

Research Article

Bi-coloring Two Complete Graphs and Resulting Strong Ramsey Numbers

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Abstract

The Ramsey number of a graph G without isolated vertices is the minimum order of a complete graph for which every red-blue edge coloring results in a monochromatic subgraph isomorphic to G . The strong Ramsey number of two non-isomorphic graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue coloring of K_n results in two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H . It is shown that for every pair F, H of non-isomorphic graphs without isolated vertices having Ramsey number 6, the strong Ramsey number of F and H is either 6 or 7.

Keywords: red-blue coloring; Ramsey number; strong Ramsey number.

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1. Introduction

The *Ramsey number* $R(G)$ of a graph G without isolated vertices is the minimum positive integer n such that every red-blue (edge) coloring of the complete graph K_n of order n results in a subgraph of K_n isomorphic to G all of whose edges are colored the same (a *monochromatic* G). There are exactly seven graphs G without isolated vertices with $R(G) = 6$, which are shown in Figure 1.1 (see [1, 13]).

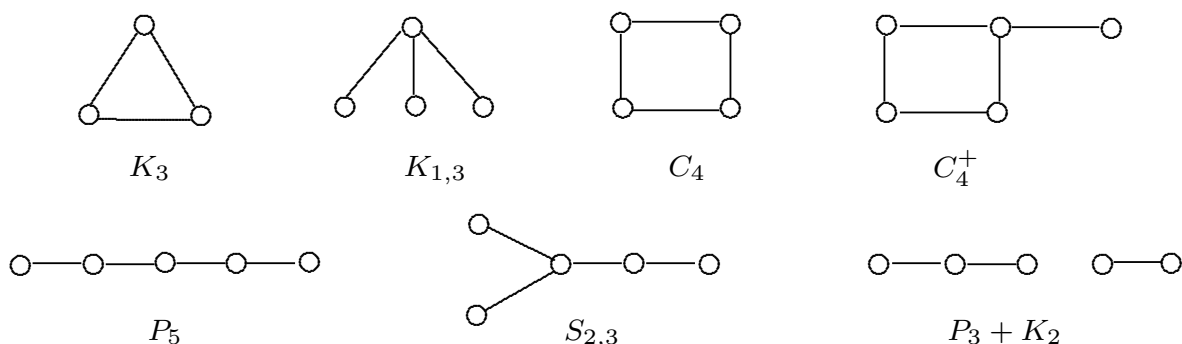


Figure 1.1: The seven graphs with Ramsey number 6.

The best known graph with Ramsey number 6 is the triangle K_3 . Every red-blue coloring of K_6 therefore produces a monochromatic triangle. In fact, every red-blue coloring of K_6 produces at least two monochromatic triangles, but not necessarily two edge-disjoint monochromatic triangles, as the red-blue coloring of K_6 shown in Figure 1.2 illustrates (where a solid edge represents a red edge and a thin edge represents a blue edge). On the other hand, every red-blue coloring of K_7 does produce two edge-disjoint monochromatic triangles.

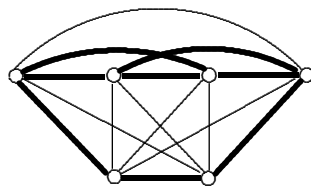


Figure 1.2: A red-blue coloring of K_6 without two edge-disjoint monochromatic triangles.

Proposition 1.1. [8] *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.*

The Ramsey number of the 4-cycle C_4 is 6 as well. As with K_3 , not every red-blue coloring of K_6 produces two edge-disjoint monochromatic 4-cycles. For example, the red-blue coloring of K_6 shown in Figure 1.3 does not contain two edge-disjoint monochromatic 4-cycles.

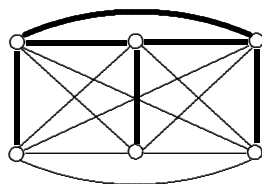


Figure 1.3: A red-blue coloring of K_6 without two edge-disjoint monochromatic 4-cycles.

On the other hand, every red-blue coloring of K_7 produces two edge-disjoint monochromatic 4-cycles (see [5]). We include a proof here for completeness. For this purpose, the following known result is presented.

Theorem 1.2. [9, 10] *For each prime p , the largest size of a graph of order $p^2 + p + 1$ that does not contain a 4-cycle is $\frac{1}{2}p(p+1)^2$.*

Theorem 1.3. [5] *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic 4-cycles.*

Proof. Let there be given a red-blue coloring of $G = K_7$. Since $R(C_4) = 6$, it follows that either the red subgraph G_r or the blue subgraph G_b of G has a 4-cycle, say G_r has a 4-cycle $H = C_4$. If G_b also has a 4-cycle, then G has two edge-disjoint monochromatic copies of C_4 . Thus, we may assume that G_b contains no 4-cycle. If the size of G_b is at most 7, then the size of G_r is at least $\binom{7}{2} - 7 = 14$. Hence, the subgraph $G_r - E(H)$ of G_r has order 7 and size at least 10. By Theorem 1.2, it follows that $G_r - E(H)$ contains a 4-cycle and so G has two edge-disjoint red 4-cycles. Hence, we may assume that the size of G_b is at least 8 and so G_b contains a cycle. Let $c(G_b)$ be the circumference (the length of a longest cycle) of G_b . Since G_b contains no 4-cycle, it follows that $c(G_b) \in \{7, 6, 5, 3\}$ resulting in four cases. A case-by-case analysis shows for every possible graph G_b in each case that G_r has two edge-disjoint 4-cycles. Therefore, G contains two edge-disjoint red 4-cycles. $\square$

