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Research Article

Laplacian Spectra and von Neumann Entropy of Generalized Splitting and Shadow-Splitting Graphs

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Abstract

Graph operations provide a setting in which structural changes can be related to spectral invariants. We study the two-parameter generalized splitting graphs $S_{p,q}(G)$ and shadow-splitting graphs $H_{p,q}(G)$. Our main result is an orthogonal reduction of their Laplacian matrices, valid for arbitrary connected base graphs, into repeated blocks and a coupled block of order $2|V(G)|$. For a connected regular base graph, this reduction yields the complete Laplacian spectra, with all multiplicity contributions specified. The corresponding von Neumann entropy formulae follow directly from these spectra and the edge counts. The case $p = 1$ recovers the known one-parameter splitting setting; our contribution lies in the common Laplacian reduction and its two-parameter spectral consequences. We also provide examples that illustrate the case $p = 1$ and verify the spectral and entropy formulae for both constructions based on the cycle of order four.

Keywords: graph operation; Laplacian spectrum; von Neumann entropy; generalized splitting graph; shadow-splitting graph.

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1. Introduction

Graph operations are fundamental tools for constructing and analysing graphs. By combining or modifying given graphs according to prescribed rules, they produce families whose structural properties can be related to those of their constituents. In spectral graph theory, they lead to a natural question: how do the eigenvalues change under a specified construction? Explicit answers connect combinatorial transformations with matrix structure and allow spectral invariants of larger graphs to be derived from information about smaller ones [3, 8]. Cvetković [7] investigated spectra of graphs formed by unary operations, and the monograph of Cvetković, Rowlinson, and Simić [8] develops this perspective systematically. The survey of Barik et al. [3] describes further spectral results for graph operations and products.

Graph operations are also relevant to network science [15]. Cvetković and Simić [9] surveyed applications of graph spectra to internet modeling, data analysis, and multiprocessor systems. These connections motivate the study of controlled graph constructions in which the effect of replication and interconnection on global spectral descriptors can be evaluated exactly. The constructions considered here are mathematical models for such an investigation; no empirical network model is assumed.

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Throughout the paper, graphs are finite, simple, and undirected. Unless otherwise stated, the underlying graph G is connected, has order $n \geq 2$ and size m, and has vertex set $V(G) = \{v_1, \ldots, v_n\}$. Write $d_i = d_G(v_i)$, and let $A(G)$ and $D(G) = \operatorname{diag}(d_1, \ldots, d_n)$ be its adjacency matrix and degree matrix. The combinatorial Laplacian is

$$L(G) = D(G) - A(G).$$

Its eigenvalues are ordered as

$$\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) > \mu_n(G) = 0.$$

The largest eigenvalue is the Laplacian spectral radius, and the smallest positive eigenvalue is the algebraic connectivity introduced by Fiedler [13]. Standard spectral graph terminology is as in [8].

The von Neumann entropy is a basic quantity in quantum information theory [22]. Braunstein, Ghosh, and Severini [5] developed a graph-theoretic framework in which the trace-normalized Laplacian is interpreted as a density matrix. Since $\operatorname{tr} L(G) = 2m$, this matrix is

$$\rho_G = \frac{L(G)}{2m}.$$

The associated entropy is

$$\begin{aligned} s(G) &= -\operatorname{tr}\bigl(\rho_G \log_2 \rho_G\bigr) \\ &= -\sum_{i=1}^{n-1} \frac{\mu_i(G)}{2m} \log_2 \frac{\mu_i(G)}{2m} \\ &= \log_2(2m) - \frac{1}{2m} \sum_{i=1}^{n-1} \mu_i(G) \log_2 \mu_i(G), \end{aligned} \tag{1}$$

where $0 \log_2 0 = 0$. We use the trace-normalized combinatorial Laplacian, not the entropy construction based on $D(G)^{-1/2} L(G) D(G)^{-1/2}$; the distinction between the two variants is discussed in [21].

