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Research Article

On the Gersenne–Leonardo Sequence

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Abstract

In this study, we introduce a new family of third-order sequences, called Gersenne–Leonardo sequences, and establish their relationship with Gersenne sequences. Gersenne sequences provide a natural generalization of second-order Mersenne-type sequences. The proposed family extends the classical Mersenne construction by incorporating a constant inhomogeneous term inspired by the Leonardo sequence, yielding a range of algebraic identities and structural properties.

Keywords: Leonardo-type sequence; Gersenne–Leonardo sequence; Tagiuri–Vajda identity.

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1. Introduction and Background

The Mersenne sequence $\{M_n\}_{n\geq 0}$ is defined by the second-order linear recurrence

$$M_n = 3M_{n-1} - 2M_{n-2}, \qquad n \geq 2,$$

with initial values $M_0 = 0$ and $M_1 = 1$. More generally, the k-Mersenne sequence $\{M_n^{(k)}\}_{n\geq 0}$ satisfies the recurrence

$$M_{n+1}^{(k)} = (k+1)M_n^{(k)} - kM_{n-1}^{(k)}, \qquad n \geq 1,$$

with initial values $M_0^{(k)} = 0$ and $M_1^{(k)} = 1$. Similarly, the Gersenne sequence $\{G_n^{(k,a,b)}\}_{n\geq 0}$, denoted more briefly by $\{G_n^{(k)}\}_{n\geq 0}$, satisfies the following recurrence:

$$G_n^{(k)} = (k+1)G_{n-1}^{(k)} - kG_{n-2}^{(k)}, \qquad n \geq 2, \tag{1}$$

with initial values $G_0^{(k)} = a$ and $G_1^{(k)} = b$, where k, a, b are fixed non-negative integers. The Gersenne sequence was introduced in [40] as a generalization of the Mersenne sequence. A straightforward computation gives its first five terms:

$$a,\ b,\ k(b-a)+b,\ (k^2+k)(b-a)+b,\ (k^3+k^2+k)(b-a)+b,\ \ldots.$$

In the special case $a = 0$ and $b = 1$, the Gersenne sequence reduces to the classical k-Mersenne sequence:

$$\{G_n^{(k,0,1)}\}_{n\geq 0} = \{M_n^{(k)}\}_{n\geq 0} = \{\upsilon_n^{(k)}\}_{n\geq 0} = \{0, 1, k+1, k^2+k+1, \ldots\}.$$

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For $k > 1$, its general term is $v_n^{(k)} = (k^n - 1)/(k - 1)$, for all $n \geq 0$. Taking $k = 2$ recovers the original Mersenne sequence $\{M_n\}_{n\geq 0}$.

Generalizations of the Mersenne sequence have been studied through several approaches, including varying the initial conditions while preserving the recurrence relation, modifying the recurrence relation while retaining the initial terms, and constructing matrix representations. These studies introduce new sequences and investigate their generalizations, extensions, and properties, often deriving identities analogous to those satisfied by the classical Mersenne sequence. The references [6,7, 9–11,13–15,19,21,24,25,27,28,33,38,40,44] discuss fundamental properties of the Mersenne sequence and related sequences, including the Lucas–Mersenne, k-Mersenne, repunit, One-kZero, and Gersenne sequences, together with their applications.

In the Leonardo sequence, each term is the sum of the preceding two terms plus one. Thus, this sequence is defined by the recurrence

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \qquad n \geq 2,$$

with initial values $Le_0 = Le_1 = 1$. It is listed as sequence `A001595` in the OEIS [42].

The paper by Catarino and Borges [8] renewed interest in Leonardo sequences. The Leonardo sequence has applications in data structures and algorithm analysis, particularly in the study of balanced trees; see, for example, [16,17,23]. Generalizations and extensions of the Leonardo sequence, together with generating functions and identities, are discussed in [2–5,18,22,29,31,32,37,41,43,47].

Polynomial sequences defined by third-order linear recurrences, including generalized Horadam–Leonardo polynomials, are studied in [1, 36, 39, 45, 46]. These constructions extend and unify several families of third-order sequences. The work [34] investigates the Bronze Leonardo, Bronze Leonardo–Lucas, and modified Bronze Leonardo sequences. Related contributions can be found in [12,14,20,26,30,35,45]. Motivated by these studies, we introduce the Gersenne–Leonardo numbers in Section 2 and investigate their properties.

2. Third-Order Gersenne Sequence

Throughout the paper, unless otherwise stated, we assume that k is an integer greater than 1.

