

Research Article

Exponential Stability for a Suspension Bridge in Thermoelasticity of Type III TheoryCarlos Raposo^{1,‡}, Edvaldo Melo Souza², Sebastião Cordeiro³, Juarez dos Santos Azevedo⁴¹*Faculty of Mathematics, Federal University of Pará, Salinópolis, Pará, Brazil*²*Department of Mathematics, Federal University of Pará, Belém, Pará, Brazil*³*Department of Mathematics, Federal University of Pará, Abaetetuba, Pará, Brazil*⁴*Institute of Science, Technology and Innovation, Federal University of Bahia, Camaçari, Bahia, Brazil***Abstract**

We consider a single-span roadbed modeled as a Timoshenko thermoelastic beam in the context of type III thermoelasticity for heat conduction, with frictional damping and suspenders. We are interested in stabilizing the system. Our study is conducted within the framework of semigroup theory. For the existence and uniqueness of the solution, we apply the Lumer–Phillips theorem, and for the exponential stability, we use the Gearhart–Huang–Prüss theorem. The results are obtained without restrictions on the system’s coefficients. In order to illustrate the system’s dissipative behavior and exponential stability, we present numerical simulations.

Keywords: Timoshenko system; semigroup theory; type III thermoelasticity; exponential stability.

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1. Introduction

In the present paper, our main goal is to study the exponential stability of the coupled system

$$\rho u_{tt} - \alpha u_{xx} - \lambda(\varphi - u) + \beta_1 \theta_{xt} + \gamma_1 u_t = 0 \quad \text{in } (0, L) \times (0, \infty), \quad (1)$$

$$\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 \varphi_t = 0 \quad \text{in } (0, L) \times (0, \infty), \quad (2)$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma_3 \psi_t = 0 \quad \text{in } (0, L) \times (0, \infty), \quad (3)$$

$$\rho_3 \theta_{tt} - \delta \theta_{xx} + \beta_1 u_{tx} - \kappa \theta_{txx} = 0 \quad \text{in } (0, L) \times (0, \infty), \quad (4)$$

subject to initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (5)$$

$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad (6)$$

$$\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad (7)$$

$$\theta(x, 0) = \theta_0(x), \quad \theta_t(x, 0) = \theta_1(x), \quad (8)$$

and Dirichlet boundary conditions

$$u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0, \quad (9)$$

where $\rho, \alpha, \lambda, \beta_1, \gamma_1, \rho_1, k, \kappa, \gamma_2, \rho_2, b, \gamma_3, \rho_3, \delta$ are positive constants whose physical meanings are well known.

The system (1)–(4) models a suspension bridge. A suspension bridge (see Figure 1.1) is a mechanical structure that carries vertical loads through the main cable, modeled as an elastic string according to Newton's second law,

$$u_{tt} - \alpha u_{xx} = 0,$$

which is coupled to the deck by suspension cables, where x denotes the distance along the centerline of the deck in its equilibrium configuration and t denotes the time variable. The functions φ and ψ denote, respectively, the transverse displacement and the rotation angle of the deck, which is modeled as a Timoshenko beam of length L .

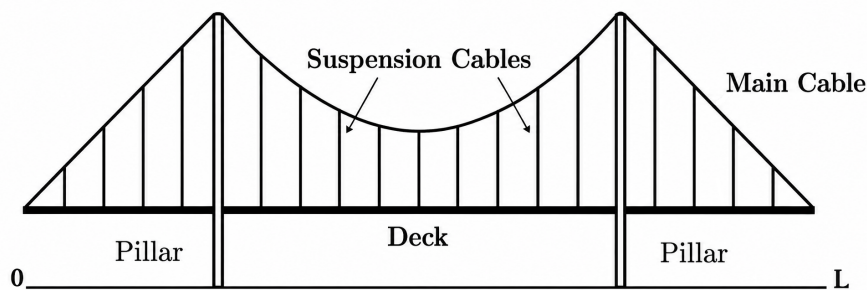


Figure 1.1: Suspension bridge. This figure is taken from the reference [28].

It is well known that Timoshenko beams, when partially damped, possess exponential stability if and only if $\frac{k}{\rho_1} = \frac{b}{\rho_2}$ (see [34]), which imposes a restriction on the system coefficients and leads to the equal wave velocity paradox. This restriction arises because the Timoshenko beam has two natural frequencies, leading to the non-physical condition known as the second spectrum. The second spectrum of frequencies was discovered in a series of analytical investigations (see [15, 21]). On the other hand, when the Timoshenko beam is completely damped, exponential stability occurs independently of the wave speed or any other relationship between the coefficients of the equations; see [30]. In this work, we consider a model completely damped by friction, thereby eliminating restrictions on the coefficients. In addition, we consider the temperature θ in the main cable, where θ is governed by the thermoelastic theory of type III.

Green and Naghdi [8–10] proposed three new thermoelastic theories in the 1990s: type I, which recovers the well-known theory based on the Fourier law, where the system is dissipative and the energy decays exponentially. However, this theory has the shortcoming of an infinite speed of propagation, which is physically unacceptable for thermoviscoelastic materials. In type II theory, heat is allowed to propagate by means of thermal waves and without dissipating energy; in this case, the system is conservative. Type III theory is the most general formulation, considering the former two as limit cases.

Recent contributions on suspension bridges can be found in [3, 12, 13, 18, 22, 29, 35, 36], and on type III thermoelasticity in [1, 2, 4, 6, 17, 19, 20, 23, 26, 31, 33, 38, 39]. Our goal is to study the stability of the model, that is, the asymptotic behavior of the solution of the system (1)–(4). The norm of the solution of a PDE is controlled by the energy of the system. Therefore, to obtain the decay rate of the solution,

we generally derive an estimate for the energy. The central idea is to establish an estimate for the total energy $E(t)$ of the model in the form

$$E(t) \leq CE(0)e^{-wt}, \quad \forall t \geq 0, \quad w > 0, \quad (10)$$

or, equivalently, to establish the exponential stability

$$\|S(t)\| \leq Ce^{-wt}, \quad \forall t \geq 0, \quad w > 0, \quad (11)$$

of the semigroup $S(t)$ generated by the system. For a long time, it remained open whether these two estimates were equivalent; however, it is now known (see [37]) that they are equivalent. In this paper, we use semigroup theory and, to prove the exponential decay of the solution, we use (11).

The paper is organized as follows: In Section 2, we present the energy of the system and prove that it is dissipative. In Section 3, we use the semigroup approach to prove the well-posedness of the system. In Section 4, we prove that the semigroup associated with the system is exponentially stable and, as a consequence, the solution decays exponentially. Finally, in Section 5, we illustrate the dissipative character and exponential stability established in the previous sections by presenting numerical experiments for a reduced one-mode approximation of the Timoshenko–thermoelastic system described by equations (1)–(4). The numerical method relies on a one-mode Galerkin approximation.

2. The Energy of the System

In this section, we define the energy of the system and prove that it is dissipative. For this goal, multiplying (1) by u_t , (2) by φ_t , (3) by ψ_t , (4) by θ_t and integrating over $(0, L)$, we obtain

$$\begin{aligned} \rho \int_0^L u_{tt}u_t dx - \alpha \int_0^L u_{xx}u_t dx - \lambda \int_0^L (\varphi - u)u_t dx + \beta_1 \int_0^L \theta_{xt}u_t dx + \gamma_1 \int_0^L |u_t|^2 dx &= 0, \\ \rho_1 \int_0^L \varphi_{tt}\varphi_t dx - k \int_0^L (\varphi_x + \psi)_x \varphi_t dx + \lambda \int_0^L (\varphi - u)\varphi_t dx + \int_0^L \gamma_2 |\varphi_t|^2 dx &= 0, \\ \rho_2 \int_0^L \psi_{tt}\psi_t dx - b \int_0^L \psi_{xx}\psi_t dx + k \int_0^L (\varphi_x + \psi)\psi_t dx + \gamma_3 \int_0^L |\psi_t|^2 dx &= 0, \\ \rho_3 \int_0^L \theta_{tt}\theta_t dx - \delta \int_0^L \theta_{xx}\theta_t dx + \beta_1 \int_0^L u_{tx}\theta_t dx - \kappa \int_0^L \theta_{txx}\theta_t dx &= 0. \end{aligned}$$

Integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned} \rho \int_0^L u_{tt}u_t dx + \alpha \int_0^L u_x u_{xt} dx - \lambda \int_0^L (\varphi - u)u_t dx - \beta_1 \int_0^L \theta_t u_{tx} dx + \gamma_1 \int_0^L |u_t|^2 dx &= 0, \\ \rho_1 \int_0^L \varphi_{tt}\varphi_t dx + k \int_0^L (\varphi_x + \psi)\varphi_{xt} dx + \lambda \int_0^L (\varphi - u)\varphi_t dx + \int_0^L \gamma_2 |\varphi_t|^2 dx &= 0, \\ \rho_2 \int_0^L \psi_{tt}\psi_t dx + b \int_0^L \psi_x \psi_{xt} dx + k \int_0^L (\varphi_x + \psi)\psi_t dx + \gamma_3 \int_0^L |\psi_t|^2 dx &= 0, \\ \rho_3 \int_0^L \theta_{tt}\theta_t dx + \delta \int_0^L \theta_x \theta_{xt} dx + \beta_1 \int_0^L u_{tx}\theta_t dx + \kappa \int_0^L |\theta_{xt}|^2 dx &= 0. \end{aligned}$$

Adding the previous equations, we obtain the following equation:

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \rho \int_0^L |u_t|^2 dx + \frac{1}{2} \frac{d}{dt} \alpha \int_0^L |u_x|^2 dx + \frac{1}{2} \frac{d}{dt} \lambda \int_0^L |\varphi - u|^2 dx + \gamma_1 \int_0^L |u_t|^2 dx \\ & + \frac{1}{2} \frac{d}{dt} \rho_1 \int_0^L |\varphi_t|^2 dx + \frac{1}{2} \frac{d}{dt} k \int_0^L |\varphi_x + \psi|^2 dx + \int_0^L \gamma_2 |\varphi_t|^2 dx \\ & + \frac{1}{2} \frac{d}{dt} \rho_2 \int_0^L |\psi_t|^2 dx + \frac{1}{2} \frac{d}{dt} b \int_0^L |\psi_x|^2 dx + \gamma_3 \int_0^L |\psi_t|^2 dx \\ & + \frac{1}{2} \frac{d}{dt} \rho_3 \int_0^L |\theta_t|^2 dx + \frac{1}{2} \frac{d}{dt} \delta \int_0^L |\theta_x|^2 dx + \kappa \int_0^L |\theta_{xt}|^2 dx = 0. \end{aligned} \quad (12)$$

We denote the norm in $L^2(0, L)$ by $\|\cdot\|$. We define the energy of the system by

$$\begin{aligned} E(t) = & \frac{1}{2} \left(\rho \|u_t\|^2 + \rho_1 \|\varphi_t\|^2 + \rho_2 \|\psi_t\|^2 + \rho_3 \|\theta_t\|^2 + \alpha \|u_x\|^2 + b \|\psi_x\|^2 + \delta \|\theta_x\|^2 \right. \\ & \left. + \lambda \|\varphi - u\|^2 + k \|\varphi_x + \psi\|^2 \right). \end{aligned} \quad (13)$$

Now, from (12), we obtain

$$E'(t) = - \left(\gamma_1 \|u_t\|^2 + \gamma_2 \|\varphi_t\|^2 + \gamma_3 \|\psi_t\|^2 + \kappa \|\theta_{tx}\|^2 \right). \quad (14)$$

Therefore, the system is dissipative.

3. Existence and Uniqueness of Solution

In this section, we establish the well-posedness using semigroup theory. Let $U = (u, v, \varphi, w, \psi, z, \theta, p)^T$, where $u_t = v$, $\varphi_t = w$, $\psi_t = z$ and $\theta_t = p$. We define the phase space

$$\mathcal{H} = [H_0^1(0, L) \times L^2(0, L)]^4.$$

Here, $\mathcal{H}$ is a Hilbert space with the inner product

$$\begin{aligned} \langle U, \bar{U} \rangle_{\mathcal{H}} = & \int_0^L \left[\rho v \bar{v} + \rho_1 w \bar{w} + \rho_2 z \bar{z} + \rho_3 p \bar{p} + \alpha u_x \bar{u}_x + \lambda (\varphi - u) (\bar{\varphi} - \bar{u}) + b \psi_x \bar{\psi}_x + \delta \theta_x \bar{\theta}_x \right. \\ & \left. + k (\varphi_x + \psi) (\bar{\varphi}_x + \bar{\psi}) \right] dx, \end{aligned}$$

where

$$U = (u, v, \varphi, w, \psi, z, \theta, p)^T, \bar{U} = (\bar{u}, \bar{v}, \bar{\varphi}, \bar{w}, \bar{\psi}, \bar{z}, \bar{\theta}, \bar{p})^T \in \mathcal{H}.$$

The norm in $\mathcal{H}$ is given by

$$\|U\|_{\mathcal{H}}^2 = \int_0^L \left[\rho |v|^2 + \rho_1 |w|^2 + \rho_2 |z|^2 + \rho_3 |p|^2 + \alpha |u_x|^2 + \lambda |\varphi - u|^2 + b |\psi_x|^2 + \delta |\theta_x|^2 + k |\varphi_x + \psi|^2 \right] dx.$$

For $U \in \mathcal{H}$, we rewrite the system (1)–(4) in a first-order Cauchy problem

$$\begin{cases} U_t - \mathcal{A}U = 0, & \text{for } t > 0, \\ U(0) = U_0, \end{cases} \quad (15)$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\alpha}{\rho} \partial_{xx} - \frac{\lambda}{\rho} I & -\frac{\gamma_1}{\rho} I & \frac{\lambda}{\rho} I & 0 & 0 & 0 & 0 & 0 & -\frac{\beta_1}{\rho} \partial_x \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ \frac{\lambda}{\rho_1} I & 0 & \frac{k}{\rho_1} \partial_{xx} - \frac{\lambda}{\rho_1} I & -\frac{\gamma_2}{\rho_1} I & \frac{k}{\rho_1} \partial_x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & -\frac{k}{\rho_2} \partial_x & 0 & \frac{b}{\rho_2} \partial_{xx} - \frac{k}{\rho_2} I & -\frac{\gamma_3}{\rho_2} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & -\frac{\beta_1}{\rho_3} \partial_x & 0 & 0 & 0 & 0 & \frac{\delta}{\rho_3} \partial_{xx} & \frac{\kappa}{\rho_3} \partial_{xx} & 0 \end{bmatrix},$$

with

$$D(\mathcal{A}) = \left\{ U \in \mathcal{H} \left| \begin{array}{l} u, \varphi, \psi \in H^2(0, L) \cap H_0^1(0, L), \\ v, w, z, p \in H_0^1(0, L), \\ \delta\theta + \kappa p \in H^2(0, L) \cap H_0^1(0, L). \end{array} \right. \right\}.$$

It follows directly from the properties of Sobolev spaces that $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Proposition 3.1. *The operator $\mathcal{A}$ is dissipative.*

Proof. For $U = (u, v, \varphi, w, \psi, z, \theta, p)^T \in D(\mathcal{A})$, we have

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} v \\ \frac{\alpha}{\rho} u_{xx} + \frac{\lambda}{\rho} (\varphi - u) - \frac{\beta_1}{\rho} p_x - \frac{\gamma_1}{\rho} v \\ w \\ \frac{k}{\rho_1} (\varphi_x + \psi)_x - \frac{\lambda}{\rho_1} (\varphi - u) - \frac{\gamma_2}{\rho_1} w \\ z \\ \frac{b}{\rho_2} \psi_{xx} - \frac{k}{\rho_2} (\varphi_x + \psi) - \frac{\gamma_3}{\rho_2} z \\ p \\ \frac{1}{\rho_3} [(\delta\theta + \kappa p)_{xx} - \beta_1 v_x] \end{pmatrix}, \begin{pmatrix} u \\ v \\ \varphi \\ w \\ \psi \\ z \\ \theta \\ p \end{pmatrix} \right\rangle$$

Hence, we have

$$\begin{aligned}
 \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = & \alpha \int_0^L u_{xx} v dx + \lambda \int_0^L (\varphi - u) v dx - \beta_1 \int_0^L p_x v dx - \gamma_1 \int_0^L v v dx \\
 & + k \int_0^L (\varphi_x + \psi)_x w dx - \lambda \int_0^L (\varphi - u) w dx - \gamma_2 \int_0^L w w dx \\
 & + b \int_0^L \psi_{xx} z dx - k \int_0^L (\varphi_x + \psi) z dx - \gamma_3 \int_0^L z z dx \\
 & - \beta_1 \int_0^L v_x p dx + \int_0^L (\delta \theta + \kappa p)_{xx} p dx + \delta \int_0^L p_x \theta_x dx \\
 & + \alpha \int_0^L v_x u_x dx + \lambda \int_0^L (w - v)(\varphi - u) dx + b \int_0^L z_x \psi_x dx \\
 & + k \int_0^L (w_x + z)(\varphi_x + \psi) dx.
 \end{aligned}$$

Integrating by parts over $(0, L)$, we have

$$\begin{aligned}
 \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = & -\alpha \int_0^L u_x v_x dx - \lambda \int_0^L (\varphi - u)(w - v) dx + \beta_1 \int_0^L p v_x dx - \gamma_1 \int_0^L v v dx \\
 & - k \int_0^L (\varphi_x + \psi)(w_x + z) dx - \gamma_2 \int_0^L w w dx - b \int_0^L \psi_x z_x dx \\
 & - \gamma_3 \int_0^L z z dx - \beta_1 \int_0^L v_x p dx - \int_0^L (\delta \theta + \kappa p)_x p_x dx + \delta \int_0^L p_x \theta_x dx \\
 & + \alpha \int_0^L v_x u_x dx + \lambda \int_0^L (w - v)(\varphi - u) dx + b \int_0^L z_x \psi_x dx \\
 & + k \int_0^L (w_x + z)(\varphi_x + \psi) dx.
 \end{aligned}$$

Consequently, we obtain

$$\langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -(\gamma_1 \|v\|^2 + \gamma_2 \|w\|^2 + \gamma_3 \|z\|^2 + \kappa \|p_x\|^2), \quad (16)$$

that is,

$$\operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = -(\gamma_1 \|v\|^2 + \gamma_2 \|w\|^2 + \gamma_3 \|z\|^2 + \kappa \|p_x\|^2) \leq 0. \quad (17)$$

Therefore, $\mathcal{A}$ is dissipative. □

We recall the Lax–Milgram theorem.

