

Research Article

A New Characterization of the Wulff Shape

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Abstract

A new characterization of the Wulff shape in terms of higher-order anisotropic mean curvatures is proved. Namely, if a closed, connected and oriented hypersurface of the Euclidean space has a constant higher-order anisotropic mean curvature H_r^F and another one H_s^F almost constant, then it is the Wulff shape, up to translations and homotheties.

Keywords: hypersurfaces; anisotropic higher mean curvatures; Wulff shape.

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1. Introduction

Let $F : \mathbb{S}^n \longrightarrow \mathbb{R}_+$ be a smooth function satisfying the following convexity assumption:

$$A_F = (\nabla dF + F \text{Id}_{|T_x \mathbb{S}^n})_x > 0, \quad (1)$$

for all $x \in \mathbb{S}^n$, where ∇dF is the Hessian of F . The Wulff shape (with respect to F) is defined by $\mathcal{W}_F = \phi(\mathbb{S}^n)$ with

$$\begin{aligned} \phi : \mathbb{S}^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto F(x)x + (\text{grad}_{|\mathbb{S}^n} F)_x. \end{aligned}$$

Now, let (M^n, g) be an n -dimensional compact, connected, oriented manifold without boundary, isometrically immersed into $\mathbb{R}^{n+1}$ by X . We denote by ν a normal unit vector field globally defined on M and the F -Weingarten operator $S_F = A_F \circ S$, where A_F is defined in (2). The eigenvalues of S_F are the anisotropic principal curvatures that we will denote $\kappa_1^F, \kappa_2^F, \dots, \kappa_n^F$. Finally, for $r \in \{1, \dots, n\}$, the r -th anisotropic mean curvature is defined by

$$H_r^F = \frac{1}{\binom{n}{r}} \sum_{i_1 < \dots < i_r} \kappa_{i_1}^F \cdots \kappa_{i_r}^F.$$

We also set $H_0^F = 1$ for convenience. Note that the Wulff shape is F -umbilical, that is $S_F = H^F \text{Id}$ and all its anisotropic principal curvatures are equal to 1 and therefore, for any $r \in \{1, \dots, n\}$, we have $H_r^F = 1$. The Wulff is the anisotropic analogue of the geodesic spheres in the classical isotropic setting. For convenience, we will denote by $\tau_F = S_F - H^F \text{Id}$ the traceless part of the F -Weingarten operator.

In the isotropic context, geodesic spheres in Euclidean spaces can be characterized among closed hypersurfaces by various properties. In particular, it is well known that geodesic spheres are the only totally umbilical closed connected hypersurfaces in Euclidean spaces. For instance, in [1] Bivens proved that if two consecutive higher-order mean curvatures are constant, then the hypersurface is a geodesic sphere. This has been extended in [6] to the constancy of the ratio of two consecutive higher-order mean curvatures and finally by Koh and Lee to the ratio of any two higher-order mean curvatures [7, 8]. As a consequence, if two higher-order mean curvatures are constant, then the hypersurface is a geodesic sphere. In [9] and [10], we were able to relax the hypothesis of the result of Bivens for the first two mean curvatures. Precisely, if the mean curvature H is constant and the second mean curvature H_2 is almost constant, the conclusion also holds. Note that the roles of H and H_2 can be exchanged. Note also that the result of Koh and Lee [7, 8] has been generalized by He and Li in the anisotropic setting [3].

A natural question is to know if we can extend the results of [9, 10] to any two higher-order mean curvatures H_r and H_s and not only H and H_2 . The aim of the present note is to give a positive answer to this question in the more general anisotropic context, in which we also give an improvement of the result of [3]. Precisely, we have the following result

Theorem 1.1. *Let n, r, s be three integers with $n \geq 2$ and $1 \leq r < s \leq n$, and let a, b be two positive real numbers. Let $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a smooth function satisfying the following convexity assumption (2) and (M^n, g) be a closed, connected and oriented Riemannian manifold of dimension n isometrically immersed into $\mathbb{R}^{n+1}$ with positive s -th anisotropic mean curvature H_s^F . Then there exists a positive constant ε_0 depending on $n, r, s, F, \min_M(H_{s;n,1}^F), \min_M(H_s^F), \|S_F\|_\infty, R$ and $\text{Vol}(M)$ so that if*

$$H_r^F = a \quad \text{and} \quad H_s^F = b + \varepsilon$$

or

$$H_r^F = a + \varepsilon \quad \text{and} \quad H_s^F = b$$

where ε is a smooth function satisfying $\|\varepsilon\|_1 \leq \varepsilon_0$, then, up to translations and homotheties, M is the Wulff shape $\mathcal{W}_F$.