For two graphs F and H without isolated vertices, the *Ramsey number* $R(F, H)$ of F and H is the minimum positive integer n such that for every red-blue coloring of K_n , there exists a subgraph of K_n isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph isomorphic to H all of whose edges are colored blue (a *blue* H). If $n = \max\{R(F), R(H)\}$, then every red-blue coloring of K_n produces monochromatic subgraphs F and H but not necessarily edge-disjoint copies of F and H . Since $R(K_3) = R(C_4) = 6$, it follows that $\max\{R(K_3), R(C_4)\} = 6$. The red-blue coloring of K_6 in Figure 1.4 does not produce edge-disjoint monochromatic subgraphs K_3 and C_4 . This observation gave rise to the problem of determining the smallest positive integer n for which every every red-blue coloring of K_n results in two *edge-disjoint* monochromatic subgraphs, either of different colors or the same color, one isomorphic to F and the other isomorphic to H .

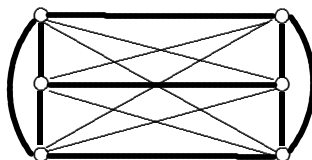


Figure 1.4: A red-blue coloring of K_6 without two edge-disjoint monochromatic triangle and 4-cycle.

The *strong Ramsey number* $\bar{R}(F, H)$ of two graphs F and H without isolated vertices is the minimum positive integer n such that every red-blue coloring of K_n results in two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H . This concept was introduced and studied in [3] and studied further in [4]. Since the Ramsey number $R(F, H)$ exists for every pair F, H of graphs by a result of Ramsey [14], the strong Ramsey number $\bar{R}(F, H)$ exists as well.

Proposition 1.4. [3] *For every two graphs F and H without isolated vertices, $\bar{R}(F, H)$ exists and*

$$\max\{R(F), R(H)\} \leq \bar{R}(F, H) \leq \max\{R(F), R(H)\} + \min\{|V(F)|, |V(H)|\}.$$

Listed below are three observations dealing with strong Ramsey numbers.

Observation 1.5. [3] *For every two graphs F and H without isolated vertices, $\bar{R}(F, H) = \bar{R}(H, F)$.*

Observation 1.6. [3] *If F is a graph without isolated vertices and $F \neq K_2$, then $\bar{R}(F, K_2) = R(F)$.*

Observation 1.7. [3] *If F, F', H , and H' are graphs without isolated vertices such that $F' \subseteq F$ or $H' \subseteq H$, then $\bar{R}(F', H) \leq \bar{R}(F, H)$ and $\bar{R}(F, H') \leq \bar{R}(F, H)$.*

For non-isomorphic graphs F and H of the same size 2, 3, or 4 without isolated vertices, it was shown in [3, 4] that $\bar{R}(F, H)$ is either $\max\{R(F), R(H)\}$ or $1 + \max\{R(F), R(H)\}$. The question was introduced as to whether this fact holds for broader pairs of graphs. By investigating red-blue colorings (or bi-colorings) of the complete graphs K_6 and K_7 , we extend this fact to the seven graphs without isolated vertices having Ramsey number 6 since it is more challenging in general to determine $\bar{R}(F, H)$ when $R(F) = R(H)$. Listed next are the known strong Ramsey numbers of the pairs of these graphs of the same size obtained in [3, 4].

Theorem 1.8. [4] *If $F \in \{K_3, C_4, C_4^+, P_3 + K_2, S_{2,3}, P_5\}$, then $\bar{R}(F, K_{1,3}) = 6$.*

Theorem 1.9. [3] *If $F \in \{K_3, S_{2,3}\}$, then $\bar{R}(F, P_3 + K_2) = 6$.*

Theorem 1.10. [4] *If $F \in \{C_4, S_{2,3}\}$, then $\bar{R}(F, P_5) = 7$ and $\bar{R}(C_4, S_{2,3}) = 6$.*

We refer to the book [2] for notation and terminology not given here. More information on Ramsey numbers can be found in [6, 7, 11, 12, 15, 16].

2. Strong Ramsey Numbers Involving C_4^+

Let C_4^+ be the graph obtained from the 4-cycle C_4 by adding a pendant edge. The graph C_4^+ is the unique graph of size 5 without isolated vertices having Ramsey number 6. Furthermore, if G is a graph of size 5 or more different from C_4^+ , then $R(G) \geq 7$ (see [1, 13]). For certain pairs of graphs involving C_4^+ , the strong Ramsey number is 7, however. First, we determine the value of the strong Ramsey number $\bar{R}(K_3, C_4)$ of the two graphs K_3 and C_4 , which corrects an error in Theorem 4.1 in [3].

Theorem 2.1. $\bar{R}(K_3, C_4) = 7$.

Proof. Since the red-blue coloring of K_6 shown in Figure 1.4 with red subgraph $K_3 \square K_2$ and blue subgraph C_6 does not produce edge-disjoint monochromatic subgraphs K_3 and C_4 , it follows that $\bar{R}(K_3, C_4) \geq 7$. Next, we show that $\bar{R}(K_3, C_4) \leq 7$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \dots, v_7\}$. Since $R(C_4) = 6$, there is a monochromatic subgraph $H = C_4$ in G , say $H = (v_1, v_2, v_3, v_4, v_1)$. If $G' = G[\{v_5, v_6, v_7\}] = K_3$ is monochromatic, then there are edge-disjoint monochromatic subgraphs K_3 and C_4 in G . Thus, we may assume that exactly two edges of G' have the same color, say v_5v_6 and v_6v_7 are red and v_5v_7 is blue. At least two of the three edges v_5v_1, v_5v_2, v_5v_3 have the same color. We consider two cases, depending on whether v_5v_1 and v_5v_3 have the same color or v_5v_1 and v_5v_2 have the same color.

