k = 1, \dots, n$, and for any $\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, $k = 1, \dots, n$, the following inequality holds

$$\sum_{i_1, j_1 \in J_1} \left(\dots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} y_{i_1+j_1+l_1, \dots, i_n+j_n+l_n} \right) \dots \right) \leq \sum_{i_1, j_1 \in J_1} \left(\dots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \dots \lambda_{i_n} \lambda_{j_n} T_2(\varphi_{i_1+j_1+l_1, \dots, i_n+j_n+l_n}) \right) \dots \right), (l_1, \dots, l_n) \in \{0, 1\}^n$$

Next, we proceed with a stronger sandwich condition on the solution T .

Theorem 4.3 (see [26], Theorem 4). *Let $F \subseteq \mathbb{R}^n$ be a closed unbounded subset, ν be a positive Borel moment determinate measure on F having finite moments of all orders, $X = L^1_\nu(F)$, $\varphi_j(t) = t^j, t \in F, j \in \mathbb{N}^n$. Let Y be an order complete Banach lattice, $(y_j)_{j \in \mathbb{N}^n}$ be a given sequence of elements in Y , T_1 and T_2 be two bounded linear operators from X to Y . Assume that there exists a sub-cone $\mathcal{P}_{++} \subseteq \mathcal{P}_+$ such that each $f \in (C_c(F))_+$ can be approximated in X by a sequence $(p_l)_l, p_l \in \mathcal{P}_{++}, p_l \geq f$ for all l . The following statements are equivalent:*

(a). *There exists a unique (bounded) linear operator $T : X \rightarrow Y, T(\varphi_j) = y_j, j \in \mathbb{N}^n, \mathbf{0} \leq T_1 \leq T \leq T_2$ on X_+ ,*

$$\|T_1\| \leq \|T\| \leq \|T_2\|;$$

(b). *For any finite subset $J_0 \subset \mathbb{N}^n$ and for any $\{\alpha_j; j \in J_0\} \subset \mathbb{R}$, the following implications hold*

$$\sum_{j \in J_0} \alpha_j \varphi_j \in \mathcal{P}_+ \implies \sum_{j \in J_0} \alpha_j T_1(\varphi_j) \leq \sum_{j \in J_0} \alpha_j y_j, \tag{6}$$

$$\sum_{j \in J_0} \alpha_j \varphi_j \in \mathcal{P}_{++} \implies \sum_{j \in J_0} \alpha_j T_1(\varphi_j) \geq 0, \sum_{j \in J_0} \alpha_j y_j \leq \sum_{j \in J_0} \alpha_j T_2(\varphi_j). \tag{7}$$

Proof. We start by observing that the first condition (7) implies the positivity of the bounded linear operator T_1 , via its continuity. Indeed, if $f \in (C_c(F))_+, p_l \in \mathcal{P}_{++}, p_l \geq f$ for all $l, p_l \rightarrow f$ in $L^1_\nu(F)$, then, according to the first condition (7), $T_1(p_l) \geq 0$ for all $l \in \mathbb{N}$ and the continuity of T_1 yields

$$T_1(f) = \lim_l T_1(p_l) \geq 0.$$

Since $(C_c(F))_+$ is dense in X_+ via measure theory reasons, the continuity of T_1 implies $T_1 \geq \mathbf{0}$ on X_+ . Thus, T_1 is a positive linear operator. Next, we define $T_0 : \mathcal{P} \rightarrow Y, T_0\left(\sum_{j \in J_0} \lambda_j \varphi_j\right) = \sum_{j \in J_0} \lambda_j y_j$, where the sums are finite and the coefficients λ_j are arbitrary real numbers. Condition (6) says that $T_0 - T_1 \geq \mathbf{0}$ on \mathcal{P}_+ . If we consider the vector subspace X_1 of X formed by all functions $\psi \in X$ having the modulus $|\psi|$ dominated by a polynomial $p \in \mathcal{P}_+$ on the entire set F , then \mathcal{P} is a majorizing subspace of X_1 and $T_0 - T_1$ is a positive linear operator on \mathcal{P} . Application of Theorem 2.1 leads to the existence of a positive linear extension $U : X_1 \rightarrow Y$, of $T_0 - T_1$. Obviously, X_1 contains $C_c(F) + \mathcal{P} : (p \in \mathcal{P} \implies |p| = \sqrt{1 \cdot p^2} = (1 + p^2)/2 \in \mathcal{P})$. Indeed, since $\varphi \in C(F) \implies |\varphi| \in (C_0(F))_+ \implies |\varphi| \leq b\mathbf{1} \in \mathcal{P}$ (according to Weierstrass' Theorem), we infer that $\varphi \in X_1$; here b is a real number. Hence, $C_0(F) \subset X_1$. Now, let $p \in \mathcal{P}$; we observe that

