

Research Article

Finite-Level Bihari Clocks for Numerical Quenching in Singular Heat EquationsThibaut K. Kouakou^{1,‡}, Nabongo Diabaté²¹*Département de Mathématiques et Informatique,
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Université Alassane Ouattara, 01 BP V 18 Bouaké 01, Côte d'Ivoire***Abstract**

This paper introduces a finite-level Bihari clock for a singular semilinear heat equation. For the Dirichlet problem with a singular power reaction, Kaplan's first-eigenfunction average and Jensen's inequality reduce the parabolic problem to a scalar differential inequality. Positivity of the resulting clock field yields an explicit upper estimate for the quenching time, while a maximum-principle comparison gives a complementary lower estimate. The construction is transferred to a centered space-semidiscrete scheme by using the discrete first eigenvalue. The numerical computation is formulated with a finite-step estimator distinct from the exact semidiscrete quenching time. The experiments compare this estimator with the clocks, identify the midpoint as the dominant node and confirm the second-order spatial convergence of the discrete upper clock.

Keywords: Bihari inequality; quenching clock; singular parabolic equation; Kaplan method; finite differences; Richardson order.

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1. Introduction

In a quenching problem, the singular event is not measured by an unbounded amplitude but by the disappearance of a positive gap. The solution stays below a finite critical level, here normalized to be one, while the reaction term becomes singular as the solution approaches that level. In this article, quenching means that a classical solution is defined on a maximal subcritical interval and satisfies

$$0 \leq u(x, t) < 1, \quad \|u(\cdot, t)\|_{\infty} \longrightarrow 1 \quad \text{as } t \uparrow T_q < \infty.$$

Thus the central quantity is the distance to the level one, rather than the size of the solution itself.

We study the one-dimensional Dirichlet problem

$$\begin{cases} u_t - u_{xx} = \lambda(1 - u)^{-q}, & 0 < x < 1, \ t > 0, \\ u(0, t) = u(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & 0 < x < 1, \end{cases} \quad (1)$$

where $q > 0$, $\lambda > 0$, and $0 \leq u_0 < 1$. Below the singular level, standard semilinear parabolic theory gives a unique maximal classical subcritical solution; the local construction follows from the parabolic theory in [10], while the comparison arguments used below rely on the maximum-principle framework of [17]. The corresponding maximal alternative, formulated for singular parabolic problems, is consistent with the general treatment in [19].

The model belongs to the classical quenching theory initiated by Kawarada's threshold equation. The early framework of Acker and Walter and Levine's survey provide the historical basis for the subject [1, 12, 14]. For the behavior of the time derivative at the singular event and for numerical quenching schemes, we use the results and techniques in [4, 5, 9]. A recent study of singular parabolic quenching is the system analysis of Wang and Zhang [20]. The present paper does not aim to describe the full quenching profile. Its purpose is to isolate a simple quantitative device that gives a computable time-to-threshold estimate.

The device is a finite-level form of Bihari's inequality. The scalar background comes from Bellman's stability lemma, the nonlinear extension of Bihari and Osgood's finite-time integrability criterion [2, 3, 16]. For related pointwise, discrete and integral-equation inequalities, we retain the references directly useful for the present clock argument [6–8, 18]. A recent Bihari inequality applicable to nonlinear fractional differential equations was established by Lan and Webb [13]. In the parabolic setting, the scalar inequality is combined with Kaplan's first-eigenfunction method and Jensen's inequality [11].

The normalized first Dirichlet eigenfunction is

$$\varphi_1(x) = \frac{\pi}{2} \sin(\pi x), \quad \int_0^1 \varphi_1(x) dx = 1, \quad -\varphi_1'' = \mu_1 \varphi_1, \quad \mu_1 = \pi^2.$$

For the weighted mean

$$Y(t) = \int_0^1 u(x, t) \varphi_1(x) dx,$$

Jensen's inequality gives

$$Y'(t) \geq \lambda(1 - Y(t))^{-q} - \mu_1 Y(t).$$

The relevant clock field is therefore

$$F(s) = \lambda(1 - s)^{-q} - \mu_1 s.$$

If F remains positive on the interval traversed by Y , the maximal subcritical time satisfies

$$T_q \leq \int_{Y(0)}^1 \frac{d\sigma}{\lambda(1 - \sigma)^{-q} - \mu_1 \sigma}.$$

This formula is the finite-level clock used throughout the paper.

The second estimate comes from the maximum principle. If $M_0 = \|u_0\|_\infty$, the solution cannot quench before

$$\frac{(1 - M_0)^{q+1}}{(q + 1)\lambda}.$$

The quenching time is therefore trapped between two explicit clocks. We then repeat the construction for the centered space-semidiscrete problem. The discrete weighted mean satisfies the analogous inequality with the discrete first eigenvalue

$$\mu_{1,h} = \frac{4}{h^2} \sin^2 \left(\frac{\pi h}{2} \right),$$

and this gives a computable discrete upper clock. The numerical experiments compare a finite-step estimator with the two theoretical clock values, follow the decay of the gaps to the level one, identify the dominant node and examine the second-order spatial behavior of the discrete upper clock.

Section 2 states the scalar finite-level clock. Section 3 applies it to the continuous singular heat equation. Section 4 develops the space-semidiscrete counterpart. Section 5 presents the numerical validation and explains how the tables and figures complement the theoretical estimates.

2. A Finite-Level Bihari Hitting Clock

We begin with the scalar device used throughout the paper. It is written for a finite endpoint because quenching is detected by the approach to a prescribed level and not by escape to infinity.

Lemma 2.1. *Let $\alpha < \beta$, and let $F \in C([\alpha, \beta), (0, \infty))$. Assume that*

$$\int_{z_0}^{\beta} \frac{d\sigma}{F(\sigma)} < \infty$$

for some $z_0 \in [\alpha, \beta)$. Let $z \in C^1([0, S))$ satisfy

$$z(0) = z_0, \quad \alpha \leq z(t) < \beta, \quad z'(t) \geq F(z(t)), \quad 0 < t < S.$$

Then z cannot remain strictly below β beyond the finite clock time

$$T_F(z_0) = \int_{z_0}^{\beta} \frac{d\sigma}{F(\sigma)}. \quad (2)$$

Equivalently, if $z(t) < \beta$ on $[0, S)$, then $S \leq T_F(z_0)$.

