

Research Article

Qualitative Analysis of a Three-Dimensional Exponential Difference System With DelayMuhammad Shabbir^{1,†}, Lubna Mustafa², Stephen Sadiq³¹Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan²Department of Mathematics, Virtual University of Pakistan, Lahore, Pakistan³Department of Mathematics, Minhaj University, Lahore, Pakistan**Abstract**

In this paper, we investigate the qualitative properties of a nonlinear exponential difference system for a three-dimensional system with delay. In particular, we study the boundedness, persistence, and invariant sets of positive solutions of the following system: $U_{m+1} = \alpha + \beta U_{m-1}e^{-V_m}$, $V_{m+1} = \gamma + \delta V_{m-1}e^{-W_m}$, $W_{m+1} = \theta + \eta W_{m-1}e^{-U_m}$, where all parameters and initial conditions are positive. Sufficient conditions ensuring boundedness, persistence, and the existence of positively invariant regions are established. Furthermore, under suitable sufficient conditions, the existence and uniqueness of the positive equilibrium point are established using fixed-point arguments.

Keywords: difference equations; exponential systems; boundedness; persistence; invariant regions; equilibrium points.

2020 Mathematics Subject Classification: 39A10, 39A30.

1. Introduction

Difference equations are known to have an important position in the modeling of discrete-time dynamical systems, and they are used to model numerous processes in biology, economics, engineering, etc. (for example, see [1, 5, 13]). Among difference equations, nonlinear difference equations have gained considerable attention due to their ability to model complex dynamics such as oscillations, chaos, and pattern formation [6, 16]. Of particular interest are difference equations with an exponential form, which are known to have considerable utility in modeling dynamical systems whose growth rates are affected by saturation, feedback, or environmental limitations [11, 14, 15, 18, 22]. Such an exponential form naturally arises in the modeling of biological and ecological systems, where competition for limited resources suppresses growth nonlinearly. Moreover, exponential-type nonlinearities also appear in various differential-difference models arising in mathematical physics, where exact solution techniques have been developed to better understand their structure [7].

Another important feature in realistic modeling is delay effects, as there exist many systems whose behavior is determined by past states as well as the current state. This fact has motivated extensive research in the study of delay difference equations and functional equations, whose dynamics display interesting properties such as stability switching, oscillations, and complex transient behavior [1, 24]. The interplay between time-delay effects and nonlinear interactions significantly enhances the complexity of dynamical systems, often leading to oscillatory behavior, multistability, and even chaotic dynamics, as reported in several studies [8, 16, 20, 25]. In the context of epidemiological modeling, discrete-time systems have recently been investigated using novel approaches to global stability, further highlighting the importance of delay and nonlinear interactions in real-world applications [23].

In recent years, considerable effort has been devoted to the qualitative analysis of nonlinear difference equations, including boundedness, persistence, periodicity, invariant regions, equilibrium behavior, and stability [3, 5, 10, 14, 17, 21, 22]. These properties are crucial for understanding the long-term behavior of solutions and for validating mathematical models of real-world phenomena. Two-dimensional systems of nonlinear difference equations, especially those involving exponential terms, have been extensively studied and are known to exhibit a wide range of dynamical behaviors such as convergence to equilibria, periodic oscillations, and even chaotic dynamics [4, 14, 15]. However, extending these results to higher-dimensional systems introduces substantial analytical challenges due to increased coupling and interaction among variables.

Three-dimensional systems are particularly important in applications involving multi-species interactions, epidemiological dynamics, and coupled economic or engineering processes. Such systems are capable of exhibiting more complex phenomena, including multistability, bifurcations, and intricate transient dynamics, which are not present in lower-dimensional models [2, 5, 6]. Despite their importance, the theoretical understanding of three-dimensional nonlinear difference systems, especially those with exponential nonlinearities and delays, remains relatively limited.

To illustrate the development of this field, consider the scalar exponential difference equation which has been widely studied due to its nonlinear and delay-induced dynamics. Various extensions of this model have been proposed to better capture interactions between multiple variables. The asymptotic nature, boundedness, stability of the nonnegative equilibrium, and periodicity of the solutions of the difference equation

$$z_{n+1} = \sigma z_n + \psi z_{n-1} e^{-z_n}, \quad n = 0, 1, 2, \dots,$$

were studied by Grove et al. [9], where the constants σ, ψ , and the initial values z_i , $i = -1, 0$, are nonnegative real numbers. Similar results were found by Papaschinopoulos et al. [18] for the following system of difference equations:

$$\begin{aligned} x_{n+1} &= a + bx_{n-1}e^{-y_n}, \\ y_{n+1} &= c + dy_{n-1}e^{-x_n}, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where the constants a, b, c and d , and the initial conditions x_i, y_i , $i = -1, 0$, are positive real numbers. Furthermore, Papaschinopoulos et al. [19] studied the asymptotic nature, persistence, and boundedness of the nonnegative solutions of the following exponential difference equations:

$$\begin{aligned} x_{n+1} &= ax_n + by_{n-1}e^{-x_n}, \\ y_{n+1} &= cy_n + dx_{n-1}e^{-y_n}, \quad n = 0, 1, 2, \dots, \end{aligned}$$

where the constants a, b, c, d , and the initial conditions x_i, y_i , $i = -1, 0$, are nonnegative real numbers.