Theorem 3.2 (Lax–Milgram Theorem). *Let $\mathbb{V}$ be a real Hilbert space, $\mathcal{B} : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ be a bilinear, continuous, and coercive form, and $\mathcal{F} : \mathbb{V} \rightarrow \mathbb{R}$ be a linear, continuous form. Then, there exists a unique $V \in \mathbb{V}$ such that $\mathcal{B}(V, \bar{V}) = \mathcal{F}(\bar{V})$ for all $\bar{V} \in \mathbb{V}$.*

Proposition 3.3. *It holds that $0 \in \rho(\mathcal{A})$; that is, for every $F \in \mathcal{H}$, there exists a unique $U \in D(\mathcal{A})$ such that $-\mathcal{A}U = F$ in $\mathcal{H}$.*

Proof. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$. Then, the equation $-\mathcal{A}U = F$ in $\mathcal{H}$ leads to

$$-v = f_1 \text{ in } H_0^1(0, L), \quad (18)$$

$$-\alpha u_{xx} - \lambda(\varphi - u) + \beta_1 p_x + \gamma_1 v = \rho f_2 \text{ in } L^2(0, L), \quad (19)$$

$$-w = f_3 \text{ in } H_0^1(0, L), \quad (20)$$

$$-k(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w = \rho_1 f_4 \text{ in } L^2(0, L), \quad (21)$$

$$-z = f_5 \text{ in } H_0^1(0, L), \quad (22)$$

$$-b\psi_{xx} + k(\varphi_x + \psi) + \gamma_3 z = \rho_2 f_6 \text{ in } L^2(0, L), \quad (23)$$

$$-p = f_7 \text{ in } H_0^1(0, L), \quad (24)$$

$$-(\delta\theta + \kappa p)_{xx} + \beta_1 v_x = \rho_3 f_8 \text{ in } L^2(0, L). \quad (25)$$

Substituting (18), (20) and (22) into (19), (21) and (23) respectively, we obtain

$$-\alpha u_{xx} - \lambda(\varphi - u) = \rho f_2 + \gamma_1 f_1 + \beta_1 f_{7,x} \text{ in } L^2(0, L), \quad (26)$$

$$-k(\varphi_x + \psi)_x + \lambda(\varphi - u) = \rho_1 f_4 + \gamma_2 f_3 \text{ in } L^2(0, L), \quad (27)$$

$$-b\psi_{xx} + k(\varphi_x + \psi) = \rho_2 f_6 + \gamma_3 f_5 \text{ in } L^2(0, L), \quad (28)$$

$$-(\delta\theta - \kappa f_7)_{xx} = \rho_3 f_8 + \beta_1 f_{1,x} \text{ in } L^2(0, L). \quad (29)$$

Multiplying (26)–(29) by $\bar{u}, \bar{\varphi}, \bar{\psi}, \sigma\bar{\theta} \in H_0^1(0, L)$, respectively, with $\sigma > 1$, integrating and using (24), we obtain

$$\alpha \int_0^L u_x \bar{u}_x dx - \lambda \int_0^L (\varphi - u) \bar{u} dx = \rho \int_0^L f_2 \bar{u} dx + \gamma_1 \int_0^L f_1 \bar{u} dx + \beta_1 \int_0^L f_{7,x} \bar{u} dx, \quad (30)$$

$$k \int_0^L (\varphi_x + \psi) \bar{\varphi}_x dx + \lambda \int_0^L (\varphi - u) \bar{\varphi} dx = \rho_1 \int_0^L f_4 \bar{\varphi} dx + \gamma_2 \int_0^L f_3 \bar{\varphi} dx, \quad (31)$$

$$b \int_0^L \psi_x \bar{\psi}_x dx + k \int_0^L (\varphi_x + \psi) \bar{\psi} dx = \rho_2 \int_0^L f_6 \bar{\psi} dx + \gamma_3 \int_0^L f_5 \bar{\psi} dx, \quad (32)$$

$$\sigma \delta \int_0^L \theta_x \bar{\theta}_x dx = \sigma \rho_3 \int_0^L f_8 \bar{\theta} dx + \sigma \beta_1 \int_0^L f_{1,x} \bar{\theta} dx + \sigma \kappa \int_0^L f_{7,x} \bar{\theta}_x dx. \quad (33)$$

By adding (30)–(33), we construct the variational problem

$$\mathcal{B}((u, \varphi, \psi, \theta), (\bar{u}, \bar{\varphi}, \bar{\psi}, \bar{\theta})) = \mathcal{F}(\bar{u}, \bar{\varphi}, \bar{\psi}, \bar{\theta}) \quad (34)$$

where

$$\begin{aligned} \mathcal{B}((u, \varphi, \psi, \theta), (\bar{u}, \bar{\varphi}, \bar{\psi}, \bar{\theta})) &= \alpha \int_0^L u_x \bar{u}_x dx + \lambda \int_0^L (\varphi - u)(\bar{\varphi} - \bar{u}) dx \\ &\quad + k \int_0^L (\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi}) dx + b \int_0^L \psi_x \bar{\psi}_x dx + \sigma \delta \int_0^L \theta_x \bar{\theta}_x dx \end{aligned}$$

and

$$\begin{aligned}\mathcal{F}(\bar{u}, \bar{\varphi}, \bar{\psi}, \bar{\theta}) = & \rho \int_0^L f_2 \bar{u} dx + \gamma_1 \int_0^L f_1 \bar{u} dx + \beta_1 \int_0^L f_{7,x} \bar{u} dx + \rho_1 \int_0^L f_4 \bar{\varphi} dx \\ & + \gamma_2 \int_0^L f_3 \bar{\varphi} dx + \rho_2 \int_0^L f_6 \bar{\psi} dx + \gamma_3 \int_0^L f_5 \bar{\psi} dx \\ & + \sigma \rho_3 \int_0^L f_8 \bar{\theta} dx + \sigma \beta_1 \int_0^L f_{1,x} \bar{\theta} dx + \sigma \kappa \int_0^L f_{7,x} \bar{\theta}_x dx.\end{aligned}$$

We define $\mathbb{V} = [H_0^1(0, L)]^4$ with the norm

$$\|V\|^2 = \alpha \|u_x\|^2 + \lambda \|\varphi - u\|^2 + k \|\varphi_x + \psi\|^2 + b \|\psi_x\|^2 + \delta \|\theta_x\|^2$$

where $V = (u, \varphi, \psi, \theta) \in \mathbb{V}$. Note that (34) can be written as

$$\mathcal{B}(V, \bar{V}) = \mathcal{F}(\bar{V}) \quad \text{for every } \bar{V} \in \mathbb{V}. \quad (35)$$

We note that $\mathcal{B}$ is symmetric and bilinear,

$$\begin{aligned}\mathcal{B} : \mathbb{V} \times \mathbb{V} & \rightarrow \mathbb{R} \\ (V, \bar{V}) & \rightarrow \mathcal{B}(V, \bar{V}).\end{aligned}$$

Now, we show that $\mathcal{B}$ is also continuous and coercive. Since

$$\begin{aligned}|\mathcal{B}(V, \bar{V})| = & \left| \alpha \int_0^L u_x \bar{u}_x dx + \lambda \int_0^L (\varphi - u)(\bar{\varphi} - \bar{u}) dx + k \int_0^L (\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi}) dx + b \int_0^L \psi_x \bar{\psi}_x dx \right. \\ & \left. + \sigma \delta \int_0^L \theta_x \bar{\theta}_x dx \right|,\end{aligned}$$

we have

$$\begin{aligned}|\mathcal{B}(V, \bar{V})| \leq & \alpha \int_0^L |u_x \bar{u}_x| dx + \lambda \int_0^L |(\varphi - u)(\bar{\varphi} - \bar{u})| dx + k \int_0^L |(\varphi_x + \psi)(\bar{\varphi}_x + \bar{\psi})| dx + b \int_0^L |\psi_x \bar{\psi}_x| dx \\ & + \sigma \delta \int_0^L |\theta_x \bar{\theta}_x| dx.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}|\mathcal{B}(V, \bar{V})| \leq & \alpha \left(\int_0^L |u_x|^2 \right)^{1/2} \left(\int_0^L |\bar{u}_x|^2 \right)^{1/2} + \lambda \left(\int_0^L |\varphi - u|^2 \right)^{1/2} \left(\int_0^L |\bar{\varphi} - \bar{u}|^2 \right)^{1/2} \\ & + k \left(\int_0^L |\varphi_x + \psi|^2 \right)^{1/2} \left(\int_0^L |\bar{\varphi}_x + \bar{\psi}|^2 \right)^{1/2} + b \left(\int_0^L |\psi_x|^2 \right)^{1/2} \left(\int_0^L |\bar{\psi}_x|^2 \right)^{1/2} \\ & + \sigma \delta \left(\int_0^L |\theta_x|^2 \right)^{1/2} \left(\int_0^L |\bar{\theta}_x|^2 \right)^{1/2},\end{aligned}$$

from which it follows that

$$|\mathcal{B}(V, \bar{V})| \leq M \|V\| \|\bar{V}\|,$$

for some $M > 0$. Therefore, $\mathcal{B}$ is continuous.

Here, we have

$$\mathcal{B}(V, \bar{V}) = \alpha \|u_x\|^2 + \lambda \|\varphi - u\|^2 + k \|\varphi_x + \psi\|^2 + b \|\psi_x\|^2 + \sigma \delta \|\theta_x\|^2.$$

Since $\sigma > 1$, it follows that $\mathcal{B}(V, \bar{V}) \geq \|V\|^2$. Therefore, $\mathcal{B}$ is coercive.