Proof. The positivity of F implies $z'(t) > 0$ as long as $z(t) < \beta$. Hence $z(t) \geq z_0$ throughout the subcritical interval. Define

$$H(r) = \int_r^{\beta} \frac{d\sigma}{F(\sigma)}, \quad z_0 \leq r < \beta.$$

The function H is finite, nonnegative and continuously differentiable on compact subintervals of $[z_0, \beta)$, with $H'(r) = -1/F(r)$. Therefore, whenever $z(t) < \beta$,

$$\frac{d}{dt}H(z(t)) = -\frac{z'(t)}{F(z(t))} \leq -1.$$

After integration over $(0, t)$, we obtain

$$0 \leq H(z(t)) \leq H(z_0) - t.$$

Thus no time $t > H(z_0)$ is compatible with the condition $z(t) < \beta$. Since $H(z_0) = T_F(z_0)$, the asserted bound follows. $\square$

Remark 2.2. *The upper endpoint β will be the quenching level. The integral in (2) should therefore be read as the time allowed by the scalar comparison before that level is reached. In the parabolic problem, the role of z is played by Kaplan's weighted mean.*

3. The Continuous Quenching Problem

We now apply the finite-level clock to (1). The assumptions on the initial datum are

$$u_0 \in C^2([0, 1]), \quad 0 \leq u_0(x) < 1, \quad u_0(0) = u_0(1) = 0, \quad u_0 \not\equiv 0. \quad (3)$$

Under (3), the local theory recalled in the introduction gives a unique maximal classical subcritical solution

$$u \in C([0, 1] \times [0, T_q)) \cap C^{2,1}((0, 1) \times (0, T_q)),$$

with values in $[0, 1)$. If $T_q < \infty$, the maximal continuation alternative implies $\|u(\cdot, t)\|_\infty \rightarrow 1$ as $t \uparrow T_q$.

The first lemma records the positivity property needed for the comparison arguments.

Lemma 3.1. *The maximal subcritical solution satisfies*

$$0 \leq u(x, t) < 1, \quad 0 \leq x \leq 1, \quad 0 \leq t < T_q.$$

Moreover, $u(x, t) > 0$ for every $0 < x < 1$ and $0 < t < T_q$.

Proof. The bound $u < 1$ is precisely the subcritical constraint. To prove nonnegativity, fix $S < T_q$. On the range of u over $[0, 1] \times [0, S]$, the map $s \mapsto \lambda(1 - s)^{-q}$ is locally Lipschitz. The zero function is a subsolution because

$$0_t - 0_{xx} = 0 \leq \lambda(1 - 0)^{-q}.$$

Since $u_0 \geq 0$ and the boundary data are zero, the parabolic comparison principle gives $u \geq 0$ on $[0, 1] \times [0, S]$. Letting S vary below T_q yields nonnegativity on the whole subcritical interval.

The solution cannot be identically zero for positive times because the source term is strictly positive. The strong parabolic maximum principle applied on each cylinder $(0, 1) \times (0, S]$, with $S < T_q$, therefore gives $u(x, t) > 0$ for $0 < x < 1$ and $0 < t \leq S$. Since $S < T_q$ is arbitrary, the conclusion follows. $\square$

The next lemma is the Kaplan reduction: the parabolic equation is converted into a one-dimensional differential inequality.

Lemma 3.2. *Let*

$$Y(t) = \int_0^1 u(x, t) \varphi_1(x) dx, \quad \varphi_1(x) = \frac{\pi}{2} \sin(\pi x).$$

Then, for every $0 < t < T_q$,

$$Y'(t) \geq -\mu_1 Y(t) + \lambda(1 - Y(t))^{-q}, \quad \mu_1 = \pi^2. \quad (4)$$

Proof. Since u is classical and φ_1 is smooth, Y is differentiable on $(0, T_q)$, and

$$Y'(t) = \int_0^1 u_t(x, t) \varphi_1(x) dx.$$

Using the equation, we obtain

$$Y'(t) = \int_0^1 u_{xx}(x, t) \varphi_1(x) dx + \lambda \int_0^1 (1 - u(x, t))^{-q} \varphi_1(x) dx.$$

Two integrations by parts give

$$\int_0^1 u_{xx}(x, t) \varphi_1(x) dx = \int_0^1 u(x, t) \varphi_1''(x) dx = -\mu_1 \int_0^1 u(x, t) \varphi_1(x) dx.$$

The boundary terms vanish because $u(0, t) = u(1, t) = 0$ and $\varphi_1(0) = \varphi_1(1) = 0$. Hence

$$Y'(t) = -\mu_1 Y(t) + \lambda \int_0^1 (1 - u(x, t))^{-q} \varphi_1(x) dx.$$

The measure $\varphi_1(x) dx$ is a probability measure on $(0, 1)$, and the function $s \mapsto (1 - s)^{-q}$ is convex on $[0, 1)$. Jensen's inequality therefore yields

$$\int_0^1 (1 - u(x, t))^{-q} \varphi_1(x) dx \geq \left(1 - \int_0^1 u(x, t) \varphi_1(x) dx\right)^{-q} = (1 - Y(t))^{-q}.$$

Substitution proves (4). □

The following result is the continuous finite-level upper clock. The positivity condition is automatic, for example, when

$$\lambda > \mu_1 \max_{0 \leq s < 1} s(1 - s)^q = \mu_1 \frac{q^q}{(q + 1)^{q+1}},$$

but only the stated condition is needed.

Theorem 3.3. *Let*

$$Y_0 = \int_0^1 u_0(x) \varphi_1(x) dx,$$

and assume that

$$F(s) := \lambda(1 - s)^{-q} - \mu_1 s > 0, \quad Y_0 \leq s < 1. \quad (5)$$

Then the solution quenches in finite time and

$$T_q \leq T_B^+(Y_0) := \int_{Y_0}^1 \frac{d\sigma}{\lambda(1 - \sigma)^{-q} - \mu_1 \sigma}. \quad (6)$$

Proof. By Lemma 3.2,

$$Y'(t) \geq F(Y(t)), \quad 0 < t < T_q.$$

Since $Y(0) = Y_0$ and $F > 0$ on $[Y_0, 1)$, the inequality implies that Y is increasing as long as $Y(t) < 1$. Hence $Y(t) \geq Y_0$ for $0 \leq t < T_q$. Moreover, $Y(t) < 1$ because $0 \leq u(x, t) < 1$ and $\varphi_1(x) dx$ has total mass one.