In the present section, we introduce the Gersenne–Leonardo sequence by adding a constant term, inspired by the Leonardo sequence, to the Gersenne recurrence. The resulting nonhomogeneous second-order recurrence can be expressed as a homogeneous third-order recurrence. After defining the sequence and specifying its initial conditions and parameters, we derive a Binet-type formula, generating functions, and algebraic identities that generalize the classical Tagiuri–Vajda, d'Ocagne, Catalan, and Cassini identities. We also examine its connections with Mersenne-type sequences and establish results concerning partial sums and limits.

Definition 2.1. *Let k, a, b, ρ be fixed non-negative integers. The sequence of Gersenne–Leonardo numbers $\{MG_n^{(k,a,b,\rho)}\}_{n\geq 0}$, denoted more briefly by $\{MG_n^{(k)}\}_{n\geq 0}$, is defined by the following recurrence relation:*

$$MG_{n+1}^{(k)} = (k+1)MG_n^{(k)} - kMG_{n-1}^{(k)} + \rho, \quad n \geq 1. \tag{2}$$

The initial values are $MG_0^{(k)} = a$ and $MG_1^{(k)} = b$.

A direct computation using Equation (2) gives the first five terms of the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$:

$$\{a,\ b,\ k(b-a)+b+\rho,\ (k^2+k)(b-a)+(k+2)\rho+b,\ (k^3+k^2+k)(b-a)+(k^2+2k+3)\rho+b,\ \ldots\}.$$

In the special case $\rho = k$, these become

$$\{a,\ b,\ k(b-a+1)+b,\ (k^2+k)(b-a+1)+k+b,\ (k^3+k^2+k)(b-a)+k^3+2k^2+3k+b,\ \ldots\}.$$

In what follows, we illustrate Definition 2.1.

Example 2.2. *(a). Setting $k = 2$, $a = 0$, and $b = \rho = 1$ in Equation (2) yields the modified 2-Mersenne sequence:*

$$\{ML_n\}_{n\geq 0} = \{0, 1, 4, 11, 26, 57, 120, 247, 502, \ldots\}.$$

This is the Gersenne–Leonardo sequence studied in [14] and listed as sequence `A000295` in the OEIS [42].

(b). Setting $k = 10$, $a = 0$, and $b = \rho = 1$ in Equation (2) yields the modified 10-Mersenne sequence, also called the modified repunit–Leonardo sequence:

$$\{MR_n\}_{n\geq 0} = \{0, 1, 12, 123, 1234, 12345, \ldots\}.$$

This sequence is listed as `A014824` in the OEIS [42].

An interesting structural property of the Gersenne–Leonardo sequence is that it satisfies a non-homogeneous second-order linear recurrence relation, given as follows:

$$MG_n^{(k)} = (k+1)MG_{n-1}^{(k)} - kMG_{n-2}^{(k)} + \rho. \tag{3}$$

By subtracting Equations (2) and (3), we obtain the third-order homogeneous recurrence relation, given as follows:

$$MG_{n+1}^{(k)} = (k+2)MG_n^{(k)} - (2k+1)MG_{n-1}^{(k)} + kMG_{n-2}^{(k)}.$$

The preceding discussion leads to the following result.

Proposition 2.3. *The Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$ satisfies the homogeneous recurrence relation*

$$MG_{n+1}^{(k)} = (k+2)MG_n^{(k)} - (2k+1)MG_{n-1}^{(k)} + kMG_{n-2}^{(k)}, \qquad n \geq 2, \tag{4}$$

with initial terms $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a)+b+\rho$.

We now present a relationship between the k-Mersenne–Leonardo sequence and the k-Mersenne sequence; specifically, we consider $a = 0$ and $b = \rho = 1$.

Proposition 2.4. *Let $\{Mv_n^{(k)}\}_{n\geq 0}$ and $\{v_n^{(k)}\}_{n\geq 0}$ denote the k-Mersenne–Leonardo and k-Mersenne sequences, respectively. Then, for every non-negative integer n, it holds that*

$$Mv_{n+1}^{(k)} - Mv_n^{(k)} = v_{n+1}^{(k)}. \tag{5}$$

Proof. We will prove this by induction on n. From the definition of k-Mersenne–Leonardo sequences, we know that $M\upsilon_0 = \upsilon_0 = 0$ and $M\upsilon_1 = \upsilon_1 = 1$. Now, with the aid of Definition 2.1, it is easy to note that $M\upsilon_2^{(k)} - M\upsilon_1^{(k)} = k + 2 - 1 = \upsilon_2^{(k)}$, which shows the validity of the result for $n = 1$. Now, assume that Equation (5) holds at indices $n - 1$ and n for $n \geq 1$. We show that it holds at $n + 1$. By Definition 2.1 and Equation (4), we have

$$\begin{aligned} M\upsilon_{n+2}^{(k)} &= (k+2)M\upsilon_{n+1}^{(k)} - (2k+1)M\upsilon_n^{(k)} + kM\upsilon_{n-1}^{(k)} \\ &= (k+1)\big(M\upsilon_{n+1}^{(k)} - M\upsilon_n^{(k)}\big) - k\big(M\upsilon_n^{(k)} - M\upsilon_{n-1}^{(k)}\big) + M\upsilon_{n+1}^{(k)} \\ &\overset{\text{hyp. ind.}}{=} (k+1)\upsilon_{n+1}^{(k)} - k\upsilon_n^{(k)} + M\upsilon_{n+1}^{(k)}. \end{aligned}$$

