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Research Article

On the Prime-Order Generations of the Projective Symplectic Group of Degree Six Over the Field of Three Elements

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Abstract

A finite group G is called (l, m, n)-generated if it is generated by two elements x and y such that the orders of x, y and xy are l, m and n, respectively. In [*Nova J. Algebra Geom.* **2** (1993) 277–285], Moori posed the problem of determining all (p, q, r)-generations of non-abelian finite simple groups, where p, q and r are prime numbers. We resolve this problem for the projective symplectic group $PSp_6(3)$, denoted by $S_6(3)$ in ATLAS notation. Specifically, we prove that, for primes $p \leq q \leq r$ dividing $|PSp_6(3)|$, the group $PSp_6(3)$ is (p, q, r)-generated if and only if $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ and $(p, q, r) \neq (2, 3, 7)$. Thus, among these prime triples, those for which $PSp_6(3)$ is not (p, q, r)-generated are precisely $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 5)$, and $(3, 3, 3)$, which fail the reciprocal-sum condition, together with $(2, 3, 7)$. Also, we determine, class by class, all triples (pX, qY, rZ) of conjugacy classes of prime-order elements that generate $PSp_6(3)$. We mainly use the structure constant method, together with other group-theoretic results. We use the GAP system and the Atlas of Finite Group Representations in our computations.

Keywords: conjugacy classes; generation; simple groups; structure constants; projective symplectic group.

2020 Mathematics Subject Classification: 20C15, 20D05, 20D06.

1. Introduction

The concept of generation of finite groups by suitable subsets is of great interest and has many applications to groups and their representations. For example, the computation of the genus of a simple group can be reduced to studying the generation of the relevant simple groups (see [27] for details). The generation of finite groups by conjugate elements has also attracted attention. In his Ph.D. thesis [25], Ward studied the generation of simple groups by conjugate involutions satisfying certain conditions. In this paper, we study the (p, q, r)-generations of the projective symplectic group $PSp_6(3)$, denoted by $S_6(3)$ in ATLAS notation, where p, q, and r are prime numbers dividing the order of $PSp_6(3)$.

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A finite group G is said to be (l,m,n)-generated if $G=\langle x,y\rangle$, with $o(x)=l$, $o(y)=m$ and $o(xy)=n$ (see [19]). Here, $[x]=lX$, $[y]=mY$ and $[z]=nZ$, where $[x]=lX$ denotes the conjugacy class of x in G, whose elements have order l. In this case, G is also a quotient group of the triangle group $T(l,m,n)$ and, by the definition of the triangle group, G is also a $(\sigma(l),\sigma(m),\sigma(n))$-generated group for any $\sigma\in S_3$. Therefore, we assume that $l\leq m\leq n$. In a series of papers [14–20], Moori and Ganief determined all possible (p,q,r)-generations of the sporadic groups J_1, J_2, J_3, HS, McL, Co_3, Co_2, and F_{i22}, where p ,q, and r are distinct primes dividing the orders of the groups. Ashrafi [1,2] did the same for the sporadic simple groups He and HN. Recently, Basheer, Motalane, and Seretlo [5,6,21] found (p,q,r) generations of the symplectic group $S_6(2)$, the alternating group A_{11}, and the Mathieu group M_{23}. Also, in [7–9], (p,q,r)-generations of the Chevalley groups $G_2(3)$ and $G_2(4)$ and the Tits group ${}^2F_4(2)'$ were determined. The motivation for the present study is outlined in these papers, and the reader is encouraged to consult them for background material as well as for fundamental computational techniques.

Throughout this paper, $PSp_6(3)$ denotes the simple projective symplectic group $Sp_6(3)/\{\pm I_6\}$ of order 4585351680. Note that the full symplectic group $Sp_6(3)$ is a double cover of $PSp_6(3)$ and all generation results in this paper concern the simple group $PSp_6(3)$.