Case 1. v_5v_1 and v_5v_3 have the same color.

- ★ If v_5v_1 and v_5v_3 are red, then v_6v_1 and v_6v_3 are blue, for if v_6v_1 say is red, then (v_1, v_5, v_6, v_1) is a red K_3 that is edge-disjoint from H .
- ★ If v_5v_1 and v_5v_3 are blue, then v_7v_1 and v_7v_3 are red, for if v_7v_1 say is blue, then (v_1, v_5, v_7, v_1) is a blue K_3 that is edge-disjoint from H .

This is illustrated in Figure 2.1, where a dotted line represents an edge in H . In either situation, regardless of the color of v_1v_3 , there is a monochromatic K_3 that is edge-disjoint from H .

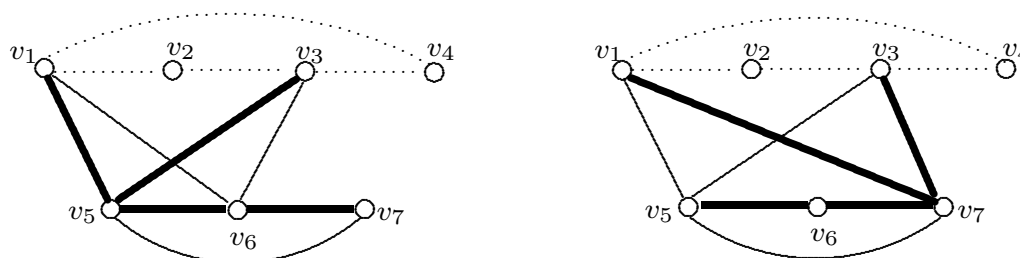


Figure 2.1: A step in the proof of Case 1 of Theorem 2.1.

Thus, we may assume that v_5v_1 and v_5v_3 have different colors and similarly that v_7v_1 and v_7v_3 have different colors.

Case 2. v_5v_1 and v_5v_2 have the same color. By Case 1 and symmetry, we may assume that v_5v_2 and v_5v_4 have different colors. Since v_5v_1 and v_5v_3 have different colors, it follows that v_5v_3 and v_5v_4 have the same color, which is different from the color of v_5v_1 and v_5v_2 . We may assume that v_5v_1 and v_5v_2 are red and v_5v_3 and v_5v_4 are blue. We proceed with the following steps.

- ★ Since v_5v_1 and v_5v_2 are red, it follows that v_6v_1 and v_6v_2 are blue, for if v_6v_2 say is red, then (v_2, v_5, v_6, v_2) is a red K_3 that is edge-disjoint from H .
- ★ Since v_5v_3 and v_5v_4 are blue, it follows that v_7v_3 and v_7v_4 are red, for if v_7v_3 say is blue, then (v_3, v_5, v_7, v_3) is a blue K_3 that is edge-disjoint from H .
- ★ Since v_7v_3 and v_7v_4 are red, it follows that v_6v_3 and v_6v_4 are blue, for if v_6v_3 say is red, then (v_3, v_6, v_7, v_3) is a red K_3 that is edge-disjoint from H . Furthermore, because v_7v_3 and v_7v_4 are red, we may assume that v_7v_1 and v_7v_2 are blue by Case 1 and symmetry.

Thus, $H' = (v_1, v_6, v_2, v_7, v_1)$ is a blue C_4 in G . We now consider the color of the 4-cycle $H = (v_1, v_2, v_3, v_4, v_1)$. If H is red, then (v_1, v_2, v_5, v_1) is a red K_3 that is edge-disjoint from H' . If H is blue, then (v_3, v_4, v_5, v_3) is a blue K_3 that is edge-disjoint from H' . This is illustrated in Figure 2.2.

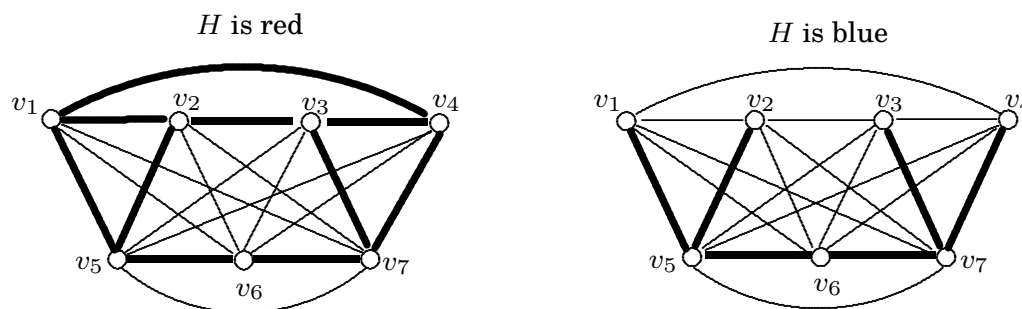


Figure 2.2: A step in the proof of Case 2 of Theorem 2.1.

Hence, there are edge-disjoint monochromatic subgraphs K_3 and C_4 in G . Therefore,

$$\bar{R}(K_3, C_4) \leq 7$$

and so $\bar{R}(K_3, C_4) = 7$. □

With the aid of Theorem 2.1, we now determine the value of the strong Ramsey number $\bar{R}(K_3, C_4^+)$.