Graph von Neumann entropy has been studied as a spectral descriptor of network complexity [21, 23]. Its mathematical investigation includes work on von Neumann and Rényi entropies and graph operations [10], random graphs [11], random multipartite graphs [14], and sharp graph-parameter bounds [18]. Unlike an invariant determined by a single extreme eigenvalue, entropy depends nonlinearly on the entire normalized Laplacian spectrum. Understanding its behaviour under a graph operation therefore requires both the transformed eigenvalues and the resulting number of edges. This makes exact spectral descriptions particularly useful.

Splitting and shadow constructions enlarge a graph by replicating its neighbourhood structure. The one-parameter k-splitting construction was introduced by Abdel-Aal [1]. Joseph and Dominic [16] use the term "generalized splitting graph" for this construction and study its topological invariants together with those of k-shadow graphs. Earlier work also considered graph energies and related spectral quantities for splitting and shadow operations [4, 6, 25]. Lu, Ma, and Zhang [19] determined the adjacency, Laplacian, and signless Laplacian spectra of two splitting-based operations and applied their results to cospectral constructions, the Kirchhoff index, and the number of spanning trees. Dudhat, Kaneria, and Popat [12] introduced the *two-parameter extension* $S_{p,q}(G)$ and the shadow-splitting construction, and derived their adjacency energies. We use the same parameters p, q for both operations: our $H_{p,q}(G)$ is their $\mathcal{H}_{c,k}(G)$ with $c = p$ and $k = q$.

The closest Laplacian results concern the one-parameter constructions. Lin and Cai [17] studied Laplacian spectral metrics, including Laplacian entropy, for one-parameter splitting and shadow graphs. In the present notation, $S_{1,q}(G)$ is exactly the q-splitting graph. Accordingly, the $p = 1$ cases of Theorem 3.1 and Corollary 3.2 recover that earlier splitting setting and are not claimed as new results. Barad and Kaneria [2] also determined the complete Laplacian and signless Laplacian spectra of m-splitting graphs of regular graphs, together with the corresponding energies. Their splitting parameter m corresponds to q here, with $p = 1$. Section 3.1 records the common specialized spectrum explicitly. The ordinary one-parameter shadow graph is related to $H_{p,q}(G)$ by deleting its q independent splitting layers; it corresponds to the formal boundary $q = 0$, outside the parameter range of our theorems.

A recent work has also investigated other invariants of the same two-parameter operations. Makadiya, Jariya, and Makadiya [20] determined their Wiener and Harary indices, while Popat and Bawani [24] studied degree-based matrices and the associated graph energies. These investigations concern distance-based invariants and degree-weighted adjacency matrices, respectively, and provide complementary accounts of the two constructions.

The principal new step of the present paper is the orthogonal Laplacian reduction in Theorem 2.4, which is valid without a regularity assumption on G. It is not a direct consequence of the adjacency-energy formulae in [12]: the original and added vertex classes contribute the different degree blocks $(q + 1)D$ and pD for $S_{p,q}(G)$, and $(p + q)D$ and pD for $H_{p,q}(G)$. These blocks must be retained in the Laplacian reduction. The theorem separates the copy-index subspaces from a coupled block of order $2n$. For a connected r-regular graph, the latter block further decomposes into 2×2 matrices indexed by the adjacency eigenvalues

$$r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -r.$$

This gives the complete two-parameter Laplacian spectra in Theorems 3.1 and 3.3. The entropy formulae in Corollaries 3.2 and 3.4 then follow immediately from the spectra and the edge counts. Thus the structural reduction and the complete spectra are the main contributions, with entropy as a direct application.

2. Graph Constructions and Orthogonal Reductions

We recall the constructions of [12], using a common pair of parameters throughout.