Khaliq et al. [12] studied the convergence rate, boundedness and persistence, and asymptotic behavior of the nonnegative solutions of the following system of exponential-type difference equations:

$$\begin{aligned}x_{n+1} &= ax_n + by_{n-1}e^{-x_n}, \\y_{n+1} &= cy_n + dz_{n-1}e^{-y_n}, \\z_{n+1} &= ez_n + fx_{n-1}e^{-z_n},\end{aligned}$$

where the constants a, b, c, d, e, f , and the initial values x_i, y_i, z_i , $i = -1, 0$, are all nonnegative real numbers.

Motivated by these developments, this paper investigates the following three-dimensional system of exponential difference equations with delay, which represents a fully coupled nonlinear delayed framework extending several existing exponential difference models:

$$\begin{aligned}U_{m+1} &= \alpha + \beta U_{m-1}e^{-V_m}, \\V_{m+1} &= \gamma + \delta V_{m-1}e^{-W_m}, \\W_{m+1} &= \theta + \eta W_{m-1}e^{-U_m},\end{aligned}\tag{1}$$

where all parameters and initial conditions are positive. The main objective of this work is to investigate the qualitative behavior of positive solutions of system (1). In particular, we establish sufficient conditions for boundedness and persistence, derive positively invariant regions, and study the existence and uniqueness of equilibrium points. Although three-dimensional exponential difference systems have previously been investigated, including the work of Khaliq et al. [12], the present system possesses a different nonlinear delayed coupling structure. Unlike the model studied in [12], each component evolves through delayed exponential interactions with constant input terms, leading to a distinct qualitative dynamical behavior.

2. Asymptotic Behavior of Solutions of System (1)

We first establish the persistence and boundedness of solutions. To this end, we compare the positive solutions of system (1) with those of a solvable system of difference equations.

Theorem 2.1. *Assume that*

$$\beta e^{-\gamma} < 1, \quad \delta e^{-\theta} < 1, \quad \eta e^{-\alpha} < 1.\tag{2}$$

Then, every positive solution of system (1) is bounded and persistent.

Proof. Since all parameters are positive, we first compute the first iterate directly from the initial history,

$$U_1 = \alpha + \beta U_{-1}e^{-V_0}, \quad V_1 = \gamma + \delta V_{-1}e^{-W_0}, \quad W_1 = \theta + \eta W_{-1}e^{-U_0}.$$

For every $m \geq 1$, positivity of the system immediately implies $U_m \geq \alpha$, $V_m \geq \gamma$, and $W_m \geq \theta$. Hence,

$$U_{m+1} \leq \alpha + \beta e^{-\gamma} U_{m-1}, \quad V_{m+1} \leq \gamma + \delta e^{-\theta} V_{m-1}, \quad W_{m+1} \leq \theta + \eta e^{-\alpha} W_{m-1}, \quad m \geq 1.$$

Now, we define the auxiliary sequences by

$$u_0 = U_0, \quad u_1 = U_1, \quad v_0 = V_0, \quad v_1 = V_1, \quad w_0 = W_0, \quad w_1 = W_1,$$

and

$$u_{m+1} = \alpha + \beta e^{-\gamma} u_{m-1}, \quad v_{m+1} = \gamma + \delta e^{-\theta} v_{m-1}, \quad w_{m+1} = \theta + \eta e^{-\alpha} w_{m-1}, \quad m \geq 1.$$

We now prove by induction, beginning at $m = 1$, that $U_m \leq u_m$, $V_m \leq v_m$, and $W_m \leq w_m$. By construction, we have

$$U_0 = u_0, \quad U_1 = u_1, \quad V_0 = v_0, \quad V_1 = v_1, \quad W_0 = w_0, \quad W_1 = w_1.$$

Hence, the comparison holds for the initial indices $m = 0, 1$. We now proceed by induction starting from $m = 1$. Assume that $U_k \leq u_k$ for all $k \leq m$. Then,

$$U_{m+1} \leq \alpha + \beta e^{-\gamma} U_{m-1} \leq \alpha + \beta e^{-\gamma} u_{m-1} = u_{m+1}.$$

Similarly, we have $V_{m+1} \leq v_{m+1}$ and $W_{m+1} \leq w_{m+1}$. Thus, the comparison holds for all m .