Theorem 2.2. $\bar{R}(K_3, C_4^+) = 7$.

Proof. Since $C_4 \subseteq C_4^+$, it follows by Observation 1.7 and Theorem 2.1 that $\bar{R}(K_3, C_4^+) \geq \bar{R}(K_3, C_4) = 7$. Next, we show that $\bar{R}(K_3, C_4^+) \leq 7$. Let there be given a red-blue coloring of $G = K_7$ with

$$V(G) = \{v_1, v_2, \dots, v_7\}.$$

Since $\bar{R}(K_3, C_4) = 7$, there are edge-disjoint monochromatic subgraphs $F = K_3$ and C_4 in G . We may assume that $H = (v_1, v_2, v_3, v_4, v_1)$ is a red C_4 . Let $G' = G - [E(F) \cup E(H)]$. Since F contains at most two vertices of H , we may further assume that F is one of the three triangles (v_5, v_6, v_7, v_5) , (v_1, v_5, v_6, v_1) , (v_1, v_3, v_5, v_1) in G shown in Figure 2.3, where a dotted line represents an edge in F .

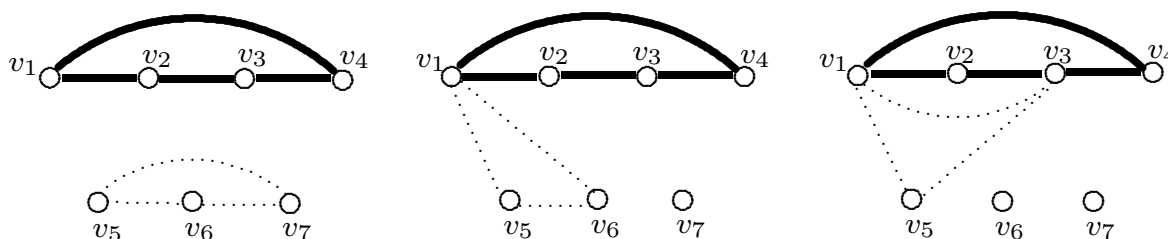


Figure 2.3: The three possible triangles F in G .

Let $X = \{v_2v_5, v_2v_6, v_4v_6, v_2v_7, v_4v_7\}$. Then $X \subseteq E(G')$.

- ★ If some edge e in X is red, then there is a red C_4^+ obtained from the 4-cycle H by adding the pendant edge e .
- ★ If no edge in X is red, then there is a blue C_4^+ obtained from the 4-cycle $(v_2, v_6, v_4, v_7, v_2)$ by adding the pendant edge v_2v_5 .

In either case, the resulting monochromatic subgraph C_4^+ is edge-disjoint from F . Hence, there are edge-disjoint monochromatic subgraphs K_3 and C_4^+ in G . Therefore, $\bar{R}(K_3, C_4^+) \leq 7$ and so

$$\bar{R}(K_3, C_4^+) = 7. \quad \square$$

We now show that $\bar{R}(F, C_4^+) = 7$ for each $F \in \{C_4, P_5, S_{2,3}\}$, beginning with $\bar{R}(C_4, C_4^+)$.

Theorem 2.3. $\bar{R}(C_4, C_4^+) = 7$.

Proof. Since $\bar{R}(C_4, P_5) = 7$ by Theorem 1.10 and $P_5 \subseteq C_4^+$, it follows by Observation 1.7 that

$$\bar{R}(C_4, C_4^+) \geq \bar{R}(C_4, P_5) = 7.$$

Thus, it remains to show that $\bar{R}(C_4, C_4^+) \leq 7$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \dots, v_7\}$. By Theorem 1.3, there are two edge-disjoint monochromatic copies F and F' of C_4 in G , where $F = (v_1, v_2, v_3, v_4, v_1)$ say. Since F' contains at most two vertices of F , we may assume that $V(F')$ is one of the three sets $\{v_1, v_5, v_6, v_7\}$, $\{v_1, v_2, v_5, v_6\}$, $\{v_1, v_3, v_5, v_6\}$. By symmetry, we may further assume that F' is one of the four possible 4-cycles $(v_1, v_5, v_6, v_7, v_1)$, $(v_1, v_5, v_2, v_6, v_1)$, $(v_1, v_5, v_6, v_3, v_1)$, $(v_1, v_5, v_3, v_6, v_1)$ shown in Figure 2.4, where an edge in F is indicated by a dotted line and an edge in F' is indicated by a dashed line.

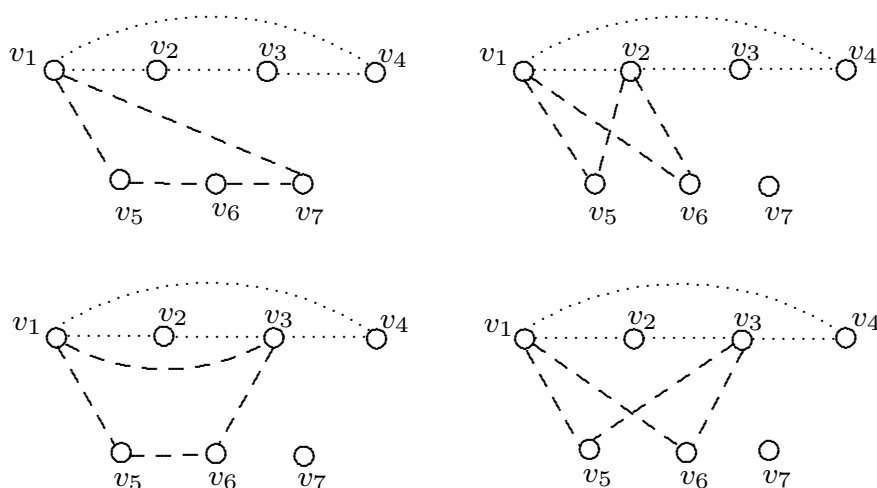


Figure 2.4: The four possible 4-cycles F' in G .