Definition 2.1 (Generalized splitting graph)**.** *Let $p, q \geq 1$ be integers. Take p vertex-disjoint copies $G^{(1)}, \ldots, G^{(p)}$ of a graph G, and denote the copy of v_i in $G^{(\ell)}$ by $v_i^{(\ell)}$. Add q pairwise disjoint independent sets*

$$U^{(k)} = \{u_1^{(k)}, \ldots, u_n^{(k)}\}, \qquad 1 \leq k \leq q.$$

Retain the edges inside each $G^{(\ell)}$, and join $u_i^{(k)}$ to $v_j^{(\ell)}$ if and only if $v_i v_j \in E(G)$. No other edges are added. The resulting graph is the (p, q)-generalized splitting graph $S_{p,q}(G)$. Equivalently,

$$N_{S_{p,q}(G)}(u_i^{(k)}) = \bigcup_{v_j \in N_G(v_i)} \{v_j^{(1)}, \ldots, v_j^{(p)}\}.$$

Definition 2.2 (Shadow-splitting graph)**.** *For integers $p, q \geq 1$, the (p, q)-shadow-splitting graph $H_{p,q}(G)$ is the graph denoted by $\mathcal{H}_{c,k}(G)$ in [12], with $c = p$ and $k = q$. It is obtained from $S_{p,q}(G)$ as follows. For each unordered edge $v_i v_j \in E(G)$ and each pair $1 \leq \ell < h \leq p$, add the two edges*

$$\{v_i^{(\ell)}, v_j^{(h)}\} \quad \textit{and} \quad \{v_j^{(\ell)}, v_i^{(h)}\}.$$

Each unordered edge is added once, and no multiple edges are introduced. Thus the neighbourhood of each $u_i^{(k)}$ *is as in Definition 2.1, whereas*

$$N_{H_{p,q}(G)}(v_i^{(\ell)}) = \bigcup_{v_j \in N_G(v_i)} \left(\{v_j^{(1)}, \dots, v_j^{(p)}\} \cup \{u_j^{(1)}, \dots, u_j^{(q)}\} \right),$$

for $1 \le i \le n$ *and* $1 \le \ell \le p$.

In particular, $S_{p,q}(G)$ is a spanning subgraph of $H_{p,q}(G)$ under the natural identification of vertices. For $p = 1$, the two constructions coincide. Figure 2.1 illustrates their layer structure when $p = q = 2$ and $G = C_4$.