We point out that we do not assume in the hypothesis that the hypersurface is embedded because in this case, the constancy of one of the higher-order mean curvatures is enough to ensure that the hypersurface is the Wulff shape. Here, the assumption on the second higher-order mean curvature will allow us to show that the hypersurface is embedded and conclude by the anisotropic Alexandrov theorem.

2. Preliminaries

We recall the basics of anisotropic mean curvatures. These facts are classical; hence, we will not recall their proofs. First, let $F : \mathbb{S}^n \rightarrow \mathbb{R}_+$ be a smooth function satisfying the following convexity assumption

$$A_F = (\nabla dF + F \text{Id}_{T_x \mathbb{S}^n})_x > 0 \tag{2}$$

at any point $x \in \mathbb{S}^n$, in the sense of quadratic forms and where ∇dF is the Hessian of F . Now, we consider the following map

$$\begin{aligned} \phi : \mathbb{S}^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto F(x)x + (\text{grad}_{\mathbb{S}^n} F)_x. \end{aligned}$$

The image $W_F = \phi(\mathbb{S}^n)$ is called the Wulff shape of F and is a smooth hypersurface of $\mathbb{R}^{n+1}$. Moreover, from the convexity condition (2), W_F is convex. Note that if F is a positive constant c , the Wulff shape is nothing else but the sphere of radius c .

Let (M^n, g) be a compact, connected, oriented Riemannian manifold without boundary isometrically immersed into $\mathbb{R}^{n+1}$ by X and denote by ν its Gauss map. The (real-valued) second fundamental form B of the immersion is defined by

$$B(Y, Z) = -\langle \bar{\nabla}_Y \nu, Z \rangle,$$

for any $Y, Z \in \Gamma(TM)$, where $\langle \cdot, \cdot \rangle$ and $\bar{\nabla}$ are respectively the Riemannian metric and the Riemannian connection of $\mathbb{R}^{n+1}$. We also denote by S the Weingarten operator, which is the $(1, 1)$ -tensor associated with B via the metric g .

We consider $N_F = \phi(\nu) : M \rightarrow W_F$, the anisotropic Gauss map of M . We set

$$S_F = -dN_F = -A_F \circ d\nu = A_F \circ S,$$

which is the anisotropic shape operator, also called F -Weingarten operator, and its eigenvalues $\kappa_1^F, \dots, \kappa_n^F$ are the anisotropic principal curvatures. Now, let us recall the definition of the anisotropic mean curvature H^F and higher-order mean curvatures H_r^F . First, at a point $x \in M$, we consider an orthonormal frame $\{e_1, \dots, e_n\}$ of $T_x M$. For all $r \in \{1, \dots, n\}$, we set

$$\sigma_r = \frac{1}{r!} \sum_{\substack{1 \leq i_1, \dots, i_r \leq n \\ 1 \leq j_1, \dots, j_r \leq n}} \epsilon \left(\begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right) S_{i_1 j_1}^F \cdots S_{i_r j_r}^F,$$

where the S_{ij}^F are the coefficients of the F -Weingarten operator. The symbol

$$\epsilon \left(\begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right)$$

is the usual permutation symbol, which is zero if the sets $\{i_1, \dots, i_r\}$ and $\{j_1, \dots, j_r\}$ are different or if there exist distinct p and q with $i_p = i_q$. For all other cases,

$$\epsilon \left(\begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right)$$

is the signature of the permutation $\left(\begin{matrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{matrix} \right)$. Equivalently, we have

$$\sigma_r = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \kappa_{i_1}^F \kappa_{i_2}^F \cdots \kappa_{i_r}^F.$$

Then, the r -th anisotropic mean curvature of the immersion is defined by

$$H_r^F = \binom{n}{r}^{-1} \sigma_r.$$

We denote simply by H^F the anisotropic mean curvature H_1^F . We also define the anisotropic umbilicity tensor $\tau_F = S_F - H^F \text{Id}$, which is the traceless part of the F -Weingarten operator. Like in the isotropic case, if $\tau_F = 0$ then all the anisotropic principal curvatures are equal and constant.

