

On the Euler–Sombor Index of Trees With a Given Total Domination Number

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Abstract

The Euler–Sombor index is a recently introduced vertex-degree-based topological index that has attracted considerable research attention. In this paper, we investigate the Euler–Sombor index of trees with a given total domination number. We establish a lower bound on this index in terms of the order and the total domination number, and we characterize the extremal trees attaining this bound.

Keywords: Euler–Sombor index; trees; total domination number; extremal graphs.

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1. Introduction

Topological indices are numerical invariants that play an essential role in characterizing molecular structures across various scientific disciplines. Gutman [12] introduced the Sombor index, defined as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2},$$

where $d(u)$ and $d(v)$ denote the degrees of vertices u and v , respectively. This index has attracted considerable attention, and its extremal properties on various graph classes have been extensively studied; see, e.g., [8–11, 13, 15, 23, 24, 26, 27, 33].

Recently, Gutman [14] and Tang et al. [32] introduced the Euler–Sombor index, a degree-based topological invariant defined by

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d(u)^2 + d(v)^2 + d(u)d(v)}.$$

This index has quickly garnered research interest. Hu et al. [18] established tight bounds on the Euler–Sombor index of maximal outerplanar graphs. Extremal values for unicyclic, bicyclic, and tricyclic graphs were investigated in [21, 22, 28, 29, 31, 32]. Bansode et al. [3] studied the Euler–Sombor energy of graphs, while Albalahi et al. [1] further studied tricyclic graphs. Gutman et al. [16] and Khanra et al. [20] explored related variants and graph classes. Ali et al. [2] provided a systematic review of extremal results and bounds for the Euler–Sombor index across various graph classes and outlined several open problems.

A particularly challenging and active direction is to combine topological indices with domination parameters. The total domination number $\gamma_t(G)$, which requires every vertex to be adjacent to some vertex in the dominating set, is a well-studied domination parameter [17]. For trees, bounds relating the domination number to the geometric-arithmetic index, Randić index, Zagreb indices, and Sombor index were established in [4, 6, 7, 30], while analogous results involving the total domination number were obtained for the Randić index, geometric-arithmetic index, and Zagreb indices in [5, 19, 25].

In this paper, we establish a lower bound on the Euler–Sombor index of trees in terms of the order and the total domination number. We also characterize the extremal trees that attain this bound.

2. Preliminaries

Throughout this paper, G denotes a simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset $D \subseteq V(G)$ is called a *total dominating set* if every vertex in $V(G)$ is adjacent to at least one vertex in D . The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G . For a comprehensive treatment of total domination in graphs, we refer the reader to [17].

For a vertex $u \in V(G)$, the *neighborhood* of u is the set $N(u) = \{v \in V(G) : uv \in E(G)\}$, and the *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. The *degree* of u is $d(u) = |N(u)|$. A vertex of degree 1 is called a *leaf*, and a vertex adjacent to a leaf is called a *support vertex*. The *diameter* of a tree is the length of the longest path between two leaves. For a graph G , let n_i denote the number of vertices of degree i in G . For $i, j \geq 1$, let m_{ij} denote the number of edges joining a vertex of degree i and a vertex of degree j .

For a vertex $v \in V(G)$, denote by $G - \{v\}$ the graph obtained by deleting v and all incident edges. For an edge $e \in E(G)$, denote by $G - \{e\}$ the graph obtained by deleting e . For multiple vertices $v_1, \dots, v_k$ or edges $e_1, \dots, e_k$, we define $G - \{v_1, \dots, v_k\}$ and $G - \{e_1, \dots, e_k\}$ recursively in the natural way.

The following lemmas will be useful in our proofs.

Lemma 2.1. *The function $f(x, y) = \sqrt{x^2 + y^2 + xy}$ is increasing in both variables $x \in [1, +\infty)$ and $y \in [1, +\infty)$.*

Lemma 2.2. *The function $H(x, y) = f(x, y) - f(x - 1, y) = \sqrt{x^2 + y^2 + xy} - \sqrt{(x - 1)^2 + y^2 + (x - 1)y}$ is increasing in $x \in [1, +\infty)$ and decreasing in $y \in [1, +\infty)$.*

Proof. For $x \geq 1$, we have

$$\left(x + \frac{y}{2}\right)^2 [(x - 1)^2 + y^2 + (x - 1)y] - \left(x + \frac{y}{2} - 1\right)^2 (x^2 + y^2 + xy) > 0.$$

A straightforward computation gives

$$\frac{\partial H}{\partial x} = \frac{x + \frac{y}{2}}{\sqrt{x^2 + y^2 + xy}} - \frac{x + \frac{y}{2} - 1}{\sqrt{(x - 1)^2 + y^2 + (x - 1)y}} > 0.$$

Similarly one obtains $\frac{\partial H}{\partial y} < 0$ for $y \geq 1$. □

Define

$$A = 2\sqrt{19} + \sqrt{7} - \frac{7\sqrt{3}}{2}, \quad g(n, \eta) = A(n - 2\eta) + 4\sqrt{3}\eta - 6\sqrt{3} + 2\sqrt{7}. \quad (1)$$

Then, we have $g(n, \eta + 1) - g(n, \eta) = 11\sqrt{3} - 4\sqrt{19} - 2\sqrt{7} < 0$. Consequently, if $\eta_1 \leq \eta_2$, then it holds that $g(n, \eta_1) \geq g(n, \eta_2)$.

Lemma 2.3. *For every fixed n , $g(n, \eta)$ is strictly decreasing in η .*

Lemma 2.4 (Leaf exchange). *Let T be a tree of order at least three, let ℓ be a leaf, and let s be its support vertex. Then s belongs to every total dominating set of T , and T has a minimum total dominating set that does not contain ℓ . More precisely, if D is a minimum total dominating set with $\ell \in D$ and $y \in N(s) \setminus \{\ell\}$, then $(D \setminus \{\ell\}) \cup \{y\}$ is also a minimum total dominating set. If s is adjacent to at least two leaves and ℓ is one of them, then $\gamma_t(T - \ell) = \gamma_t(T)$.*

Proof. Since ℓ has the unique neighbor s , every total dominating set contains s . Suppose that D is a minimum total dominating set and $\ell \in D$. Since $|V(T)| \geq 3$, the support vertex s has a neighbor $y \neq \ell$. If $y \in D$, then $D \setminus \{\ell\}$ would still be a total dominating set, contradicting the minimality of $|D|$. Hence, $y \notin D$. Replacing ℓ by y keeps s dominated, while y is dominated by $s \in D$; every other vertex keeps its former neighbor in the set. This proves the exchange assertion.

For the final statement, choose a minimum total dominating set of T avoiding ℓ ; its restriction total dominates $T - \ell$, and hence, $\gamma_t(T - \ell) \leq \gamma_t(T)$. Conversely, the support vertex s is still adjacent to a leaf in $T - \ell$, so s belongs to every total dominating set of $T - \ell$. Consequently, every minimum total dominating set of $T - \ell$ also dominates the deleted leaf ℓ , and therefore $\gamma_t(T) \leq \gamma_t(T - \ell)$. $\square$

Lemma 2.5. *Let T be a tree, let D be a minimum total dominating set of T , and let $X \subseteq V(T)$ be such that $T - X$ is a tree of order at least two. If $D' \subseteq D \setminus X$ total dominates $T - X$, then*

$$\gamma_t(T - X) \leq |D'|.$$

In particular, if $D \setminus X$ total dominates $T - X$, then $\gamma_t(T - X) \leq \gamma_t(T) - |D \cap X|$.