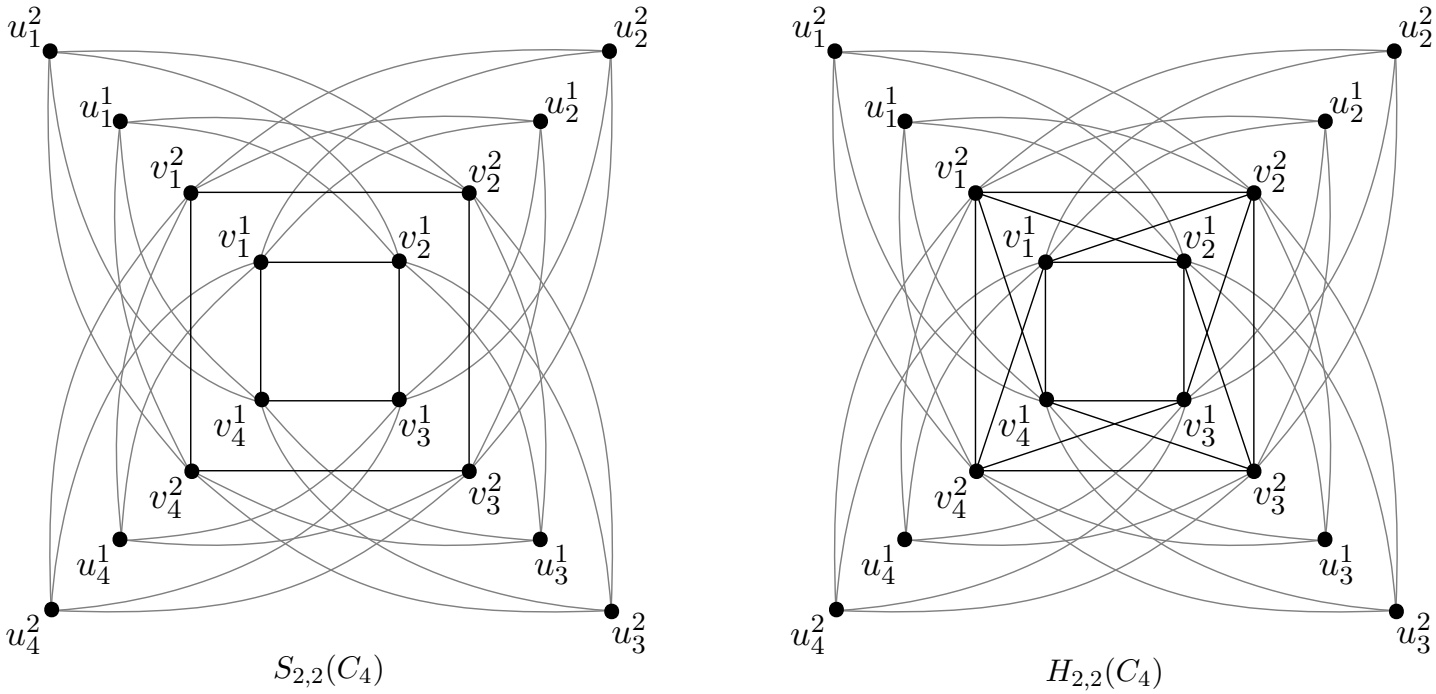


Figure 2.1: The graphs $S_{2,2}(C_4)$ and $H_{2,2}(C_4)$.

Lemma 2.3. *Let* G *be connected of order* $n \ge 2$ *and size* m*, and let* $p, q \ge 1$*. Both* $S_{p,q}(G)$ *and* $H_{p,q}(G)$ *are connected and have order* $N = (p+q)n$*. Their vertex degrees are*

$$d_{S_{p,q}(G)}(v_i^{(\ell)}) = (q+1)d_i, \quad d_{S_{p,q}(G)}(u_i^{(k)}) = pd_i, \quad d_{H_{p,q}(G)}(v_i^{(\ell)}) = (p+q)d_i, \quad d_{H_{p,q}(G)}(u_i^{(k)}) = pd_i.$$

Consequently,

$$|E(S_{p,q}(G))| = p(2q+1)m, \qquad |E(H_{p,q}(G))| = p(p+2q)m. \tag{2}$$

Proof. The vertex counts and degree formulae follow from the definitions. The handshaking identity gives (2). Since G is connected and $n \ge 2$, every vertex of G has a neighbour. Each added vertex therefore has a neighbour in every original copy. Each copy is connected, and $q \ge 1$, so all copies and all added vertices lie in one component of $S_{p,q}(G)$. The graph $H_{p,q}(G)$ is a spanning supergraph of this connected graph and is therefore connected as well. □

Write $A = A(G)$ and $D = D(G)$. Let I_k be the identity matrix of order k, let $J_{p\times q}$ be the all-ones matrix of size $p \times q$, and put $J_p = J_{p\times p}$. The symbol $\otimes$ denotes the Kronecker product. With the original layers followed by the added layers, the two Laplacian matrices are

$$L(S_{p,q}(G)) = \begin{pmatrix} I_p \otimes ((q+1)D - A) & -J_{p\times q} \otimes A \\ -J_{q\times p} \otimes A & I_q \otimes pD \end{pmatrix} \tag{3}$$

and

$$L(H_{p,q}(G)) = \begin{pmatrix} I_p \otimes ((p+q)D) - J_p \otimes A & -J_{p\times q} \otimes A \\ -J_{q\times p} \otimes A & I_q \otimes pD \end{pmatrix}. \tag{4}$$

Theorem 2.4. *Let G be a connected graph of order $n \geq 2$, and let $p, q \geq 1$. Define*

$$\mathcal{B}_S = \begin{pmatrix} (q+1)D - A & -\sqrt{pq}\,A \\ -\sqrt{pq}\,A & pD \end{pmatrix} \quad \textit{and} \quad \mathcal{B}_H = \begin{pmatrix} (p+q)D - pA & -\sqrt{pq}\,A \\ -\sqrt{pq}\,A & pD \end{pmatrix}.$$

Then

$$L(S_{p,q}(G)) \sim_{\text{orth}} \big(I_{p-1} \otimes ((q+1)D - A)\big) \oplus \big(I_{q-1} \otimes pD\big) \oplus \mathcal{B}_S, \tag{5}$$

$$L(H_{p,q}(G)) \sim_{\text{orth}} \big(I_{p-1} \otimes ((p+q)D)\big) \oplus \big(I_{q-1} \otimes pD\big) \oplus \mathcal{B}_H. \tag{6}$$

A direct-sum term is omitted when its copy multiplicity is zero. In particular, the remaining coupled block has order $2n$ in both cases.