Now, we recall the anisotropic version of the well-known Hsiung–Minkowski formulas [5] (see [3] for a proof). For any $r \in \{0, \dots, n-1\}$, we have

$$\int_M \left(F(\nu) H_r^F + H_{r+1}^F \langle X, \nu \rangle \right) dv_g = 0. \quad (3)$$

We also recall the following classical inequalities between higher-order anisotropic mean curvatures. First, for any $j \in \{1, \dots, n-1\}$, we have

$$(H_j^F)^2 - H_{j+1}^F H_{j-1}^F \geq 0 \quad (4)$$

with equality at a point p , provided $H_j^F(p) \neq 0$, if and only if all the anisotropic principal curvatures at p are equal. Hence, if $H_j^F > 0$, equality occurs everywhere if and only if M is the Wulff shape, up to translations and homotheties.

Moreover, if $H_r^F > 0$ for some $r \in \{2, \dots, n\}$, then for all $j \in \{1, \dots, r\}$, $H_j^F > 0$, and

$$(H_r^F)^{\frac{1}{r}} \leq (H_{r-1}^F)^{\frac{1}{r-1}} \leq (H_2^F)^{\frac{1}{2}} \leq H^F.$$

We conclude the review of higher-order anisotropic mean curvatures by giving a more precise version of inequality (4). For this, we give the definition of the function $H_{r+1;n,1}^F$ appearing in the statement of the result. Let $l \in \{2, \dots, n\}$ and $i, j \in \{1, \dots, n\}$ be three integers with $i > j$. Then, we can define the extrinsic curvature term $H_{l;i,j}^F$ by

$$H_{l;i,j}^F = \frac{\partial H_l^F}{\partial \kappa_i^F \partial \kappa_j^F}$$

where H_l^F is the l -th anisotropic mean curvature and κ_i^F, κ_j^F are the i -th and j -th anisotropic principal curvatures (ordered as $\kappa_1^F \leq \kappa_2^F \leq \dots \leq \kappa_n^F$). The notation $\frac{\partial H_l^F}{\partial \kappa_i^F \partial \kappa_j^F}$ is to understand in the sense that H_l^F is defined from the anisotropic principal curvature by

$$H_l^F = \frac{1}{\binom{n}{l}} \sigma_l(\kappa_1^F, \dots, \kappa_n^F)$$

and thus

$$H_{l;i,j}^F = \frac{1}{\binom{n}{l}} \sigma_{l-2}(\kappa_1^F, \dots, \kappa_{j-1}^F, \kappa_{j+1}^F, \dots, \kappa_{i-1}^F, \kappa_{i+1}^F, \dots, \kappa_n^F).$$

In the sequel, we will only be interested in

$$H_{s;n,1}^F = \frac{1}{\binom{n}{s}} \sigma_{s-2}(\kappa_2^F, \dots, \kappa_{n-1}^F).$$

In [11], Scheuer showed that if H_{s-1}^F is positive then $H_{s;n,1}^F$ is also positive. Note that the proof of this fact is by contradiction so that we do not have an explicit inequality between H_{s-1}^F and $H_{s;n,1}^F$. Note also that Scheuer gives the proof for the isotropic case, but the proof in the anisotropic case is almost the same with the following adaptation. First, note that $S_F = A_F S$ is generally not self-adjoint with respect to the induced metric, although it has real eigenvalues. However, since A_F is positive definite due to the convexity assumption, we can write

$$S_F = A_F^{1/2} \left(A_F^{1/2} S A_F^{1/2} \right) A_F^{-1/2}.$$

The middle operator is symmetric and so its traceless norm and the traceless norm of S_F , that is $\|\tau_F\|$, are equivalent, with constants depending on the ellipticity ratio of A_F , that is on F . Now, we can state the following inequality

$$(H_k^F)^2 - H_{k+1}^F H_{k-1}^F \geq c_{n,F} \|\tau_F\|^2 (H_{k+1;n,1}^F)^2 \quad (5)$$

where $c_{n,F}$ is a constant depending only on n and F . One can find the proof in [11] for the isotropic case; the proof for the anisotropic case is once again the same.

We conclude the preliminaries section by recalling that the extrinsic radius $R(M)$ of M is defined by

$$R(M) = \inf\{\rho > 0 \mid \exists x \in \mathbb{R}^{n+1} \text{ s.t. } X(M) \subset B(x, \rho)\},$$

where X is the immersion of M into $\mathbb{R}^{n+1}$. By a slight abuse of notation, we denote it by $R(M)$; however, this radius depends not only on M but also on the immersion X . Since the immersion considered in this paper is fixed, this notation does not lead to any ambiguity.