The result now follows with the aid of Equation (1). □

2.1. The Binet Formula and Generating Functions

In this subsection, we present the characteristic equation of the Gersenne–Leonardo sequence and provide Binet's formula. Additionally, we present the exponential generating function and the ordinary generating function of the Gersenne–Leonardo sequence.

The characteristic equation associated with the Gersenne–Leonardo sequence, given by Equation (4), is as follows:

$$x^3 - (k+2)x^2 + (2k+1)x - k = 0. \tag{6}$$

Its roots are $x_1 = x_2 = 1$ (a double root), and $x_3 = k$. Based on these roots, the general solution of the recurrence relation is given by

$$MG_n^{(k)} = \frac{ck^n + dn(1)^n + e(1)^n}{(k-1)^2}, \tag{7}$$

where $c, d,$ and e are constants determined by the initial conditions of the sequence.

The Binet-type formula is presented in the next result.

Proposition 2.1.1 (Binet's Formula)**.** *For every non-negative integer n, it holds that*

$$MG_n^{(k)} = \frac{c \cdot k^n + dn + e}{(k-1)^2}, \tag{8}$$

with initial elements $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a) + b + \rho$, where $c = (b-a)(k-1) + \rho$, $d = -\rho(k-1)$, and $e = a(k^2-k) - b(k-1) - \rho$.

Proof. The proof is immediate since the general solution is Equation (7). □

The special case $a = 0$, $b = 1$ and $\rho = 1$ is denoted as

$$MG_n^{(k)} = M\upsilon_n^{(k)} = \frac{k^n - \rho n - 1}{k-1} + \frac{\rho(k^n - 1)}{(k-1)^2}, \tag{9}$$

and is called the generalized Modified k-Mersenne sequence.

Table 2.1 shows some examples of the k-Mersenne–Leonardo sequence.

Our next result presents the ordinary generating function for the Gersenne–Leonardo sequence.

Proposition 2.1.2. *The ordinary generating function for the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$, denoted by $\mathcal{G}(x)$, is*

$$\mathcal{G}(x) = \frac{a + \big(b - (k+2)a\big)x + \big((k+1)a - b + \rho\big)x^2}{1 - (k+2)x + (2k+1)x^2 - kx^3}, \tag{10}$$

with initial values $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a) + b + \rho$.

Table 2.1: Some examples of the k-Mersenne–Leonardo sequence.

k	Sequence	Name	ID catalog [42]
2	$0, 1, 4, 11, 26, 57, 120, 247, 502, \ldots$	Mersenne–Leonardo	A000295
3	$0, 1, 5, 18, 58, 179, 543, 1636, \ldots$	3-Mersenne–Leonardo	A000340
4	$0, 1, 6, 27, 112, 453, 1818, 7279, \ldots$	4-Mersenne–Leonardo	A014825
5	$0, 1, 7, 38, 194, 975, 4881, 24412, \ldots$	5-Mersenne–Leonardo	A014827
6	$0, 1, 8, 51, 310, 1865, 11196, 67183, \ldots$	6-Mersenne–Leonardo	A014829
7	$0, 1, 9, 66, 466, 3267, 22875, 160132, \ldots$	7-Mersenne–Leonardo	A014830
8	$0, 1, 10, 83, 668, 5349, 42798, 342391, \ldots$	8-Mersenne–Leonardo	A014831
9	$0, 1, 11, 102, 922, 8303, 74733, \ldots$	9-Mersenne–Leonardo	A014832
10	$0, 1, 12, 123, 1234, 12345, 123456, \ldots$	Repunit-Leonardo	A014824
11	$0, 1, 13, 146, 1610, 17715, 194871, \ldots$	11-Mersenne–Leonardo	A014881
$\vdots$	$\vdots$	$\vdots$	
k	$0, 1, k+(\rho+1), (k^2+k)+(k+2)\rho+1, \ldots$	k-Mersenne–Leonardo	