The clock integral is finite. Indeed, as $s \uparrow 1$,

$$F(s) = \lambda(1 - s)^{-q} - \mu_1 s \sim \lambda(1 - s)^{-q}, \quad \frac{1}{F(s)} \sim \frac{(1 - s)^q}{\lambda}.$$

Since $q > 0$, the last function is integrable near $s = 1$. Applying Lemma 2.1 with $\beta = 1$ and $z = Y$, we conclude that the Kaplan mean cannot remain below one beyond $T_B^+(Y_0)$.

If the parabolic solution were subcritical on a longer interval, then $Y(t) < 1$ on that interval, contradicting the scalar clock. Hence the maximal subcritical time is finite and satisfies (6). By the maximal continuation alternative recalled in the introduction, if $\|u(\cdot, t)\|_\infty$ failed to approach one as $t \uparrow T_q$, then the solution would remain uniformly separated from the singular level and the local parabolic theory would extend it beyond T_q . Therefore

$$\|u(\cdot, t)\|_\infty \longrightarrow 1 \quad \text{as } t \uparrow T_q,$$

and the finite maximal time obtained above is the quenching time. □

The lower clock is independent of the eigenfunction argument and follows from a pointwise maximum comparison.

Proposition 3.4. *Let $M_0 = \|u_0\|_\infty$. If quenching occurs at $T_q < \infty$, then*

$$T_q \geq T^- := \frac{(1 - M_0)^{q+1}}{(q+1)\lambda}. \quad (7)$$

Proof. Set $M(t) = \|u(\cdot, t)\|_\infty$ for $0 < t < T_q$. By Lemma 3.1, the solution is nonnegative and satisfies the homogeneous Dirichlet boundary condition. Therefore, when a positive maximum is attained, it is attained in the interior of $(0, 1)$. At an interior maximum point x_t , one has $u_{xx}(x_t, t) \leq 0$. The usual upper right Dini derivative estimate for the spatial maximum gives

$$D^+ M(t) \leq \lambda(1 - M(t))^{-q}, \quad 0 < t < T_q.$$

Compare M with the scalar solution m defined by

$$m'(t) = \lambda(1 - m(t))^{-q}, \quad m(0) = M_0.$$

Solving this equation gives

$$1 - m(t) = \left((1 - M_0)^{q+1} - (q+1)\lambda t \right)^{1/(q+1)}$$

as long as the expression inside the parenthesis is positive. The scalar comparison principle for Dini differential inequalities gives $M(t) \leq m(t)$ on every compact subinterval of this time interval. Therefore $M(t) < 1$ there, and quenching cannot occur before the time in (7). $\square$

Corollary 3.5. *Under the assumptions of Theorem 3.3, the quenching time satisfies*

$$\frac{(1 - M_0)^{q+1}}{(q+1)\lambda} \leq T_q \leq \int_{Y_0}^1 \frac{d\sigma}{\lambda(1 - \sigma)^{-q} - \mu_1 \sigma}.$$

Proof. The lower bound is Proposition 3.4, and the upper bound is Theorem 3.3. $\square$

4. Space-Semidiscrete Finite-Level Clock

We next pass to a centered finite-difference approximation in space. Let $I \geq 2$, $h = 1/I$, and $x_i = ih$, $1 \leq i \leq I-1$. The semidiscrete system is

$$\begin{cases} \frac{dU_i}{dt} = \delta_h^2 U_i + \lambda(1 - U_i)^{-q}, & 1 \leq i \leq I-1, \\ U_0(t) = U_I(t) = 0, \\ U_i(0) = u_0(x_i), & 1 \leq i \leq I-1, \end{cases} \quad (8)$$

where

$$\delta_h^2 U_i = \frac{U_{i+1} - 2U_i + U_{i-1}}{h^2}.$$

Let $T_{q,h}$ be the maximal time on which $0 \leq U_i(t) < 1$ for all interior nodes. If $T_{q,h} < \infty$ and $\max_{1 \leq i \leq I-1} U_i(t) \rightarrow 1$ as $t \uparrow T_{q,h}$, we say that the semidiscrete solution quenches at $T_{q,h}$.

The discrete first eigenpair of the Dirichlet second-difference operator is

$$\mu_{1,h} = \frac{4}{h^2} \sin^2 \left(\frac{\pi h}{2} \right), \quad \psi_i = \sin(\pi x_i).$$

We use the normalized weights

$$\varphi_i^h = \frac{\psi_i}{h \sum_{j=1}^{I-1} \psi_j}, \quad h \sum_{i=1}^{I-1} \varphi_i^h = 1,$$

and define

$$Y_h(t) = h \sum_{i=1}^{I-1} U_i(t) \varphi_i^h.$$

The system in (8) is cooperative, and the zero vector is a subsolution. The discrete comparison principle therefore gives $U_i(t) \geq 0$ on the whole subcritical interval. Consequently, the loss of subcriticality can only occur through the approach of at least one component to the level one.

Lemma 4.1. *For every $0 < t < T_{q,h}$, the discrete Kaplan mean satisfies*

$$Y_h'(t) \geq -\mu_{1,h} Y_h(t) + \lambda(1 - Y_h(t))^{-q}. \quad (9)$$

Proof. Since the semidiscrete solution is differentiable on its maximal subcritical interval, we may differentiate Y_h term by term:

$$Y_h'(t) = h \sum_{i=1}^{I-1} \frac{dU_i}{dt}(t) \varphi_i^h.$$

Using (8) gives

$$Y_h'(t) = h \sum_{i=1}^{I-1} \delta_h^2 U_i(t) \varphi_i^h + \lambda h \sum_{i=1}^{I-1} (1 - U_i(t))^{-q} \varphi_i^h.$$

The vector $\psi_i = \sin(\pi x_i)$ satisfies

$$-\delta_h^2 \psi_i = \mu_{1,h} \psi_i, \quad \psi_0 = \psi_I = 0.$$

Since φ^h is a positive normalization of ψ , it satisfies the same eigenvalue equation. The Dirichlet second-difference operator is symmetric for the discrete inner product

$$\langle V, W \rangle_h = h \sum_{i=1}^{I-1} V_i W_i.$$

Consequently,

$$h \sum_{i=1}^{I-1} \delta_h^2 U_i(t) \varphi_i^h = h \sum_{i=1}^{I-1} U_i(t) \delta_h^2 \varphi_i^h = -\mu_{1,h} Y_h(t).$$