$$1 + p^2 - 2|p| = (1 - |p|)^2 \geq 0$$

which implies that

$$|p| = \frac{1 + p^2}{2} \in \mathcal{P}.$$

By using the definition of X_1 , one has $\mathcal{P} \subset X_1$. Consequently, $C_c(F) + \mathcal{P} \subset X_1$. Going back to the positive linear extension $U : X_1 \rightarrow Y$ of $T_0 - T_1$, we conclude that $\hat{T}_0 = U + T_1 : X_1 \rightarrow Y$ is an extension of $T_0, \hat{T}_0 \geq T_1$ on $(X_1)_+$, and $\hat{T}_0(p) = T_0(p) \leq T_2(p)$ for all $p \in \mathcal{P}_{++}$, according to the last requirement (7). Our first conclusion is:

$$T_1(p) \leq \hat{T}_0(p) \leq T_2(p) \text{ for all } p \in \mathcal{P}_{++}, \hat{T}_0(\psi) \geq T_1(\psi) \geq 0, \psi \in (X_1)_+. \tag{8}$$

Our next goal is to prove the continuity of \hat{T}_0 on $C_c(F)$. Let $(f_l)_{l \geq 0}$ be a sequence of nonnegative continuous compactly supported functions, such that $f_l \rightarrow \mathbf{0}$ in X_1 , and a sequence of polynomials $p_l \geq f_l \geq \mathbf{0}, p_l \in \mathcal{P}_{++}$ for all l , such that the following convergence result holds: $\|p_l - f_l\|_1 \rightarrow 0, l \rightarrow \infty$. Consequently,

$$\|p_l\|_1 \leq \|p_l - f_l\|_1 + \|f_l\|_1 \rightarrow 0, l \rightarrow \infty.$$

Now, (8) and the continuity of T_1 and T_2 yield:

$$0 \leftarrow T_1(p_l) \leq \hat{T}_0(p_l) \leq T_2(p_l) \rightarrow 0,$$

hence $\hat{T}_0(p_l) \rightarrow 0$. It results:

$$0 \leq T_1(f_l) \leq \hat{T}_0(f_l) \leq \hat{T}_0(p_l) \rightarrow 0,$$

hence $\hat{T}_0(f_l) \rightarrow 0$. If $(g_n)_{n \geq 0}$ is an arbitrary sequence of compactly supported and continuous functions such that $g_n \rightarrow 0$ in X_1 , then $g_n^+ \rightarrow 0, g_n^- \rightarrow 0$. According to what we already have proved, we can write $\hat{T}_0(g_n^+) \rightarrow 0$ and $\hat{T}_0(g_n^-) \rightarrow 0$, which further yield $\hat{T}_0(g_n) \rightarrow 0$. This proves the continuity of \hat{T}_0 on $C_c(F)$ and the subspace $C_c(F)$ is dense in X . Moreover, $(C_c(F))_+$ is dense in X_+ . Hence, there exists a unique continuous linear extension $T \in B(X, Y)$ of \hat{T}_0 . For $f \in (C_c(F))_+$, passing to the limit in (8) when $l \rightarrow \infty, p_l \rightarrow f, p_l \in \mathcal{P}_{++}$ for all l and using the continuity of the three operators T_1, T, T_2 , it results $0 \leq T_1(f) \leq T(f) \leq T_2(f)$. Next, we apply the density of $(C_c(F))_+$ in X_+ (see [29]), which yields

$$0 \leq T_1 \leq T \leq T_2 \text{ on } X_+, \|T_1\| \leq \|T\| \leq \|T_2\|, T(\varphi_j) = T_0(\varphi_j) = y_j, j \rightarrow \mathbb{N}^n.$$