To justify the convergence of the auxiliary recurrence, let $c = \beta e^{-\gamma}$ with $0 < c < 1$. The auxiliary equation can be written as $u_{m+1} = \alpha + cu_{m-1}$. Since the recurrence advances by two indices, we consider its even and odd subsequences separately. Define

$$x_n = u_{2n}, \quad y_n = u_{2n+1}, \quad n \geq 0.$$

Then, the recurrence becomes $x_{n+1} = \alpha + cx_n$ and $y_{n+1} = \alpha + cy_n$. Each of these is a first-order affine difference equation. Since $0 < c < 1$, both sequences are contractions. Hence,

$$\lim_{n \rightarrow \infty} x_n = \frac{\alpha}{1-c} \quad \text{and} \quad \lim_{n \rightarrow \infty} y_n = \frac{\alpha}{1-c}.$$

Therefore,

$$\lim_{m \rightarrow \infty} u_m = \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Applying the same argument to the auxiliary sequences

$$v_{m+1} = \gamma + \delta e^{-\theta} v_{m-1} \quad \text{and} \quad w_{m+1} = \theta + \eta e^{-\alpha} w_{m-1},$$

yields

$$\lim_{m \rightarrow \infty} v_m = \frac{\gamma}{1 - \delta e^{-\theta}}, \quad \text{and} \quad \lim_{m \rightarrow \infty} w_m = \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Consequently, we have $U_m \leq u_m$, $V_m \leq v_m$, and $W_m \leq w_m$, which together with the above limits, imply

$$\limsup_{m \rightarrow \infty} U_m \leq \frac{\alpha}{1 - \beta e^{-\gamma}}, \quad \limsup_{m \rightarrow \infty} V_m \leq \frac{\gamma}{1 - \delta e^{-\theta}}, \quad \text{and} \quad \limsup_{m \rightarrow \infty} W_m \leq \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Therefore, all positive solutions of system (1) are bounded. Furthermore, since $U_m \geq \alpha$, $V_m \geq \gamma$, and $W_m \geq \theta$ for every $m \geq 1$, it follows immediately that

$$\liminf_{m \rightarrow \infty} U_m \geq \alpha > 0, \quad \liminf_{m \rightarrow \infty} V_m \geq \gamma > 0, \quad \text{and} \quad \liminf_{m \rightarrow \infty} W_m \geq \theta > 0.$$

Hence, every positive solution is persistent. □

The boundedness and persistence established in Theorem 2.1 provide fundamental information about the long-term behavior of positive solutions. In particular, every positive solution remains confined to a bounded subset of the positive orthant and stays uniformly away from the coordinate planes. These estimates naturally motivate the construction of a positively invariant region for system (1), which serves as the basis for the subsequent analysis of equilibrium points and qualitative properties of the system.

Theorem 2.2. Assume that (2) holds and let

$$\Omega = \left[\alpha, \frac{\alpha}{1 - \beta e^{-\gamma}} \right] \times \left[\gamma, \frac{\gamma}{1 - \delta e^{-\theta}} \right] \times \left[\theta, \frac{\theta}{1 - \eta e^{-\alpha}} \right].$$

Then the phase space $\Omega \times \Omega$ is positively invariant under system (1). That is, if $(U_{m-1}, V_{m-1}, W_{m-1}) \in \Omega$ and $(U_m, V_m, W_m) \in \Omega$, then $(U_{m+1}, V_{m+1}, W_{m+1}) \in \Omega$.

Proof. Assume that $(U_{m-1}, V_{m-1}, W_{m-1}) \in \Omega$ and $(U_m, V_m, W_m) \in \Omega$. Since

$$\alpha \leq U_{m-1} \leq \frac{\alpha}{1 - \beta e^{-\gamma}} \quad \text{and} \quad V_m \geq \gamma,$$

we obtain

$$U_{m+1} = \alpha + \beta U_{m-1} e^{-V_m} \leq \alpha + \beta \frac{\alpha}{1 - \beta e^{-\gamma}} e^{-\gamma} = \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Moreover, $U_{m+1} \geq \alpha$. Therefore,

$$\alpha \leq U_{m+1} \leq \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Similarly, since

$$\gamma \leq V_{m-1} \leq \frac{\gamma}{1 - \delta e^{-\theta}} \quad \text{and} \quad W_m \geq \theta,$$

we obtain

$$\gamma \leq V_{m+1} \leq \frac{\gamma}{1 - \delta e^{-\theta}}.$$

Also, since

$$\theta \leq W_{m-1} \leq \frac{\theta}{1 - \eta e^{-\alpha}} \quad \text{and} \quad U_m \geq \alpha,$$

we have

$$\theta \leq W_{m+1} \leq \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Consequently, we conclude that $(U_{m+1}, V_{m+1}, W_{m+1}) \in \Omega$. Hence, $\Omega \times \Omega$ is positively invariant under system (1). $\square$

Having established the positively invariant phase space $\Omega \times \Omega$ for system (1), we now investigate its equilibrium structure. Since every positive solution starting in $\Omega \times \Omega$ remains there for all future iterations, the qualitative analysis may be restricted to this compact phase space. This invariance property provides a suitable framework for establishing the existence and uniqueness of equilibrium points.