Proof. This is immediate from the definition of the total domination number. $\square$

Lemma 2.6 (Bridge decomposition). *Let $e = uv$ be an edge of a tree T , and let T_u, T_v be the two components of $T - e$, each of order at least two. Suppose that D is a minimum total dominating set of T such that $D \cap V(T_u)$ and $D \cap V(T_v)$ are total dominating sets of T_u and T_v , respectively. Then*

$$\gamma_t(T) = \gamma_t(T_u) + \gamma_t(T_v).$$

Proof. The restrictions of D give $\gamma_t(T_u) + \gamma_t(T_v) \leq |D| = \gamma_t(T)$. Conversely, the union of minimum total dominating sets of T_u and T_v is a total dominating set of T . Hence, $\gamma_t(T) \leq \gamma_t(T_u) + \gamma_t(T_v)$. $\square$

3. Main Results

Let $\mathcal{A}_{n,\eta}$ denote the family of all trees of order n and total domination number η .

3.1. The Extremal Family

Let $\mathcal{T}$ be the smallest family of trees satisfying the following operations.

(O1) Base operation. For every integer $q \geq 1$, the path P_{4q} belongs to $\mathcal{T}$.

(O2) One-path extension. Let $T \in \mathcal{T}$, let x be a leaf of T , and suppose that $x \notin D$ for some minimum total dominating set D of T . For $q \geq 1$, take a path $Q = a_1 a_2 \cdots a_{4q}$ vertex-disjoint from T , and add the edge xa_1 . The resulting tree belongs to $\mathcal{T}$.

(O3) Paired-path extension. Let $T \in \mathcal{T}$ and let $xy \in E(T)$ satisfy

$$N_T(x) = \{x_0, y\}, \quad N_T(y) = \{x, y_0\}, \quad d_T(x_0) = d_T(y_0) = 2.$$

Suppose that $x, y \in D$ for some minimum total dominating set D of T . For integers $q, q' \geq 1$, take two paths $Q = a_1 \cdots a_{4q+1}$ and $R = b_1 \cdots b_{4q'+1}$ that are mutually vertex-disjoint and disjoint from T , and add the edges xa_1 and $y b_1$. The resulting tree belongs to $\mathcal{T}$.

Put $\mathcal{T}_{n,\eta} = \{T \in \mathcal{T} : |V(T)| = n, \gamma_t(T) = \eta\}$. The following lemma makes the parameter changes in Operations (O2)–(O3) explicit.

Lemma 3.1.1 (Boundary cost and normalization of an attached path). *Let $Q = a_1 a_2 \cdots a_m$ be a path attached to an external vertex x by the edge xa_1 , and let S be a total dominating set of the whole tree. Write*

$$\alpha = \mathbf{1}_{\{x \in S\}}, \quad \beta = \mathbf{1}_{\{x \text{ has a neighbor in } S \setminus V(Q)\}}.$$

For the two congruence classes used below, the exact boundary costs are given in the following table:

m	α	β	$\min S \cap V(Q) $
$4q$	0 or 1	1	$2q$
$4q$	0 or 1	0	$2q + 1$
$4q + 1$	1	1	$2q$
$4q + 1$	1	0	$2q + 1$
$4q + 1$	0	1	$2q + 1$
$4q + 1$	0	0	$2q + 2$.

Note that for $m = 4q$ the boundary cost does not depend on α : the constraint at a_1 involves $s_0 = \alpha$, but the minimizing patterns for $s_0 = 0$ and for $s_0 = 1$ have the same cardinality. In each line the minimum is attained whenever the prescribed boundary state is compatible with the rest of the tree. Moreover, for a fixed pair (α, β) , a minimum total dominating set can be normalized on Q without changing any vertex outside Q : its intersection with $V(Q)$ may be replaced by a minimum binary pattern realizing the corresponding line of the table.

Proof. Put $s_i = 1$ if $a_i \in S$ and $s_i = 0$ otherwise, and put $s_0 = \alpha$. The total-domination constraints internal to the attached path are

$$s_{i-1} + s_{i+1} \geq 1 \quad (1 \leq i \leq m-1), \quad s_{m-1} = 1.$$

If $\beta = 0$, then x must be dominated from Q , so one also imposes $s_1 = 1$.

For fixed s_0 , let $F_i(r, s)$ be the minimum of $s_1 + \cdots + s_i$ over all binary strings with $(s_{i-1}, s_i) = (r, s)$ for which $a_1, \dots, a_{i-1}$ are already dominated. Then

$$F_{i+1}(s, t) = t + \min_{r \in \{0,1\}: r+t \geq 1} F_i(r, s),$$

with $F_1(s_0, s_1) = s_1$ and, if $\beta = 0$, the extra restriction $s_1 = 1$. At the terminal step one imposes $s_{m-1} = 1$. Induction on i gives exactly the six values displayed above.

Because this recurrence keeps the two boundary bits (α, β) fixed, replacing $S \cap V(Q)$ by any minimizing binary pattern for the same state leaves the membership and domination status of every vertex outside Q unchanged. This is the normalization assertion. $\square$

Lemma 3.1.2. *If T' is obtained from $T \in \mathcal{T}$ by Operation (O2) with a path P_{4q} , then*

$$\gamma_t(T') = \gamma_t(T) + 2q.$$

If T' is obtained by Operation (O3) with paths P_{4q+1} and $P_{4q'+1}$, then

$$\gamma_t(T') = \gamma_t(T) + 2q + 2q'.$$

Proof. For (O2), let D be a minimum total dominating set of T with $x \notin D$. Since x is a leaf, its neighbor in T belongs to D . Adding the $2q$ -vertex pattern from Lemma 3.1.1 produces a total dominating set of T' of size $\gamma_t(T) + 2q$.

Conversely, let S be a minimum total dominating set of T' . If the old neighbor of x belongs to S , then $S \cap V(T)$ total dominates T , and Lemma 3.1.1 contributes at least $2q$ new vertices. If that neighbor is not in S , then $a_1 \in S$ is needed to dominate x ; adding the old neighbor to $S \cap V(T)$ produces a total dominating set of T , while Lemma 3.1.1 contributes at least $2q + 1$ vertices of the attached path. In both cases $|S| \geq \gamma_t(T) + 2q$.

For (O3), start with a minimum total dominating set D of T containing x and y . The two $2q$ - and $2q'$ -vertex extremal path patterns yield the upper bound. For the reverse inequality, let S be a minimum total dominating set of T' and put $r = |\{x, y\} \setminus S|$. By Lemma 3.1.1, the two attached paths together contribute at least $2q + 2q' + r$ vertices. Moreover, the set $(S \cap V(T)) \cup (\{x, y\} \setminus S)$ is a total dominating set of T : after inserting the missing vertices among x, y , these two adjacent vertices dominate one another, and all other old vertices retain a neighbor in the set. Therefore, $|S \cap V(T)| + r \geq \gamma_t(T)$, and hence, $|S| \geq \gamma_t(T) + 2q + 2q'$. $\square$

Lemma 3.1.3 (Structure of the extremal family). *Let $T \in \mathcal{T}_{n,\eta}$. Then, the following conclusions hold.*

- (i) $\Delta(T) \leq 3$.
- (ii) Every leaf is adjacent to a vertex of degree 2.
- (iii) Every vertex of degree 3 has exactly two neighbors of degree 2 and exactly one neighbor of degree 3; in particular, the degree-3 vertices occur in adjacent pairs.
- (iv) $n_3 = n - 2\eta$, $n_1 = n - 2\eta + 2$, and $n_2 = 4\eta - n - 2$.
- (v) $m_{12} = n_1$, $m_{23} = 2n_3$, and $m_{33} = \frac{n_3}{2}$.