For an arbitrary $\phi \in X$, the following inequalities hold via the preceding remarks:

$$\pm T(\phi) = T(\pm\phi) \leq T(|\phi|) \leq T_2(|\phi|) \implies |T(\phi)| \leq T_2(|\phi|) \implies \|T(\phi)\| \leq \|T_2(|\phi|)\| \leq \|T_2\| \|\phi\|.$$

Therefore, $\|T\| \leq \|T_2\|$ and similarly, $\|T_1\| \leq \|T\|$. The uniqueness of the solution T follows by the density of polynomials in X via continuity of the linear operator T . This ends the proof. \square

Our next goal is to give a result for the Markov moment problem in the space $L_\nu^1(\mathbb{R})$, where ν is a moment determinate measure on \mathbb{R} having finite moments $\int_{\mathbb{R}} t^k d\nu$ of all orders $k \in \mathbb{N}$.

Corollary 4.3. *Let $X = L_\nu^1(\mathbb{R})$, where ν is a moment determinate positive measure on \mathbb{R} . Assume that Y is an arbitrary order complete Banach lattice and $(y_n)_{n \geq 0}$ is a given sequence having its terms in Y . Let T_1 and T_2 be two linear operators from X to Y such that $0 \leq T_1 \leq T_2$ on X_+ . The following statements are equivalent:*

(a). *There exists a unique bounded linear operator T from X to Y , $T_1 \leq T \leq T_2$ on X_+ , $\|T_1\| \leq \|T\| \leq \|T_2\|$, such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$.*

(b). *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$ then*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j T_1(\varphi_{i+j}) \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j T_2(\varphi_{i+j}).$$

Corollary 4.4. *Let ν be a moment determinate positive Borel measure on \mathbb{R} , with finite moments of all orders. Assume that h_1 and h_2 are two functions in $L_\nu^\infty(\mathbb{R})$, such that $0 \leq h_1 \leq h_2$ almost everywhere. Let $(y_n)_{n \geq 0}$ be a given sequence of real numbers. The following statements are equivalent:*

(a). *There exists $h \in L_\nu^\infty(\mathbb{R})$, such that $h_1 \leq h \leq h_2$ ν -almost everywhere, $\int_{\mathbb{R}} t^j h(t) d\nu = y_j$ for all $j \in \mathbb{N}$.*

(b). *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$ then*

$$\sum_{i,j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}} t^{i+j} h_1(t) d\nu \leq \sum_{i,j \in J_0} \lambda_i \lambda_j y_{i+j} \leq \sum_{i,j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}} t^{i+j} h_2(t) d\nu.$$

In cases of Corollaries 4.3 and 4.4, we have $\mathcal{P}_{++} = \mathcal{P}_+$. Going further to the multidimensional case, when nonnegative polynomials are not usually expressible as sums of squares, the inclusion $\mathcal{P}_{++} \subset \mathcal{P}_+$ is strict. In the case of $F = \mathbb{R}_+^n$ (respectively $F = \mathbb{R}^n$), $n \geq 2$, the cone \mathcal{P}_{++} consists of all polynomials which are sums of products of the form

$$p_1 \otimes \cdots \otimes p_n,$$

where each $p_j, j = 1, \dots, n$, is a nonnegative polynomial on \mathbb{R}_+ (respectively on \mathbb{R}), hence is expressible by means of sums of squares of polynomials of one variable. Proceeding this way, the last conditions of (7) can be written in terms of quadratic forms (see Corollary 4.5 stated as follows).