3. Proof of Theorem 1.1

The strategy of the proof is to show that the hypotheses imply the almost umbilicity of the hypersurface, thereby allowing the following result to be applied to obtain embeddedness:

Proposition 3.1 (see [2]). *Let $n \geq 2$, $p > n$, and let M be a closed hypersurface of $\mathbb{R}^{n+1}$. Assume that there exists a constant $c_0 > 0$ such that $\|S_F\|_p \leq c_0$ and $\text{Vol}(M) = \text{Vol}(\mathcal{W})$. Then, for every $\eta > 0$ sufficiently small, there exists a constant $\delta > 0$ depending on η , n , p , c_0 and $\mathcal{W}$ such that if $\|\tau_F\|_p \leq \delta$, then M admits an anisotropic radial parametrization*

$$\psi : \mathcal{W} \longrightarrow M, \quad \psi(x) = x + u(x)\nu_{\mathcal{W}}(x),$$

with $\|u\|_{C^1} \leq \eta$. In particular, for η sufficiently small, M is embedded.

It is crucial to note that, for this result, the hypersurface is only assumed to be immersed and the embeddedness is obtained as a consequence of the almost umbilicity assumption. Indeed, for the almost umbilicity assumption, the authors proved the existence of an anisotropic radial parametrization, which implies the embeddedness. Note also that if we do not assume that $\text{Vol}(M) = \text{Vol}(\mathcal{W})$, then $\delta > 0$ depends in addition on $\text{Vol}(M)$. Precisely, we will use this result for $p = n + 1$. For this, we will show that the L^{n+1} -norm of $\|\tau_F\|$ is small. First, we remark that

$$\begin{aligned} \|\tau_F\|_{n+1}^{2(n+1)} &= \left(\frac{1}{V(M)} \int_M \|\tau_F\|^{(n+1)} dv_g \right)^2 \\ &= \left(\frac{1}{V(M)} \int_M \|\tau_F\|^n \cdot \|\tau_F\| dv_g \right)^2, \end{aligned}$$

and by the Cauchy-Schwarz inequality, we get

$$\|\tau_F\|_{n+1}^{2(n+1)} \leq \frac{1}{V(M)^2} \left(\int_M \|\tau_F\|^{2n} dv_g \right) \left(\int_M \|\tau_F\|^2 dv_g \right).$$

From this, we deduce immediately that

$$\|\tau_F\|_{n+1}^{2(n+1)} \leq \frac{1}{V(M)} \|S_F\|_\infty^{2n} \left(\int_M \|\tau_F\|^2 dv_g \right). \quad (6)$$

Hence, the problem reduces to control the L^2 -norm of $\|\tau_F\|$. For this, we begin by giving the following lemma

Lemma 3.2. *Let $r, s \in \{1, \dots, n\}$ with $r < s$. Let (M^n, g) be a closed, connected and oriented Riemannian manifold of dimension n isometrically immersed into the Euclidean space $\mathbb{R}^{n+1}$ so that the s -th anisotropic mean curvature H_s^F is positive everywhere on M . Then, there exists a positive constant K depending on $n, r, s, F, \min_M(H_{s;n,1}^F), \min_M(H_s^F)$ and $\|S_F\|_\infty$ so that*

$$\|\tau_F\|^2 \leq K (H_{s-1}^F H_r^F - H_s^F H_{r-1}^F).$$

Proof: First of all, note that if $s = 2$ and $r = 1$, then $H_{s-1}^F H_r^F - H_s^F H_{r-1}^F$ is nothing else but $(H^F)^2 - H_2^F$ and the desired estimate follows directly from (5) with $k = 1$, since $H_{2;n,1}^F = \binom{n}{2}^{-1}$.

Now, we can consider the case $s > 2$. We start from inequality (5), that is

$$(H_k^F)^2 - H_{k+1}^F H_{k-1}^F \geq c_{n,F} \|\tau_F\|^2 (H_{k+1;n,1}^F)^2.$$

Since $H_s^F > 0$, all the functions H_k^F are also positive for $k \in \{1, \dots, s-1\}$. Thus, dividing by $H_k^F H_{k+1}^F$, (5) becomes

$$\frac{H_k^F}{H_{k+1}^F} - \frac{H_{k-1}^F}{H_k^F} \geq c_{n,F} \|\tau_F\|^2 \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F}. \quad (7)$$