Proof. Choose orthogonal matrices $Q_p \in \mathbb{R}^{p\times p}$ and $Q_q \in \mathbb{R}^{q\times q}$ whose first columns are $\mathbf{1}_p/\sqrt{p}$ and $\mathbf{1}_q/\sqrt{q}$, respectively. If $e_1^{(k)}$ is the first standard basis vector of $\mathbb{R}^k$, then

$$Q_p^T J_p Q_p = p\, e_1^{(p)} (e_1^{(p)})^T \quad \text{and} \quad Q_p^T J_{p\times q} Q_q = \sqrt{pq}\, e_1^{(p)} (e_1^{(q)})^T.$$

Set

$$U_p = Q_p \otimes I_n, \qquad U_q = Q_q \otimes I_n, \qquad U = \operatorname{diag}(U_p, U_q).$$

The orders of U_p, U_q, U are $pn, qn, (p+q)n$, respectively. Write a Laplacian in (3) or (4) as

$$L = \begin{pmatrix} W & B \\ B^T & F \end{pmatrix}.$$

The orthogonal conjugate of L is

$$U^T L U = \begin{pmatrix} U_p^T W U_p & U_p^T B U_q \\ U_q^T B^T U_p & U_q^T F U_q \end{pmatrix}.$$

In particular, the transformed off-diagonal block is

$$U_p^T B U_q = -\sqrt{pq}\, e_1^{(p)} (e_1^{(q)})^T \otimes A.$$

Thus only the first copy-index coordinate from each class remains coupled.

For $S_{p,q}(G)$, each of the other $p-1$ original-layer coordinates contributes the block $(q+1)D - A$, and each of the other $q-1$ added-layer coordinates contributes pD. The two first coordinates give $\mathcal{B}_S$.

For $H_{p,q}(G)$, the term involving J_p acts only on the first original-layer coordinate. On that coordinate the diagonal block is $(p+q)D - pA$; on its orthogonal complement it is $(p+q)D$. The added-layer contributions are unchanged, and the two first coordinates give $\mathcal{B}_H$. A simultaneous permutation of rows and columns groups the uncoupled coordinates first and the two coupled coordinates last, yielding (5) and (6). □

3. Complete Laplacian Spectra and Entropy Formulae

In this section, G is a connected r-regular graph of order $n \geq 2$, so $r \geq 1$. Let $r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n \geq -r$ be its adjacency eigenvalues, repeated according to multiplicity. All spectrum lists below are multisets: contributions from different rows and indices are added whenever their numerical values coincide. Rows with zero multiplicity contribute no eigenvalues.

Theorem 3.1. *Let G be a connected r-regular graph of order $n \geq 2$, with adjacency eigenvalues*

$$r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n.$$

Let $p, q \geq 1$ be integers and define, for $1 \leq i \leq n$,

$$\alpha_i = (q+1)r - \lambda_i,$$
$$\beta_i^{\pm} = \frac{\alpha_i + pr \pm \sqrt{(\alpha_i - pr)^2 + 4pq\lambda_i^2}}{2}. \tag{7}$$

The complete Laplacian spectrum of $S_{p,q}(G)$ is given in the following table:

Eigenvalue	*Multiplicity contribution*
α_i, *for each* $1 \leq i \leq n$	$p-1$ *for each index* i
pr	$n(q-1)$
β_i^+, *for each* $1 \leq i \leq n$	1 *for each index* i
β_i^-, *for each* $1 \leq i \leq n$	1 *for each index* i

Moreover,

$$\alpha_1 = qr, \qquad \beta_1^+ = (p+q)r, \qquad \beta_1^- = 0,$$

and every eigenvalue in the list other than β_1^- is positive.