Clearly, $\mathcal{F}$ is linear,

$$\begin{aligned} \mathcal{F} : \mathbb{V} &\rightarrow \mathbb{C} \\ \bar{V} &\rightarrow \mathcal{F}(\bar{V}). \end{aligned}$$

In what follows, we show that $\mathcal{F}$ is continuous. Since

$$\begin{aligned} \mathcal{F}(\bar{V}) = & \rho \int_0^L f_2 \bar{u} dx + \gamma_1 \int_0^L f_1 \bar{u} dx + \beta_1 \int_0^L f_{7,x} \bar{u} dx + \rho_1 \int_0^L f_4 \bar{\varphi} dx \\ & + \gamma_2 \int_0^L f_3 \bar{\varphi} dx + \rho_2 \int_0^L f_6 \bar{\psi} dx + \gamma_3 \int_0^L f_5 \bar{\psi} dx \\ & + \sigma \rho_3 \int_0^L f_8 \bar{\theta} dx + \sigma \beta_1 \int_0^L f_{1,x} \bar{\theta} dx + \sigma \kappa \int_0^L f_{7,x} \bar{\theta}_x dx, \end{aligned}$$

we have

$$\begin{aligned} |\mathcal{F}(\bar{V})| \leq & \rho \int_0^L |f_2 \bar{u}| dx + \gamma_1 \int_0^L |f_1 \bar{u}| dx + \beta_1 \int_0^L |f_{7,x} \bar{u}| dx + \rho_1 \int_0^L |f_4 \bar{\varphi}| dx \\ & + \gamma_2 \int_0^L |f_3 \bar{\varphi}| dx + \rho_2 \int_0^L |f_6 \bar{\psi}| dx + \gamma_3 \int_0^L |f_5 \bar{\psi}| dx \\ & + \sigma \rho_3 \int_0^L |f_8 \bar{\theta}| dx + \sigma \beta_1 \int_0^L |f_{1,x} \bar{\theta}| dx + \sigma \kappa \int_0^L |f_{7,x} \bar{\theta}_x| dx. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |\mathcal{F}(\bar{V})| \leq & \rho \|f_2\| \|\bar{u}\| + \gamma_1 \|f_1\| \|\bar{u}\| + \beta_1 \|f_{7,x}\| \|\bar{u}\| + \rho_1 \|f_4\| \|\bar{\varphi}\| \\ & + \gamma_2 \|f_3\| \|\bar{\varphi}\| + \rho_2 \|f_6\| \|\bar{\psi}\| + \gamma_3 \|f_5\| \|\bar{\psi}\| \\ & + \sigma \rho_3 \|f_8\| \|\bar{\theta}\| + \sigma \beta_1 \|f_{1,x}\| \|\bar{\theta}\| + \sigma \kappa \|f_{7,x}\| \|\bar{\theta}_x\|. \end{aligned}$$

Hence, there exists a constant $N > 0$ such that

$$|\mathcal{F}(\bar{V})| \leq N \|\bar{V}\|.$$

Therefore, $\mathcal{F}$ is continuous.

Now, according to Theorem 3.2, there exists a unique solution $(u, \varphi, \psi, \theta) \in \mathbb{V}$ of equation (35). From (26)–(29), we have $u, \varphi, \psi \in H^2(0, L)$ and $\delta \theta - \kappa f_7 \in H^2(0, L)$. Since $v = -f_1, w = -f_3, z = -f_5$ and $p = -f_7$, we conclude that $U \in D(\mathcal{A})$. $\square$

Now, we consider the following result that follows from the Lumer–Phillips theorem (see [14, Theorem 1.2.4, p. 3]).

Lemma 3.4. *Let $\mathcal{A}$ be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. If $\mathcal{A}$ is dissipative and $0 \in \rho(\mathcal{A})$, then $\mathcal{A}$ is the infinitesimal generator of a C_0 -semigroup of contractions in $\mathcal{H}$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$.*

From Propositions 3.1 and 3.3 and the fact that $D(\mathcal{A})$ is dense in $\mathcal{H}$, it follows by Lemma 3.4 that $\mathcal{A}$ is the infinitesimal generator of the C_0 -semigroup of contractions $S(t) = e^{\mathcal{A}t}$.

We are now in a position to present our result on the existence and uniqueness of the solution.

Theorem 3.5. *For $U_0 \in \mathcal{H}$, there is a unique solution $U(t) \in C([0, +\infty), \mathcal{H})$ to the problem (15). Furthermore, if $U_0 \in \mathcal{D}(\mathcal{A})$, then $U(t) \in C([0, \infty[; \mathcal{D}(\mathcal{A})) \cap C^1([0, \infty[; \mathcal{H})$.*

Proof. Given that $\mathcal{A}$ is the infinitesimal generator of the C_0 -semigroup of contractions $S(t) = e^{\mathcal{A}t}$, it follows from the theory of semigroups (see [24, p. 100]) that $U(t) = S(t)U_0$ is the unique solution of the Cauchy problem (15) under the conditions of Theorem 3.5. $\square$

4. Exponential Stability

In this section, we prove that the semigroup $S(t) = e^{\mathcal{A}t}$ associated with the system (1)–(9) is exponentially stable. First, we present a result about the necessary and sufficient conditions for exponential stability of a C_0 -semigroup in a Hilbert space $\mathcal{H}$. This result was established independently by Gearhart [7] and Huang [11] (see also [25]). The following statement is due to Huang.

Theorem 4.1. *Let $\{S(t)\}_{t \geq 0}$ be a C_0 -semigroup of contractions of linear operators on a Hilbert space $\mathcal{H}$ with infinitesimal generator $\mathcal{A}$. Then $\{S(t)\}_{t \geq 0}$ is exponentially stable if and only if*

$$\{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$$

and

$$\sup_{\beta \in \mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty,$$

where $\rho(\mathcal{A})$ denotes the resolvent set of the operator $\mathcal{A}$.

The resolvent equation in the Hilbert space $\mathcal{H}$ is defined by

$$(i\beta I - \mathcal{A})U = F, \quad (36)$$

$U = (u, v, \varphi, w, \psi, z, \theta, p)^T \in D(\mathcal{A})$, $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$ and $\beta \in \mathbb{R}$. From the resolvent equation, we obtain

$$\begin{pmatrix} i\beta u \\ i\beta v \\ i\beta \varphi \\ i\beta w \\ i\beta \psi \\ i\beta z \\ i\beta \theta \\ i\beta p \end{pmatrix} - \begin{pmatrix} v \\ \frac{\alpha}{\rho} u_{xx} + \frac{\lambda}{\rho} (\varphi - u) - \frac{\beta_1}{\rho} p_x - \frac{\gamma_1}{\rho} v \\ w \\ \frac{k}{\rho_1} (\varphi_x + \psi)_x - \frac{\lambda}{\rho_1} (\varphi - u) - \frac{\gamma_2}{\rho_1} w \\ z \\ \frac{b}{\rho_2} \psi_{xx} - \frac{k}{\rho_2} (\varphi_x + \psi) - \frac{\gamma_3}{\rho_2} z \\ p \\ \frac{\delta}{\rho_3} \theta_{xx} - \frac{\beta_1}{\rho_3} v_x + \frac{\kappa}{\rho_3} p_{xx} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{pmatrix}. \quad (37)$$

By (37), we have the following system:

$$\begin{aligned}
 i\beta u - v &= f_1 \text{ in } H_0^1(0, L), \\
 i\beta \rho v - \alpha u_{xx} - \lambda(\varphi - u) + \beta_1 p_x + \gamma_1 v &= \rho f_2 \text{ in } L^2(0, L), \\
 i\beta \varphi - w &= f_3 \text{ in } H_0^1(0, L), \\
 i\beta \rho_1 w - k(\varphi_x + \psi)_x + \lambda(\varphi - u) + \gamma_2 w &= \rho_1 f_4 \text{ in } L^2(0, L), \\
 i\beta \psi - z &= f_5 \text{ in } H_0^1(0, L), \\
 i\beta \rho_2 z - b\psi_{xx} + k(\varphi_x + \psi) + \gamma_3 z &= \rho_2 f_6 \text{ in } L^2(0, L), \\
 i\beta \theta - p &= f_7 \text{ in } H_0^1(0, L), \\
 i\beta \rho_3 p - \delta \theta_{xx} + \beta_1 v_x - \kappa p_{xx} &= \rho_3 f_8 \text{ in } L^2(0, L).
 \end{aligned}$$

Lemma 4.2. *The resolvent set of the operator $\mathcal{A}$ contains the imaginary axis, that is,*

$$\{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}). \quad (38)$$

Proof. We argue by contradiction. Suppose that (38) is false. As $0 \in \rho(\mathcal{A})$, by an argument in [14, p. 25], there exists a sequence $\beta_n \in \mathbb{R}$ with $\beta_n \rightarrow w$, where $w \in \mathbb{R}$, and a sequence of complex vector functions $U_n \in D(\mathcal{A})$ with $\|U_n\|_{\mathcal{H}} = 1$ such that $\|(i\beta_n I - \mathcal{A})U_n\|_{\mathcal{H}} \rightarrow 0$ as $n \rightarrow \infty$, that is,

$$i\beta_n u_n - v_n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (39)$$

$$i\beta_n \rho v_n - \alpha u_{n,xx} - \lambda(\varphi_n - u_n) + \beta_1 p_{n,x} + \gamma_1 v_n \rightarrow 0 \text{ in } L^2(0, L), \quad (40)$$

$$i\beta_n \varphi_n - w_n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (41)$$

$$i\beta_n \rho_1 w_n - k(\varphi_{n,x} + \psi_n)_x + \lambda(\varphi_n - u_n) + \gamma_2 w_n \rightarrow 0 \text{ in } L^2(0, L), \quad (42)$$

$$i\beta_n \psi_n - z_n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (43)$$

$$i\beta_n \rho_2 z_n - b\psi_{n,xx} + k(\varphi_{n,x} + \psi_n) + \gamma_3 z_n \rightarrow 0 \text{ in } L^2(0, L), \quad (44)$$

$$i\beta_n \theta_n - p_n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (45)$$

$$i\beta_n \rho_3 p_n - \delta \theta_{n,xx} + \beta_1 v_{n,x} - \kappa p_{n,xx} \rightarrow 0 \text{ in } L^2(0, L). \quad (46)$$