Proof. The generating function of the Gersenne–Leonardo sequence is given as follows:

$$\mathcal{G}(x) = \sum_{n=0}^{\infty} MG_n^{(k)} x^n.$$

Using the equations for $-(k+2)x\mathcal{G}(x)$, $(2k+1)x^2\mathcal{G}(x)$ and $-kx^3\mathcal{G}(x)$, we have

$$\begin{aligned}
\mathcal{G}(x) &= MG_0^{(k)} + MG_1^{(k)}x + MG_2^{(k)}x^2 + MG_3^{(k)}x^3 + \cdots + MG_n^{(k)}x^n + \cdots, \\
-(k+2)x\mathcal{G}(x) &= -(k+2)MG_0^{(k)}x - (k+2)MG_1^{(k)}x^2 - (k+2)MG_2^{(k)}x^3 - \cdots - (k+2)MG_n^{(k)}x^{n+1} - \cdots, \\
(2k+1)x^2\mathcal{G}(x) &= (2k+1)MG_0^{(k)}x^2 + (2k+1)MG_1^{(k)}x^3 + (2k+1)MG_2^{(k)}x^4 + \cdots + (2k+1)MG_n^{(k)}x^{n+2} + \cdots, \\
-kx^3\mathcal{G}(x) &= -kMG_0^{(k)}x^3 - kMG_1^{(k)}x^4 - kMG_2^{(k)}x^5 - \cdots - kMG_n^{(k)}x^{n+3} - \cdots.
\end{aligned}$$

By adding both sides of these equations and using Equation (4), we obtain the required result. □

Figure 2.1 illustrates Proposition 2.1.2 by displaying the power series expansion of the generating function for the 2-Mersenne–Leonardo sequence, obtained by setting $k = 2$, $a = 0$, and $b = \rho = 1$ in Equation (10). The expansion was computed using Maxima, a free and open-source computer algebra system for symbolic and numerical computations.

```
(%i44)
gr(x) := (x)/(-2·x^3 + 5·x^2 - 4·x + 1);

taylor(gr(x), x, 0, 30);
```

(%o43) $gr(x) := \dfrac{x}{-2x^3 + 5x^2 + -4x + 1}$

(%o44)/T/ $x + 4x^2 + 11x^3 + 26x^4 + 57x^5 + 120x^6 + 247x^7 + 502x^8 + 1013x^9 + 2036x^{10} + 4083x^{11} + 8178x^{12} + 16369x^{13} + 32752x^{14} + 65519x^{15} + 131054x^{16} + 262125x^{17} + 524268x^{18} + 1048555x^{19} + 2097130x^{20} + 4194281x^{21} + 8388584x^{22} + 16777191x^{23} + 33554406x^{24} + 67108837x^{25} + 134217700x^{26} + 268435427x^{27} + 536870882x^{28} + 1073741793x^{29} + 2147483616x^{30} + \ldots$

Figure 2.1: Power series expansion of the generating function for the 2-Mersenne–Leonardo sequence.

The exponential generating function of a sequence $\{a_n\}_{n\geq 0}$, denoted by $E_{a_n}(x)$, is a power series of the form

$$a_0 + a_1 x + \frac{a_2 x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}.$$

In the next result, we consider the Binet formula (8), and obtain the exponential generating function of the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$.

Proposition 2.1.3. *For all $n \geq 0$, the exponential generating function for the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$, denoted by $\mathcal{E}(x)$, is*

$$\mathcal{E}(x) = \frac{1}{(k-1)^2}\big(c\mathbf{e}^{kx} + (dx+e)\mathbf{e}^{x}\big),$$

with initial terms $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a) + b + \rho$, where $c = (b-a)(k-1) + \rho$, $d = -\rho(k-1)$, and $e = a(k^2-k) - b(k-1) - \rho$.

Proof. We note that

$$\begin{aligned}\mathcal{E}(x) =& \sum_{n=0}^{\infty} \frac{MG_n^{(k)} x^n}{n!} = \sum_{n=0}^{\infty} \frac{ck^n + (dn+e)}{(k-1)^2 n!} x^n \\ =& \sum_{n=0}^{\infty} \frac{ck^n}{(k-1)^2 n!} x^n + \sum_{n=0}^{\infty} \frac{dn+e}{(k-1)^2 n!} x^n \\ =& \frac{1}{(k-1)^2}\big(c\mathbf{e}^{kx} + (dx+e)\mathbf{e}^{x}\big),\end{aligned}$$

as required. □

Now, we derive the corresponding Poisson generating function.

Corollary 2.1.4. *For all $n > 0$, the Poisson generating function for the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$ is*

$$\mathcal{P}(x) = \frac{1}{(k-1)^2}(c\mathbf{e}^{(k-1)x} + dx + e),$$

with initial terms $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a) + b + \rho$, where $c = (b-a)(k-1) + \rho$, $d = -\rho(k-1)$, and $e = a(k^2-k) - b(k-1) - \rho$.