Proof. We use structural induction on the construction of $\mathcal{T}$. All assertions are immediate for P_{4q} . Operation (O2) creates no degree-3 vertex and turns the attachment leaf into a degree-2 vertex, so (i)–(iii) are preserved. By Lemma 3.1.2,

$$(n + 4q) - 2(\eta + 2q) = n - 2\eta,$$

so the identity $n_3 = n - 2\eta$ is preserved as well.

Under Operation (O3), the adjacent degree-2 vertices x, y become adjacent degree-3 vertices. Because their other old neighbors x_0, y_0 have degree 2 and the first new path vertices also have degree 2, each of x, y has exactly two degree-2 neighbors and one degree-3 neighbor. The strengthened local condition in (O3) also prevents a later application of (O3) from converting one of these two degree-2 neighbors into another degree-3 neighbor. Thus, (i)–(iii) are preserved. Furthermore, n_3 increases by 2, while

$$(n + 4q + 4q' + 2) - 2(\eta + 2q + 2q') = n - 2\eta + 2.$$

Hence, $n_3 = n - 2\eta$ follows by induction. The formulas for n_1, n_2 now follow from

$$n_1 + n_2 + n_3 = n, \quad n_1 + 2n_2 + 3n_3 = 2(n - 1).$$

Finally, (ii)–(iii) imply $m_{12} = n_1$, $m_{23} = 2n_3$, and $m_{33} = n_3/2$. $\square$

Lemma 3.1.4. *For every $T \in \mathcal{T}_{n,\eta}$, $\text{EU}(T) = g(n, \eta)$.*

Proof. By Lemma 3.1.3, the only edge types are $(1, 2)$, $(2, 2)$, $(2, 3)$, $(3, 3)$, and

$$m_{12} = n - 2\eta + 2, \quad m_{23} = 2(n - 2\eta), \quad m_{33} = \frac{n - 2\eta}{2},$$

with $m_{22} = n - 1 - m_{12} - m_{23} - m_{33}$. Since

$$f(1, 2) = \sqrt{7}, \quad f(2, 2) = 2\sqrt{3}, \quad f(2, 3) = \sqrt{19}, \quad f(3, 3) = 3\sqrt{3},$$

a direct substitution gives (1). □

3.2. The Lower Bound

We now prove the lower bound. We use strong induction because several reductions delete more than one vertex.

Theorem 3.2.1. *Let T be a tree of order $n \geq 3$ and total domination number $\gamma_t = \gamma_t(T)$. Also, let $A = 2\sqrt{19} + \sqrt{7} - \frac{7\sqrt{3}}{2}$. Then*

$$\text{EU}(T) \geq g(n, \gamma_t) = A(n - 2\gamma_t) + 4\sqrt{3}\gamma_t - 6\sqrt{3} + 2\sqrt{7},$$

where equality throughout holds if and only if $T \in \mathcal{T}_{n,\gamma_t}$.

Proof. We verified the theorem computationally for every non-isomorphic tree of order n with $3 \leq n \leq 19$. The computation uses a linear-time dynamic program on a rooted tree to obtain $\gamma_t(T)$, and then evaluates $\text{EU}(T)$ directly. No counterexample occurs; equality occurs, in this range, exactly for P_4, P_8, P_{12}, P_{16} .

Assume now that $n \geq 20$ and that the statement, including the equality characterization, holds for every tree of order m with $3 \leq m < n$. Let $P = x_1x_2 \cdots x_{d+1}$ be a diametral path, where $d = \text{diam}(T)$. Thus, $d = 2$ if and only if $T \cong S_n$, where S_n denotes the star of order n .

We shall repeatedly use the following elementary consequence of the choice of P . We say that an off-spine branch at x_i has *height* h if a longest path from x_i into the branch has h edges. If an off-spine path starting at x_i reaches a leaf y after h edges and is otherwise disjoint from P , then $h \leq i - 1$ and $h \leq d + 1 - i$. Indeed, otherwise the path from y to one of the two ends of P would be longer than P . In particular, off-spine branches at x_2, x_3, x_4, x_5 have heights at most 1, 2, 3, 4, respectively.

For $T = S_n$, we have $\gamma_t(S_n) = 2$ and

$$\Phi(n) := \text{EU}(S_n) - g(n, 2) = (n - 1)\sqrt{n^2 - n + 1} - A(n - 4) - 2\sqrt{3} - 2\sqrt{7}.$$

Since

$$\Phi''(n) = \frac{8n^3 - 12n^2 + 15n - 7}{4(n^2 - n + 1)^{3/2}} > 0 \quad (n \geq 5),$$

Φ' is increasing on $[5, \infty)$. However, $\Phi'(5) > 0$ and $\Phi(5) > 0$, whence $\Phi(n) > 0$ for every $n \geq 5$.

Henceforth, $d \geq 3$. We first establish a list of strict configurations. In every claim, the reduced tree has order at least 3 because $n \geq 20$, so the induction hypothesis is applicable.

Claim 3.2.2. *Let x_2 be a support vertex with*

$$N(x_2) = \{x_1, x_3, u_1, \dots, u_k\}, \quad d(x_1) = 1, \quad d(x_3) \geq 2,$$

and suppose $u_1, \dots, u_k$ are leaves with $k \geq 2$. Then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Put $T_1 = T - x_1$ and $r = d_T(x_2) \geq 4$. Because x_2 still has the leaf u_1 in T_1 , it is forced into every total dominating set of both T and T_1 ; hence, $\gamma_t(T_1) = \gamma_t(T)$. Therefore,

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_1) + f(1, r) + (r - 2)H(r, 1) + H(r, d(x_3)) \\ &> g(n - 1, \gamma_t) + \sqrt{21} + 2(\sqrt{21} - \sqrt{13}) > g(n, \gamma_t), \end{aligned}$$

where Lemma 2.2 is used in the last step. □

Claim 3.2.3. Suppose $d(x_2) = 3$, $N(x_2) = \{x_1, x_3, u_1\}$, $d(x_1) = d(u_1) = 1$, and $2 \leq d(x_3) \leq 3$. Then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Let $T_2 = T - u_1$. As x_1 remains a leaf adjacent to x_2 , deleting u_1 does not change the total domination number: $\gamma_t(T_2) = \gamma_t$. Hence,

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_2) + f(1, 3) + f(3, 1) - f(2, 1) + f(3, d(x_3)) - f(2, d(x_3)) \\ &\geq g(n - 1, \gamma_t) + \sqrt{13} + (\sqrt{13} - \sqrt{7}) + (\sqrt{27} - \sqrt{19}) > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.4. Suppose $d(x_2) = 3$, $N(x_2) = \{x_1, x_3, u_1\}$, $d(x_1) = d(u_1) = 1$, and $d(x_3) \geq 4$. If T has a minimum total dominating set D with $|N(x_3) \cap D| \geq 2$, then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. By Lemma 2.4, choose a minimum total dominating set D with $x_1, u_1 \notin D$. Then $x_2 \in D$, and since x_2 must be dominated, $x_3 \in D$. Set $T_3 = T - \{x_1, u_1\}$ and $D' = D \setminus \{x_2\}$. Then $|D'| = \gamma_t - 1$, and D' totally dominates T_3 : x_2 is dominated by $x_3 \in D'$, while x_3 keeps a neighbor in D' because $|N(x_3) \cap D| \geq 2$. Hence, $\gamma_t(T_3) \leq \gamma_t - 1$.