Some generation properties of $PSp_6(3)$ are already known. Pellegrini [22] proved that $PSp_6(q)$ is $(2,3)$-generated for every q. Vincent and Zalesski [24] observed that $PSp_6(3)$ is not a Hurwitz group, that is, it is not $(2,3,7)$-generated. These two facts do not contradict each other. The $(2,3)$-generation only requires $G=\langle x,y\rangle$ with $o(x)=2$ and $o(y)=3$, with no condition on $o(xy)$, whereas being a Hurwitz group requires in addition that $o(xy)=7$. We show in Proposition 4.2.6 that $PSp_6(3)$ is $(2B,3G,13A)$-generated, giving an explicit $(2,3)$-generation with $o(xy)=13$, while Propositions 4.2.1 and 4.2.2 show that no such generating pair has $o(xy)=7$. Moreover, the standard generators a,b of $PSp_6(3)$ given in the online ATLAS [26] satisfy $a\in 2A$, $o(b)=5$ and $o(ab)=13$, and hence an explicit $(2,5,13)$-generating pair is also known; in [26], $PSp_6(3)$ is denoted by $S_6(3)$. The new contribution of the present paper is the complete classification of the prime-order generations of $PSp_6(3)$. For every triple (pX,qY,rZ) of conjugacy classes of elements of prime order with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$, we decide whether or not $PSp_6(3)$ is (pX,qY,rZ)-generated. In particular, the results for (p,q,r) with p,q,r odd, and all the class-by-class non-generation results, are new.

In this paper, we determine all (p,q,r)-generations of the projective symplectic group $PSp_6(3)$. We follow the methods used in the papers [2, 4–8, 14–18, 21]. For the conjugacy classes and maximal subgroups of $PSp_6(3)$, we follow the notation of [12]. Additionally, Table 3.1 is prepared using information given in [12]. Note that, in general, if G is a $(2,2,n)$-generated group, then G is a dihedral group and therefore G is not simple. Also, by a theorem of Conder [10], if G is a non-abelian (l,m,n)-generated group, then either $G\cong A_5$ or $\frac{1}{l}+\frac{1}{m}+\frac{1}{n}<1$. Thus, for our purpose of determining (p,q,r)-generations of $PSp_6(3)$, the only cases we need to consider are when $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$. Our findings on (p,q,r)-generations of $PSp_6(3)$ are summarized in Theorem 1.1.

Theorem 1.1. *Let $p\leq q\leq r$ be primes dividing $|PSp_6(3)|$. Then the projective symplectic group $PSp_6(3)$ is (p,q,r)-generated if and only if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}<1$ and $(p,q,r)\neq(2,3,7)$. More precisely, if pL, qM and rN are conjugacy classes of $PSp_6(3)$ with $p\leq q\leq r$, then, up to interchanging the first two classes when $p=q$, the group $PSp_6(3)$ is (pL,qM,rN)-generated if and only if (pL,qM,rN) belongs to the set* $\{(2B,3G,13X)$, $(2B,5A,5A)$, $(2B,5A,7A)$, $(2X,5A,13K)$, $(2X,7A,7A)$, $(2X,7A,13K)$, $(2X,13K,13L)$, $(3Y,3G,5A)$, $(3Z,3G,7A)$, $(3Z,3G,13K)$, $(3U,5A,5A)$, $(3U,5A,7A)$, $(3C,5A,13K)$, $(3U,5A,13K)$, $(3V,7A,7A)$, $(3V,7A,13X)$, $(3V,13K,13L)$, $(5A,5A,5A)$, $(5A,5A,7A)$, $(5A,5A,13K)$, $(5A,7A,13K)$, $(5A,13K,13L)$, $(7A,7A,7A)$, $(7A,7A,13K)$, $(7A,13K,13L)$, $(13K,13L,13M)\}$ *for all $X,K,L,M\in\{A,B\}$, $Y\in\{E,F,G\}$, $Z\in\{C,E,F,G\}$, $U\in\{D,E,F,G\}$ and $V\in\{A,B,C,D,E,F,G\}$.*

2. Preliminaries

Let G be a finite group. Let $C_1, C_2, \cdots, C_k$ (not necessarily distinct) be conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes, respectively, where $k \geq 3$. For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as the *class algebra constant* or *structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G using the following equation:

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}. \tag{1}$$

Also, for a fixed $g_k \in C_k$, we denote by $\Delta^*_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle. \tag{2}$$

Definition 2.1. *If $\Delta^*_G(C_1, C_2, \cdots, C_k) > 0$, then the group G is said to be $(C_1, C_2, \cdots, C_k)$-generated.*

For any subgroup H of G containing the fixed element $g_k \in C_k$, let $\Sigma_H(C_1, C_2, \cdots, C_k) := \Sigma(H)$ be the number of distinct tuples $(h_1, h_2, \cdots, h_{k-1})$ which are in $C_1 \times C_2 \times \cdots \times C_{k-1}$ such that

$$h_1 h_2 \cdots h_{k-1} = g_k \quad \text{and} \quad \langle h_1, h_2, \cdots, h_{k-1} \rangle \leq H. \tag{3}$$

The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, c_2, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Finally, for conjugacy classes $c_1, c_2, \cdots, c_k$ of a proper subgroup H of G and a fixed $g_k \in c_k$, let $\Sigma^*_H(c_1, c_2, \cdots, c_k) := \Sigma^*(H)$ represents the number of tuples $(h_1, h_2, \cdots, h_{k-1}) \in c_1 \times c_2 \times \cdots \times c_{k-1}$ such that $h_1 h_2 \cdots h_{k-1} = g_k$ and $\langle h_1, h_2, \cdots, h_{k-1} \rangle = H$.