To conclude this subsection, we determine the limit of the ratio

$$\frac{MG_{n+1}^{(k)}}{MG_n^{(k)}},$$

where $MG_n^{(k)}$ is the n-th term of the Gersenne–Leonardo sequence. Again, using Binet's formula (8), we obtain another property of Gersenne–Leonardo sequences $\{MG_n^{(k)}\}_{n\geq 0}$, which is stated in the following proposition.

Proposition 2.1.5. *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. If $c \neq 0$, then*

$$\lim_{n\to\infty} \frac{MG_{n+1}^{(k)}}{MG_n^{(k)}} = k. \tag{11}$$

Proof. We note that

$$\lim_{n\to\infty}\frac{MG_{n+1}^{(k)}}{MG_n^{(k)}}=\lim_{n\to\infty}\frac{c\,k^{n+1}+d(n+1)+e}{c\,k^n+dn+e}$$
$$=\lim_{n\to\infty}\frac{c\,k}{c+\frac{dn+e}{k^n}}+\frac{\frac{d(n+1)+e}{k^n}}{c+\frac{dn+e}{k^n}}=k,$$

because $\lim_{n\to\infty}\frac{n+\rho}{k^n}=0$ for some fixed non-zero integers k and ρ. □

The following result follows from Equation (11) and standard limit laws.

Corollary 2.1.6. *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. If $c\neq 0$, then*

$$\lim_{n\to\infty}\frac{MG_n^{(k)}}{MG_{n+1}^{(k)}}=\frac{1}{k}.$$

2.2. Some Identities

In this subsection, we establish identities for the Gersenne–Leonardo sequence, including extensions of the classical Tagiuri–Vajda, d'Ocagne, Catalan, and Cassini identities to third-order recurrences. We also derive additional identities that describe the algebraic structure of the Gersenne–Leonardo sequence.

The bilinear form associated with the Gersenne–Leonardo sequence is given in the following proposition.

Proposition 2.2.1. *Let h,l,v,t be non-negative integers, where $h+l=v+t$. Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Then,*

$$MG_h^{(k)}MG_l^{(k)}-MG_v^{(k)}MG_t^{(k)}=\frac{ce(k^h+k^l-k^v-k^t)+cd(lk^h+hk^l-vk^t-tk^v)+d^2(hl-vt)}{(k-1)^4},$$

with initial terms $MG_0^{(k)}=a$, $MG_1^{(k)}=b$, and $MG_2^{(k)}=k(b-a)+b+\rho$, where $c=(b-a)(k-1)+\rho$, $d=-\rho(k-1)$, and $e=a(k^2-k)-b(k-1)-\rho$.

Proof. Using Equation (8), we have

$$\begin{aligned}MG_h^{(k)}MG_l^{(k)}&=\frac{ck^h+dh+e}{(k-1)^2}\cdot\frac{ck^l+dl+e}{(k-1)^2}\\&=\frac{c^2k^{h+l}+cdlk^h+cek^h+cdhk^l+d^2hl+deh+cek^l+del+e^2}{(k-1)^4}.\end{aligned}\tag{12}$$

Similarly, we obtain

$$\begin{aligned}MG_v^{(k)}MG_t^{(k)}&=\frac{ck^v+dv+e}{(k-1)^2}\cdot\frac{ck^t+dt+e}{(k-1)^2}\\&=\frac{c^2k^{v+t}+cdtk^v+cek^v+cdvk^t+d^2vt+dev+cek^t+det+e^2}{(k-1)^4}.\end{aligned}\tag{13}$$

By combining Equations (12) and (13), we have

$$MG_h^{(k)}MG_l^{(k)}-MG_v^{(k)}MG_t^{(k)}=\frac{ce(k^h+k^l-k^v-k^t)+cd(lk^h+hk^l-vk^t-tk^v)+d^2(hl-vt)}{(k-1)^4},$$

as required. □

We now recall the Binet-type formula for the Gersenne sequence.

Lemma 2.2.2 (Proposition 1, [40]). *For every non-negative integer n, it holds that*

$$G_n^{(k)} = \frac{ck^n + d}{k-1}, \tag{14}$$

where $G_0^{(k)} = a$, $G_1^{(k)} = b$, $c = b - a$, and $d = ka - b$ with $k > 1$.

The special case $a = 0$ and $b = 1$ is denoted as

$$G_n^{(k,0,1)} = \upsilon_n^{(k)} = \frac{k^n - 1}{k-1}, \tag{15}$$

which represents the general term of the k-Mersenne sequence.

Using Proposition 2.2.1, we obtain the following result.