Taking the inner product of $(i\beta_n I - \mathcal{A})U_n$ with U_n in $\mathcal{H}$, and then taking its real part yields

$$\operatorname{Re}\langle (i\beta_n I - \mathcal{A})U_n, U_n \rangle_{\mathcal{H}} = \gamma_1 \|v_n\|^2 + \gamma_2 \|w_n\|^2 + \gamma_3 \|z_n\|^2 + \kappa \|p_{n,x}\|^2 \rightarrow 0.$$

Hence

$$v_n, w_n, z_n, p_{n,x} \rightarrow 0 \text{ in } L^2(0, L),$$

and, by the Poincaré inequality, $p_n \rightarrow 0$ in $L^2(0, L)$. Since $0 \in \rho(\mathcal{A})$, the limiting frequency w is nonzero. From (45) we therefore obtain $\theta_{n,x} \rightarrow 0$ in $L^2(0, L)$. Taking the L^2 inner products of (40), (42), and (44) with u_n , φ_n and ψ_n , respectively, summing, integrating by parts, and taking real parts, we obtain

$$\alpha \|u_{n,x}\|^2 + \lambda \|\varphi_n - u_n\|^2 + k \|\varphi_{n,x} + \psi_n\|^2 + b \|\psi_{n,x}\|^2 \rightarrow 0.$$

Indeed, $\{\beta_n\}$ is bounded and the remaining terms tend to zero by the preceding convergences. Consequently, $\|U_n\|_{\mathcal{H}} \rightarrow 0$, contradicting $\|U_n\|_{\mathcal{H}} = 1$. Therefore $\{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A})$. $\square$

Lemma 4.3. *It holds that*

$$\sup_{\beta \in \mathbb{R}} \|(i\beta I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty. \quad (47)$$

Proof. If (47) is not true, then there exists a sequence $\beta^n \rightarrow \infty$. Without loss of generality, we may assume that $\beta^n > 0$. There also exists a sequence of complex vectors $F^n \in \mathcal{H}$ and a corresponding sequence $U^n = (u^n, v^n, \varphi^n, w^n, \psi^n, z^n, \theta^n, p^n)^T \in D(\mathcal{A})$, with $\|U^n\|_{\mathcal{H}} = 1$, defined by

$$U^n = (i\beta^n I - \mathcal{A})^{-1} F^n, \quad (48)$$

such that

$$\frac{\|(i\beta^n I - \mathcal{A})^{-1} F^n\|_{\mathcal{H}}}{\|F^n\|_{\mathcal{H}}} > n, \quad \forall n > n_0,$$

or, equivalently,

$$\|F^n\|_{\mathcal{H}} < \frac{\|(i\beta^n I - \mathcal{A})^{-1} F^n\|_{\mathcal{H}}}{n} = \frac{\|U^n\|_{\mathcal{H}}}{n} = \frac{1}{n}, \quad \forall n > n_0.$$

Hence, it follows that $F^n \rightarrow 0$ in $\mathcal{H}$, and from (48), we have

$$\|(i\beta^n I - \mathcal{A}) U^n\|_{\mathcal{H}} \rightarrow 0 \quad \text{when } n \rightarrow \infty. \quad (49)$$

Taking the inner product of $(i\beta^n I - \mathcal{A}) U^n$ with U^n in $\mathcal{H}$, we have

$$\langle (i\beta^n I - \mathcal{A}) U^n, U^n \rangle_{\mathcal{H}} = \langle F^n, U^n \rangle_{\mathcal{H}},$$

that is, $i\beta^n \|U^n\|_{\mathcal{H}}^2 - \langle \mathcal{A} U^n, U^n \rangle_{\mathcal{H}} = \langle F^n, U^n \rangle_{\mathcal{H}}$. Taking the real part and using (17), we obtain

$$\gamma_1 \|v^n\|^2 + \gamma_2 \|w^n\|^2 + \gamma_3 \|z^n\|^2 + \kappa \|p_x^n\|^2 = \Re \langle F^n, U^n \rangle_{\mathcal{H}}.$$

Since U^n is bounded and $F^n \rightarrow 0$ in $\mathcal{H}$, we deduce that

$$v^n \rightarrow 0 \text{ in } L^2(0, L), \quad (50)$$

$$w^n \rightarrow 0 \text{ in } L^2(0, L), \quad (51)$$

$$z^n \rightarrow 0 \text{ in } L^2(0, L), \quad (52)$$

$$p_x^n \rightarrow 0 \text{ in } L^2(0, L). \quad (53)$$

From (49), we obtain

$$i\beta^n u^n - v^n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (54)$$

$$i\beta^n \rho v^n - \alpha u_{xx}^n - \lambda(\varphi^n - u^n) + \beta_1 p_x^n + \gamma_1 v^n \rightarrow 0 \text{ in } L^2(0, L), \quad (55)$$

$$i\beta^n \varphi^n - w^n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (56)$$

$$i\beta^n \rho_1 w^n - k(\varphi_x^n + \psi^n)_x + \lambda(\varphi^n - u^n) + \gamma_2 w^n \rightarrow 0 \text{ in } L^2(0, L), \quad (57)$$

$$i\beta^n \psi^n - z^n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (58)$$

$$i\beta^n \rho_2 z^n - b\psi_{xx}^n + k(\varphi_x^n + \psi^n) + \gamma_3 z^n \rightarrow 0 \text{ in } L^2(0, L), \quad (59)$$

$$i\beta^n \theta^n - p^n \rightarrow 0 \text{ in } H_0^1(0, L), \quad (60)$$

$$i\beta^n \rho_3 p^n - \delta \theta_{xx}^n + \beta_1 v_x^n - \kappa p_{xx}^n \rightarrow 0 \text{ in } L^2(0, L). \quad (61)$$

Using (50) in (54), (51) in (56), (52) in (58) and (53) in (60), we have

$$\beta^n u^n \rightarrow 0 \text{ in } L^2(0, L),$$

$$\beta^n \varphi^n \rightarrow 0 \text{ in } L^2(0, L),$$

$$\beta^n \psi^n \rightarrow 0 \text{ in } L^2(0, L),$$

$$\beta^n \theta_x^n \rightarrow 0 \text{ in } L^2(0, L).$$

Taking into account that $\beta^n \rightarrow \infty$, it follows that

$$u^n \rightarrow 0 \text{ in } L^2(0, L), \quad (62)$$

$$\varphi^n \rightarrow 0 \text{ in } L^2(0, L), \quad (63)$$

$$\psi^n \rightarrow 0 \text{ in } L^2(0, L), \quad (64)$$

$$\theta_x^n \rightarrow 0 \text{ in } L^2(0, L). \quad (65)$$

From (60), it follows that $p^n \in H_0^1(0, L)$. Therefore, applying the Poincaré inequality and using (53), we have

$$p^n \rightarrow 0 \text{ in } L^2(0, L). \quad (66)$$

Taking the L^2 inner products of (55), (57), and (59) with u_n , φ_n , and ψ_n , respectively, summing, integrating by parts, and taking real parts, we obtain

$$\alpha \|u_{n,x}\|^2 + \lambda \|\varphi_n - u_n\|^2 + k \|\varphi_{n,x} + \psi_n\|^2 + b \|\psi_{n,x}\|^2 \rightarrow 0.$$

For example,

$$|\beta_n(u_n, v_n)| \leq \|v_n\| \|\beta_n u_n\| \rightarrow 0$$

by (50) and (54), and the analogous terms containing w_n , φ_n and z_n , ψ_n are treated in the same way. All other terms tend to zero by (50)–(66). Hence, $\|U_n\|_{\mathcal{H}} \rightarrow 0$, contradicting $\|U^n\|_{\mathcal{H}} = 1$. This completes the proof of (47). $\square$

Now, we are in a position to present our main result.

Theorem 4.4. *The semigroup $S(t) = e^{At}$ generated by $\mathcal{A}$ is exponentially stable, where $t \geq 0$.*

Proof. By Lemmas 4.2 and 4.3, exponential stability follows immediately from Theorem 4.1. $\square$

5. Numerical Simulations

To illustrate the system's dissipative behavior and exponential stability established in the previous sections, we consider a reduced one-mode approximation of the Timoshenko–thermoelastic system given by (1)–(4). The numerical approach is based on a Galerkin modal approximation, which is widely used in the modeling of elastic and structural systems [5, 16, 27].