Corollary 4.5. *Let $\nu = \nu_1 \times \cdots \times \nu_n, n \geq 2, \nu_j$ being an M -determinate (moment determinate) positive regular Borel measure on $\mathbb{R}, j = 1, \dots, n, X = L_\nu^1(\mathbb{R}^n), \varphi_j(t) = t^j, t \in \mathbb{R}^n, j \in \mathbb{N}^n$. Additionally, assume that ν_j has finite moments of all orders, $j = 1, \dots, n$. Let Y be an order complete Banach lattice, $(y_j)_{j \in \mathbb{N}^n}$ be a given sequence of elements in Y , and T_1, T_2 be two bounded linear operators from X to Y . The following statements are equivalent:*

(a). *There exists a unique (bounded) linear operator $T : X \rightarrow Y$, $T(\varphi_j) = y_j$, $j \in \mathbb{N}^n$, $\mathbf{0} \leq T_1 \leq T \leq T_2$ on X_+ ,*

$$\|T_1\| \leq \|T\| \leq \|T_2\|.$$

(b). *For any finite subset $J_0 \subset \mathbb{N}^n$ and for any $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, the following implication holds:*

$$\sum_{j \in J_0} \lambda_j \varphi_j \in \mathcal{P}_+ \implies \sum_{j \in J_0} \lambda_j T_1(\varphi_j) \leq \sum_{j \in J_0} \lambda_j y_j;$$

and for any finite subsets $J_k \subset \mathbb{N}$, $k = 1, \dots, n$, and for any $\{\lambda_{j_k}\}_{j_k \in J_k} \subset \mathbb{R}$, the following inequalities hold:

$$0 \leq \sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} T_1(\varphi_{i_1+j_1, \dots, i_n+j_n}) \right) \cdots \right);$$

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} y_{i_1+j_1, \dots, i_n+j_n} \right) \cdots \right) \leq$$

$$\sum_{i_1, j_1 \in J_1} \left(\cdots \left(\sum_{i_n, j_n \in J_n} \lambda_{i_1} \lambda_{j_1} \cdots \lambda_{i_n} \lambda_{j_n} T_2(\varphi_{i_1+j_1, \dots, i_n+j_n}) \right) \cdots \right).$$

The last consequences of Theorem 4.3 refer to the Markov moment problem on \mathbb{R}_+ .

Corollary 4.6. *Let $X = L^1_\nu(\mathbb{R}_+)$, where ν is a moment determinate positive Borel measure on \mathbb{R}_+ . Assume that Y is an arbitrary order complete Banach lattice, and $(y_n)_{n \geq 0}$ is a given sequence having its terms in Y . Let T_1 and T_2 be two linear operators from X to Y such that $\mathbf{0} \leq T_1 \leq T_2$ on X_+ . As usual, we denote $\varphi_j(t) = t^j$, $j \in \mathbb{N}$, $t \in \mathbb{R}_+$. The following statements are equivalent:*

(a). *There exists a unique bounded linear operator T from X to Y , $T_1 \leq T \leq T_2$ on X_+ , $\|T_1\| \leq \|T\| \leq \|T_2\|$, such that $T(\varphi_n) = y_n$ for all $n \in \mathbb{N}$.*

(b). *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$ then*

$$\sum_{i, j \in J_0} \lambda_i \lambda_j T_1(\varphi_{i+j+k}) \leq \sum_{i, j \in J_0} \lambda_i \lambda_j y_{i+j+k} \leq \sum_{i, j \in J_0} \lambda_i \lambda_j T_2(\varphi_{i+j+k}), \quad k \in \{0, 1\}.$$

Corollary 4.7. *Let ν be a moment determinate positive Borel measure on \mathbb{R}_+ , with finite moments of all orders. Assume that h_1, h_2 are two functions in $L^\infty_\nu(\mathbb{R}_+)$, such that $\mathbf{0} \leq h_1 \leq h_2$ almost everywhere. Let $(y_n)_{n \geq 0}$ be a given sequence of real numbers. The following statements are equivalent:*