Theorem 2.2. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H):H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular,*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|}, \tag{4}$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [14] and [16, Theorem 2.1]. □

The number $h(g, H)$ in Theorem 2.2 is useful in giving a lower bound for $\Delta^*_G(C_1, C_2, \cdots, C_k)$, namely $\Delta^*_G(C_1, C_2, \cdots, C_k) \geq \Theta_G(C_1, C_2, \cdots, C_k)$, where

$$\Theta_G(C_1, C_2, \cdots, C_k) = \Delta_G(C_1, C_2, \cdots, C_k) - \sum_H h(g_k, H)\Sigma_H(C_1, C_2, \cdots, C_k), \tag{5}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1, C_2, \cdots, C_k$.

Remark 2.3. *Every $(k-1)$-tuple $(g_1, \ldots, g_{k-1})$ with $g_1 \cdots g_{k-1} = g_k$ that generates a proper subgroup of G lies in some maximal subgroup of G containing g_k, and for each representative H, there are exactly $h(g_k, H)$ conjugates of H containing g_k. Hence, the number of such tuples is at most $\sum_H h(g_k, H)\, \Sigma_H(C_1, \ldots, C_k)$, which gives $\Delta^*_G \geq \Theta_G$. This bound remains valid regardless of how the maximal subgroups intersect. Intersections only cause some tuples to be subtracted more than once, which makes the bound smaller but does not affect its validity. Therefore, in all our proofs of generation, we ignore intersections of maximal subgroups. Note that, if $\Theta_G \leq 0$, then no conclusion can be drawn; but, in all our generation proofs, we have $\Theta_G > 0$. Intersections are taken into account only in proofs of non-generation, where an exact count is needed.*

The following results are in some cases useful in establishing generations and non-generation for finite groups.

Lemma 2.4 (see Lemma 2.5 in [3])**.** *If G is a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Lemma 2.5. *Let G be a finite centerless group. If $\Delta_G(C_1, C_2, \cdots, C_k) < |C_G(g_k)|$ and $g_k \in C_k$, then $\Delta^*_G(C_1, C_2, \cdots, C_k) = 0$ and G is not $(C_1, C_2, \cdots, C_k)$-generated.*

Proof. See [3, Lemma 2.7]. □

Lemma 2.6. *Let H be a proper subgroup of G containing the fixed element $g_k \in C_k$. If*

$$\Sigma_H(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k),$$

then $\Delta^*_G(C_1, \cdots, C_k) = 0$ and G is not $(C_1, \cdots, C_k)$-generated.

Proof. The number $\Sigma_H(C_1, \cdots, C_k)$ counts those tuples $(g_1, \ldots, g_{k-1}) \in C_1 \times \cdots \times C_{k-1}$ with $g_1 \cdots g_{k-1} = g_k$ and $\langle g_1, \ldots, g_{k-1} \rangle \leq H$. If $\Sigma_H(C_1, \cdots, C_k) = \Delta_G(C_1, \cdots, C_k)$, then every such tuple generates a subgroup of the proper subgroup H, and therefore no tuple generates G. □

Theorem 2.7 (see [23])**.** *Let G be a transitive permutation group generated by permutations $g_1, g_2, \cdots, g_s$ acting on a set of n elements such that $g_1 g_2 \cdots g_s = 1_G$. If the generator g_i has exactly c_i cycles for $1 \leq i \leq s$, then $\sum_{i=1}^{s} c_i \leq (s-2)n + 2$.*