Let $L > 0$ and let $k_0 = \pi/L$ denote the first wavenumber. Since (9) requires u, φ, ψ, θ to vanish at both endpoints, a one-mode ansatz compatible with these boundary conditions is naturally chosen from the family $\{\sin(nk_0x)\}_{n \geq 1}$, which consists of the eigenfunctions of $-\partial_{xx}$ subject to homogeneous Dirichlet boundary conditions. If the same mode number n is assigned to all fields, orthogonality eliminates both the shear coupling $k(\varphi_x + \psi)$ and the thermoelastic coupling $\beta_1\theta_{xt}$ after Galerkin projection, thereby removing the coupling effects that we wish to illustrate. We therefore retain a single-mode ansatz for each field, while assigning different Dirichlet-compatible mode numbers to the pair (u, φ) and to the pair (ψ, θ) :

$$\begin{aligned} u(x, t) &\approx U(t) \sin(k_0x), & \varphi(x, t) &\approx \Phi(t) \sin(k_0x), \\ \psi(x, t) &\approx P(t) \sin(2k_0x), & \theta(x, t) &\approx T(t) \sin(2k_0x), \end{aligned}$$

such that $u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = \theta(0, t) = \theta(L, t) = 0$ holds, in agreement with (9).

Projecting the PDE system onto these basis functions and using

$$\int_0^L \sin^2(k_0x) dx = \int_0^L \sin^2(2k_0x) dx = \frac{L}{2}, \quad \int_0^L k_0 \cos(k_0x) \sin(2k_0x) dx = \frac{4}{3} = \frac{\mu L}{2}, \quad \mu := \frac{8}{3L},$$

we obtain the following system of ordinary differential equations:

$$\begin{aligned} \rho U'' + \gamma_1 U' + (\alpha k_0^2 + \lambda)U - \lambda\Phi - \mu\beta_1 T' &= 0, \\ \rho_1 \Phi'' + \gamma_2 \Phi' + (k k_0^2 + \lambda)\Phi + \mu k P - \lambda U &= 0, \\ \rho_2 P'' + \gamma_3 P' + (4b k_0^2 + k)P + \mu k \Phi &= 0, \\ \rho_3 T'' + 4\kappa k_0^2 T' + 4\delta k_0^2 T + \mu\beta_1 U' &= 0. \end{aligned}$$

Introducing the first-order variables $V = U'$, $W = \Phi'$, $Z = P'$, and $Q = T'$, the system can be written as

$$X'(t) = A X(t), \quad X = (U, V, \Phi, W, P, Z, T, Q)^\top \in \mathbb{R}^8,$$

where the matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{\alpha k_0^2 + \lambda}{\rho} & -\frac{\gamma_1}{\rho} & \frac{\lambda}{\rho} & 0 & 0 & 0 & 0 & \frac{\mu\beta_1}{\rho} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{\lambda}{\rho_1} & 0 & -\frac{k k_0^2 + \lambda}{\rho_1} & -\frac{\gamma_2}{\rho_1} & -\frac{\mu k}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\mu k}{\rho_2} & 0 & -\frac{4b k_0^2 + k}{\rho_2} & -\frac{\gamma_3}{\rho_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{\mu\beta_1}{\rho_3} & 0 & 0 & 0 & 0 & -\frac{4\delta k_0^2}{\rho_3} & -\frac{4\kappa k_0^2}{\rho_3} \end{pmatrix}.$$

All simulations are performed using dimensionless parameters, with time expressed in normalized units associated with the characteristic time scale of the system. The following parameter values are used throughout the simulations:

$$\begin{aligned} L &= 1, \quad k_0 = \frac{\pi}{L}, \quad \mu = \frac{8}{3L} = \frac{8}{3}, \\ \rho &= \rho_1 = \rho_2 = \rho_3 = 1, \quad \alpha = 1, \quad b = 1, \quad \delta = 1, \\ \lambda &= 1, \quad k = 1, \quad \gamma_2 = 0.5, \quad \gamma_3 = 0.5, \quad \kappa = 15. \end{aligned}$$

All simulations start from the initial state

$$X(0) = (U, V, \Phi, W, P, Z, T, Q)(0) = (1, 0, 0.5, 0, 0.2, 0, 0.1, 0).$$

The parameters γ_1 and β_1 are varied to investigate the influence of mechanical damping and thermoelastic coupling on the system dynamics. We stress that only $\gamma_1, \gamma_2, \gamma_3$ and κ are genuine dissipation mechanisms, since these are exactly the coefficients appearing in the energy identity (14); β_1 , by contrast, is a conservative coupling parameter that cancels out of (14), redistributes energy between the mechanical and thermal components, and therefore affects the *observed* decay rate only indirectly, as illustrated below. To quantify this behavior, we consider the modal energy defined in (13), which can be written as

$$\begin{aligned} E_{\text{modal}}(t) &= \frac{1}{2} \left[\rho V(t)^2 + \rho_1 W(t)^2 + \rho_2 Z(t)^2 + \rho_3 Q(t)^2 + \alpha k_0^2 U(t)^2 + 4b k_0^2 P(t)^2 + 4\delta k_0^2 T(t)^2 \right. \\ &\quad \left. + \lambda (\Phi(t) - U(t))^2 + k k_0^2 \Phi(t)^2 + k P(t)^2 + 2\mu k \Phi(t) P(t) \right]. \end{aligned}$$

A direct computation — differentiating E_{modal} along the system above — shows that the modal energy obeys the same dissipation law as the continuous energy (14):

$$E'_{\text{modal}}(t) = - \left(\gamma_1 V(t)^2 + \gamma_2 W(t)^2 + \gamma_3 Z(t)^2 + 4\kappa k_0^2 Q(t)^2 \right) \leq 0.$$

The two β_1 -terms cancel identically, independently of β_1 . This is the discrete counterpart of the argument used above to explain why β_1 is a conservative coupling rather than a dissipation mechanism. Moreover, at $L = 1$, the coupling quadratic form $k(k_0^2 \Phi^2 + P^2 + 2\mu \Phi P)$ is positive definite, since $2\mu = 16/3 \approx 5.33 < 2k_0 = 2\pi \approx 6.28$. Thus, E_{modal} remains a genuine positive energy. We stress that E_{modal} is $E(t)$ from (13) evaluated on the ansatz and then rescaled by $2/L$ for convenience, so that $E(t) = \frac{L}{2} E_{\text{modal}}(t)$; at $L = 1$, this gives $E_{\text{modal}}(t) = 2E(t)$. Since every plot below shows the normalized ratio $E_{\text{modal}}(t)/E_{\text{modal}}(0) = E(t)/E(0)$, this constant factor cancels and does not affect any of the figures. However, the absolute values of dE_{modal}/dt quoted in the discussion of Figure 5.4 refer to E_{modal} , rather than to $E(t)$, and are therefore twice the corresponding values of dE/dt .

The dissipative behavior is illustrated in Figure 5.1, where the normalized modal energy is shown for different values of the damping coefficient γ_1 and the thermoelastic coupling parameter β_1 . Solid lines correspond to $\beta_1 = 0$, whereas dashed lines correspond to $\beta_1 = 1$. Increasing γ_1 leads to a markedly faster decay rate: for instance, $E(5)/E(0)$ drops from 0.38 ($\gamma_1 = 0.1$) to 0.04 ($\gamma_1 = 2$). The effect of β_1 alone (solid vs. dashed curves, $\beta_1 = 0$ vs. $\beta_1 = 1$) is, by comparison, modest and not uniformly one-signed for this value of κ , consistent with β_1 being a conservative coupling rather than an intrinsic dissipation mechanism (cf. (14)).

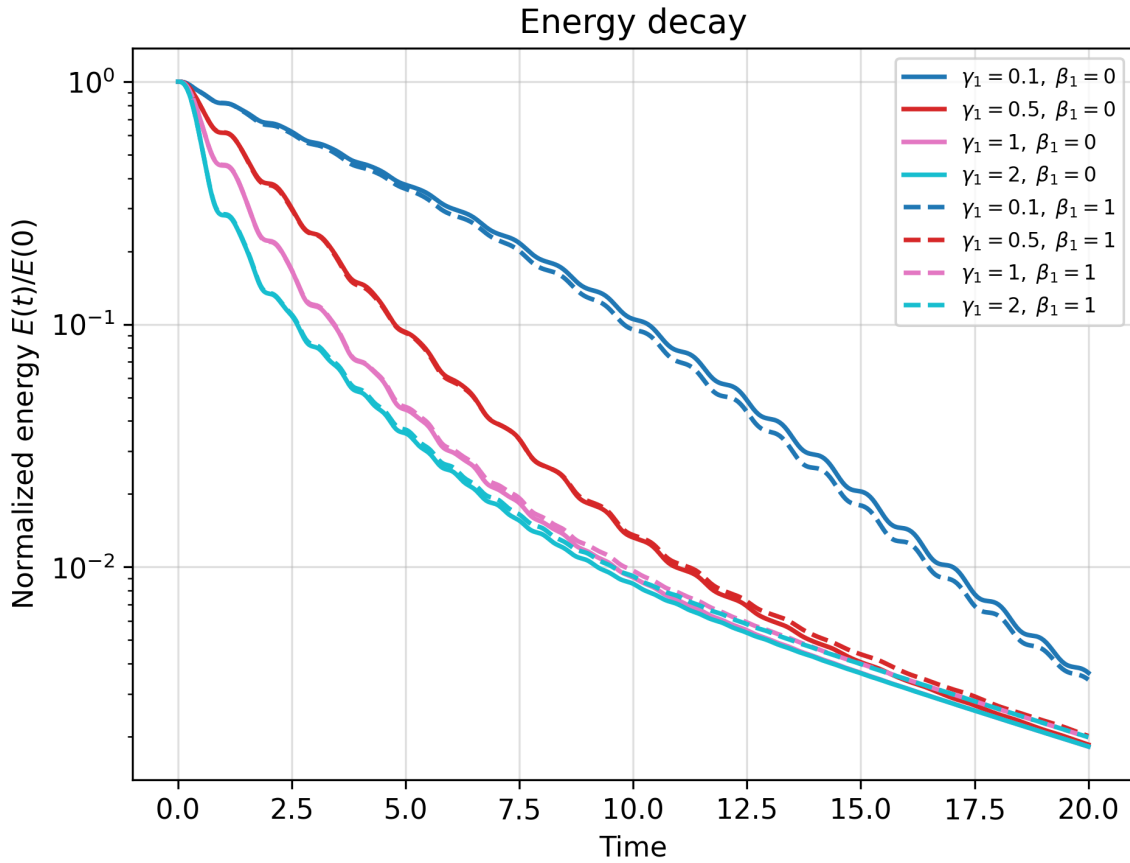


Figure 5.1: Normalized modal energy $E(t)/E(0)$ for different values of the damping coefficient γ_1 and thermoelastic coupling β_1 . Solid lines correspond to $\beta_1 = 0$, while dashed lines correspond to $\beta_1 = 1$.