Proof. Choose an orthogonal matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^T A P = \Lambda := \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Since $D = rI_n$, the first term of (5) contributes α_i with multiplicity $p-1$ for each i, and the second contributes pr with multiplicity $n(q-1)$.

For the remaining $2n$-dimensional block, set $\widehat{P} = \operatorname{diag}(P, P)$. Then

$$\widehat{P}^T \mathcal{B}_S \widehat{P} = \begin{pmatrix} (q+1)rI_n - \Lambda & -\sqrt{pq}\,\Lambda \\ -\sqrt{pq}\,\Lambda & prI_n \end{pmatrix}.$$

After a simultaneous permutation of rows and columns, grouping the two coordinates associated with each λ_i, this matrix is the direct sum of

$$C_i^S = \begin{pmatrix} \alpha_i & -\sqrt{pq}\,\lambda_i \\ -\sqrt{pq}\,\lambda_i & pr \end{pmatrix}, \qquad 1 \leq i \leq n.$$

The characteristic polynomial of C_i^S is $t^2 - (\alpha_i + pr)t + \alpha_i pr - pq\lambda_i^2$, whose roots are (7). The multiplicity contributions sum to

$$n(p-1) + n(q-1) + 2n = (p+q)n.$$

Substitution of $\lambda_1 = r$ gives the three specified values. To verify the positivity statement explicitly, note that $\alpha_i \geq qr > 0$ and

$$\beta_i^+ \beta_i^- = p(r - \lambda_i)\big((q+1)r + q\lambda_i\big).$$

For $i \geq 2$, the first factor $r - \lambda_i$ is positive, while $(q+1)r + q\lambda_i \geq r > 0$. Both roots are real and have positive sum, so both are positive. This also covers bipartite underlying graphs, for which $\lambda_n = -r$. □

Corollary 3.2. *Under the hypotheses and notation of Theorem 3.1,*

$$s(S_{p,q}(G)) = \log_2\big(pnr(2q+1)\big) - \frac{q-1}{2q+1}\log_2(pr) - \frac{(p-1)q}{pn(2q+1)}\log_2(qr) - \frac{p+q}{pn(2q+1)}\log_2\big((p+q)r\big)$$
$$- \frac{1}{pnr(2q+1)}\sum_{i=2}^{n}\big((p-1)\alpha_i \log_2 \alpha_i + \beta_i^+ \log_2 \beta_i^+ + \beta_i^- \log_2 \beta_i^-\big). \tag{8}$$

Proof. Put $f(x) = x\log_2 x$ for $x > 0$, with $f(0) = 0$, and $T_S = 2|E(S_{p,q}(G))|$. By Lemma 2.3 and $2m = nr$,

$$T_S = pnr(2q+1).$$

The spectrum in Theorem 3.1 and (1) give

$$s(S_{p,q}(G)) = \log_2 T_S - \frac{1}{T_S}\Big[n(q-1)f(pr) + \sum_{i=1}^{n}\big((p-1)f(\alpha_i) + f(\beta_i^+) + f(\beta_i^-)\big)\Big].$$

Separating the $i = 1$ terms and using $\alpha_1 = qr$, $\beta_1^+ = (p+q)r$, and $f(\beta_1^-) = 0$ yields (8). □

Theorem 3.3. *Let G be a connected r-regular graph of order $n \geq 2$, with adjacency eigenvalues*

$$r = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_n.$$

Let $p, q \geq 1$ be integers and define, for $1 \leq i \leq n$,

$$\eta_i^{\pm} = \frac{(2p+q)r - p\lambda_i \pm \sqrt{(qr - p\lambda_i)^2 + 4pq\lambda_i^2}}{2}. \tag{9}$$

The complete Laplacian spectrum of $H_{p,q}(G)$ is given in the following table:

Eigenvalue	*Multiplicity contribution*
$(p+q)r$	$n(p-1)$
pr	$n(q-1)$
η_i^+, *for each* $1 \leq i \leq n$	1 *for each index* i
η_i^-, *for each* $1 \leq i \leq n$	1 *for each index* i

Moreover,

$$\eta_1^+ = (p+q)r, \qquad \eta_1^- = 0,$$

and all the other listed eigenvalues are positive. In particular, the contribution $n(p-1)$ in the first row does not include the additional occurrence $\eta_1^+ = (p+q)r$.