3. The Projective Symplectic Group $PSp_6(3)$

In this section, we apply the results discussed in Section 2 to the non-abelian finite projective symplectic group $PSp_6(3)$. We determine the (p, q, r)-generations of $PSp_6(3)$, where p, q and r are primes dividing $|PSp_6(3)|$ and are not necessarily distinct. The group $PSp_6(3) = Sp_6(3)/\{\pm I_6\}$ is a simple group of order $4585351680 = 2^9 \times 3^9 \times 5 \times 7 \times 13$, and according to [12], it has exactly 74 conjugacy classes of its elements and 11 conjugacy classes of its maximal subgroups. Representatives of the maximal subgroups can be taken as follows:

$$M_1 = 3^{1+4}_{+} : 2U_4(2), \quad M_2 = 3^6 : L_3(3), \quad M_3 = 3^{3+4} : 2(S_4 \times A_4), \quad M_4 = 2(A_4 \times U_4(2)), \quad M_5 = 2^{2+6} : 3^3 : S_3,$$

$$M_6 = L_2(27) : 3, \quad M_7 = (2 \times U_3(3))\cdot 2, \quad M_8 = L_3(3) : 2, \quad M_9 = L_2(13), \quad M_{10} = L_2(13), \quad M_{11} = A_5.$$

According to the electronic ATLAS of Wilson et al. [26], $PSp_6(3)$ can be generated in terms of permutations on 364 points. Generators a and b can be taken as follows:

$$
\begin{aligned}
a \;=\; & (1,2)(3,5)(4,7)(6,10)(8,13)(11,17)(12,19)(14,22)(16,25)(18,28)(21,31)(23,34)(24,35)(26,38)\\
& (27,40)(29,43)(30,45)(32,48)(33,47)(36,53)(37,54)(39,57)(41,60)(42,50)(44,64)(46,67)(49,70)\\
& (51,68)(52,74)(56,79)(59,82)(62,86)(63,88)(65,91)(69,94)(71,97)(73,98)(75,87)(76,93)(77,102)\\
& (78,104)(80,107)(81,109)(83,112)(84,114)(85,116)(89,118)(92,121)(96,125)(99,129)(100,131)\\
& (101,133)(105,138)(106,139)(108,142)(110,126)(111,145)(113,148)(115,151)(117,152)(119,155)\\
& (122,158)(123,159)(124,161)(127,157)(128,165)(130,162)(132,169)(134,170)(135,171)(136,173)\\
& (141,163)(143,181)(146,184)(147,186)(149,189)(150,190)(153,176)(156,177)(164,196)(167,201)\\
& (168,203)(172,208)(174,211)(178,215)(179,217)(180,207)(182,220)(183,221)(185,224)(187,227)\\
& (188,229)(191,232)(192,231)(193,234)(194,236)(195,238)(197,241)(198,243)(199,244)\\
& (200,245)(202,242)(206,249)(209,253)(210,255)(213,258)(216,261)(219,256)(222,266)(223,268)\\
& (225,270)(230,275)(233,277)(235,267)(240,283)(246,289)(247,291)(248,293)(251,259)(252,297)\\
& (257,296)(260,300)(262,302)(263,282)(264,301)(265,304)(269,273)(271,272)(274,278)(276,308)\\
& (279,314)(280,315)(281,295)(284,319)(286,321)(287,318)(292,325)(294,305)(298,330)(307,327)\\
& (309,320)(311,312)(313,335)(316,331)(317,340)(322,343)(323,332)(324,338)(326,347)(333,345)\\
& (334,350)(336,352)(337,342)(341,344)(346,353)(348,357)(351,358)(359,361)(360,362)(363,364),
\end{aligned}
$$

$$
\begin{aligned}
b \;=\; & (1,3,6,11,18)(2,4,8,14,23)(5,9,15,24,36)(7,12,20,30,46)(10,16,26,39,58)(13,21,32,49,71)\\
& (17,27,41,61,85)(19,29,44,65,92)(22,33,50,72,86)(25,37,55,78,105)(28,42,62,87,117)\\
& (31,47,68,88,74)(34,51,73,99,130)(35,52,75,100,132)(38,56,80,108,143)(40,59,83,113,149)\\
& (43,63,89,45,66)(48,69,95,124,162)(53,76,101,134,158)(54,77,103,136,174)\\
& (57,81,110,144,182)(60,84,115,152,192)(64,90,119,97,127)(67,93,122,118,154)\\
& (70,96,126,164,199)(79,106,140,178,216)(82,111,146,185,225)(91,120,156,195,239)\\
& (94,123,160,196,240)(98,128,166,121,157)(102,135,172,209,254)(104,137,175,212,257)\\
& (107,141,179,218,263)(109,142,180,169,205)(112,147,187,228,273)(114,150,131,168,204)\\
& (116,153,193,235,279)(125,163,198,211,256)(129,167,202,220,264)(133,139,177,214,260)\\
& (138,176,213,259,299)(145,183,222,267,305)(148,188,230,190,232)(151,191,233,270,308)\\
& (155,194,237,281,317)(159,161,197,242,285)(165,200,246,290,315)(170,206,250,238,282)\\
& (171,207,251,296,329)(173,210,221,265,253)(181,219,236,280,316)(184,223,269,307,336)\\
& (186,226,271,309,337)(189,231,276,312,319)(201,247,292,304,334)(203,248,294,327,258)\\
& (208,252,261,301,331)(215,217,262,303,333)(224,244,287,255,298)(227,272,229,274,310)\\
& (234,278,313,338,353)(241,284,320,341,350)(243,286,322,344,297)(245,288,323,345,356)\\
& (249,295,328,289,300)(266,275,311,291,324)(268,306,335,351,359)(277,283,318,321,342)\\
& (293,326,314,339,354)(302,332,349,340,355)(325,346,352,330,348)(357,358,360,363,361),
\end{aligned}
$$