To better understand the role of dissipation mechanisms, we compare four representative regimes: a baseline case ($\gamma_1 = 1, \beta_1 = 1$), a thermoelastic-dominated regime ($\gamma_1 = 1, \beta_1 = 30$), a mechanically-dominated damping regime ($\gamma_1 = 30, \beta_1 = 1$), and a combined regime ($\gamma_1 = 30, \beta_1 = 30$), as shown in Figure 5.2. The results show that γ_1 enhances the decay rate in the moderate range explored in Figure 5.1 ($E(5)/E(0)$ decreases from 0.38 at $\gamma_1 = 0.1$ to 0.04 at $\gamma_1 = 2$), whereas increasing γ_1 further to 30 no longer accelerates the decay. At $\beta_1 = 1$ and $t = 5$, the mechanically dominated regime gives $E/E(0) \approx 0.056$, which is marginally *slower* than the baseline value of 0.046; thus, the reduced system is over-damped in this range, and increasing γ_1 beyond a certain point no longer increases the decay rate.

The role of β_1 is similarly non-monotone. For the fixed value $\kappa = 15$ used here, a large value of β_1 transiently *slows down* the observed decay relative to the corresponding $\beta_1 = 1$ curve (e.g. at $t = 5$, $E/E(0) \approx 0.12$ for $\beta_1 = 30$, compared with ≈ 0.05 for $\beta_1 = 1$, when $\gamma_1 = 1$), because energy transferred to the thermal component through the coupling must still be dissipated through κ , which introduces its own time scale. This is consistent with β_1 being a conservative, redistributive coupling rather than a dissipation mechanism in itself: the total energy remains monotonically decreasing throughout (we verified that $E'(t) \leq 0$ numerically along every curve, as required by Proposition 3.1), but the *rate* at which either γ_1 or β_1 affects this decrease is not monotone and should not be interpreted as meaning that “more damping/coupling always means faster decay.”

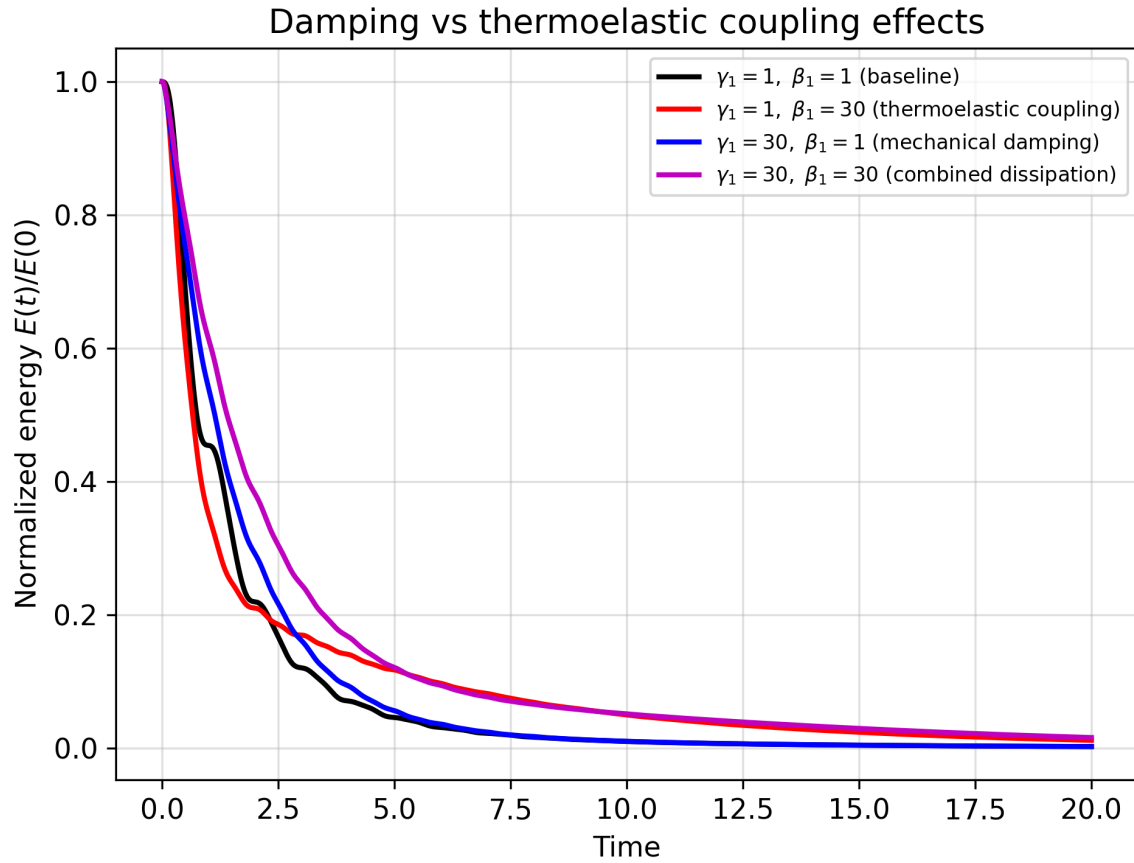


Figure 5.2: Normalized modal energy $E(t)/E(0)$ for four regimes of dissipation: the baseline case ($\gamma_1 = 1, \beta_1 = 1$), thermoelastic-dominated case ($\gamma_1 = 1, \beta_1 = 30$), mechanically-dominated damping ($\gamma_1 = 30, \beta_1 = 1$), and combined regime ($\gamma_1 = 30, \beta_1 = 30$).

The mechanical and thermal parts of E_{modal} plotted in Figure 5.3 are defined, consistently with the split of the ansatz above, by

$$E_m(t) = \frac{1}{2} \left[\rho V^2 + \rho_1 W^2 + \rho_2 Z^2 + \alpha k_0^2 U^2 + 4b k_0^2 P^2 + \lambda (\Phi - U)^2 + k k_0^2 \Phi^2 + k P^2 + 2\mu k \Phi P \right],$$

$$E_t(t) = \frac{1}{2} \left[\rho_3 Q^2 + 4\delta k_0^2 T^2 \right],$$

such that $E_m(t) + E_t(t) = E_{\text{modal}}(t)$.

Additional insight is provided in Figure 5.3, where the mechanical and thermal contributions to the normalized modal energy are shown separately. In this analysis, we focus on the two extreme regimes: the baseline case ($\gamma_1 = 1, \beta_1 = 1$) and the combined case ($\gamma_1 = 30, \beta_1 = 30$), which exhibit the most pronounced differences in energy transfer dynamics. The results show that part of the mechanical energy is transferred to the thermal component due to thermoelastic coupling. Since $T(0) = 0.1 \neq 0$, E_t does *not* start at zero: $E_t(0) = \frac{1}{2} \cdot 4\delta k_0^2 T(0)^2 \approx 0.026 E(0)$; it then increases only marginally, peaking at about $0.027 E(0)$ near $t \approx 0.8$ in the baseline regime, and far more visibly, to about $0.066 E(0)$ near $t \approx 5$, in the combined regime, before being drained by κ . This behavior reflects the role of the thermal field as a temporary reservoir for mechanical energy. For the baseline regime, E_m decays faster (e.g. $E_m(2.5)/E(0) \approx 0.15$ compared with ≈ 0.25 in the combined regime), which reflects the weaker net transfer into the thermal channel; its profile is, however, visibly more oscillatory, with the solid curve showing small steps and kinks that are absent from the dashed curve corresponding to the combined regime. In both regimes $E_m + E_t = E_{\text{modal}}$ decreases monotonically, in agreement with Proposition 3.1.

The roles of the parameters are distinct: γ_1 is what increases the mechanical dissipation rate; β_1 only redistributes energy from the mechanical into the thermal channel, without dissipating it; and it is κ alone that drains the thermal channel. The larger peak of E_t in the combined regime is a joint effect of a larger γ_1 pushing more energy through the system *and* a larger β_1 routing more of it through the thermal channel, rather than from β_1 acting as a dissipative term itself.