On $[0, T_{q,h})$, all interior components satisfy $0 \leq U_i(t) < 1$. The weights $h\varphi_i^h$ are positive and sum to one, so they define a discrete probability measure. Since $s \mapsto (1 - s)^{-q}$ is convex on $[0, 1)$, Jensen's inequality yields

$$h \sum_{i=1}^{I-1} (1 - U_i(t))^{-q} \varphi_i^h \geq \left(1 - h \sum_{i=1}^{I-1} U_i(t) \varphi_i^h \right)^{-q} = (1 - Y_h(t))^{-q}.$$

Combining the diffusion identity and this Jensen estimate gives (9). □

Theorem 4.2. *Let*

$$Y_{h,0} = h \sum_{i=1}^{I-1} u_0(x_i) \varphi_i^h,$$

and assume that

$$F_h(s) := \lambda(1 - s)^{-q} - \mu_{1,h}s > 0, \quad Y_{h,0} \leq s < 1.$$

Then the semidiscrete solution quenches in finite time and

$$T_{q,h} \leq T_{B,h}^+ := \int_{Y_{h,0}}^1 \frac{d\sigma}{\lambda(1 - \sigma)^{-q} - \mu_{1,h}\sigma}. \quad (10)$$

Proof. By Lemma 4.1,

$$Y'_h(t) \geq F_h(Y_h(t)), \quad 0 < t < T_{q,h}.$$

Since $Y_h(0) = Y_{h,0}$ and $F_h > 0$ on $[Y_{h,0}, 1)$, the differential inequality prevents Y_h from crossing below $Y_{h,0}$. Thus $Y_h(t) \geq Y_{h,0}$ for $0 \leq t < T_{q,h}$. The subcritical condition also gives $Y_h(t) < 1$.

The clock integral is finite because, as $s \uparrow 1$,

$$F_h(s) = \lambda(1-s)^{-q} - \mu_{1,h}s \sim \lambda(1-s)^{-q}, \quad \frac{1}{F_h(s)} \sim \frac{(1-s)^q}{\lambda}.$$

Lemma 2.1, applied with $z = Y_h$ and $\beta = 1$, implies that Y_h cannot remain below one beyond $T_{B,h}^+$. If the semidiscrete solution remained subcritical after this time, then all components would satisfy $U_i(t) < 1$, and their convex combination $Y_h(t)$ would also remain below one, a contradiction. Hence the maximal subcritical time is finite and satisfies (10).

It remains to identify this finite maximal time as a quenching time. Suppose that the nodal maximum does not tend to one. Then there exist $\varepsilon > 0$ and a sequence $t_n \uparrow T_{q,h}$ such that

$$0 \leq U_i(t_n) \leq 1 - \varepsilon, \quad 1 \leq i \leq I - 1.$$

After extraction of a subsequence, $U(t_n)$ converges to a vector U^* in the compact set $[0, 1 - \varepsilon]^{I-1}$. The vector field in (8) is locally Lipschitz in a neighborhood of this compact set. The local existence time can therefore be chosen uniformly for all sufficiently large n , and uniqueness extends the original solution beyond $T_{q,h}$, a contradiction. Hence $\max_{1 \leq i \leq I-1} U_i(t) \rightarrow 1$ as $t \uparrow T_{q,h}$. $\square$

Proposition 4.3. Let $M_{h,0} = \max_{1 \leq i \leq I-1} U_i(0)$. If the semidiscrete solution quenches at $T_{q,h} < \infty$, then

$$T_{q,h} \geq T_h^- := \frac{(1 - M_{h,0})^{q+1}}{(q+1)\lambda}.$$

Proof. Set $M_h(t) = \max_{1 \leq i \leq I-1} U_i(t)$ and choose an index $i(t)$ at which the maximum is attained. Since the boundary values are zero and the interior maximum is nonnegative,

$$\delta_h^2 U_{i(t)}(t) = \frac{U_{i(t)+1}(t) - 2U_{i(t)}(t) + U_{i(t)-1}(t)}{h^2} \leq 0.$$

Hence the upper right Dini derivative of the maximum satisfies

$$D^+ M_h(t) \leq \lambda(1 - M_h(t))^{-q}, \quad 0 < t < T_{q,h}.$$

Let m_h solve

$$m'_h(t) = \lambda(1 - m_h(t))^{-q}, \quad m_h(0) = M_{h,0}.$$

Then

$$1 - m_h(t) = ((1 - M_{h,0})^{q+1} - (q+1)\lambda t)^{1/(q+1)}$$

while the right-hand side is positive. The scalar comparison principle gives $M_h(t) \leq m_h(t)$ before the scalar reaching time. Consequently all components remain below one for $0 \leq t < T_h^-$, and semidiscrete quenching cannot occur before that time. $\square$

Corollary 4.4. Under the assumptions of Theorem 4.2, the exact semidiscrete quenching time satisfies

$$\frac{(1 - M_{h,0})^{q+1}}{(q+1)\lambda} \leq T_{q,h} \leq \int_{Y_{h,0}}^1 \frac{d\sigma}{\lambda(1 - \sigma)^{-q} - \mu_{1,h}\sigma}.$$

Proof. The result follows from Proposition 4.3 and Theorem 4.2. $\square$

The next proposition concerns the Bihari upper clock produced by the discrete Kaplan reduction; it is not a convergence theorem for the time-discrete estimator introduced later.