with $o(a) = 2, o(b) = 5$ and $o(ab) = 13$. These are the standard generators of $PSp_6(3)$ given in [26]. For the conjugacy classes and maximal subgroups of $PSp_6(3)$, we have computed the values of $h(g, M_i)$. These values are listed in Table 3.1.

Table 3.1: The values $h(g, M_i)$, $1 \leq i \leq 11$, for non-identity classes and maximal subgroups of $PSp_6(3)$.

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}
$2A$	44	160	200	91	1125	0	1620	6480	0	0	0
$2B$	0	0	28	63	189	288	568	504	2016	2016	6048
$3A$	121	40	400	891	3645	0	0	0	0	0	0
$3B$	121	40	400	891	3645	0	0	0	0	0	0
$3C$	40	4	49	189	486	0	486	0	0	0	0
$3D$	40	76	103	135	243	0	0	972	0	0	0
$3E$	13	13	49	81	108	216	0	0	0	0	0
$3F$	13	13	49	81	108	216	0	0	0	0	0
$3G$	4	16	13	0	27	54	27	54	162	162	324
$4A$	40	0	40	91	45	0	540	0	0	0	0
$4B$	8	0	16	3	13	0	84	48	0	0	0
$4C$	4	16	4	7	5	0	24	72	0	0	0
$5A$	4	0	0	1	0	0	0	0	0	0	48
$6A$	41	40	80	91	45	0	0	0	0	0	0
$6B$	41	40	80	91	45	0	0	0	0	0	0
$6C$	17	16	56	19	117	0	0	0	0	0	0
$6D$	17	16	56	19	117	0	0	0	0	0	0
$6E$	14	4	17	19	36	0	0	0	0	0	0
$6F$	14	4	17	19	36	0	0	0	0	0	0
$6G$	14	4	17	19	9	0	0	0	0	0	0
$6H$	14	4	17	19	9	0	0	0	0	0	0
$6I$	8	4	17	13	6	0	54	0	0	0	0
$6J$	8	28	23	7	3	0	0	108	0	0	0
$6K$	5	7	11	7	12	0	0	0	0	0	0
$6L$	5	7	11	7	12	0	0	0	0	0	0
$6M$	0	0	1	9	18	0	10	0	0	0	0
$6N$	5	1	5	13	24	0	0	0	0	0	0
$6O$	5	1	5	13	24	0	0	0	0	0	0
$6P$	0	0	1	0	3	6	7	6	6	6	0
$7A$	0	0	0	0	0	1	1	0	6	6	0
$8A$	4	8	2	1	1	0	6	12	0	0	0
$8B$	0	0	2	1	1	0	6	4	0	0	0
$9A$	13	4	13	9	0	0	0	0	0	0	0
$9B$	13	4	13	9	0	0	0	0	0	0	0
$9C$	4	7	4	9	0	0	0	0	0	0	0
$9D$	4	7	4	9	0	0	0	0	0	0	0
$9E$	4	1	4	9	0	0	0	0	0	0	0
$9F$	4	1	4	9	0	0	0	0	0	0	0
$9G$	1	1	1	0	3	6	0	0	0	0	0
$9H$	1	1	1	0	3	6	0	0	0	0	0
$10A$	4	0	0	1	0	0	0	0	0	0	0
$12A$	13	0	4	19	9	0	0	0	0	0	0
$12B$	13	0	4	19	9	0	0	0	0	0	0
$12C$	4	0	1	1	6	0	18	0	0	0	0
$12D$	1	4	4	7	5	0	0	0	0	0	0
$12E$	1	4	4	7	5	0	0	0	0	0	0
$12F$	5	0	4	3	1	0	0	0	0	0	0
$12G$	5	0	4	3	1	0	0	0	0	0	0
$12H$	4	0	7	7	3	0	0	0	0	0	0
$12I$	4	4	1	1	2	0	6	0	0	0	0