If F and F' have different colors, then regardless of the color of the edge v_4v_5 , there are edge-disjoint monochromatic subgraphs C_4 and C_4^+ in G . To see this, we may assume that F is red and F' is blue. If v_4v_5 is red, then the graph C_4^+ obtained from F by adding the pendant edge v_4v_5 is edge-disjoint from $F' = C_4$. Similarly, if v_4v_5 is blue, then the graph C_4^+ obtained from F' by adding the pendant edge v_4v_5 is edge-disjoint from $F = C_4$. Thus, we may assume that F and F' have the same color, say red.

- ★ If $F' = (v_1, v_5, v_6, v_7, v_1)$, then each edge in $X = \{v_5v_2, v_5v_3, v_6v_2, v_6v_3, v_6v_4\}$ is blue, for otherwise there are edge-disjoint red subgraphs C_4 and C_4^+ in G . Then there is a blue C_4^+ with $E(C_4^+) = X$ that is edge-disjoint from F .
- ★ If $F' = (v_1, v_5, v_2, v_6, v_1)$, then each edge in $Y = \{v_5v_3, v_5v_4, v_6v_3, v_6v_4, v_6v_7\}$ is blue, for otherwise there are edge-disjoint red subgraphs C_4 and C_4^+ in G . Then there is a blue C_4^+ with $E(C_4^+) = Y$ that is edge-disjoint from F .
- ★ If $F' = (v_1, v_5, v_6, v_3, v_1)$ or $F' = (v_1, v_5, v_3, v_6, v_1)$, then each edge in $Z = \{v_5v_2, v_5v_4, v_6v_2, v_6v_4, v_6v_7\}$ is blue, for otherwise there are edge-disjoint red subgraphs C_4 and C_4^+ in G . Then there is a blue C_4^+ with $E(C_4^+) = Z$ that is edge-disjoint from F .

Hence, there are edge-disjoint monochromatic subgraphs C_4 and C_4^+ in G . Therefore, $\bar{R}(C_4, C_4^+) \leq 7$ and so $\bar{R}(C_4, C_4^+) = 7$. $\square$

Theorem 2.4. $\bar{R}(P_5, C_4^+) = 7$.

Proof. Since $\bar{R}(P_5, C_4^+) \geq \bar{R}(P_5, C_4) = 7$ by Theorem 1.10, it remains to show that $\bar{R}(P_5, C_4^+) \leq 7$. Let there be given a red-blue coloring of $G = K_7$ with $V(G) = \{v_1, v_2, \dots, v_7\}$. Since $R(C_4^+) = 6$, there is a monochromatic subgraph $H = C_4^+$ in G . We may assume that H is obtained from the 4-cycle $(v_1, v_2, v_3, v_4, v_1)$ by adding the pendant edge v_4v_7 . Since $G' = G[\{v_1, v_3, v_5, v_6, v_7\}] = K_5$ and $R(P_4) = 5$, there is a monochromatic $F' = P_4$ in G' , where F' is red say. By symmetry, we may assume that $V(F')$ is one of the three sets $\{v_1, v_5, v_6, v_7\}$, $\{v_1, v_3, v_5, v_6\}$, $\{v_1, v_3, v_5, v_7\}$. We consider these three cases.

Case 1. $V(F') = \{v_1, v_5, v_6, v_7\}$. We may assume that F' is one of the six possible 4-paths (v_1, v_5, v_6, v_7) , (v_1, v_5, v_7, v_6) , (v_1, v_7, v_6, v_5) , (v_5, v_1, v_6, v_7) , (v_5, v_1, v_7, v_6) , (v_7, v_1, v_5, v_6) in G' shown in Figure 2.5.

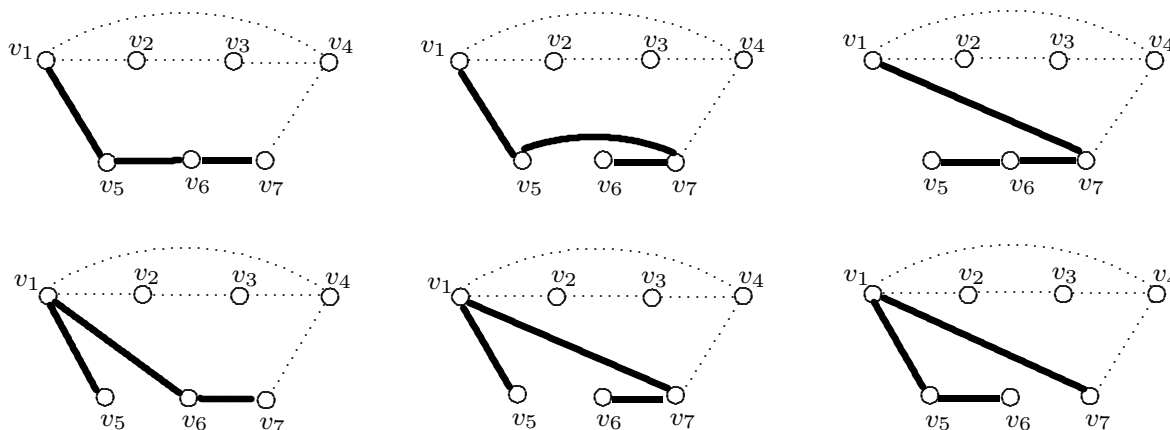


Figure 2.5: The six possible 4-paths F' in Case 1 of Theorem 2.4.