Thus, by summing equation (7) for k from r to $s-1$, we get

$$\frac{H_{s-1}^F}{H_s^F} - \frac{H_{r-1}^F}{H_r^F} \geq c_{n,F} \|\tau_F\|^2 \sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F}, \quad (8)$$

and so

$$\frac{H_{s-1}^F}{H_s^F} H_r^F - H_{r-1}^F \geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F} \right) H_r^F, \quad (9)$$

which gives, after multiplying by H_s^F ,

$$\begin{aligned} H_{s-1}^F H_r^F - H_s^F H_{r-1}^F &\geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F} \right) H_r^F H_s^F \\ &\geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F} \right) (H_s^F)^{\frac{r}{s}} H_s^F \\ &\geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{H_k^F H_{k+1}^F} \right) (H_s^F)^{\frac{r+s}{s}}. \end{aligned} \quad (10)$$

Note that, in the second line, we have used $(H_s^F)^{\frac{1}{s}} \leq (H_r^F)^{\frac{1}{r}}$.

Moreover, we have $H_k^F H_{k+1}^F \leq \|S_F\|_\infty^{2k+1}$ which gives

$$H_{s-1}^F H_r^F - H_s^F H_{r-1}^F \geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{(H_{k+1;n,1}^F)^2}{\|S_F\|_\infty^{2k+1}} \right) (H_s^F)^{\frac{r+s}{s}}. \quad (11)$$

In addition, since H_s^F is positive, all the functions H_k^F are also positive for $k \in \{1, \dots, s-1\}$ and thus, as mentioned in the preliminaries section, the functions $H_{k+1;n,1}^F$ are also positive. Furthermore, since they are normalized symmetric polynomials evaluated for $\kappa_2^F, \dots, \kappa_{n-1}^F$, they also satisfy the Maclaurin inequality, up to a normalization constant; that is, for $l > 2$,

$$(H_{l;n,1}^F)^{\frac{1}{l-2}} \geq a_{n,l} (H_{l+1;n,1}^F)^{\frac{1}{l-1}},$$

where $a_{n,l}$ is a positive constant depending only on n and l , and so for $k > 1$,

$$(H_{k+1;n,1}^F)^{\frac{1}{k-1}} \geq b_{n,k,s} (H_{s;n,1}^F)^{\frac{1}{s-2}},$$

where $b_{n,k,s}$ is a positive constant depending only on n , k and s . Note that the exponents come from the fact that $H_{l;n,1}^F$ is the symmetric polynomial of degree $l-2$.

In the case where $r = 1$, the term for $k = r = 1$ is $H_{2;n,1}^F = \binom{n}{2}^{-1}$, which is a positive constant depending only on n and can be absorbed into $c_{n,F}$, after decreasing $c_{n,F}$ if necessary, and we set $b_{n,1,s} = 1$. Thus, (11) gives

$$H_{s-1}^F H_r^F - H_s^F H_{r-1}^F \geq c_{n,F} \|\tau_F\|^2 \left(\sum_{k=r}^{s-1} \frac{b_{n,k,s}^{2(k-1)} (H_{s;n,1}^F)^{\frac{2(k-1)}{s-2}}}{\|S_F\|_\infty^{2k+1}} \right) (H_s^F)^{\frac{r+s}{s}} \geq C \|\tau_F\|^2, \quad (12)$$

where

$$C = c_{n,F} \min_{r \leq k \leq s-1} \left(b_{n,k,s}^{2(k-1)} \right) \frac{\min(H_s^F)^{\frac{r+s}{s}}}{\|S_F\|_\infty^3} \sum_{k=r}^{s-1} \left(\frac{\min(H_{s;n,1}^F)^{\frac{1}{s-2}}}{\|S_F\|_\infty} \right)^{2(k-1)}.$$

This concludes the proof of the lemma by setting $K = \frac{1}{C}$, which depends only on n , r , s , F , $\min_M(H_{s;n,1}^F)$, $\min_M(H_s^F)$ and $\|S_F\|_\infty$. $\square$

Now, we estimate $H_{s-1}^F H_r^F - H_s^F H_{r-1}^F$ in order to show from Lemma 3.2 that M is almost anisotropically umbilical and then conclude with Proposition 3.1.