	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}	M_{11}
$12J$	2	0	1	3	4	0	12	0	0	0	0
$12K$	2	0	1	3	4	0	12	0	0	0	0
$12L$	2	0	1	3	1	0	0	12	0	0	0
$12M$	1	1	1	1	2	0	0	0	0	0	0
$12N$	1	1	1	1	2	0	0	0	0	0	0
$13A$	0	2	0	0	0	1	0	1	1	1	0
$13B$	0	2	0	0	0	1	0	1	1	1	0
$14A$	0	0	0	0	0	0	1	1	0	0	0
$15A$	1	0	0	1	0	0	0	0	0	0	0
$15B$	1	0	0	1	0	0	0	0	0	0	0
$18A$	5	4	5	1	0	0	0	0	0	0	0
$18B$	5	4	5	1	0	0	0	0	0	0	0
$18C$	2	1	2	1	0	0	0	0	0	0	0
$18D$	2	1	2	1	0	0	0	0	0	0	0
$18E$	2	1	2	1	0	0	0	0	0	0	0
$18F$	2	1	2	1	0	0	0	0	0	0	0
$20A$	0	0	0	1	0	0	0	0	0	0	0
$24A$	1	2	2	1	1	0	0	0	0	0	0
$24B$	1	2	2	1	1	0	0	0	0	0	0
$30A$	1	0	0	1	0	0	0	0	0	0	0
$30B$	1	0	0	1	0	0	0	0	0	0	0
$36A$	1	0	1	1	0	0	0	0	0	0	0
$36B$	1	0	1	1	0	0	0	0	0	0	0

4. $(2, q, r)$-Generations

The $(2, q, r)$-generations of the group $PSp_6(3)$ fall into five cases: $(2, 2, r)$-, $(2, 3, r)$-, $(2, 5, r)$-, $(2, 7, r)$-, and $(2, 13, r)$-generations.

4.1. $(2, 2, r)$-Generations

Proposition 4.1.1. *Let $X, Y, L, M \in \{A, B\}$ and $Z \in \{A, B, C, D, E, F, G\}$. The group $PSp_6(3)$ is not $(2X, 2Y, tK)$-generated, where $tK \in \{2L, 3Z, 5A, 7A, 13M\}$.*

Proof. The group $PSp_6(3)$ is not $(2X, 2Y, tK)$-generated; otherwise, it would be a dihedral group and thus would not be simple. □

4.2. $(2, 3, r)$-Generations

The condition $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} < 1$ implies that r must be 7 or 13. We therefore consider the cases $(2X, 3Y, 7A)$ and $(2X, 3Y, 13Z)$ for $X, Z \in \{A, B\}$ and $Y \in \{A, B, C, D, E, F, G\}$.

Proposition 4.2.1. *The group $PSp_6(3)$ is not $(2A, 3Y, 7A)$-generated for every $Y \in \{A, B, C, D, E, F, G\}$.*

Proof. From Table 4.1, we have $\Delta_{PSp_6(3)}(2A, 3Y, 7A) = 0$ for $Y \in \{A, B, C, D, E, F\}$. We conclude, by Definition 2.1, that $PSp_6(3)$ is not $(2A, 3Y, 7A)$-generated for $Y \in \{A, B, C, D, E, F\}$. Moreover, we have $\Delta_{PSp_6(3)}(2A, 3G, 7A) = 7 < 14 = |C_{PSp_6(3)}(7A)|$, and by Lemma 2.5, we conclude that $PSp_6(3)$ is not $(2A, 3G, 7A)$-generated. □