- (1) If $F' = (v_1, v_5, v_6, v_7)$, then each of the edges v_1v_3, v_2v_7, v_3v_7 is blue, for if v_1v_3 say is red, then $(v_3, v_1, v_5, v_6, v_7)$ is a red P_5 that is edge-disjoint from H . Then (v_1, v_3, v_7, v_2) is a blue P_4 . This implies that v_2v_5 and v_2v_4 are red, producing a red $P_5 = (v_4, v_2, v_5, v_6, v_7)$ that is edge-disjoint from H .
- (2) If $F' = (v_1, v_5, v_7, v_6)$, then each of the edges $v_1v_3, v_2v_6, v_3v_6, v_4v_6$ is blue, for if v_4v_6 say is red, then $(v_1, v_5, v_7, v_6, v_4)$ is a red P_5 that is edge-disjoint from H . Then (v_1, v_3, v_6, v_2) is a blue P_4 . This implies that v_2v_5 and v_2v_4 are red, producing a red $P_5 = (v_4, v_2, v_5, v_7, v_6)$ that is edge-disjoint from H .
- (3) If $F' = (v_1, v_7, v_6, v_5)$, then each of the edges $v_1v_3, v_2v_5, v_3v_5, v_4v_5$ is blue, producing a blue

$$P_4 = (v_1, v_3, v_5, v_2).$$

This implies that v_2v_4 and v_2v_7 are red. Then $(v_4, v_2, v_7, v_6, v_5)$ is a red P_5 that is edge-disjoint from H .

- (4) If $F' = (v_5, v_1, v_6, v_7)$, then each of the edges $v_2v_5, v_3v_5, v_4v_5, v_2v_7, v_3v_7$ is blue, producing a blue

$$P_5 = (v_4, v_5, v_2, v_7, v_3)$$

that is edge-disjoint from H .

- (5) If $F' = (v_5, v_1, v_7, v_6)$, then each of the edges $v_2v_5, v_3v_5, v_4v_5, v_2v_6, v_3v_6, v_4v_6$ is blue, producing a blue $P_5 = (v_4, v_5, v_2, v_6, v_3)$ that is edge-disjoint from H .

- (6) If $F' = (v_7, v_1, v_5, v_6)$, then each of the edges $v_2v_6, v_3v_6, v_4v_6, v_2v_7, v_3v_7$ is blue, producing a blue

$$P_5 = (v_4, v_6, v_2, v_7, v_3)$$

that is edge-disjoint from H .

Case 2. $V(F') = \{v_1, v_3, v_5, v_6\}$. We may assume that F' is one of the four possible 4-paths (v_3, v_1, v_5, v_6) , (v_5, v_1, v_3, v_6) , (v_1, v_5, v_6, v_3) , (v_5, v_1, v_6, v_3) in G' shown in Figure 2.6.

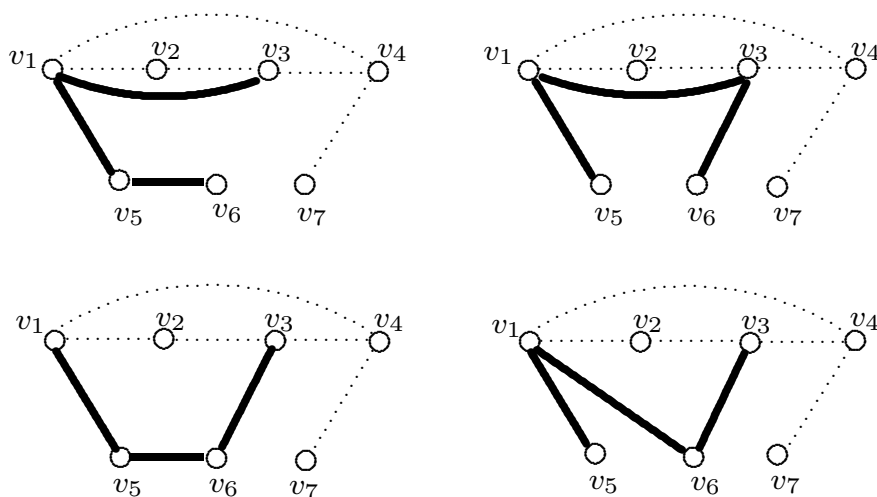


Figure 2.6: The four possible 4-paths F' in Case 2 of Theorem 2.4.