Proof. Since $D = rI_n$, the first two terms of (6) contribute $(p+q)r$ with multiplicity $n(p-1)$ and pr with multiplicity $n(q-1)$. Diagonalizing A by the matrix P used above gives

$$\widehat{P}^T \mathcal{B}_H \widehat{P} = \begin{pmatrix} (p+q)rI_n - p\Lambda & -\sqrt{pq}\,\Lambda \\ -\sqrt{pq}\,\Lambda & prI_n \end{pmatrix}.$$

A simultaneous permutation of rows and columns reduces it to the direct sum of

$$C_i^H = \begin{pmatrix} (p+q)r - p\lambda_i & -\sqrt{pq}\,\lambda_i \\ -\sqrt{pq}\,\lambda_i & pr \end{pmatrix}, \qquad 1 \le i \le n.$$

The characteristic equation is

$$\big((p+q)r - p\lambda_i - t\big)(pr - t) - pq\lambda_i^2 = 0,$$

which has the two roots (9). Again the multiplicity contributions total $(p+q)n$.

At $i = 1$, these roots are $(p+q)r$ and 0. For $i \ge 2$,

$$\eta_i^+ \eta_i^- = p(r - \lambda_i)\big((p+q)r + q\lambda_i\big) > 0,$$

because $(p+q)r + q\lambda_i \ge pr > 0$. Their sum is

$$(2p+q)r - p\lambda_i \ge (p+q)r > 0.$$

Both roots are therefore positive. The repeated eigenvalues are positive as well. □

Corollary 3.4. *Under the hypotheses and notation of Theorem 3.3,*

$$\begin{aligned} s(H_{p,q}(G)) = \log_2\big(pnr(p+2q)\big) - \frac{(p-1)(p+q)}{p(p+2q)}\log_2\big((p+q)r\big) - \frac{q-1}{p+2q}\log_2(pr) \\ - \frac{1}{pnr(p+2q)}\Big[(p+q)r\log_2\big((p+q)r\big) + \sum_{i=2}^{n}\big(\eta_i^+\log_2\eta_i^+ + \eta_i^-\log_2\eta_i^-\big)\Big]. \end{aligned} \tag{10}$$

Proof. Here $T_H = 2|E(H_{p,q}(G))| = pnr(p+2q)$. Applying (1) to the multiset in Theorem 3.3 gives

$$s(H_{p,q}(G)) = \log_2 T_H - \frac{1}{T_H}\Big[n(p-1)f\big((p+q)r\big) + n(q-1)f(pr) + \sum_{i=1}^{n}\big(f(\eta_i^+) + f(\eta_i^-)\big)\Big],$$

where $f(x) = x\log_2 x$ and $f(0) = 0$. Separating $\eta_1^+ = (p+q)r$ and $\eta_1^- = 0$ yields (10). Coincident eigenvalues cause no difficulty because the corresponding multiset contributions are added. □

3.1. Examples

Example 3.1.1. When $p = 1$, there is only one original layer, so Definition 2.2 adds no edges. Hence

$$H_{1,q}(G) = S_{1,q}(G),$$

and this common graph is the q-splitting graph. For a connected r-regular base graph, both Theorems 3.1 and 3.3 give the eigenvalue r with multiplicity contribution $n(q-1)$, together with

$$\gamma_i^{\pm} = \frac{(q+2)r - \lambda_i \pm \sqrt{(qr - \lambda_i)^2 + 4q\lambda_i^2}}{2}, \qquad 1 \le i \le n.$$