Proposition 4.5. *Assume that $u_0 \in C^4([0, 1])$, $u_0(0) = u_0(1) = 0$, and that (5) holds. Then, for all sufficiently small h , the discrete upper clock $T_{B,h}^+$ is well defined and*

$$T_{B,h}^+ \longrightarrow T_B^+(Y_0) \quad \text{as } h \rightarrow 0.$$

Moreover, for the centered grid used above,

$$T_{B,h}^+ = T_B^+(Y_0) + O(h^2).$$

Proof. Taylor's formula gives

$$\mu_{1,h} = \frac{4}{h^2} \sin^2\left(\frac{\pi h}{2}\right) = \pi^2 + O(h^2) = \mu_1 + O(h^2).$$

It remains to estimate the discrete initial Kaplan mean. Put

$$\psi(x) = \sin(\pi x), \quad S = \int_0^1 \psi(x) dx = \frac{2}{\pi}, \quad N = \int_0^1 u_0(x) \psi(x) dx.$$

Then $Y_0 = N/S$. With

$$S_h = h \sum_{j=1}^{I-1} \psi(x_j), \quad N_h = h \sum_{j=1}^{I-1} u_0(x_j) \psi(x_j),$$

the composite trapezoidal rule gives $S_h = S + O(h^2)$ and $N_h = N + O(h^2)$, because the endpoint contributions vanish. Since S_h is bounded away from zero and $Y_{h,0} = N_h/S_h$,

$$Y_{h,0} - Y_0 = \frac{S(N_h - N) - N(S_h - S)}{SS_h} = O(h^2).$$

Define

$$G(\eta, \mu) = \int_{\eta}^1 \frac{d\sigma}{\lambda(1-\sigma)^{-q} - \mu\sigma}.$$

The positivity assumption (5), together with the asymptotic behavior of the denominator near $\sigma = 1$, implies that G is continuously differentiable in a neighborhood of (Y_0, μ_1) . Near one, the integrand is controlled by a multiple of $(1-\sigma)^q$, and its derivative with respect to μ is controlled by a multiple of $(1-\sigma)^{2q}$; both functions are integrable because $q > 0$.

For all sufficiently small h , the pair $(Y_{h,0}, \mu_{1,h})$ belongs to this neighborhood. Hence the mean value theorem yields

$$T_{B,h}^+ - T_B^+(Y_0) = G(Y_{h,0}, \mu_{1,h}) - G(Y_0, \mu_1) = O(|Y_{h,0} - Y_0| + |\mu_{1,h} - \mu_1|) = O(h^2).$$

This completes the proof. $\square$

5. Numerical Validation of the Finite-Level Clock

The numerical tests compare the finite-step IMEX estimator with the lower and upper clocks, examine the second-order spatial behavior of the discrete upper clock and identify the dominant node. Adaptive time stepping is standard in numerical singular-time computations; the present use follows the

finite-difference treatment of Nakagawa [15], with a stopping criterion adapted to quenching. The exact semidiscrete time $T_{q,h}$ remains the quantity enclosed theoretically by Corollary 4.4; the time-discrete output below is denoted differently.

We take $q = 1$, $\lambda = 20$, and $u_0(x) = 0.2 \sin(\pi x)$. Then $M_0 = 0.2$, and the lower clock is

$$T^- = \frac{(1 - 0.2)^2}{2\lambda} = 0.016.$$

The continuous Kaplan mean is

$$Y_0 = \int_0^1 0.2 \sin(\pi x) \frac{\pi}{2} \sin(\pi x) dx = \frac{0.2\pi}{4}.$$

Since $\lambda = 20 > \mu_1/4$, the field $F(s) = 20(1 - s)^{-1} - \pi^2 s$ is positive on $[0, 1)$. The continuous Bihari upper clock is $T_B^+ \approx 0.0197902715$.

For the time integration of (8), we use the adaptive IMEX Euler update

$$\frac{U_i^{n+1} - U_i^n}{\Delta t_n} = \delta_h^2 U_i^{n+1} + \frac{\lambda}{(1 - U_i^n)^q}, \quad 1 \leq i \leq I - 1.$$

Diffusion is treated implicitly, while the singular source is evaluated at the old time level. The adaptive step is

$$\Delta t_n = \min \left\{ 2 \times 10^{-5}, 0.20 \frac{(1 - \|U^n\|_\infty)^{q+1}}{(q+1)\lambda} \right\}.$$

The computation stops at the first index for which $\Delta t_n \leq 10^{-16}$. No competing gap threshold is used. From the final computed state we form the finite-step estimator

$$\widehat{T}_{q,h} = t_n + \frac{(1 - \|U^n\|_\infty)^{q+1}}{(q+1)\lambda}.$$

The added term is the remaining time of the reaction-only scalar equation starting from the last computed maximum. It is therefore an estimator and not an identity for the exact semidiscrete time $T_{q,h}$. We use the mesh sequence $I = 16, 32, 64, 128, 256, 512$.

Table 5.1: Finite-step estimator and Bihari clock values for the adaptive IMEX computation.

I	$\widehat{T}_{q,h}$	T_h^-	$T_{B,h}^+$	steps	final gap	x_h^*
16	0.0173033150	0.0160000000	0.0197601577	985	1.2776×10^{-7}	0.500000
32	0.0173030699	0.0160000000	0.0197827554	985	1.3165×10^{-7}	0.500000
64	0.0173032096	0.0160000000	0.0197883932	985	1.3613×10^{-7}	0.500000
128	0.0173032893	0.0160000000	0.0197898020	985	1.4050×10^{-7}	0.500000
256	0.0173033189	0.0160000000	0.0197901541	986	1.3013×10^{-7}	0.500000
512	0.0173033283	0.0160000000	0.0197902421	986	1.3353×10^{-7}	0.500000

For every mesh in Table 5.1, the observed estimator satisfies $T_h^- < \widehat{T}_{q,h} < T_{B,h}^+$. This numerical placement should not be confused with the rigorous enclosure of the exact $T_{q,h}$ in Corollary 4.4. The final gaps are all slightly larger than 10^{-7} , which is consistent with the small-step stopping condition becoming active first. The detected dominant node is always $x_h^* = 0.5$, in agreement with the symmetry of the initial datum.

Table 5.2: Spatial convergence of the discrete Bihari upper clock.

	I	$Y_{h,0}$	$\mu_{1,h}$	$T_{B,h}^+$	s_B
	16	0.1575862454	9.8379364335	0.0197601577	–
	32	0.1572059193	9.8616797753	0.0197827554	–
	64	0.1571111815	9.8676227672	0.0197883932	2.0030
	128	0.1570875185	9.8691089628	0.0197898020	2.0007
	256	0.1570816040	9.8694805396	0.0197901541	2.0002
	512	0.1570801255	9.8695734356	0.0197902421	2.0000

Table 5.2 confirms Proposition 4.5. The discrete mean $Y_{h,0}$ converges to the continuous Kaplan mean, the eigenvalue $\mu_{1,h}$ converges to π^2 , and the upper clock approaches T_B^+ . The Richardson order is computed from

$$s_B = \log_2 \left(\frac{|T_{B,4h}^+ - T_{B,2h}^+|}{|T_{B,2h}^+ - T_{B,h}^+|} \right).$$

It is a spatial order associated with mesh refinement and not a temporal order of the IMEX integrator.