By induction and Lemma 2.3, $\text{EU}(T_3) \geq g(n - 2, \gamma_t - 1)$. Since

$$\text{EU}(T) = \text{EU}(T_3) + [f(3, d(x_3)) - f(1, d(x_3))] + 2f(3, 1) > g(n - 2, \gamma_t - 1) + 2\sqrt{13} > g(n, \gamma_t),$$

the claim follows. □

Claim 3.2.5. Suppose $d(x_2) = 2$, $N(x_2) = \{x_1, x_3\}$, $d(x_1) = 1$, and $d(x_3) \geq 2$. If a minimum total dominating set D satisfies $|N(x_3) \cap D| \geq 2$, then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. The leaf x_1 forces $x_2 \in D$. By Lemma 2.4, we may choose D with $x_3 \in D$ and $x_1 \notin D$. Since $|N(x_3) \cap D| \geq 2$, the set $D \setminus \{x_2\}$ total dominates $T_4 = T - x_1$. Hence, $\gamma_t(T_4) \leq \gamma_t - 1$. Therefore,

$$\text{EU}(T) = \text{EU}(T_4) + f(2, 1) + f(2, d(x_3)) - f(1, d(x_3)) \geq g(n - 1, \gamma_t - 1) + \sqrt{7} > g(n, \gamma_t).$$

□

Claim 3.2.6. Suppose $d(x_2) = 2$, $N(x_2) = \{x_1, x_3\}$, $d(x_1) = 1$, and

$$N(x_3) = \{x_2, x_4, u_1, \dots, u_k\}, \quad d(x_3) \geq 4,$$

where all u_i are leaves. Then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Since $k \geq 2$, deleting one leaf, say u_1 , does not change the total domination number. With $T_5 = T - u_1$, Lemma 2.2 gives

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_5) + f(1, d(x_3)) + H(d(x_3), d(x_4)) + H(d(x_3), 2) + (d(x_3) - 3)H(d(x_3), 1) \\ &\geq g(n - 1, \gamma_t) + f(1, 4) + H(4, 2) + H(4, 1) > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.7. Suppose $d(x_2) = 2$, $N(x_2) = \{x_1, x_3\}$, $d(x_1) = 1$, and

$$N(x_3) = \{x_2, x_4, u_1\}, \quad d(u_1) = 1, \quad d(x_4) \leq 3.$$

Then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Let $T_6 = T - u_1$. We claim $\gamma_t(T_6) = \gamma_t(T)$. The inequality $\gamma_t(T_6) \leq \gamma_t(T)$ follows by choosing a minimum total dominating set of T without the leaf u_1 . Conversely, let S be a minimum total dominating set of T_6 . The leaf x_1 forces $x_2 \in S$. If $x_3 \in S$, then S already total dominates T . If $x_3 \notin S$, then $x_1 \in S$ is needed to dominate x_2 , and $(S \setminus \{x_1\}) \cup \{x_3\}$ total dominates T . Thus, $\gamma_t(T) \leq \gamma_t(T_6)$. Now,

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_6) + f(1, 3) + H(3, d(x_4)) + f(2, 3) - f(2, 2) \\ &\geq g(n-1, \gamma_t) + f(1, 3) + H(3, 3) + H(3, 2) > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.8. Let $x_1x_2x_3x_4$ be a path with $N(x_3) = \{x_2, x_4, u_1\}$, $d(x_1) = d(u_1) = 1$, $d(x_2) = 2$, and suppose $d(x_4) \geq 4$. If a minimum total dominating set D satisfies $|N(x_4) \cap D| \geq 2$, then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Both x_2 and x_3 are support vertices, and hence, belong to D . Put $T_7 = T - \{x_1, x_2, x_3, u_1\}$. The hypothesis on $N(x_4) \cap D$ ensures that $D \cap V(T_7)$ total dominates T_7 , so $\gamma_t(T_7) \leq \gamma_t - 2$. Therefore,

$$\begin{aligned} \text{EU}(T) &\geq g(n-4, \gamma_t-2) + f(4, 3) + f(3, 1) + f(3, 2) + f(2, 1) \\ &= g(n, \gamma_t) - 8\sqrt{3} + \sqrt{37} + \sqrt{13} + \sqrt{19} + \sqrt{7} > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.9. Let $x_1x_2x_3x_4$ be a path with $d(x_1) = 1$, $d(x_2) = d(x_3) = 2$, and

$$N(x_4) = \{x_3, x_5, u_1, \dots, u_k\}, \quad k \geq 1.$$

If a minimum total dominating set D satisfies $|N(x_4) \cap D| \geq 2$, then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. We may choose D with $x_2, x_3 \in D$ and $x_1 \notin D$. Set $T_8 = T - \{x_1, x_2, x_3\}$. Because x_4 has a neighbor in D other than x_3 , the set $D \cap V(T_8)$ total dominates T_8 , so $\gamma_t(T_8) \leq \gamma_t - 2$. Hence, we have

$$\text{EU}(T) = \text{EU}(T_8) + f(2, 1) + f(2, 2) + f(d(x_4), 2) + H(d(x_4), d(x_5)) + \sum_{i=1}^k H(d(x_4), d(u_i)).$$

Since $d(x_4) \geq 3$, we obtain

$$\text{EU}(T) \geq g(n-3, \gamma_t-2) + \sqrt{7} + 2\sqrt{3} + \sqrt{19} = g(n, \gamma_t) + 3\sqrt{19} + 2\sqrt{7} - \frac{19}{2}\sqrt{3} > g(n, \gamma_t).$$

□

Claim 3.2.10. Let $x_1x_2x_3x_4x_5$ be a path with $d(x_1) = 1$, $d(x_2) = d(x_3) = d(x_4) = 2$, $d(x_5) \geq 4$, and $N(x_5) = \{x_4, u_1, \dots, u_t\}$. Then $\text{EU}(T) > g(n, \gamma_t)$.

Proof. Choose a minimum total dominating set D with $x_2, x_3 \in D$ and $x_1 \notin D$. If $x_4 \in D$, then Claim 3.2.5 applies. Hence, we may assume $x_4 \notin D$. The vertex x_5 has a neighbor in D other than x_4 . Thus, $D \setminus \{x_2, x_3\}$ totally dominates $T_9 = T - \{x_1, x_2, x_3, x_4\}$ and $\gamma_t(T_9) \leq \gamma_t - 2$. Hence,

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_9) + f(2, 1) + 2f(2, 2) + f(d(x_5), 2) + \sum_{i=1}^t H(d(x_5), d(u_i)) \\ &\geq g(n-4, \gamma_t-2) + f(4, 2) + 2f(2, 2) + f(2, 1) > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.11. *Let $x_1x_2x_3x_4x_5$ be a path with $d(x_1) = 1$, $d(x_2) = d(x_3) = d(x_4) = 2$, and $d(x_5) = 3$. If some minimum total dominating set D contains x_5 , then $\text{EU}(T) > g(n, \gamma_t)$.*

Proof. Choose a minimum total dominating set D with $x_2, x_3 \in D$ and $x_1 \notin D$. Set $T_{10} = T - \{x_1, x_2, x_3\}$. Since $x_5 \in D$, the remaining vertex x_4 is dominated after the deletion, so $\gamma_t(T_{10}) \leq \gamma_t - 2$. Therefore,

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_{10}) + f(2, 1) + 2f(2, 2) + f(3, 2) - f(3, 1) \\ &\geq g(n - 3, \gamma_t - 2) + \sqrt{7} + 4\sqrt{3} + \sqrt{19} - \sqrt{13} > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.12. *Suppose T contains the path $x_1x_2x_3x_4x_5x_6x_7$ and a pendent path $x_5z_1z_2z_3$, and suppose that $N(x_6) = \{x_5, x_7, u_1, \dots, u_k\}$, where $d(x_1) = d(z_3) = 1$, $d(x_2) = d(x_3) = d(x_4) = d(z_1) = d(z_2) = 2$, and $d(x_5) = 3$. Then $\text{EU}(T) > g(n, \gamma_t)$.*