Indeed, substituting $p = 1$ gives $\beta_i^{\pm} = \eta_i^{\pm} = \gamma_i^{\pm}$, while both terms with multiplicity $n(p-1)$ disappear. There are $n(q-1) + 2n = (q+1)n$ eigenvalues, counted with multiplicity, and their sum is $nr(2q+1)$. Consequently, with $f(x) = x\log_2 x$ and $f(0) = 0$, both Corollaries 3.2 and 3.4 reduce to

$$s(S_{1,q}(G)) = s(H_{1,q}(G)) = \log_2\big(nr(2q+1)\big) - \frac{n(q-1)f(r) + \sum_{i=1}^{n}\big(f(\gamma_i^+) + f(\gamma_i^-)\big)}{nr(2q+1)}.$$

This displays explicitly the one-parameter splitting setting considered in [2,17]; the entropy comparison concerns [17].

Example 3.1.2. Consider the graphs shown in Figure 2.1. Let $G = C_4$ and $p = q = 2$. Then $n = 4$, $r = 2$, $m = 4$, and the adjacency spectrum of G is $\{2, 0, 0, -2\}$. Substitution in Theorems 3.1 and 3.3, followed by aggregation of coincident eigenvalues, gives Table 3.1. A superscript $[a]$ indicates multiplicity a; each sign in a $\pm$ expression contributes one eigenvalue. Entropies use logarithms to base 2 and are rounded to six decimal places.

Table 3.1: Laplacian spectra and entropies of the graphs in Figure 2.1.

Quantity	$S_{2,2}(C_4)$	$H_{2,2}(C_4)$
Laplacian spectrum	$0^{[1]},\ 4^{[7]},\ 6^{[4]},\ 8^{[2]},\ (6\pm 2\sqrt{5})^{[1]}$	$0^{[1]},\ 4^{[6]},\ 8^{[7]},\ (8\pm 4\sqrt{2})^{[1]}$
Number of eigenvalues	16	16
Number of edges	40	48
Sum of eigenvalues	$80 = 2\cdot 40$	$96 = 2\cdot 48$
Von Neumann entropy	3.791202	3.768442

With $f(x) = x\log_2 x$ and $f(0) = 0$, the exact entropy expressions underlying the table are

$$s(S_{2,2}(C_4)) = \log_2 80 - \frac{104 + 24\log_2 6 + f(6-2\sqrt{5}) + f(6+2\sqrt{5})}{80},$$

$$s(H_{2,2}(C_4)) = \log_2 96 - \frac{216 + f(8-4\sqrt{2}) + f(8+4\sqrt{2})}{96}.$$

4. Concluding Remarks

Theorem 2.4 provides a common orthogonal reduction for the Laplacian matrices of $S_{p,q}(G)$ and $H_{p,q}(G)$ without assuming that the connected base graph is regular. The reductions retain the distinct degree contributions of the original and added layers and separate repeated blocks from a coupled block of order $2n$. For a connected regular base graph, Theorems 3.1 and 3.3 determine the complete Laplacian spectra, including all multiplicity contributions. Together with the edge counts in Lemma 2.3, these spectra yield the trace-normalized von Neumann entropy formulae in Corollaries 3.2 and 3.4. Examples 3.1.1 and 3.1.2 illustrate the common one-parameter specialization and the spectral and entropy calculations.

The reductions also suggest extensions beyond regular base graphs. Here the blocks involving both A and D require further analysis. A natural direction is to obtain explicit spectra or entropy bounds for suitable nonregular families, such as biregular bipartite graphs, by identifying invariant subspaces adapted to both adjacency and vertex degrees. Weighted constructions that preserve the layer structure and analogous reductions for the normalized Laplacian are further possible extensions.

The explicit entropy formulae also allow questions about the parameters p and q to be formulated precisely. For a fixed base graph, keeping $p+q$ fixed preserves the order of the constructed graphs and leads to extremal problems for the allocation of original and added layers. It would be useful to determine which parameter pairs maximize or minimize entropy and to identify conditions on G under which the entropies of the two constructions can be compared.

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