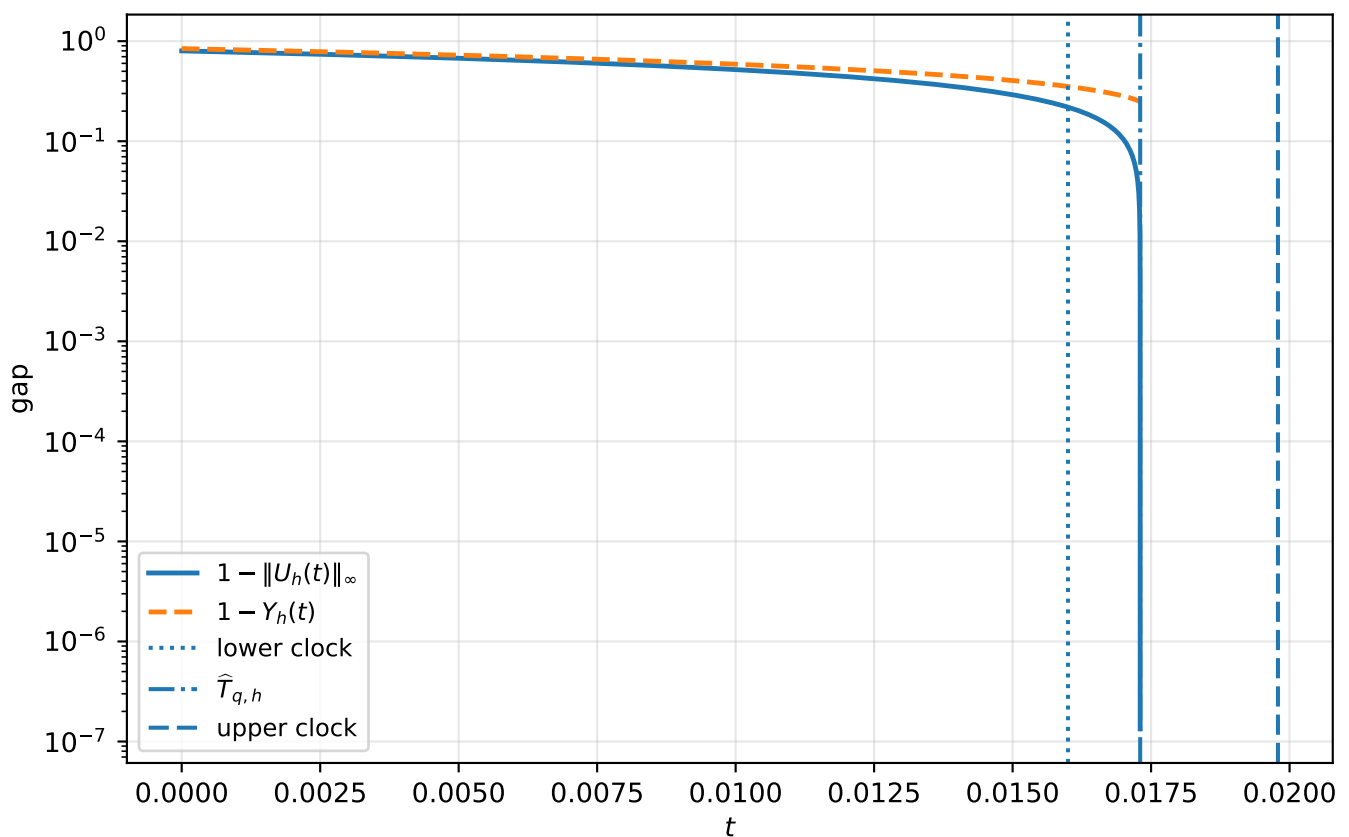


Figure 5.1: Computed gaps for $I = 512$. Both $1 - \|U_h(t)\|_\infty$ and $1 - Y_h(t)$ terminate at the final computed time. The lower clock, the estimator $\hat{T}_{q,h}$ and the Bihari upper clock are shown only as vertical reference lines.

Figure 5.1 distinguishes the computed quantities from the clock values. In particular, the discrete Kaplan mean is not continued beyond the final computed state, while the upper clock lies to the right as a theoretical reference value.