Table 4.1: Structure constants of $PSp_6(3)$ for $(2A, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(2A, 2A, rX)$	90	0	0	0	0	54	0	0	3	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 2B, rX)$	0	63	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3C, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3D, rX)$	320	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3E, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3F, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2A, 3G, rX)$	1920	0	0	0	0	0	0	0	81	0	7	13	13
$\Delta_{PSp_6(3)}(2A, 5A, rX)$	0	0	0	0	0	0	0	0	0	10	0	52	52
$\Delta_{PSp_6(3)}(2A, 7A, rX)$	0	0	0	0	0	0	0	0	486	0	350	520	520
$\Delta_{PSp_6(3)}(2A, 13A, rX)$	0	0	0	0	0	0	0	0	972	480	560	754	754
$\Delta_{PSp_6(3)}(2A, 13B, rX)$	0	0	0	0	0	0	0	0	972	480	560	754	754
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Proposition 4.2.2. *The group $PSp_6(3)$ is not $(2B, 3Y, 7A)$-generated for every $Y \in \{A, B, C, D, E, F, G\}$.*

Proof. From Table 4.2, we have $\Delta_{PSp_6(3)}(2B, 3Y, 7A) = 0$ for $Y \in \{A, B, C, D\}$, and therefore, the group $PSp_6(3)$ is not $(2B, 3Y, 7A)$-generated for $Y \in \{A, B, C, D\}$. Calculations with GAP [13] yield $\Delta_{PSp_6(3)}(2B, 3Y, 7A) = 14$ for $Y \in \{E, F\}$. Only the maximal subgroup M_6 has intersection with all the classes $2B$, $3Y$ and $7A$ of $PSp_6(3)$ for $Y \in \{E, F\}$. On the other hand, further calculations with GAP give $\Sigma(M_6) = 14 = \Delta_{PSp_6(3)}(2B, 3Y, 7A)$. By Lemma 2.6, $\Delta^*_{PSp_6(3)}(2B, 3Y, 7A) = 0$, and hence $PSp_6(3)$ is not $(2B, 3Y, 7A)$-generated for $Y \in \{E, F\}$.

Table 4.2: Structure constants of $PSp_6(3)$ for $(2B, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(2B, 2A, rX)$	0	63	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2B, 2B, rX)$	1620	504	0	0	972	486	0	0	81	30	14	13	13
$\Delta_{PSp_6(3)}(2B, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2B, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2B, 3C, rX)$	0	112	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2B, 3D, rX)$	0	112	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(2B, 3E, rX)$	0	0	0	0	0	0	729	0	0	0	14	13	13
$\Delta_{PSp_6(3)}(2B, 3F, rX)$	0	0	0	0	0	0	0	729	0	0	14	13	13
$\Delta_{PSp_6(3)}(2B, 3G, rX)$	0	2016	0	0	0	0	0	0	648	240	168	156	156
$\Delta_{PSp_6(3)}(2B, 5A, rX)$	0	6048	0	0	0	0	0	0	1944	1680	1708	1586	1586
$\Delta_{PSp_6(3)}(2B, 7A, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\Delta_{PSp_6(3)}(2B, 13A, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\Delta_{PSp_6(3)}(2B, 13B, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

From Table 4.2, we have $\Delta_{PSp_6(3)}(2B, 3G, 7A) = 168$. For a fixed $z \in 7A$, our computations with GAP show that every proper $(2B, 3G, 7A)$-generated subgroup of $PSp_6(3)$ containing z is isomorphic to $PSL(2, 13)$. More precisely, z is contained in exactly 12 conjugates of $PSL(2, 13)$, and each of these contributes $\Sigma(PSL(2, 13)) = 14$ pairs $(x, y) \in 2B \times 3G$ with $xy = z$. Hence, the total number of such pairs generating a proper subgroup of $PSp_6(3)$ is $12 \times 14 = 168$. Consequently, it follows that $\Delta^*_{PSp_6(3)}(2B, 3G, 7A) = 168 - 12 \times 14 = 0$, and therefore $PSp_6(3)$ is not $(2B, 3G, 7A)$-generated. □

Remark 4.2.3. *By Propositions 4.2.1 and 4.2.2, the group $PSp_6(3)$ is not $(2, 3, 7)$-generated, that is, it is not a Hurwitz group. This fact was first proved by Vincent and Zalesski [24]; our propositions confirmed this fact. In contrast, the symplectic group $S_6(2) = Sp_6(2)$ is a Hurwitz group (see [6]).*

Proposition 4.2.4. *The group $PSp_6(3)$ is not $(2A, 3Y, 13Z)$-generated for all $Y \in \{A, B, C, D, E, F, G\}$ and $Z \in \{A, B\}$.*

Proof. By Table 4.1, we have $\Delta_{PSp_6(3)}(2A, 3Y, 13Z) = 0$ for $Y \in \{A, B, C, D, E, F\}$ and $Z \in \{A, B\}$, and therefore the non-generation follows immediately.