Theorem 2.2.3 (Tagiuri–Vajda's Identity). *Let m, r, s be non-negative integers, and let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Then,*

$$MG_{m+r}^{(k)} MG_{m+s}^{(k)} - MG_m^{(k)} MG_{m+r+s}^{(k)} = -k^m \frac{(k-1)c(e+dm)\upsilon_r^{(k)}\upsilon_s^{(k)} - cd\big(s\upsilon_r^{(k)} + r\upsilon_s^{(k)}\big)}{(k-1)^3} + \frac{d^2 rs}{(k-1)^4}.$$

with initial terms $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, and $MG_2^{(k)} = k(b-a) + b + \rho$, where $c = (b-a)(k-1) + \rho$, $d = -\rho(k-1)$, and $e = a(k^2-k) - b(k-1) - \rho$.

Proof. By taking $h = m + r$, $l = m + s$, $t = m$ and $v = m + r + s$ in Proposition 2.2.1, we obtain

$$\begin{aligned}
&MG_{m+r}^{(k)} MG_{m+s}^{(k)} - MG_m^{(k)} MG_{m+r+s}^{(k)} \\
=&\frac{ce\big(k^{m+r} + k^{m+s} - k^m - k^{m+r+s}\big) + cdm\big(k^{m+r} + k^{m+s} - k^m - k^{m+r+s}\big) + cds(k^{m+r} - k^m) + cdr(k^{m+s} - k^m) + d^2rs}{(k-1)^4} \\
\overset{(15)}{=}&\frac{-ce(k-1)^2 k^m \upsilon_r^{(k)}\upsilon_s^{(k)} - cdm(k-1)^2 k^m \upsilon_r^{(k)}\upsilon_s^{(k)} + cds(k^{m+r} - k^m) + cdr(k^{m+s} - k^m) + d^2rs}{(k-1)^4} \\
=&-k^m \frac{(k-1)c(e+dm)\upsilon_r^{(k)}\upsilon_s^{(k)} - cd\big(s\upsilon_r^{(k)} + r\upsilon_s^{(k)}\big)}{(k-1)^3} + \frac{d^2rs}{(k-1)^4},
\end{aligned}$$

which proves the desired result. □

The following identities are derived using Theorem 2.2.3.

Proposition 2.2.4 (d'Ocagne's identity). *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Let m and n be non-negative integers with $m \geq n$. Then,*

$$MG_m^{(k)} \cdot MG_{n+1}^{(k)} - MG_n^{(k)} \cdot MG_{m+1}^{(k)} = -ck^n \frac{(k-1)(e+dn)\upsilon_{m-n}^{(k)} - d\big(\upsilon_{m-n}^{(k)} + (m-n)\big)}{(k-1)^3} + \frac{d^2(m-n)}{(k-1)^4},$$

where $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, $c = (b-a)(k-1) + \rho$, $d = -\rho(k-1)$, $e = a(k^2-k) - b(k-1) - \rho$ and $\{\upsilon_m^{(k)}\}_{m\geq 0}$ is the generalized k-Mersenne sequence.

Proof. By taking $m = n$, $r = m - n$ and $s = 1$ in Theorem 2.2.3, we have

$$MG_m^{(k)} \cdot MG_{n+1}^{(k)} - MG_n^{(k)} \cdot MG_{m+1}^{(k)} = -ck^n \frac{(k-1)(e+dn)\upsilon_{m-n}^{(k)}\upsilon_1^{(k)} - d\big(\upsilon_{m-n}^{(k)} + (m-n)\upsilon_1^{(k)}\big)}{(k-1)^3} + \frac{d^2(m-n)}{(k-1)^4}.$$

Since $\upsilon_1^{(k)} = 1$, we obtain the desired result. □

We now recall the following result concerning the family of Mersenne-type sequences.

Lemma 2.2.5 (Proposition 2, [11]). *Let k, m, n, and s be arbitrary non-negative integers. Then,*

$$\left(G_{m+n}^{(k)} + G_{m+s}^{(k)}\right) - \left(G_m^{(k)} + G_{m+n+s}^{(k)}\right) = -f(k-1)\cdot k^m \upsilon_n^{(k)} \upsilon_s^{(k)},$$

where $f = b - a$, and $\{G_n^{(k)}\}_{n\geq 0}$ and $\{\upsilon_n^{(k)}\}_{n\geq 0}$ denote the Gersenne and k-Mersenne sequences, respectively.

The k-Mersenne sequence $\{\upsilon_n^{(k)}\}_{n\geq 0}$ can be extended to negative indices:

$$\upsilon_{-n}^{(k)} = -\frac{\upsilon_n^{(k)}}{k^n}, \quad n \geq 1. \tag{16}$$

Similar to Proposition 2.2.4, we have the Catalan identity given in the following proposition.