- (1) If $F' = (v_3, v_1, v_5, v_6)$, then each of the edges $v_3v_7, v_2v_6, v_4v_6, v_6v_7$ is blue. Then (v_2, v_6, v_7, v_3) is a blue P_4 . This implies that v_2v_5, v_2v_4, v_3v_5 are red, producing a red $P_5 = (v_4, v_2, v_5, v_3, v_1)$ that is edge-disjoint from H .
- (2) If $F' = (v_5, v_1, v_3, v_6)$, then each of the edges $v_5v_2, v_5v_4, v_5v_7, v_6v_2, v_6v_4, v_6v_7$ is blue, producing a blue $P_5 = (v_7, v_5, v_2, v_6, v_4)$ that is edge-disjoint from H .
- (3) If $F' = (v_1, v_5, v_6, v_3)$, then v_1v_7 and v_3v_7 are blue. We now consider the color of the edge v_3v_5 . If v_3v_5 is red, then (v_1, v_5, v_3, v_6) is a red P_4 . This implies that each of the edges v_6v_2, v_6v_4, v_6v_7 is blue. Since $P_4 = (v_2, v_6, v_7, v_3)$ is blue, it follows that v_2v_5 and v_2v_4 are red, producing a red $P_5 = (v_4, v_2, v_5, v_3, v_6)$ that is edge-disjoint from H . If v_3v_5 is blue, then (v_1, v_7, v_3, v_5) is a blue P_4 . Thus, we may assume that each of the edges v_5v_2, v_5v_4, v_5v_7 is red. Since (v_2, v_5, v_6, v_3) is a red P_4 , we may assume that v_1v_3 is blue and so (v_5, v_3, v_1, v_7) is a blue P_4 . This implies that v_7v_2 and v_7v_6 are red, producing a red $P_5 = (v_4, v_5, v_6, v_7, v_2)$ that is edge-disjoint from H .
- (4) If $F' = (v_5, v_1, v_6, v_3)$, then each of the edges $v_5v_2, v_5v_4, v_5v_7, v_3v_1, v_3v_7$ is blue, producing a blue

$$P_5 = (v_1, v_3, v_7, v_5, v_4)$$

that is edge-disjoint from H .

Case 3. $V(F') = \{v_1, v_3, v_5, v_7\}$. We may assume that F' is one of the six possible 4-paths (v_5, v_1, v_3, v_7) , (v_1, v_3, v_7, v_5) , (v_3, v_1, v_5, v_7) , (v_1, v_5, v_3, v_7) , (v_1, v_5, v_7, v_3) , (v_5, v_1, v_7, v_3) in G' shown in Figure 2.7.

- (1) If $F' = (v_5, v_1, v_3, v_7)$, then each of the edges $v_5v_2, v_5v_4, v_5v_6, v_7v_2, v_7v_6$ is blue, producing a blue

$$P_5 = (v_4, v_5, v_6, v_7, v_2)$$

that is edge-disjoint from H .

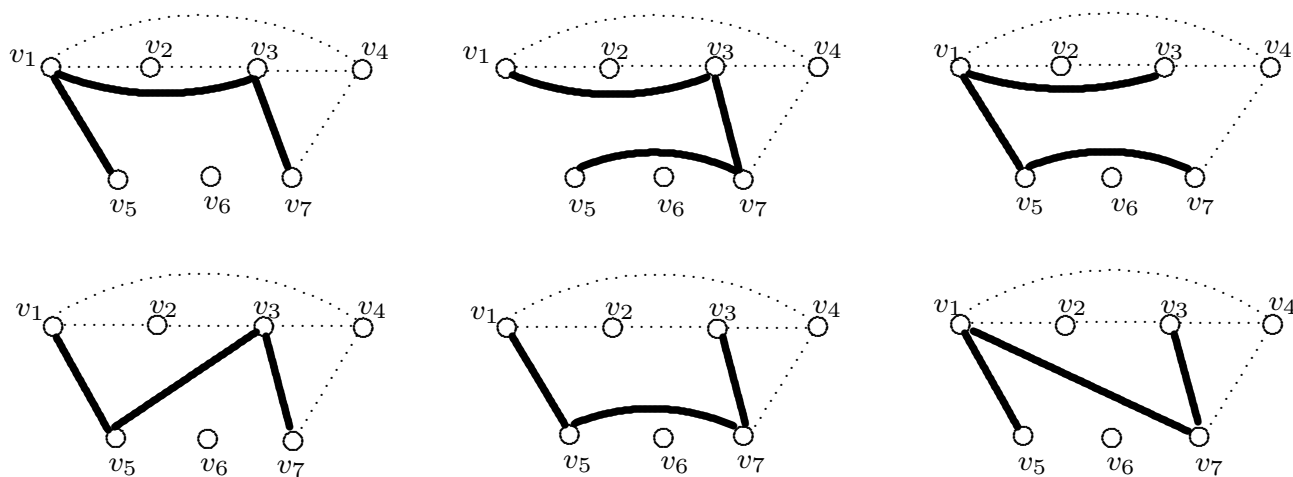


Figure 2.7: The six possible 4-paths F' in Case 3 of Theorem 2.4.

- (2) If $F' = (v_1, v_3, v_7, v_5)$, then each of the edges $v_5v_2, v_5v_4, v_5v_6, v_1v_6$ is blue. Then (v_1, v_6, v_5, v_2) is a blue P_4 . This implies that v_2v_4 and v_2v_7 are red, producing a red $P_5 = (v_4, v_2, v_7, v_3, v_1)$ that is edge-disjoint from H .
- (3) If $F' = (v_3, v_1, v_5, v_7)$, then each of the edges v_3v_6, v_7v_2, v_7v_6 is blue. Then (v_2, v_7, v_6, v_3) is a blue P_4 . This implies that v_2v_4 and v_2v_5 are red, producing a red $P_5 = (v_4, v_2, v_5, v_1, v_3)$ that is edge-disjoint from H .
- (4) If $F' = (v_1, v_5, v_3, v_7)$, then each of the edges v_1v_6, v_7v_2, v_7v_6 is blue. Then (v_1, v_6, v_7, v_2) is a blue P_4 . This implies that v_2v_4, v_2v_5, v_1v_3 are red, producing a red $P_5 = (v_4, v_2, v_5, v_3, v_1)$ that is edge-disjoint from H .
- (5) If $F' = (v_1, v_5, v_7, v_3)$, then v_1v_6 and v_3v_6 are blue. We now consider the color of the edge v_3v_5 . If v_3v_5 is red, then (v_1, v_5, v_3, v_7) is a red P_4 . This implies that each of the edges v_7v_2 and v_6v_7 is blue. Since $P_4 = (v_2, v_7, v_6, v_3)$ is blue, it follows that v_2v_5 and v_2v_4 are red, producing a red $P_5 = (v_4, v_2, v_5, v_3, v_7)$ that is edge-disjoint from H . If v_3v_5 is blue, then (v_1, v_6, v_3, v_5) is a blue P_4 . Thus, each of the edges v_5v_2, v_5v_4, v_1v_7 is red, producing a red $P_5 = (v_4, v_5, v_1, v_7, v_3)$ that is edge-disjoint from H .
- (6) If $F' = (v_5, v_1, v_7, v_3)$, then each of the edges $v_5v_2, v_5v_4, v_5v_6, v_3v_6$ is blue. Then (v_2, v_5, v_6, v_3) is a blue P_4 . This implies that v_3v_1, v_2v_4, v_2v_7 are red, producing a red $P_5 = (v_4, v_2, v_7, v_1, v_3)$ that is edge-disjoint from H .