(a). *There exists $h \in L^\infty_\nu(\mathbb{R}_+)$, such that $h_1 \leq h \leq h_2$ ν -almost everywhere and $\int_{\mathbb{R}_+} t^j h(t) d\nu = y_j$ for all $j \in \mathbb{N}$.*

(b). *If $J_0 \subset \mathbb{N}$ is a finite subset and $\{\lambda_j; j \in J_0\} \subset \mathbb{R}$, then*

$$\sum_{i, j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_1(t) d\nu \leq \sum_{i, j \in J_0} \lambda_i \lambda_j y_{i+j+k} \leq \sum_{i, j \in J_0} \lambda_i \lambda_j \int_{\mathbb{R}_+} t^{i+j+k} h_2(t) d\nu, \quad k \in \{0, 1\}.$$

5. On the truncated moment problem

This section is subdivided into two parts, namely: (i) General type results and existence of solutions (ii) Perturbations of moments and the corresponding perturbations of solutions.

General type results and existence of solutions

The truncated moment problem is important in mathematics because it involves only a finite number of moments (of limited order), which are assumed to be known (or given, or measurable); therefore, it can be related to optimization problems [12] as well as to constructive methods for finding solutions [9, 10, 25, 26] (see [25, 26] for the existence of a polynomial solution). We start by recalling the truncated (reduced) Markov moment problems on a closed, bounded or unbounded subset F of \mathbb{R}^n , where $n \geq 1$ is an integer. We denote by $\mathbb{R}_d[t_1, \dots, t_n]$ the real vector subspace of all polynomial functions \mathcal{P} of n real variables, with real coefficients, generated by $t^k = t_1^{k_1} \cdots t_n^{k_n}$, $k_i \in \{0, 1, \dots, d\}$, $i = 1, \dots, n$, where $d \geq 1$ is a fixed integer. The dimension of this subspace is clearly equal to $N = (d + 1)^n$. Given a finite set $\{m_k\}_{0 \leq k_i \leq d; i=1, \dots, n}$ of real numbers and a

positive Borel measure μ on F with finite absolute moments of all orders smaller than or equal to d (i.e., $\int_F |t|^k d\mu < \infty$ for all $k = (k_1, \dots, k_n) \in \mathbb{N}^n$ with $k_i \leq d, i = 1, \dots, n$), one studies the existence and, eventually construction or approximation of a Lebesgue measurable real function $h \in L^\infty_\mu(F)$, satisfying the moment conditions

$$\int_F t^k h(t) d\mu = m_k, \quad k_i \leq d, \quad i = 1, \dots, n,$$

and $0 \leq h \leq b \cdot 1$, for some positive number b . An application of Theorem 3.1 and standard measure theory arguments (see [29]) lead to the following existence type results for scalar-valued solution of the truncated moment problem. Let $F \subseteq \mathbb{R}^n$ be a closed unbounded subset and μ be a positive regular Borel measure on F .

Theorem 5.1 (see [25], Theorem 2.2). *Assume that all absolute moments $\int_F |t|^k d\mu < \infty$ for $k = (k_1, \dots, k_n)$ with $k_i \leq d, i = 1, \dots, n$, are finite. Let $\{m_k\}_{k_i \leq d; i=1, \dots, n}$ be a given (finite) sequence of real numbers. The following statements are equivalent:*

(a). *There exists $h \in L^\infty_\mu(F)$, $0 \leq h \leq 1$, $\int_F t^k h(t) d\mu = m_k, k_i \leq d, i = 1, \dots, n$.*

(b). *For any family of scalars $\{a_k\}_{k_i \leq d; i=1, \dots, n}$ the inequality*

$$\sum_{k_i \leq d; i=1, \dots, n} a_k \varphi_k \leq \varphi \in L^1_\mu(F)$$

implies

$$\sum_{k_i \leq d; i=1, \dots, n} a_k m_k \leq \int_F |\varphi(t)| d\mu = \|\varphi\|_{1, \mu}.$$