Lemma 2.3. Assume that (2) holds and define

$$F(U, V, W) = \left(\alpha + \beta U e^{-V}, \gamma + \delta V e^{-W}, \theta + \eta W e^{-U} \right).$$

Then, the mapping $F : \Omega \rightarrow \Omega$ is well defined. In particular, $F(\Omega) \subseteq \Omega$.

Proof. Let $(U, V, W) \in \Omega$. Then, we have

$$\alpha \leq U \leq \frac{\alpha}{1 - \beta e^{-\gamma}}, \quad \gamma \leq V \leq \frac{\gamma}{1 - \delta e^{-\theta}}, \quad \text{and} \quad \theta \leq W \leq \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Since $V \geq \gamma$, we have $e^{-V} \leq e^{-\gamma}$. Hence,

$$\alpha + \beta U e^{-V} \leq \alpha + \beta \frac{\alpha}{1 - \beta e^{-\gamma}} e^{-\gamma} = \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Moreover, we have $\alpha + \beta U e^{-V} \geq \alpha$. Therefore,

$$\alpha \leq \alpha + \beta U e^{-V} \leq \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Similarly, using $W \geq \theta$, we obtain

$$\gamma \leq \gamma + \delta V e^{-W} \leq \frac{\gamma}{1 - \delta e^{-\theta}},$$

and using $U \geq \alpha$, we have

$$\theta \leq \theta + \eta W e^{-U} \leq \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Consequently, $F(U, V, W) \in \Omega$. Since the choice of $(U, V, W) \in \Omega$ was arbitrary, we conclude that $F(\Omega) \subseteq \Omega$. Hence, the mapping F is well defined on Ω . $\square$

Theorem 2.4. Assume that (2) holds and define

$$F(U, V, W) = \left(\alpha + \beta U e^{-V}, \gamma + \delta V e^{-W}, \theta + \eta W e^{-U} \right).$$

If

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_{\infty} < 1,$$

then system (1) possesses a unique equilibrium point in Ω .

Proof. By Lemma 2.3, the mapping $F : \Omega \rightarrow \Omega$ is well defined and continuous. Since Ω is compact and convex, Brouwer's fixed-point theorem guarantees the existence of at least one fixed point of F in Ω . The Jacobian matrix of F is

$$J_F(U, V, W) = \begin{pmatrix} \beta e^{-V} & -\beta U e^{-V} & 0 \\ 0 & \delta e^{-W} & -\delta V e^{-W} \\ -\eta W e^{-U} & 0 & \eta e^{-U} \end{pmatrix}.$$

Since the entries of $J_F(U, V, W)$ are continuous on the compact set Ω , the quantity

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_{\infty}$$

is well defined. By assumption,

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_{\infty} < 1.$$

Therefore, by the mean-value theorem for vector-valued mappings, we have

$$\|F(X) - F(Y)\|_{\infty} \leq q \|X - Y\|_{\infty}, \quad 0 < q < 1,$$

for all $X, Y \in \Omega$. Hence, F is a contraction on Ω . Therefore, Banach's fixed-point theorem guarantees that F possesses a unique fixed point in Ω . Consequently, system (1) possesses a unique equilibrium point in Ω . $\square$

Corollary 2.5. Assume that condition (2) holds. Also, assume that

$$\beta e^{-\gamma} \left(1 + \frac{\alpha}{1 - \beta e^{-\gamma}} \right) < 1, \quad (3)$$

$$\delta e^{-\theta} \left(1 + \frac{\gamma}{1 - \delta e^{-\theta}} \right) < 1, \quad (4)$$

and

$$\eta e^{-\alpha} \left(1 + \frac{\theta}{1 - \eta e^{-\alpha}} \right) < 1. \quad (5)$$

Then

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_{\infty} < 1.$$

Moreover, system (1) possesses a unique positive equilibrium point.

Proof. For every $(U, V, W) \in \Omega$, the infinity norm of the Jacobian matrix is given by

$$\|J_F(U, V, W)\|_{\infty} = \max\{R_1, R_2, R_3\},$$

where

$$R_1 = \beta e^{-V} + \beta U e^{-V}, \quad R_2 = \delta e^{-W} + \delta V e^{-W}, \quad R_3 = \eta e^{-U} + \eta W e^{-U}.$$

Since $(U, V, W) \in \Omega$, we have

$$U \leq \frac{\alpha}{1 - \beta e^{-\gamma}} \quad \text{and} \quad V \geq \gamma,$$

which implies $e^{-V} \leq e^{-\gamma}$. Hence,

$$R_1 \leq \beta e^{-\gamma} \left(1 + \frac{\alpha}{1 - \beta e^{-\gamma}} \right).$$

Similarly, we have

$$R_2 \leq \delta e^{-\theta} \left(1 + \frac{\gamma}{1 - \delta e^{-\theta}} \right) \quad \text{and} \quad R_3 \leq \eta e^{-\alpha} \left(1 + \frac{\theta}{1 - \eta e^{-\alpha}} \right).$$