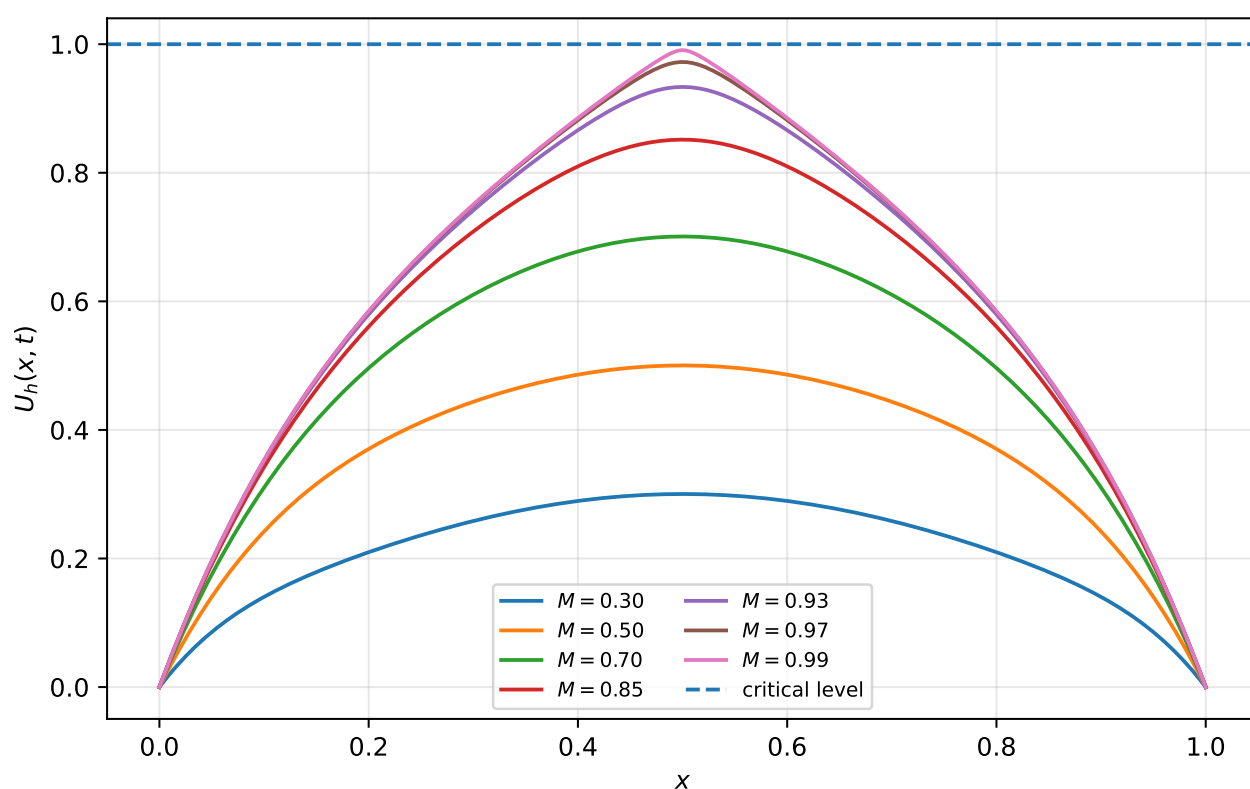


Figure 5.2: Symmetric numerical profiles for $I = 512$ at selected maximum levels. The dominant node remains the midpoint $x = 0.5$.

Figure 5.2 shows that the numerical profiles preserve the symmetry of the data and approach the critical level first at the central node.

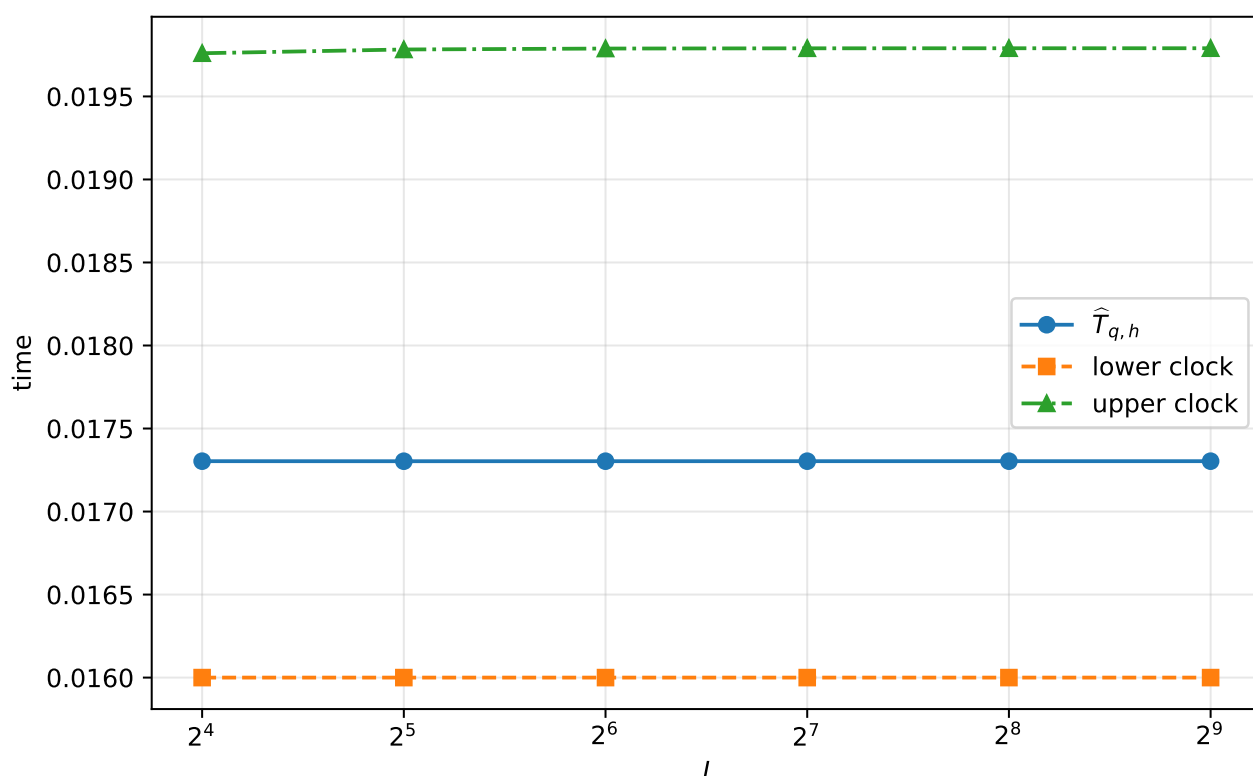


Figure 5.3: Finite-step estimator and clock values along the mesh sequence. The plotted estimator lies between the two clocks for every reported mesh.

Figure 5.3 is the graphical counterpart of Table 5.1. It displays the computed estimator separately from the exact semidiscrete time appearing in the theoretical statement.