Theorem 5.2 (see [25], Theorem 2.4). *Let $p \in (1, \infty)$, $X = L^p_\mu(F)$, $\{m_k\}_{k_i \leq d; i=1, \dots, n}$ be a given (finite) sequence of real numbers, and q be the conjugate of p . The following statements are equivalent:*

(a). *There exists $h \in L^q_\mu(F)$, $0 \leq h(t)$ a.e., $\|h\|_{q, \mu} \leq 1$, $\int_F t^k h(t) d\mu = m_k, k_i \leq d, i = 1, \dots, n$.*

(b). *For any family of scalars $\{a_k\}_{k_i \leq d; i=1, \dots, n}$ the inequality*

$$\sum_{k_i \leq d; i=1, \dots, n} a_k \varphi_k \leq \varphi \in L^p_\mu$$

implies

$$\sum_{k_i \leq d; i=1, \dots, n} a_k m_k \leq \|\varphi\|_{p, \mu}.$$

Perturbations of moments and the corresponding perturbations of solutions

Assume now that the moments m_k , where $k = (k_1, \dots, k_n) \in \mathbb{N}_0^n, 0 \leq k_i \leq d, i \in \{1, 2, \dots, n\}$ are exact, determined in a training stage, when no external influence can occur. Also, assume that the moments in real time stage, denoted by v_k , where $k_i \leq d, i = 1, \dots, n$, can be measured, but errors may occur due to external influences. The vector subspace of polynomials involved this way has the dimension $(d + 1)^n$. Assume now that there exists h, w in $L^2_\mu(F)$ such that

$$\int_S t^k h(t) d\mu = m_k, \quad \int_S t^k w(t) d\mu = v_k$$

for all multi-index $k \in \mathbb{N}_0^n$ having all components $0 \leq k_i \leq d, i = 1, \dots, n$. Let $\{e_k\}_{0 \leq k_i \leq d}$ be the orthogonal system of polynomials having unit norms, obtained from the system $\{\varphi_k\}_{0 \leq k_i \leq d}$ via Gram-Schmidt process, in the space $L^2_\mu(F)$. Denote by $P_d : L^2_\mu(F) \rightarrow \text{Span}\{e_k; k_i \in \{0, \dots, d\}, i = 1, \dots, n\}$ the orthogonal projection. It holds that

$$e_k = \sum_{l \leq k} c_{l,k} \varphi_l,$$

where $l \leq k$ means $l_i \leq k_i, i = 1, \dots, n$, and the coefficients $c_{l,k}$ are known from Gram-Schmidt process. It yields

$$\langle P_d(h) - P_d(w), e_k \rangle = \sum_{l \leq k} c_{l,k} \langle P_d(h) - P_d(w), \varphi_l \rangle = \sum_{l \leq k} c_{l,k} (m_l - v_l).$$

Thus,

$$\|P_d(w) - P_d(h)\|_{2, \mu}^2 = \sum_{k_i \leq d} \langle P_d(h) - P_d(w), e_k \rangle^2 = \sum_{k_i \leq d} \left(\sum_{l \leq k} c_{l,k} (m_l - v_l) \right)^2. \tag{9}$$

Hence, one determines the integral mean of the square $(P_d(w) - P_d(h))^2$ on the left hand side of (9), in terms of the squares of the errors $|m_k - v_k|, (k_i \leq d \text{ for all } i \in \{1, \dots, n\})$.

6. Conclusion

Some Hahn-Banach type results are reviewed and their recently published applications are discussed. Also, their direct applications to concrete spaces are stated and some of them are proved. One of the applications of Theorem 3.2 consists of proving an unexpected sandwich theorem (see Theorems 3.6 and 3.7). An example of a Mazur-Orlicz type theorem on concrete spaces is emphasized in Theorem 3.5. On the other hand, existence and uniqueness of the solutions for the classical Markov full moment problem are emphasized, by means of the polynomial approximation on unbounded subsets, accompanied by the extension result stated in Theorem 2.1. In the last part of the paper (Section 5), a few aspects of the truncated moment problem are discussed.

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