Therefore,

$$\|J_F(U, V, W)\|_{\infty} \leq \max \left\{ \beta e^{-\gamma} \left(1 + \frac{\alpha}{1 - \beta e^{-\gamma}} \right), \delta e^{-\theta} \left(1 + \frac{\gamma}{1 - \delta e^{-\theta}} \right), \eta e^{-\alpha} \left(1 + \frac{\theta}{1 - \eta e^{-\alpha}} \right) \right\}.$$

By conditions (3)–(5), each of the above quantities is strictly less than one. Consequently,

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_{\infty} < 1.$$

It remains to show that every positive equilibrium belongs to Ω . Let (U^*, V^*, W^*) be any positive equilibrium of system (1). Then $U^* = \alpha + \beta U^* e^{-V^*}$. Since $V^* \geq \gamma$, we have $U^* \leq \alpha + \beta e^{-\gamma} U^*$, and hence,

$$U^* \leq \frac{\alpha}{1 - \beta e^{-\gamma}}.$$

Similarly, we have

$$V^* \leq \frac{\gamma}{1 - \delta e^{-\theta}} \quad \text{and} \quad W^* \leq \frac{\theta}{1 - \eta e^{-\alpha}}.$$

Moreover, from the equilibrium equations and positivity, we obtain $U^* \geq \alpha$, $V^* \geq \gamma$, and $W^* \geq \theta$. Therefore, $(U^*, V^*, W^*) \in \Omega$. By Theorem 2.4, the mapping F has a unique fixed point in Ω . Since every positive equilibrium belongs to Ω , system (1) possesses a unique positive equilibrium point. $\square$

The uniqueness of the equilibrium point implies that system (1) possesses a single steady-state configuration within the invariant region Ω . This property is important from both theoretical and practical perspectives because it excludes the possibility of multiple equilibrium states under the imposed assumptions.

Remark 2.6. *The conditions in Corollary 2.5 provide explicit sufficient parameter inequalities guaranteeing*

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_\infty < 1.$$

Hence, the contraction hypothesis in Theorem 2.4 can be verified directly from the model parameters without computing the Jacobian norm explicitly.

3. Numerical Simulations

To support the theoretical findings, we present numerical simulations of system (1). The simulations illustrate the boundedness, persistence, and positively invariant region established in Section 2. In addition, for the selected parameter values, the computed trajectories appear to approach the unique equilibrium point. This observed behavior is presented only as a numerical observation and not as a theoretical consequence of the results proved in this paper.

Example 3.1. Let the parameters be chosen as

$$\alpha = 1.2, \quad \beta = 0.8, \quad \gamma = 1.1, \quad \delta = 0.7, \quad \theta = 1.3, \quad \eta = 0.6.$$

Then, we note that the conditions of Theorem 2.1 are satisfied. Moreover, the sufficient conditions of Corollary 2.5 are also satisfied. In particular, the corresponding row-sum bounds are

$$R_1 \approx 0.70184, \quad R_2 \approx 0.45009, \quad R_3 \approx 0.46747,$$

all of which are less than one. Hence,

$$\sup_{(U,V,W) \in \Omega} \|J_F(U, V, W)\|_\infty < 1,$$

and therefore the existence and uniqueness of the positive equilibrium follow from Corollary 2.5. Taking the initial values

$$U_{-1} = 0.5, \quad U_0 = 1.0, \quad V_{-1} = 0.6, \quad V_0 = 1.2, \quad W_{-1} = 0.7, \quad W_0 = 1.1,$$

the numerical solution of system (1) is presented in Figure 3.1. The first two iterates are computed directly from system (1). We obtain

$$(U_1, V_1, W_1) = (1.320478, 1.239806, 1.454509) \quad \text{and} \quad (U_2, V_2, W_2) = (1.431552, 1.296153, 1.476225).$$

A direct verification shows that $(U_1, V_1, W_1) \in \Omega$ and $(U_2, V_2, W_2) \in \Omega$ because

$$\Omega = \left[\alpha, \frac{\alpha}{1 - \beta e^{-\gamma}} \right] \times \left[\gamma, \frac{\gamma}{1 - \delta e^{-\theta}} \right] \times \left[\theta, \frac{\theta}{1 - \eta e^{-\alpha}} \right].$$

Therefore, $((U_1, V_1, W_1), (U_2, V_2, W_2)) \in \Omega \times \Omega$. Consequently, by Theorem 2.2, the trajectory remains in the positively invariant region for all $m \geq 2$. The regenerated numerical results show that all three sequences remain positive and bounded throughout the computation. Furthermore, after the first iteration,

$$U_m > \alpha, \quad W_m > \theta, \quad m \geq 1,$$

which is fully consistent with the defining equations of system (1). The trajectories remain inside the positively invariant region established in Theorem 2.2. The numerical solution also appears to approach the unique equilibrium whose existence and uniqueness are established in Theorem 2.4.