Proof. If some minimum total dominating set contains x_5 , Claim 3.2.11 already gives the desired strict inequality. We may therefore assume that every minimum total dominating set avoids x_5 . The leaf z_3 forces z_2 into every total dominating set; by Lemma 2.4 we may avoid z_3 , and then z_1 must belong to the set in order to dominate z_2 . Since z_1 already dominates x_5 , Lemma 3.1.1 normalizes the pendent P_4 segment $x_4x_3x_2x_1$ without increasing the cardinality. Hence, we may choose a minimum total dominating set D with $x_2, x_3, z_1, z_2 \in D$ and $x_1, x_4, x_5, z_3 \notin D$. Put $T_{11} = T - \{x_1, x_2, x_3, x_4, x_5, z_1, z_2, z_3\}$. Since $x_5 \notin D$, deleting x_5 does not remove a dominator of x_6 ; hence, $D \cap V(T_{11})$ total dominates T_{11} and $\gamma_t(T_{11}) \leq \gamma_t - 4$. Hence, we have

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_{11}) + f(d_T(x_6), 3) + 2f(3, 2) + 3f(2, 2) + 2f(2, 1) + H(d_T(x_6), d_T(x_7)) + \sum_{i=1}^k H(d_T(x_6), d_T(u_i)) \\ &> g(n - 8, \gamma_t - 4) + f(2, 3) + 2f(3, 2) + 3f(2, 2) + 2f(2, 1) \\ &= g(n, \gamma_t) - 16\sqrt{3} + 3\sqrt{19} + 3\sqrt{12} + 2\sqrt{7} > g(n, \gamma_t). \end{aligned}$$

□

Claim 3.2.13. *Suppose T contains the path $x_1x_2x_3x_4x_5x_6$ and a pendent path $x_5z_1z_2z_3z_4$, such that $d(x_1) = d(z_4) = 1$, $d(x_2) = d(x_3) = d(x_4) = d(z_1) = d(z_2) = d(z_3) = 2$, and $d(x_5) = 3$. Then $\text{EU}(T) > g(n, \gamma_t)$.*

Proof. If some minimum total dominating set contains x_5 , Claim 3.2.11 applies. Hence, assume that every minimum total dominating set avoids x_5 . A finite boundary-state check, equivalently two applications of the recurrence in Lemma 3.1.1 at the common boundary vertex x_5 , shows that a minimum total dominating set D can be normalized so that the two pendent P_4 segments use the pairs x_2, x_3 and z_2, z_3 , respectively. In this normalized set neither x_4 nor z_1 belongs to the set; therefore, x_6 must belong to D in order to dominate x_5 . Put $T_{12} = T - \{x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4\}$. Then $D \cap V(T_{12})$ total dominates T_{12} , so $\gamma_t(T_{12}) \leq \gamma_t - 4$. Consequently, we have

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_{12}) + 2f(3, 2) + 4f(2, 2) + 2f(2, 1) + f(3, d_T(x_6)) - f(1, d_T(x_6)) \\ &> g(n - 8, \gamma_t - 4) + 2f(3, 2) + 4f(2, 2) + 2f(2, 1) \\ &= g(n, \gamma_t) - 16\sqrt{3} + 2\sqrt{19} + 4\sqrt{12} + 2\sqrt{7} > g(n, \gamma_t). \end{aligned}$$

□

Lemma 3.2.14 (Universal-neighbor branch reduction). *Assume the induction hypothesis of Theorem 3.2.1. Let v be a spine vertex of degree at least 3, let p be one of its spine neighbors, and let*

$$Q = a_1 a_2 \cdots a_h \quad (1 \leq h \leq 7)$$

be an off-spine pendent path attached to v by the edge va_1 , whose internal vertices have degree 2 and whose last vertex is a leaf. Suppose that $S \cap N(v) = \{p\}$ for every minimum total dominating set S of T . Then, unless $\text{EU}(T) > g(n, \gamma_t(T))$, one has $h \in \{1, 5\}$.

Proof. The hypothesis implies $a_1 \notin S$ for every minimum total dominating set S . Fix the membership of v and note that v is already dominated by $p \in S$. The four-state recurrence in Lemma 3.1.1 shows that, for $h \equiv 2, 3 \pmod{4}$, there is a minimum pattern on Q of the same cardinality with $a_1 \in S$. Replacing the old pattern by this one leaves all vertices outside Q unchanged and produces a minimum total dominating set with a second member of $N(v)$, contrary to the hypothesis. Thus, for $1 \leq h \leq 7$, only $h \in \{1, 4, 5\}$ remains.

If $h = 4$, normalize S on Q . Exactly two vertices of Q are then used, and deleting Q leaves a total dominating set of $T - Q$ of size at most $\gamma_t(T) - 2$, because v remains dominated by p . Writing $r = d_T(v) \geq 3$ and discarding the positive degree-drop terms at v , we obtain

$$\begin{aligned} \text{EU}(T) &\geq g(n-4, \gamma_t-2) + f(r, 2) + 2f(2, 2) + f(2, 1) \\ &\geq g(n-4, \gamma_t-2) + \sqrt{19} + 4\sqrt{3} + \sqrt{7} \\ &= g(n, \gamma_t) + \sqrt{19} + \sqrt{7} - 4\sqrt{3} \\ &> g(n, \gamma_t). \end{aligned}$$

Hence, $h = 4$ is strict, and only $h = 1$ or $h = 5$ can occur in a non-strict configuration. $\square$

The next lemma is the structural step that makes the preceding claims exhaustive. It is separated from the numerical estimates because it depends only on total domination and the fact that P is diametral.

Lemma 3.2.15 (Structural reduction). *Assume none of Claims 3.2.2–3.2.13 applies to T . Then the diametral path $P = x_1 \cdots x_{d+1}$ can be oriented so that*

$$d \geq 7, \quad d(x_2) = d(x_3) = d(x_4) = d(x_5) = 2, \quad (2)$$

and there is a minimum total dominating set D satisfying

$$x_2, x_3, x_6 \in D, \quad x_1, x_4, x_5 \notin D. \quad (3)$$

Moreover every vertex in $N(x_6) \setminus \{x_5, x_7\}$ has degree at most 2. If $d(x_6) = 3$ and the unique off-spine neighbor w has degree 2, then, in a non-strict configuration, the component of $T - x_6$ containing w is a pendent path of five edges.

(For later spine vertices we do not make a branch classification from a single chosen minimum total dominating set. Whenever the stronger condition

$$S \cap N(v) = \{p\} \quad \text{for every minimum total dominating set } S$$

is available, the branch lengths are instead controlled by Lemma 3.2.14. This distinction between an existential and a universal minimum-set condition is essential for total domination.)

Proof. We give the elimination explicitly. Since x_1 is an end of a diametral path, every neighbor of x_2 other than x_3 is a leaf. If $d(x_2) \geq 4$, Claim 3.2.2 applies. If $d(x_2) = 3$ and $d(x_3) \leq 3$, Claim 3.2.3 applies. Suppose $d(x_2) = 3$ and $d(x_3) \geq 4$. By Lemma 2.4, choose a minimum total dominating set D with the two leaf neighbors x_1, u_1 of x_2 excluded. Then $x_2 \in D$, and $x_3 \in D$ is forced in order to dominate x_2 . If $|N(x_3) \cap D| \geq 2$, Claim 3.2.4 applies.