First, we consider the case where $H_r^F = a$ and $H_s^F = b + \varepsilon$. We have

$$\begin{aligned} \int_M (H_{s-1}^F H_r^F - H_s^F H_{r-1}^F) F(\nu) dv_g &= \int_M (a H_{s-1}^F - b H_{r-1}^F - \varepsilon H_{r-1}^F) F(\nu) dv_g \\ &= a \int_M H_{s-1}^F F(\nu) dv_g - b \int_M H_{r-1}^F F(\nu) dv_g - \int_M \varepsilon H_{r-1}^F F(\nu) dv_g \\ &= -a \int_M H_s^F \langle X, \nu \rangle dv_g + b \int_M H_r^F \langle X, \nu \rangle dv_g - \int_M \varepsilon H_{r-1}^F F(\nu) dv_g \end{aligned} \quad (13)$$

by the Hsiung–Minkowski formulas. Using the assumption again, we get

$$\begin{aligned} &\int_M (H_{s-1}^F H_r^F - H_s^F H_{r-1}^F) F(\nu) dv_g \\ &= -ab \int_M \langle X, \nu \rangle dv_g - a \int_M \varepsilon \langle X, \nu \rangle dv_g + ab \int_M \langle X, \nu \rangle dv_g - \int_M \varepsilon H_{r-1}^F F(\nu) dv_g \\ &= -a \int_M \varepsilon \langle X, \nu \rangle dv_g - \int_M \varepsilon H_{r-1}^F F(\nu) dv_g \\ &= - \int_M \varepsilon (H_r^F \langle X, \nu \rangle + H_{r-1}^F F(\nu)) dv_g \end{aligned} \quad (14)$$

Now, we use Lemma 3.2 to get

$$\begin{aligned}
 \int_M \|\tau_F\|^2 F(\nu) dv_g &\leq K \int_M (H_{s-1}^F H_r^F - H_s^F H_{r-1}^F) F(\nu) dv_g \\
 &= -K \int_M \varepsilon (H_r^F \langle X, \nu \rangle + H_{r-1}^F F(\nu)) dv_g \\
 &\leq K \int_M |\varepsilon| (H_r^F \|X\| + H_{r-1}^F F(\nu)) dv_g \\
 &\leq A_1 \int_M |\varepsilon| dv_g
 \end{aligned} \tag{15}$$

where $A_1 = K(\|S_F\|_\infty^r R + \|S_F\|_\infty^{r-1} \|F\|_\infty)$ is a constant depending on $n, r, s, \min_M(H_{s;n,1}^F), \min_M(H_s^F), \|S_F\|_\infty, R$ and F .

Hence, using (6) and (15), we obtain

$$\|\tau_F\|_{n+1}^{2(n+1)} \leq A_2 \|\varepsilon\|_1, \tag{16}$$

with

$$A_2 = \frac{A_1 \|S_F\|_\infty^{2n}}{\inf(F)},$$

which is also a constant depending on $n, r, s, \min_M(H_{s;n,1}^F), \min_M(H_s^F), \|S_F\|_\infty, R$ and F .

Now, we fix $\eta > 0$ sufficiently small. By Proposition 3.1 with $p = n + 1$, there exists $\delta > 0$ depending on η, n, F, S_F and $\text{Vol}(M)$ so that if $\|\tau_F\|_{n+1} \leq \delta$, then M is embedded. We set

$$\varepsilon_0 = \frac{\delta^{2(n+1)}}{A_2}$$

so that if $\|\varepsilon\|_1 \leq \varepsilon_0$, then (16) gives $\|\tau_F\|_{n+1} \leq \delta$ and so M is embedded. Note that due to its form $\frac{\delta^{2(n+1)}}{A_2}$ and the dependence of δ and A_2, ε_0 depends on $n, r, s, F, \min_M(H_{s;n,1}^F), \min_M(H_s^F), \|S_F\|_\infty, R$ and $\text{Vol}(M)$. We conclude by applying the anisotropic Alexandrov theorem for higher-order mean curvature proved by He, Li, Ma and Ge [4]. Since M is embedded with constant H_r^F , it follows that M is the Wulff shape up to translations and homotheties.

We finish by noting that the proof of the second case, that is, $H_r^F = a + \varepsilon$ and $H_s^F = b$ is analogous. The only difference is that, with this assumption, relation (14) is replaced by

$$\int_M (H_{s-1}^F H_r^F - H_s^F H_{r-1}^F) F(\nu) dv_g = \int_M \varepsilon (H_s^F \langle X, \nu \rangle + H_{s-1}^F F(\nu)) dv_g,$$

when using Hsiung–Minkowski formulas. The rest of the proof is the same: we obtain embeddedness and then conclude with the Alexandrov theorem for constant anisotropic s -th mean curvature, since $H_s^F = b$ is constant.

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