Proposition 2.2.6 (Catalan's identity). *Let $\{MG_n^{(k)}\}_{n\geq 0}$ denote the Gersenne–Leonardo sequence. Let n and r be non-negative integers such that $n \geq r$. Then,*

$$\left(MG_n^{(k)}\right)^2 - MG_{n+r}^{(k)} MG_{n-r}^{(k)} = -ck^{n-r}\frac{(k-1)\big(e+d(n-r)\big)\left(\upsilon_r^{(k)}\right)^2 - 2dr\,\upsilon_r^{(k)}}{(k-1)^3} + \frac{d^2r^2}{(k-1)^4},$$

where $MG_0^{(k)} = a$, $MG_1^{(k)} = b$, $c = (b-a)(k-1)+\rho$, $d = -\rho(k-1)$, $e = a(k^2-k) - b(k-1) - \rho$ and $\{\upsilon_m^{(k)}\}_{m\geq 0}$ is the generalized k-Mersenne sequence.

Proof. Taking $m = n - r$ and $s = r$ in Theorem 2.2.3, and using Equation (16), we have

$$\begin{aligned}\left(MG_n^{(k)}\right)^2 - MG_{n+r}^{(k)}\cdot MG_{n-r}^{(k)} &= \left(MG_n^{(k)}\right)^2 - MG_{n+r}^{(k)} MG_{n-r}^{(k)} \\ &= -ck^{n-r}\frac{(k-1)\big(e+d(n-r)\big)\left(\upsilon_r^{(k)}\right)^2 - 2dr\upsilon_r^{(k)}}{(k-1)^3} + \frac{d^2r^2}{(k-1)^4},\end{aligned}$$

as required. □

Applying the Catalan identity with $r = 1$ yields the following result.

Corollary 2.2.7 (Cassini's identity). *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Then, for every positive integer n, it holds that*

$$\left(MG_n^{(k)}\right)^2 - MG_{n+1}^{(k)} MG_{n-1}^{(k)} = -ck^{n-1}\frac{(k-1)\big(e+d(n-1)\big) - 2d}{(k-1)^3} + \frac{d^2}{(k-1)^4}.$$

2.3. Sum of Terms Involving the Gersenne–Leonardo Sequence

In this subsection, we study the partial sums of the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$. For each non-negative integer n, the corresponding partial sum is

$$\sum_{i=0}^{n} MG_i^{(k)} = MG_0^{(k)} + MG_1^{(k)} + \cdots + MG_n^{(k)}.$$

Proposition 2.3.1. *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Let $\{\upsilon_n^{(k)}\}_{n\geq 0}$ be the k-Mersenne sequence. Then, the following identities hold:*

(a). $\displaystyle\sum_{i=0}^{n} MG_i^{(k)} = \frac{c\upsilon_{n+1}^{(k)} + e(n+1)}{(k-1)^2} + \frac{d(n+1)n}{2(k-1)^2},$

(b). $\displaystyle\sum_{i=0}^{n} MG_{2i}^{(k)} = \frac{c}{(k+1)(k-1)^2}\upsilon_{2n+2}^{(k)} + \frac{(dn+e)(n+1)}{(k-1)^2},$

(c). $\displaystyle\sum_{i=0}^{n} MG_{2i+1}^{(k)} = \frac{ck}{(k+1)(k-1)^2}\upsilon_{2n+2}^{(k)} + \frac{(dn+d+e)(n+1)}{(k-1)^2}.$

Proof. (a). Using the definition of partial sums of the Gersenne–Leonardo sequence and employing Equation (8), we obtain

$$\begin{aligned}\sum_{i=0}^{n} MG_i^{(k)} &= MG_0^{(k)} + MG_1^{(k)} + MG_2^{(k)} + \cdots + MG_n^{(k)}\\ &= \frac{(ck^0 + d\cdot 0 + e) + (ck^1 + d + e) + \cdots + (ck^n + dn + e)}{(k-1)^2}\\ &= \frac{c(k^0 + k^1 + \cdots + k^n) + d(1+2+\cdots+n) + e(n+1)}{(k-1)^2}\\ &= \frac{c\left(\frac{k^{n+1}-1}{k-1}\right) + d\left(\frac{n(n+1)}{2}\right) + e(n+1)}{(k-1)^2}.\end{aligned}$$

Now, by applying Equation (15), we obtain

$$\sum_{i=0}^{n} MG_i^{(k)} = \frac{c\upsilon_{n+1}^{(k)} + e(n+1)}{(k-1)^2} + \frac{d(n+1)n}{2(k-1)^2},$$

which establishes the desired result.

(b). The sum of the even-indexed terms of the Gersenne–Leonardo sequence is given as follows:

$$\begin{aligned}\sum_{i=0}^{n} MG_{2i}^{(k)} &= MG_0^{(k)} + MG_2^{(k)} + MG_4^{(k)} + \cdots + MG_{2n}^{(k)}\\ &= \frac{(ck^0 + d\cdot 0 + e) + (ck^2 + 2d + e) + \cdots + (ck^{2n} + 2nd + e)}{(k-1)^2}\\ &= \frac{c(k^0 + k^2 + \cdots + k^{2n}) + d(0+2+\cdots+2n) + e(n+1)}{(k-1)^2}\\ &= \frac{c\left(\frac{k^{2(n+1)}-1}{k^2-1}\right) + dn(n+1) + e(n+1)}{(k-1)^2},\end{aligned}$$

which together with Equation (15) yields the desired identity.