Assume therefore that $N(x_3) \cap D = \{x_2\}$. Every off-spine branch at x_3 has height at most two. Hence, if an off-spine neighbor v of x_3 were not a leaf, then v would be adjacent to a leaf and would consequently belong to every total dominating set, contradicting $N(x_3) \cap D = \{x_2\}$. Thus, every off-spine neighbor of x_3 is a leaf. Since $d(x_3) \geq 4$, at least two such leaves occur. Delete one of them, say u . By Lemma 2.4 we may assume $u \notin D$, and hence, D total dominates $T' = T - u$, so $\gamma_t(T') \leq \gamma_t$. With $r = d_T(x_3) \geq 4$ and with another leaf still adjacent to x_3 , we have

$$\text{EU}(T) - \text{EU}(T') \geq f(r, 1) + H(r, 3) + H(r, 1) \geq f(4, 1) + H(4, 3) + H(4, 1) > A.$$

Thus, $\text{EU}(T) > g(n - 1, \gamma_t) + A = g(n, \gamma_t)$. Therefore, $d(x_2) = 2$ in every non-strict configuration.

Now, x_2 is forced into every total dominating set. Replacing x_1 by x_3 when necessary, we may choose a minimum total dominating set D with $x_2, x_3 \in D$ and $x_1 \notin D$. If x_3 has a second neighbor in a minimum total dominating set, Claim 3.2.5 applies. If x_3 has an off-spine leaf and degree at least 4, Claim 3.2.6 applies; if $d(x_3) = 3$ and the next spine degree is at most 3, Claim 3.2.7 applies; and if the next spine degree is at least 4 with an additional neighboring dominator, Claim 3.2.8 applies.

It remains to exclude the case in which $d(x_3) = 3$, the off-spine neighbor v of x_3 is a leaf, and $d(x_4) \geq 4$. Suppose first that x_4 has an off-spine neighbor w of degree at least 2. The component of $T - x_4$ containing w has height at most three. If w is a support vertex, then w belongs to every total dominating set. Otherwise, a support vertex of that component is adjacent to w , and Lemma 2.4 applied to one of its leaf neighbors yields a minimum total dominating set avoiding that leaf; since the support vertex must be dominated, that set contains w . In both cases some minimum total dominating set D satisfies $\{x_3, w\} \subseteq N(x_4) \cap D$, and Claim 3.2.8 applies.

Hence, every off-spine neighbor of x_4 is a leaf. Since $d(x_4) \geq 4$, at least two such leaves occur. Delete one of them, say a , and put $T' = T - a$. The vertex x_4 remains adjacent to a leaf in T' , so Lemma 2.4 gives $\gamma_t(T') = \gamma_t(T)$. Writing $r = d_T(x_4) \geq 4$ and using Lemma 2.2, and discarding the nonnegative term $H(r, d(x_5))$, we obtain

$$\begin{aligned} \text{EU}(T) - \text{EU}(T') &= f(r, 1) + H(r, 3) + H(r, d(x_5)) + (r - 3)H(r, 1) \\ &\geq f(4, 1) + H(4, 3) + H(4, 1) = 2\sqrt{21} - \sqrt{13} + \sqrt{37} - 3\sqrt{3} > A. \end{aligned}$$

Therefore, we have

$$\text{EU}(T) > g(n - 1, \gamma_t) + A = g(n, \gamma_t).$$

This completes the elimination at x_3 . Hence, in the remaining case, $d(x_3) = 2$.

At x_4 , suppose $d(x_4) \geq 3$, so that x_4 has at least one off-spine neighbor. If x_4 has a leaf neighbor, then x_4 is a support vertex and belongs to every total dominating set; since x_2 also belongs to every total dominating set, every minimum total dominating set D satisfies $\{x_2, x_4\} \subseteq N(x_3) \cap D$, and Claim 3.2.5 applies. If x_4 has an off-spine neighbor of degree at least 2, the same leaf exchange as at x_3 produces a minimum total dominating set D with $|N(x_4) \cap D| \geq 2$, and Claim 3.2.9 applies. Hence, $d(x_4) = 2$ in every non-strict configuration.

Apply the same argument at x_5 . A vertex of degree at least 4 at x_5 is excluded by Claim 3.2.10; and a degree-3 vertex x_5 belonging to a minimum total dominating set is excluded by Claim 3.2.11. The two possible short off-spine branches at a degree-3 x_5 are excluded by Claims 3.2.12 and 3.2.13. A leaf attached to x_5 forces x_5 into every total dominating set, contrary to the assumption of this branch; and for a pendent two-edge path $x_5 z_1 z_2$, the vertex z_1 belongs to every total dominating set, so the leaf exchange $z_2 \mapsto x_5$ of Lemma 2.4 yields a minimum total dominating set containing x_5 , again excluded by Claim 3.2.11. Therefore, $d(x_4) = d(x_5) = 2$.

To see (3), start from a minimum total dominating set D with $x_2, x_3 \in D$ and $x_1 \notin D$. Since Claim 3.2.5 does not apply and $x_2 \in D$, we have $x_4 \notin D$. Then $x_6 \in D$ (to dominate x_5). If $x_5 \in D$, then, as $d(x_6) \geq 2$, swapping x_5 for any other neighbor of x_6 yields a minimum total dominating set satisfying (3); otherwise, (3) already holds. The checks for P_5, P_6, P_7 (already included in the small-order verification) imply $d \geq 7$.

It remains to justify the local assertion at x_6 . Any off-spine component at x_6 has height at most five. If the first off-spine neighbor has degree at least 3, orient a longest path in that component away from x_6 ; the resulting terminal support configuration is one of Claims 3.2.2–3.2.8, and the inequality is strict. Thus, the first off-spine neighbor has degree at most 2.

Suppose now that $d(x_6) = 3$ and that its off-spine neighbor w has degree 2. After excluding an internal branching vertex in the same way, the component through w is a pendent path of length $h \leq 5$. The four boundary states at x_6 and x_7 are evaluated by the recurrence of Lemma 3.1.1. For $h = 4$, deleting the branch after normalization gives

$$\text{EU}(T) \geq g(n-4, \gamma_t-2) + \sqrt{19} + 4\sqrt{3} + \sqrt{7} > g(n, \gamma_t).$$

For $h = 2, 3$, the equal-cost boundary switch at the branch, followed by the bridge decomposition at $x_6 x_7$ whenever the restriction on the x_7 -side is a total dominating set, gives a strict positive excess; in the complementary boundary state the unique outside dominator is propagated to x_7 and Lemma 3.2.14 applies. Since the spine is finite, this state propagation terminates either at a bridge-decomposable state or at the opposite end, where the same calculation is strict. Thus, $h = 2, 3, 4$ cannot occur in a non-strict configuration, leaving only $h = 5$. This is precisely the local assertion needed below. No analogous statement is made for an arbitrary later spine vertex unless the universal hypothesis of Lemma 3.2.14 has first been established. $\square$

By Lemma 3.2.15, it remains to treat the following cases.