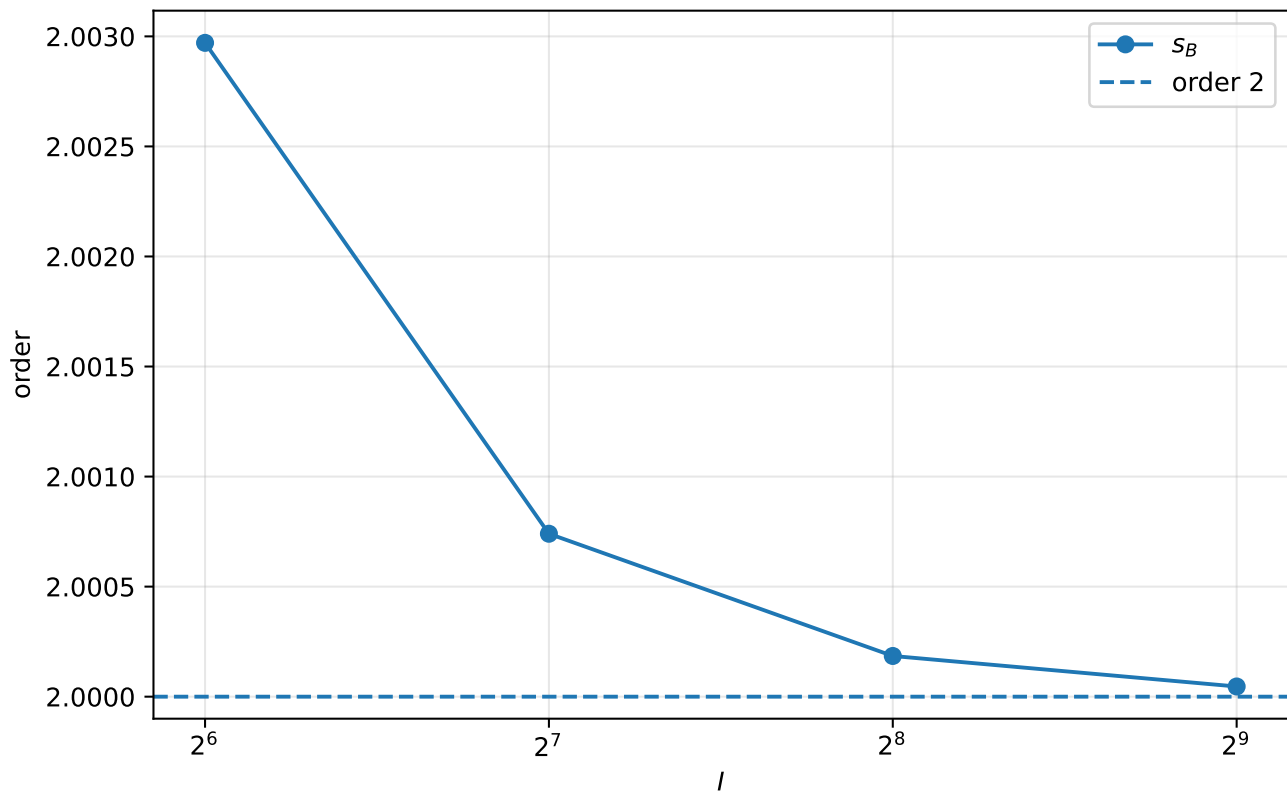


Figure 5.4: Richardson order of the discrete Bihari upper clock. The values approach the second-order reference line.

Figure 5.4 confirms the second-order behavior established in Proposition 4.5.

6. Conclusion

This work has developed a finite-level Bihari clock for quenching in a singular semilinear heat equation. Kaplan’s eigenfunction reduction and Jensen’s inequality give a scalar differential inequality for the weighted mean, and the finite-level clock yields an explicit upper estimate for the quenching time. A maximum-principle comparison gives the complementary lower clock, so the continuous and exact semidiscrete quenching times are bounded between computable quantities. The discrete upper clock converges to its continuous counterpart with second-order spatial accuracy. In the time-discrete experiments, a finite-step estimator distinct from the exact semidiscrete time lies between the reported clock values, the computed gaps terminate at the final numerical time, the midpoint is the dominant node and the discrete upper clock exhibits the predicted second-order behavior. Natural continuations include weighted singular reactions, nonlocal diffusion operators and refined localization criteria for quenching sets.

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References

- [1] A. Acker, W. Walter, The quenching problem for nonlinear parabolic differential equations, In: W. N. Everitt, B. D. Sleeman (Eds.), *Ordinary and Partial Differential Equations*, Springer, Berlin, 1976, 1–12.
- [2] R. Bellman, The stability of solutions of linear differential equations, *Duke Math. J.* **10** (1943) 643–647.
- [3] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, *Acta Math. Acad. Sci. Hungar.* **7** (1956) 81–94.
- [4] T. K. Boni, On quenching of solutions for some semilinear parabolic equations of second order, *Bull. Belg. Math. Soc. Simon Stevin* **7** (2000) 73–95.
- [5] T. K. Kouakou, T. K. Boni, R. K. Kouakou, An adaptive scheme to handle the phenomenon of quenching for a localized semilinear heat equation with Neumann boundary conditions, *Comput. Methods Appl. Math.* **9** (2009) 339–353.
- [6] G. Butler, T. Rogers, A generalization of a lemma of Bihari and applications to pointwise estimates for integral equations, *J. Math. Anal. Appl.* **33** (1971) 77–81.
- [7] L. Chen, N. A. Abdul Rahman, New generalizations of Gronwall–Bellman–Bihari-type integral inequalities, *Axioms* **14** (2025) #929.
- [8] C. Corduneanu, *Principles of Differential and Integral Equations*, Chelsea, New York, 1977.
- [9] K. Deng, H. A. Levine, On the blowup of u_t at quenching, *Proc. Amer. Math. Soc.* **106** (1989) 1049–1056.
- [10] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer, Berlin, 1981.
- [11] S. Kaplan, On the growth of solutions of quasi-linear parabolic equations, *Comm. Pure Appl. Math.* **16** (1963) 305–330.
- [12] H. Kawarada, On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$, *Publ. Res. Inst. Math. Sci.* **10** (1974/75) 729–736.
- [13] K. Lan, J. R. L. Webb, A new Bihari inequality and initial value problems of first order fractional differential equations, *Fract. Calc. Appl. Anal.* **26** (2023) 962–988.
- [14] H. A. Levine, The phenomenon of quenching: a survey, In: V. Lakshmikantham (Ed.), *Trends in the Theory and Practice of Non-Linear Analysis*, North-Holland, Amsterdam, 1985, 275–286.
- [15] T. Nakagawa, Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$, *Appl. Math. Optim.* **2** (1975) 337–350.
- [16] W. F. Osgood, Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx = f(x, y)$ ohne Hinzunahme der Cauchy–Lipschitz’schen Bedingung, *Monatsh. Math. Phys.* **9** (1898) 331–345.
- [17] M. H. Protter, H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, 1967.
- [18] Y. Qin, *Integral and Discrete Inequalities and Their Applications: Linear Inequalities*, Birkhäuser, Cham, 2016.
- [19] P. Quittner, P. Souplet, *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*, 2nd Edition, Birkhäuser, Cham, 2019.
- [20] Q. Wang, Y. Zhang, Asymptotic and quenching behaviors of semilinear parabolic systems with singular nonlinearities, *Commun. Pure Appl. Anal.* **21** (2022) 797–816.