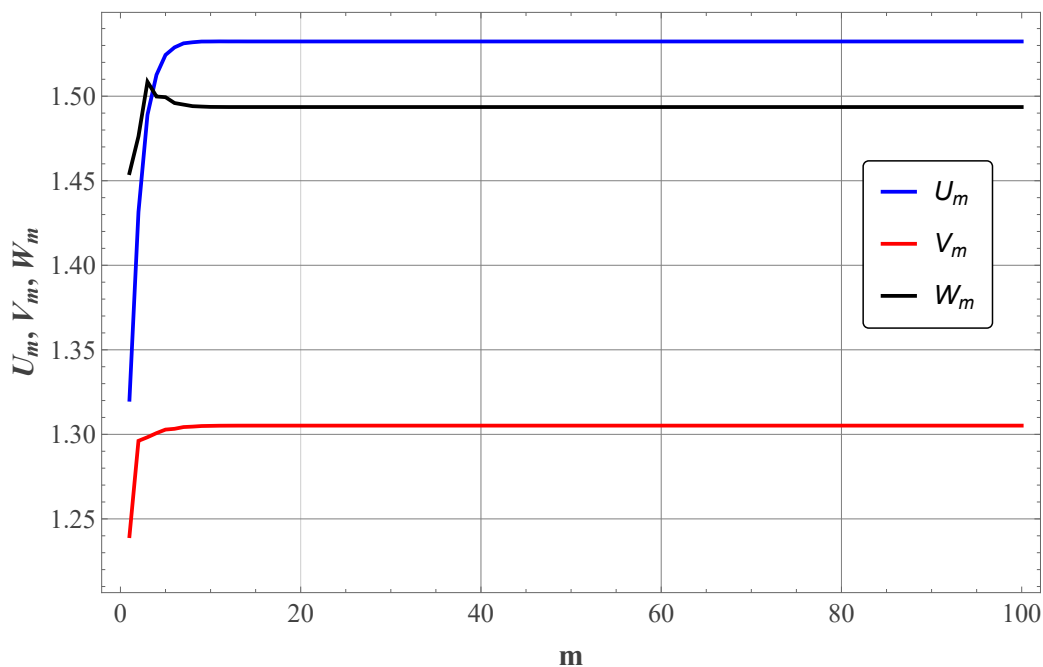


Figure 3.1: Numerical solution of system (1) for the parameter values and initial conditions of Example 3.1. The displayed trajectories begin with the first computed iterates ($m = 1$). The computed solutions remain positive and bounded throughout the simulation and stay inside the positively invariant region predicted by Theorem 2.2.

Example 3.2. Consider the following values of the parameters:

$$\alpha = 1, \quad \beta = 0.5, \quad \gamma = 1, \quad \delta = 0.4, \quad \theta = 1, \quad \eta = 0.3.$$

The simulations in this case were performed with the following initial history:

$$U_{-1} = 1, \quad U_0 = 1, \quad V_{-1} = 1, \quad V_0 = 1, \quad W_{-1} = 1, \quad W_0 = 1.$$

We verify the assumptions of Theorem 2.1:

$$\beta e^{-\gamma} = 0.5e^{-1} \approx 0.1839 < 1, \quad \delta e^{-\theta} = 0.4e^{-1} \approx 0.1472 < 1, \quad \eta e^{-\alpha} = 0.3e^{-1} \approx 0.1104 < 1.$$

Furthermore, the sufficient conditions of Corollary 2.5 are also satisfied. Indeed, the corresponding row-sum bounds are

$$R_1 \approx 0.40934, \quad R_2 \approx 0.31969, \quad R_3 \approx 0.23442,$$

which are all strictly less than one. Hence, Corollary 2.5 guarantees the existence and uniqueness of the positive equilibrium for the considered parameter set. The numerical simulations illustrate the theoretical boundedness established in Theorem 2.1. Moreover, for the selected parameter values, the computed trajectories appear to approach the unique equilibrium point (see Figure 3.2). Direct computation shows that the first two computed state vectors belong to the invariant region Ω . Therefore, by Theorem 2.2, all subsequent iterates remain inside Ω .