Case 1: $d(x_6) = 2$. Let $T_{13} = T - \{x_1, x_2, x_3, x_4\}$. From (3), $D \setminus \{x_2, x_3\}$ total dominates T_{13} , and hence, $\gamma_t(T_{13}) \leq \gamma_t - 2$. Conversely, x_5 is a leaf of T_{13} . Hence, x_6 belongs to every total dominating set of T_{13} ; adding x_2, x_3 to such a set total dominates T . Thus, $\gamma_t(T_{13}) = \gamma_t - 2$. Moreover, we have

$$\text{EU}(T) = \text{EU}(T_{13}) + 4f(2, 2) \geq g(n-4, \gamma_t-2) + 8\sqrt{3} = g(n, \gamma_t).$$

Case 2: $d(x_6) \geq 4$. Write $N(x_6) = \{x_5, x_7, y_1, \dots, y_q\}$ with $q \geq 2$. By Lemma 3.2.15, $d(y_i) \leq 2$. Put $T_{14} = T - \{x_1, \dots, x_5\}$. The set $D \setminus \{x_2, x_3\}$ total dominates T_{14} , so $\gamma_t(T_{14}) \leq \gamma_t - 2$. Hence,

$$\text{EU}(T) > g(n-5, \gamma_t-2) + 3f(2, 2) + f(2, 1) + f(4, 2) + 2H(4, 2) > g(n, \gamma_t).$$

Case 3: $d(x_6) = 3$. Write $N(x_6) = \{x_5, x_7, w\}$.

Case 3.1: $d(w) = 1$ and $d(x_7) \leq 3$. With $T_{15} = T - \{x_1, \dots, x_5\}$, the same pruning gives $\gamma_t(T_{15}) \leq \gamma_t - 2$, and

$$\text{EU}(T) \geq g(n - 5, \gamma_t - 2) + f(3, 2) + 3f(2, 2) + f(3, 1) + H(3, 3) > g(n, \gamma_t).$$

Case 3.2: $d(w) = 1$ and $d(x_7) \geq 4$. Write $N(x_7) = \{x_6, x_8, u_1, \dots, u_k\}$. If some minimum total dominating set D satisfies $|D \cap N(x_7)| \geq 2$, let $T_{16} = T - \{x_1, x_2, x_3, x_4, x_5, x_6, w\}$. The set $D \cap V(T_{16})$ total dominates T_{16} and $\gamma_t(T_{16}) \leq \gamma_t - 3$. Hence, we have

$$\text{EU}(T) > g(n - 7, \gamma_t - 3) + 3f(2, 2) + f(3, 1) + f(3, 2) + f(2, 1) + f(4, 3) > g(n, \gamma_t).$$

We therefore assume that every minimum total dominating set D satisfies $D \cap N(x_7) = \{x_6\}$. Suppose that some off-spine neighbor, say u_1 , has degree 2. By Lemma 3.2.14, its branch is $x_7 u_1 w_1 w_2 w_3 w_4$ such that $d(u_1) = d(w_1) = d(w_2) = d(w_3) = 2$, $d(w_4) = 1$. We first normalize the chosen minimum total dominating set. Since w is a leaf adjacent to x_6 , the vertex x_6 belongs to every total dominating set. If $w \in D$, Lemma 2.4 permits the exchange $w \mapsto x_7$; if $w \notin D$, then $x_5 \notin D$ from (3), so $x_7 \in D$ is already necessary to dominate x_6 . Hence, a minimum total dominating set may be chosen with $x_7 \in D$. Lemma 3.1.1 then normalizes the displayed P_5 branch so that exactly two of its vertices belong to D .

Let $T' = T - \{u_1, w_1, w_2, w_3, w_4\}$. The normalized set loses exactly two vertices and still total dominates T' , since x_7 remains dominated by $x_6 \in D$. Thus, $\gamma_t(T') \leq \gamma_t - 2$. Writing $r = d_T(x_7) \geq 4$ and using Lemma 2.2, we have

$$\begin{aligned} \text{EU}(T) &\geq \text{EU}(T') + f(r, 2) + 3f(2, 2) + f(2, 1) + H(r, 3) + H(r, d(x_8)) \\ &> g(n - 5, \gamma_t - 2) + f(4, 2) + 3f(2, 2) + f(2, 1) + H(4, 3) \\ &= g(n, \gamma_t) + \sqrt{37} - \sqrt{27} + \sqrt{28} - 2\sqrt{19} + \frac{3\sqrt{3}}{2} \\ &> g(n, \gamma_t). \end{aligned}$$

Therefore, no degree-2 off-spine neighbor occurs in a non-strict configuration. Thus, all u_i are leaves. Deleting one of them gives $T_{17} = T - u_1$ with $\gamma_t(T_{17}) \leq \gamma_t$, and

$$\text{EU}(T) > g(n - 1, \gamma_t) + H(4, 3) + f(4, 1) > g(n, \gamma_t).$$

Case 3.3: $d(w) = 2$. By Lemma 3.2.15, the branch through w is $x_6 w v_1 v_2 v_3 v_4$ such that $d(v_4) = 1$ and $d(w) = d(v_1) = d(v_2) = d(v_3) = 2$. We divide the discussion into three subcases according to $d(x_7)$ and the position of a minimum total dominating set around x_7 .

Case 3.3.1: $d(x_7) = 2$. Let $T_{18} = T - \{x_1, \dots, x_5\}$. Then $\gamma_t(T_{18}) \leq \gamma_t - 2$ and

$$\text{EU}(T) \geq g(n - 5, \gamma_t - 2) + 3f(3, 2) + f(2, 2) + f(2, 1) > g(n, \gamma_t).$$

Case 3.3.2: $d(x_7) \geq 3$, and some minimum total dominating set D satisfies $|D \cap N(x_7)| \geq 2$. We write $N(x_7) = \{x_6, x_8, u_1, \dots, u_k\}$ and put $T_{19} = T - \{x_1, x_2, x_3, x_4, x_5, x_6, w, v_1, v_2, v_3, v_4\}$. Using Lemma 3.1.1 successively on the initial pendent segment and on the P_5 branch at x_6 , we may normalize a minimum total dominating set so that exactly five deleted vertices belong to it. Hence, $\gamma_t(T_{19}) \leq \gamma_t - 5$, and

$$\text{EU}(T) > g(n - 11, \gamma_t - 5) + f(3, 3) + 2f(3, 2) + 6f(2, 2) + 2f(2, 1) > g(n, \gamma_t).$$

Case 3.3.3: $D \cap N(x_7) = \{x_6\}$ for every minimum total dominating set D . We distinguish according to the degrees of the off-spine neighbors u_i of x_7 .

Case 3.3.3.1: some off-spine neighbor u_1 of x_7 is a leaf. If $d(x_7) \geq 4$, then deleting the leaf gives T_{20} with $\gamma_t(T_{20}) \leq \gamma_t$, and the estimate $\text{EU}(T) > g(n-1, \gamma_t) + f(4, 1) + H(4, 3) > g(n, \gamma_t)$ is strict. If $d(x_7) = 3$, $d(u_1) = 1$, and $d(x_8) \leq 3$, then delete the leaf together with the pendent five-edge branch at x_6 ; this yields a tree T_{21} with $\gamma_t(T_{21}) \leq \gamma_t - 2$. Keeping all local edge contributions gives

$$\begin{aligned} \text{EU}(T) &\geq g(n-6, \gamma_t-2) + 2f(3, 3) + f(3, 2) + f(2, 2) + f(3, 1) + f(2, 1) \\ &= g(n, \gamma_t) + 7\sqrt{3} + \sqrt{13} - 3\sqrt{19} - \sqrt{7} \\ &> g(n, \gamma_t). \end{aligned}$$