Hence, there are edge-disjoint monochromatic P_5 and C_4^+ in G . Therefore, $\bar{R}(P_5, C_4^+) \leq 7$ and so $\bar{R}(P_5, C_4^+) = 7$. $\square$

Applying the same technique used in the proof of Theorem 2.4, the strong Ramsey number $\bar{R}(S_{2,3}, C_4^+)$ can be determined.

Theorem 2.5. $\bar{R}(S_{2,3}, C_4^+) = 7$.

3. Closing Remarks

In this section, we show that if F and H are two graphs having Ramsey number 6, then $\bar{R}(F, H)$ is either 6 or 7. Let $S = \{K_3, K_{1,3}, C_4, C_4^+, P_5, S_{2,3}, P_3 + K_2\}$ be the set of the seven graphs without isolated vertices with Ramsey number 6 shown in Figure 1.1.

Theorem 3.1. *If $F, H \in S$ and $F \neq H$, then $\bar{R}(F, H) \in \{6, 7\}$.*

Proof. If $K_{1,3} \in \{F, H\}$, then $\bar{R}(F, H) = 6$ by Theorem 1.8. If F and H have the same size, then $\bar{R}(F, H) \in \{6, 7\}$ by Theorems 1.9 and 1.10. Thus, we may assume that $K_{1,3} \notin \{F, H\}$ and F and H are of different size. We consider two cases, depending on whether $C_4^+ \in \{F, H\}$ or $C_4^+ \notin \{F, H\}$.

Case 1. $C_4^+ \in \{F, H\}$, say $H = C_4^+$. Then $F \in \{K_3, C_4, P_5, S_{2,3}, P_3 + K_2\}$. If $F \in \{K_3, C_4, P_5, S_{2,3}\}$, then $\bar{R}(F, C_4^+) = 7$ by Theorems 2.2, 2.3, 2.4, and 2.5. If $F = P_3 + K_2$, then $F \subseteq P_5$ and so

$$\bar{R}(F, C_4^+) \leq \bar{R}(P_5, C_4^+) = 7$$

by Theorem 2.4 and Observation 1.7.

Case 2. $C_4^+ \notin \{F, H\}$. Thus, F and H have size 3 or 4. Since F and H have different sizes, we may assume that F has size 3 and H has size 4. Thus, $F \in \{K_3, P_3 + K_2\}$ and $H \in \{C_4, P_5, S_{2,3}\}$.

★ First, suppose that $F = K_3$. Since $H \subseteq C_4^+$ for each $H \in \{C_4, P_5, S_{2,3}\}$, it follows that

$$\bar{R}(K_3, H) \leq \bar{R}(K_3, C_4^+) = 7$$

by Theorem 2.2 and Observation 1.7.

★ Next, suppose that $F = P_3 + K_2$. Since $P_3 + K_2 \subseteq P_5$ and $H \subseteq C_4^+$ for each $H \in \{C_4, P_5, S_{2,3}\}$, it follows that $\bar{R}(P_3 + K_2, H) \leq \bar{R}(P_5, C_4^+) = 7$ by Theorem 2.4 and Observation 1.7. $\square$

We saw that red-blue colorings of the complete graph K_6 have a property that is stronger than the property stated in the definition of the Ramsey number of a graph having Ramsey number 6. By Theorems 1.8 and 1.9, it follows for several pairs of non-isomorphic graphs F and H in the set S of the seven graphs with Ramsey number 6, that every red-blue coloring of K_6 produces two edge-disjoint monochromatic subgraphs, one isomorphic to F and the other isomorphic to H . From this fact, a question arises:

For three given graphs F_1, F_2, F_3 in S , can every red-blue coloring of K_6 or K_7 produce three pairwise edge-disjoint monochromatic subgraphs F_1, F_2, F_3 ?

If the answer to this question is yes, then three graphs in S can be extended to four or more graphs in S . If the answer to this question is no, then another question arises.

For three given graphs F_1, F_2, F_3 in S , what is the minimum positive integer n such that every red-blue coloring of K_n produces three pairwise edge-disjoint monochromatic subgraphs isomorphic to F_1, F_2, F_3 ?

Of course, there is the more general question of determining the minimum positive integer n such that every red-blue coloring of K_n produces pairwise edge-disjoint monochromatic subgraphs isomorphic to the graphs in a given set of two or more non-isomorphic graphs.

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