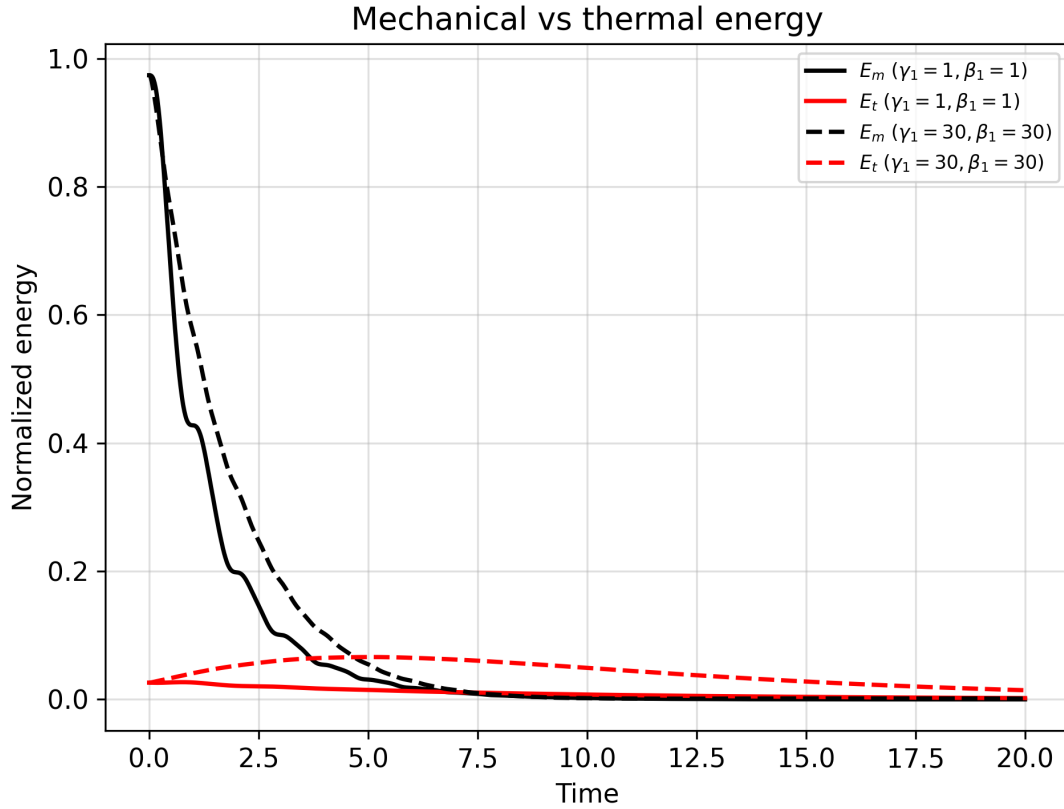


Figure 5.3: Mechanical (E_m) and thermal (E_t) components of the normalized modal energy for two representative regimes: the baseline case ($\gamma_1 = 1$, $\beta_1 = 1$) and the combined case ($\gamma_1 = 30$, $\beta_1 = 30$). Thermoelastic coupling transiently transfers energy from the mechanical to the thermal field, while the total energy $E_m + E_t$ remains monotonically decreasing.

Finally, Figure 5.4 shows the time derivative of the modal energy for the same two representative regimes: the baseline case ($\gamma_1 = 1$, $\beta_1 = 1$) and the combined case ($\gamma_1 = 30$, $\beta_1 = 30$). In both cases, the derivative remains nonpositive throughout the simulation, in agreement with the theoretical energy dissipation law (14). Both regimes exhibit an oscillatory transient with a pronounced initial trough. The baseline regime reaches a deeper minimum, $dE_{\text{modal}}/dt \approx -7.5$ at $t \approx 0.4$, than the combined regime, whose minimum is approximately -3.9 at $t \approx 0.3$ (recall that $E_{\text{modal}} = 2E(t)$ at $L = 1$, so the corresponding values of dE/dt are half of these). During this initial transient ($t \lesssim 5$), the oscillations in the combined regime are visibly smaller in amplitude than those in the baseline regime; however, this behavior does not persist throughout the simulation since the already small oscillations in dE_{modal}/dt cross over and become marginally larger in the combined regime for $t \gtrsim 5$. Correspondingly, dE_{modal}/dt approaches zero for $t \gtrsim 5$ in the baseline regime, but only for $t \gtrsim 7.5$ in the combined regime, which continues to release energy at a non-negligible rate ($dE_{\text{modal}}/dt \approx -0.23$ at $t = 5$) after the baseline regime has essentially settled. This indicates that the combination of strong mechanical damping and thermoelastic coupling smooths the *initial* dissipation profile rather than simply accelerating the energy release throughout the simulation. In both cases, the curves remain nonpositive at all times, confirming dissipativity, which we verified numerically not only for these two regimes but also for every curve shown in Figures 5.1–5.4.

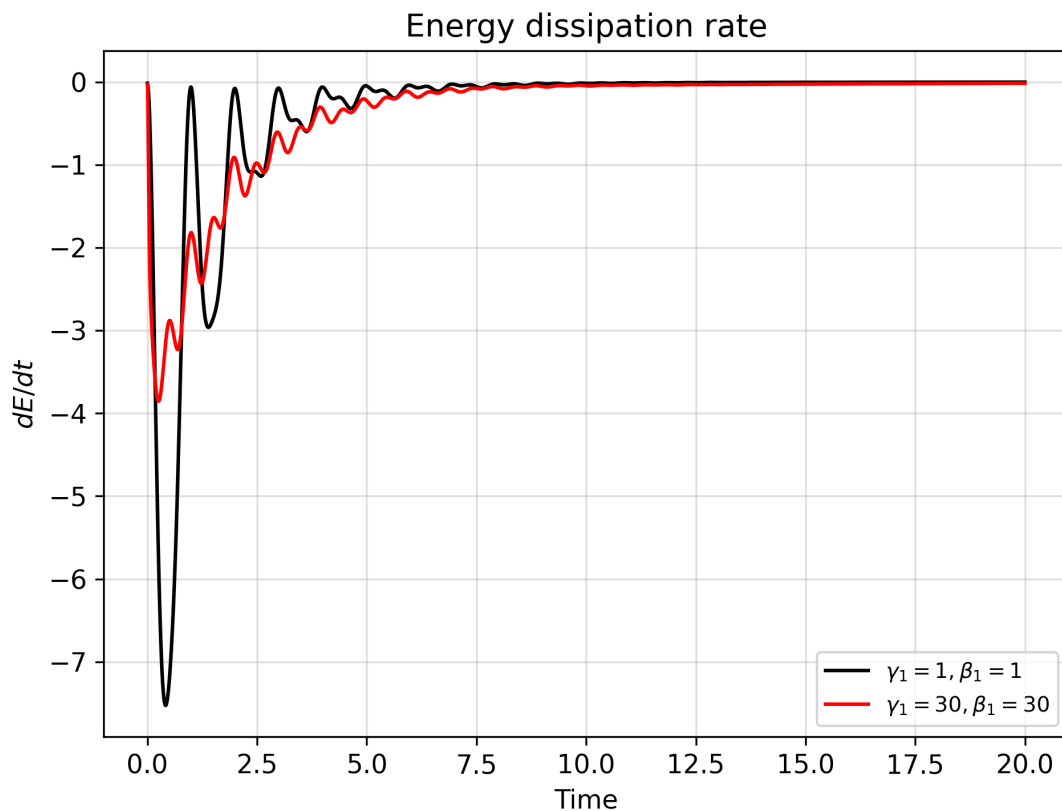


Figure 5.4: Time derivative of the modal energy for two representative regimes: the baseline case ($\gamma_1 = 1, \beta_1 = 1$) and the combined case ($\gamma_1 = 30, \beta_1 = 30$). Both curves remain nonpositive throughout, confirming the monotonic decrease of the total energy; the combined regime shows smaller-amplitude oscillations during the initial transient, but settles more slowly than the baseline.

6. Conclusion

We studied a suspension bridge model in which the deck is described by a Timoshenko beam and the main cable is governed by type III thermoelasticity, with frictional damping acting on all three mechanical components and thermal dissipation in the cable. Using semigroup theory, we established the well-posedness of the system through the Lumer–Phillips theorem and proved its exponential stability by means of the Gearhart–Huang–Prüss theorem. The exponential stability is obtained *without any restriction on the system coefficients*, thereby avoiding the equal wave velocity condition $\frac{k}{\rho_1} = \frac{b}{\rho_2}$, which typically arises in partially damped Timoshenko systems. Hence, no relation between the wave speeds is required, and the so-called second spectrum paradox is avoided. This contrasts with previous results for Timoshenko-type systems with type III thermoelasticity, where only polynomial decay is obtained when the equal wave velocity condition is not satisfied [32].

It is worth emphasizing what the present contribution does *not* claim. As recalled in the introduction, a Timoshenko beam that is completely damped by friction is already exponentially stable independently of any thermal effect (see [30]). In the present model, all three mechanical equations include frictional damping, so the mechanical subsystem alone is exponentially stable. What we prove here is that this exponential stability *persists* under coupling with the main cable through the suspension term $\lambda(\varphi - u)$ and with the type III thermal field through the conservative coupling β_1 . In other words, neither the suspension coupling nor the type III thermal coupling destabilizes the fully damped Timoshenko beam. The thermal dissipation parameter κ contributes directly to the energy identity, while the conservative parameter β_1 redistributes energy between the mechanical and thermal channels; however, the overall decay is not driven by the thermal field alone.

This study complements, and is distinct from, the recent work [17], in which the type III thermal field is coupled to the shear force of the Timoshenko beam. In the present model, the thermal field is instead coupled to the main cable equation through the term $\beta_1\theta_{xt}$. This represents a different physical configuration, leading to a distinct coupling structure and requiring a separate stability analysis.

Numerical simulations based on a reduced one-mode-per-field Galerkin approximation confirm the dissipative character of the system and illustrate the exponential decay predicted by the theoretical analysis. The results show that the suspension bridge model remains exponentially stable under frictional damping when the main cable is governed by type III thermoelasticity.

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