Assume now $d(x_7) = 3$, $d(u_1) = 1$, and $d(x_8) \geq 4$. Write $N(x_8) = \{x_7, x_9, z_1, \dots, z_p\}$. If a minimum total dominating set D satisfies $|D \cap N(x_8)| \geq 2$, cut the bridge x_7x_8 and let T_{22}, T_{23} be the components containing x_7, x_8 , respectively. Here $x_7 \in D$ because it is the support vertex of the leaf u_1 , while $D \cap N(x_7) = \{x_6\}$ implies $x_8 \notin D$. Thus, the restriction of D total dominates T_{22} . The condition $|D \cap N(x_8)| \geq 2$ guarantees that, after the loss of the neighbor $x_7 \in D$, x_8 is still dominated inside T_{23} . By Lemma 2.6, we have $\gamma_t(T) = \gamma_t(T_{22}) + \gamma_t(T_{23})$. Put $n_i = |V(T_i)|$ and $\eta_i = \gamma_t(T_i)$ for $i \in \{22, 23\}$. Then $n_{22} + n_{23} = n$ and, by Lemma 2.6, $\eta_{22} + \eta_{23} = \gamma_t$. Both components have order at least 3, so strong induction applies. Since g is affine, we have $g(n_{22}, \eta_{22}) + g(n_{23}, \eta_{23}) = g(n, \gamma_t) - 6\sqrt{3} + 2\sqrt{7}$. Substituting the local edge corrections at x_7x_8 gives

$$\text{EU}(T) > g(n, \gamma_t) - 6\sqrt{3} + 2\sqrt{7} + \sqrt{37} + \sqrt{13} + \sqrt{27} - \sqrt{7} - \sqrt{19} > g(n, \gamma_t).$$

If no such minimum total dominating set exists, Lemma 3.2.14 reduces every off-spine pendent path at x_8 to a leaf or a five-edge canonical branch, and Claims 3.2.12–3.2.13 then give a strict inequality.

Case 3.3.3.2: every off-spine neighbor of x_7 has degree 2. By Lemma 3.2.14, every such branch is a pendent five-edge path in a non-strict configuration. If at least two such branches occur, the strict alternatives above apply; hence, $d(x_7) = 3$. Denote the unique branch by $x_7u_1w_1w_2w_3w_4$ such that $d(u_1) = d(w_1) = d(w_2) = d(w_3) = 2$, $d(w_4) = 1$.

Subcase $d(x_8) = 2$ (the equality reduction). Put $T_{24} = T - \{w, v_1, v_2, v_3, v_4, u_1, w_1, w_2, w_3, w_4\}$. By Lemma 3.1.1, the two pendent P_5 branches may be normalized simultaneously. If one of x_6, x_7 is absent from the chosen minimum total dominating set, the corresponding branch must pay at least one additional vertex; replacing that extra boundary vertex by the missing endpoint produces a minimum total dominating set with $x_6, x_7 \in D$ and exactly two vertices on each deleted branch. Restricting this normalized set to T_{24} gives $\gamma_t(T_{24}) \leq \gamma_t - 4$, and

$$\begin{aligned} \text{EU}(T) &= \text{EU}(T_{24}) + f(3, 3) - f(2, 2) + 2(f(2, 3) - f(2, 2)) + 2f(3, 2) + 6f(2, 2) + 2f(2, 1) \\ &\geq g(n-10, \gamma_t-4) + f(3, 3) + 4f(3, 2) + 3f(2, 2) + 2f(2, 1) \\ &= g(n, \gamma_t). \end{aligned}$$

Subcase $d(x_8) \geq 3$. Suppose first that some minimum total dominating set D satisfies $|D \cap N(x_8)| \geq 2$. Cut the bridge x_7x_8 and let T_{25}, T_{26} be the components containing x_7, x_8 , respectively. The same restriction argument as above proves additivity of γ_t . The component T_{25} is the fixed 17-vertex tree consisting of the initial spine together with the two pendent P_5 branches. We record the exact value of its total domination number. For any total dominating set S of T_{25} , put

$$a = \mathbf{1}_{\{x_6 \notin S\}}, \quad b = \mathbf{1}_{\{x_7 \notin S\}}.$$

The two P_5 branches incident with x_6 contribute at least $2(2+a)$ vertices in total, while the P_5 branch incident with x_7 contributes at least $2+b$, by Lemma 3.1.1. Moreover $|S \cap \{x_6, x_7\}| = 2 - a - b$. Hence,

$$|S| \geq 2(2+a) + (2+b) + (2-a-b) = 8+a \geq 8.$$

On the other hand, $\{x_2, x_3, x_6, x_7, v_2, v_3, w_2, w_3\}$ is a total dominating set of size 8. Thus, $\gamma_t(T_{25}) = 8$. Therefore, $|V(T_{26})| = n - 17 \geq 3$ and, by Lemma 2.6, $\gamma_t(T_{26}) = \gamma_t - 8$. Strong induction and the local edge calculation yield

$$\text{EU}(T) > g(n - 17, \gamma_t - 8) + 3f(2, 3) + 3f(2, 1) + 2f(3, 3) + 9f(2, 2) > g(n, \gamma_t).$$

If no minimum total dominating set has a second neighbor of x_8 in it, Lemma 3.2.14 again reduces every off-spine pendent path at x_8 to a leaf or a five-edge branch; the leaf and multiple-branch alternatives are among the strict cases already treated.

This completes the proof of the lower bound in Theorem 3.2.1.

In what follows, we discuss the equality characterization for the lower bound in Theorem 3.2.1. All claims and all cases above are strict except Case 1 and the subcase producing T_{24} . We verify that these two non-strict reductions are exactly Operations (O2) and (O3). In Case 1, equality implies $\text{EU}(T_{13}) = g(n - 4, \gamma_t - 2)$. By the induction hypothesis, $T_{13} \in \mathcal{T}_{n-4, \gamma_t-2}$. In T_{13} , the vertex x_5 is a leaf. Every total dominating set contains x_6 ; moreover, if a minimum total dominating set contains x_5 , replacing x_5 by x_7 preserves its cardinality and total domination. Hence, T_{13} has a minimum total dominating set avoiding x_5 . The original tree T is obtained from T_{13} by attaching the path $x_4x_3x_2x_1 \cong P_4$ at x_5 . Thus, all hypotheses of Operation (O2) are satisfied, and $T \in \mathcal{T}$.

In the subcase involving T_{24} , equality in the chain of inequalities forces $\text{EU}(T_{24}) = g(n - 10, \gamma_t - 4)$ and $\gamma_t(T_{24}) = \gamma_t - 4$. Hence, $T_{24} \in \mathcal{T}_{n-10, \gamma_t-4}$ by induction. In T_{24} the adjacent vertices x_6, x_7 both have degree 2, with $N_{T_{24}}(x_6) = \{x_5, x_7\}$, $N_{T_{24}}(x_7) = \{x_6, x_8\}$, and $d_{T_{24}}(x_5) = d_{T_{24}}(x_8) = 2$. The normalization part of Lemma 3.1.1 gives a minimum total dominating set D of T with $x_6, x_7 \in D$ and exactly two vertices on each deleted P_5 branch. Because equality forces $\gamma_t(T_{24}) = \gamma_t - 4$, its restriction to T_{24} is a minimum total dominating set containing x_6, x_7 . Now, the tree T is obtained by attaching the two paths $wv_1v_2v_3v_4$ and $u_1w_1w_2w_3w_4$ at x_6 and x_7 , respectively. Therefore, every hypothesis of Operation (O3) is satisfied, and $T \in \mathcal{T}$.

Conversely, if $T \in \mathcal{T}_{n, \gamma_t}$, then Lemma 3.1.4 gives $\text{EU}(T) = g(n, \gamma_t)$. This completes the proof of Theorem 3.2.1. $\square$

4. Concluding Remarks

In this paper, we have established a lower bound on the Euler–Sombor index of trees with a given total domination number. This bound is valid for all trees, and we have characterized the extremal trees that attain it; these are precisely the trees in the recursively defined family $\mathcal{T}_{n, \gamma_t}$. For trees outside this family, the Euler–Sombor index is strictly larger than the bound. A natural direction for future work is to extend this investigation to other classes of graphs with given total domination number.

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