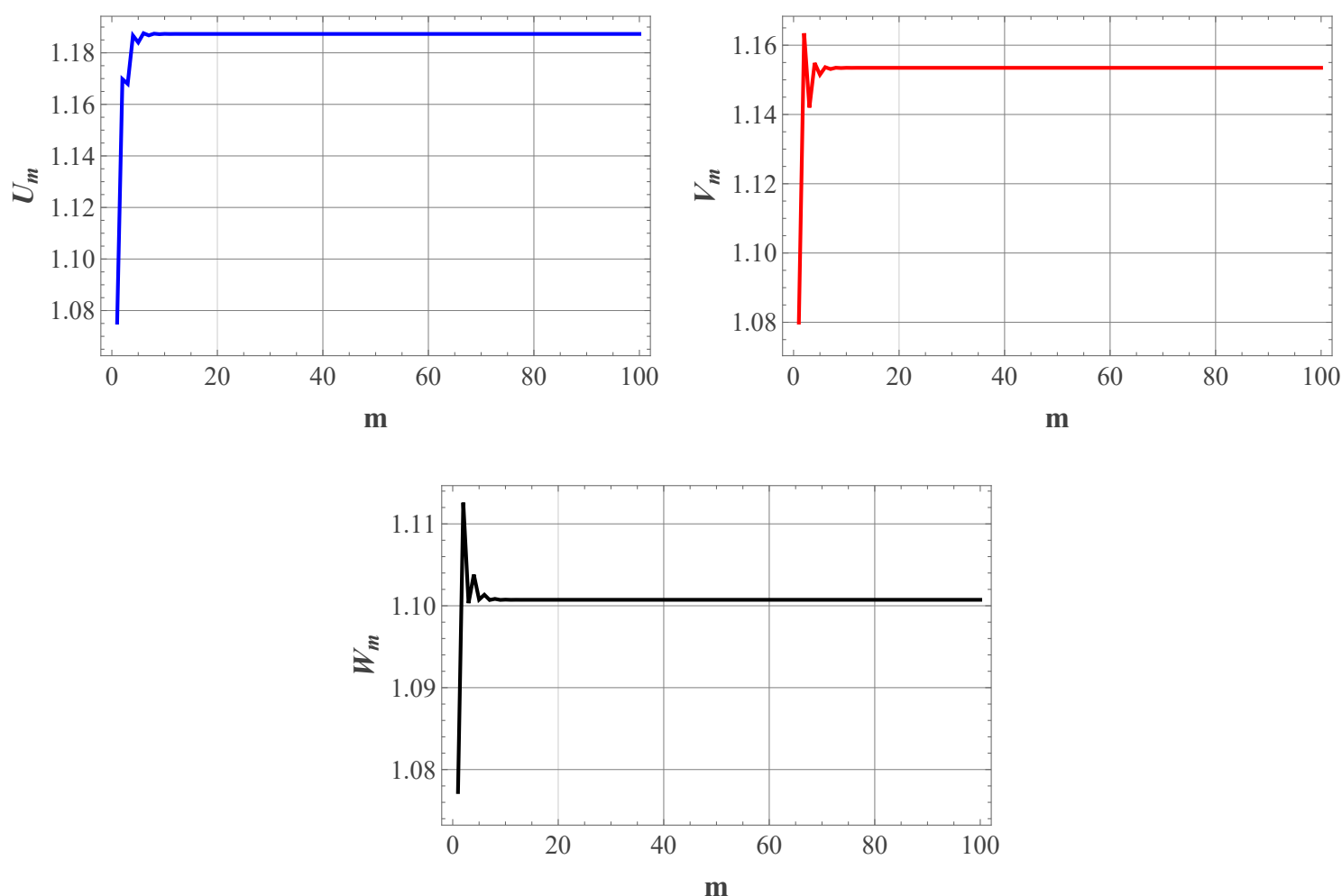


Figure 3.2: Time evolution of the sequences U_m , V_m , and W_m for the parameter values of Example 3.2. The displayed trajectories begin with the first computed iterates ($m = 1$). The computed trajectories remain positive and bounded throughout the simulation and appear to approach the unique equilibrium point guaranteed by Corollary 2.5.

Example 3.3. In this example, unless otherwise stated, the same initial history as in Example 3.2 is used. We vary parameter β while keeping others fixed. For the selected values $\beta = 0.4, 0.7, 0.95$, the first row-sum bounds in Corollary 2.5 are approximately 0.31969, 0.60435, 0.88673, respectively, while the remaining two row-sum bounds remain below one. Therefore, the contraction condition of Corollary 2.5 continues to hold for all parameter values considered.

The following observations are made (see Figure 3.3):

- Increasing β increases the transient oscillations.
- For the tested parameter values, all trajectories remain bounded, and the numerical solutions appear to approach the equilibrium point for the parameter values considered.
- As the parameter values approach the boundary of the contraction condition in Corollary 2.5, the transient phase becomes longer before the numerical trajectories appear to approach the equilibrium point.