(c). Here, we have

$$\begin{aligned}\sum_{i=0}^{n} MG_{2i+1}^{(k)} =& MG_1^{(k)} + MG_3^{(k)} + MG_5^{(k)} + \cdots + MG_{2n+1}^{(k)} \\ =& \frac{(ck + d\cdot 1 + e) + (ck^3 + 3d + e) + \cdots + (ck^{2n+1} + (2n+1)d + e)}{(k-1)^2} \\ =& \frac{c(k + k^3 + \cdots + k^{2n+1}) + d(1 + 3 + \cdots + (2n+1)) + e(n+1)}{(k-1)^2} \\ =& \frac{ck\left(\frac{k^{2(n+1)}-1}{k^2-1}\right) + d(n+1)^2 + e(n+1)}{(k-1)^2}.\end{aligned}$$

Now, an argument analogous to that used in item (b) yields the required result. □

The next result is a direct consequence of Proposition 2.3.1.

Proposition 2.3.2. *Let $\{MG_n^{(k)}\}_{n\geq 0}$ be the Gersenne–Leonardo sequence. Then, the following identities hold:*

(a). $\displaystyle\sum_{i=0}^{2n+1}(-1)^i MG_i^{(k)} = -\frac{c}{(k^2-1)}\upsilon_{2n+2}^{(k)} - \frac{d(n+1)}{(k-1)^2},$

(b). $\displaystyle\sum_{i=0}^{2n}(-1)^i MG_i^{(k)} = \frac{c}{(k-1)(k^2-1)}\left(\upsilon_{2n+2}^{(k)} - k\upsilon_{2n}^{(k)}\right) + \frac{dn+e}{(k-1)^2}.$

Proof. (a). Since $2n+1$ is an odd natural number, the final term has a minus sign; thus,

$$\begin{aligned}\sum_{i=0}^{2n+1}(-1)^i MG_i^{(k)} =& MG_0^{(k)} - MG_1^{(k)} + MG_2^{(k)} - MG_3^{(k)} + \cdots + MG_{2n}^{(k)} - MG_{2n+1}^{(k)} \\ =& (MG_0^{(k)} + MG_2^{(k)} + \cdots + MG_{2n}^{(k)}) - (MG_1^{(k)} + MG_3^{(k)} + \cdots + MG_{2n+1}^{(k)}) \\ =& \sum_{i=0}^{n} MG_{2i}^{(k)} - \sum_{i=0}^{n} MG_{2i+1}^{(k)}.\end{aligned}$$

Now, using items (b) and (c) of Proposition 2.3.1, we have

$$\sum_{i=0}^{2n+1}(-1)^i MG_i^{(k)} = -\frac{c}{(k^2-1)}\upsilon_{2n+2}^{(k)} - \frac{d(n+1)}{(k-1)^2}.$$

(b). The proof is analogous to that of item (a). □

For a sequence $\{a_n\}_{n\geq 0}$, we define the backward difference operator Δ by

$$\Delta a_n = a_n - a_{n-1}, \qquad n \geq 1.$$

Let $\{S_n\}_{n\geq 0}$ be the sequence of partial sums of the Gersenne–Leonardo sequence $\{MG_n^{(k)}\}_{n\geq 0}$, defined by

$$S_n = \sum_{j=0}^{n} MG_j^{(k)}, \qquad n \geq 0.$$

Proposition 2.3.3. *Let* $\{MG_n^{(k)}\}_{n\geq 0}$ *be the Gersenne–Leonardo sequence and* $S_n = \sum_{j=0}^{n} MG_j^{(k)}$. *Then, the following identities hold:*

(a). $\Delta S_n = \dfrac{c\cdot k^n + dn + e}{(k-1)^2}$ *for* $n \geq 1$,

(b). $\Delta^2 S_n = \dfrac{ck^{n-1}}{k-1} + \dfrac{d}{(k-1)^2}$ *for* $n \geq 2$,

(c). $\Delta^3 S_n = ck^{n-2}$ *for* $n \geq 3$.

Proof. (a). We note that

$$\Delta S_n = S_n - S_{n-1} \ = \sum_{j=0}^{n} MG_j^{(k)} - \sum_{j=0}^{n-1} MG_j^{(k)} = MG_n^{(k)},$$

which together with (8) yields the required identity.

(b). It follows from the definition of Δ and item (a).

(c). This identity is obtained by a straightforward calculation. □

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