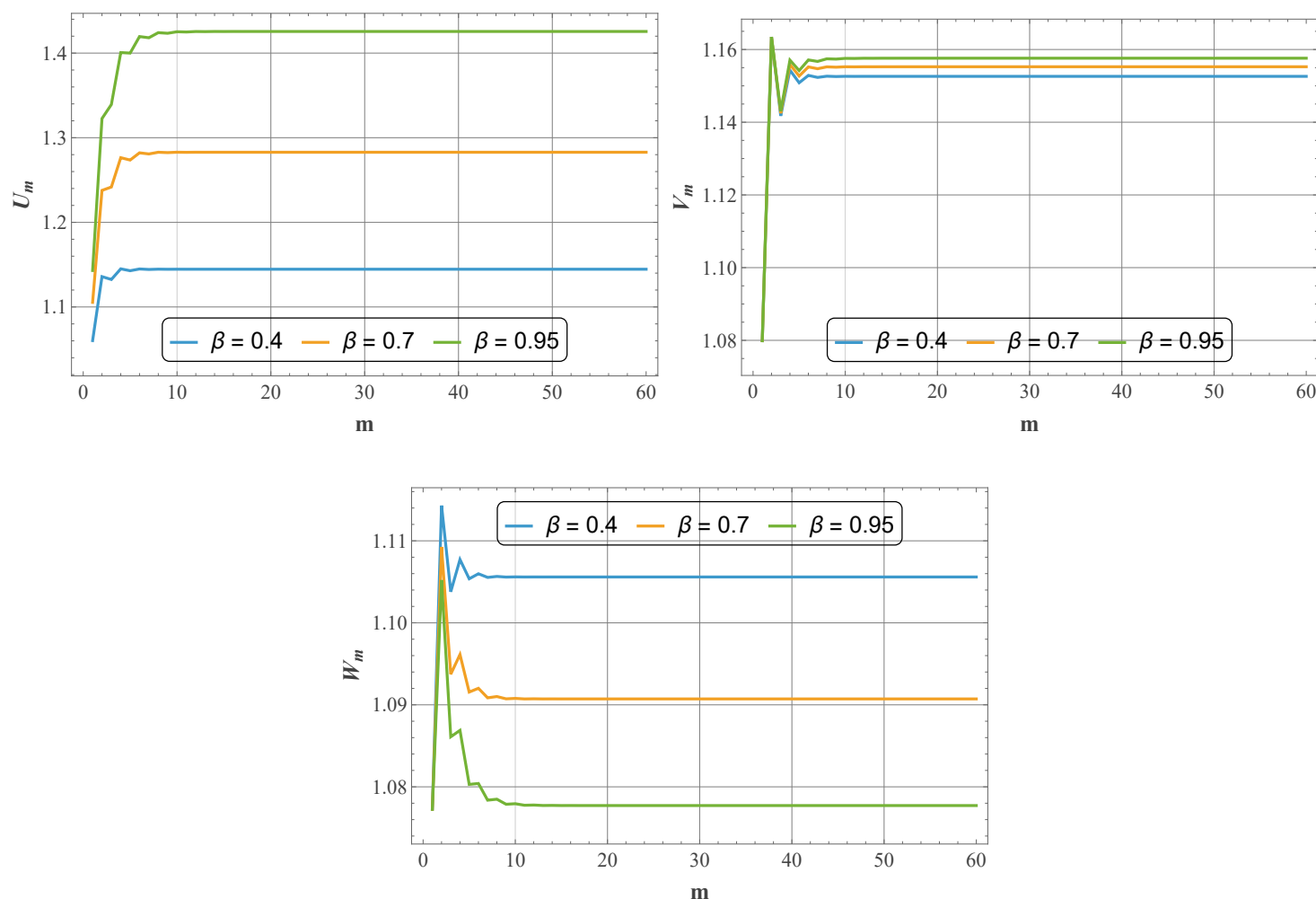


Figure 3.3: Effect of varying the parameter β on the numerical dynamics of system (1). The displayed trajectories begin with the first computed iterates ($m = 1$). Increasing β produces more pronounced transient oscillations, while all computed trajectories remain positive and bounded throughout the simulation.

The numerical experiments in Examples 3.1–3.3 are consistent with the theoretical results established in Section 2. In particular, all computed trajectories remain positive, bounded, and inside the positively invariant region obtained in Theorem 2.2. For the parameter values considered, the numerical solutions appear to approach the unique equilibrium point. This observed behavior is based solely on numerical evidence and should not be interpreted as a proof of asymptotic convergence or stability. The simulations further indicate that increasing the parameter β leads to more pronounced transient oscillations before the numerical trajectories appear to settle near the equilibrium point.

4. Conclusion

In this paper, we investigated several qualitative properties of a three-dimensional nonlinear exponential difference system with delay. Sufficient conditions ensuring the boundedness and persistence of positive solutions were established, and a positively invariant region was derived. Furthermore, the existence and uniqueness of a positive equilibrium point were obtained by means of fixed-point arguments. The obtained results contribute to the qualitative theory of nonlinear delayed difference equations by providing a rigorous analysis of boundedness, persistence, invariant regions, and equilibrium structure for a class of delayed exponential systems. Numerical simulations were presented to illustrate the theoretical boundedness and persistence results and to investigate the influence of the model parameters on the solution behavior. For the examples considered, the computed trajectories appear to approach the equilibrium point. This observation is based solely on numerical experiments and does not constitute a theoretical proof of asymptotic convergence or stability.

Future research may focus on the investigation of periodic solutions, bifurcation phenomena, chaotic dynamics, and fractional-order extensions of the model considered here, using Caputo-type operators. The development of rigorous criteria for local and global asymptotic stability, as well as the computation of Lyapunov exponents and other dynamical indicators, also remains an interesting direction for further study.

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