Again, from Table 4.1, we have $\Delta_{PSp_6(3)}(2A, 3G, 13Z) = 13$ for $Z \in \{A, B\}$. Thus, for a fixed $z \in 13Z$, there are exactly 13 pairs $(x, y) \in 2A \times 3G$ with $xy = z$. The maximal subgroups of $PSp_6(3)$ having conjugacy classes which fuse to $2A$, $3G$ and $13Z$ are M_2 and M_8, with $h(13Z, M_2) = 2$ and $h(13Z, M_8) = 1$. The intersection $M_2 \cap M_8 \cong PSL(3, 3)$ has also the required fusion. Through some GAP verifications, we found that 13 pairs $(x, y) \in 2A \times 3G$ lie in M_2, M_8 as well as in their intersection. In fact, we found that each of these 13 pairs generates the intersection $PSL(3, 3)$, which is a proper subgroup of $PSp_6(3)$. Hence, $PSp_6(3)$ is not $(2A, 3G, 13Z)$-generated for $Z \in \{A, B\}$. □

Remark 4.2.5. *The maximal subgroup M_2 does not need to be considered separately. Since $\Sigma(M_2) = 13$ as well, it contains the same 13 pairs, all of which lie in $M_2 \cap M_8 \cong PSL(3, 3)$. Subtracting its contribution in addition to that of M_8 would count these pairs twice.*

Proposition 4.2.6. *Let $Z \in \{A, B\}$. The group $PSp_6(3)$ is not $(2B, 3Y, 13Z)$-generated for every $Y \in \{A, B, C, D, E, F\}$, while it is $(2B, 3G, 13Z)$-generated.*

Proof. By Table 4.2, we have $\Delta_{PSp_6(3)}(2B, 3Y, 13Z) = 0$ for $Y \in \{A, B, C, D\}$ and $Z \in \{A, B\}$, and therefore the non-generation occurs.

For the triple $(2B, 3Y, 13Z)$ with $Y \in \{E, F\}$ and $Z \in \{A, B\}$, we observe in Table 4.3 that only M_6 intersect with all the classes $2B$, $3Y$ and $13Z$ of $PSp_6(3)$. We obtain $\Delta_{PSp_6(3)}(2B, 3Y, 13Z) = 13$ and $\Sigma(M_6) = 13$. By Table 3.1, we have $h(13Z, M_6) = 1$. From Lemma 2.6, it follows that

$$\Delta^*_{PSp_6(3)}(2B, 3Y, 13Z) = \Delta_{PSp_6(3)}(2B, 3Y, 13Z) - \Sigma(M_6) = 13 - 13 = 0,$$

which shows that $(2B, 3Y, 13Z)$ does not generate $PSp_6(3)$ for $Y \in \{E, F\}$ and $Z \in \{A, B\}$.

Now, we consider the triple $(2B, 3G, 13Z)$ with the structure constant $\Delta_{PSp_6(3)}(2B, 3G, 13Z) = 156$ for $Z \in \{A, B\}$. From Table 4.3, we see that the maximal subgroups M_6, M_8, M_9 and M_{10} are the only ones with classes that fuse into the classes $2B, 3G$ and $13Z$. However, $\Sigma(M_6) = \Sigma(M_8) = 0$, and therefore these two subgroups make no contribution, and by Remark 2.3, intersections of the maximal subgroups may be ignored. We found that $\Sigma(M_9) = \Sigma(M_{10}) = 13$ and $h(13Z, M_9) = h(13Z, M_{10}) = 1$ for every $Z \in \{A, B\}$. Therefore, we have

$$\Delta^*_{PSp_6(3)}(2B, 3G, 13Z) \geq \Delta_{PSp_6(3)}(2B, 3G, 13A) - 1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10}) = 156 - 1(13) - 1(13) = 130 > 0.$$

Thus, $PSp_6(3)$ is $(2B, 3G, 13Z)$-generated for $Z \in \{A, B\}$. □

Remark 4.2.7. *The result in Proposition 4.2.6 is consistent with the $(2, 3)$-generation of $PSp_6(3)$ proved by Pellegrini [22]; that is, every $(2, 3)$-generating pair (x, y) of $PSp_6(3)$ with $o(xy)$ prime satisfies $o(xy) = 13$.*

4.3. $(2, 5, r)$-Generations

Due to the condition $\frac{1}{2} + \frac{1}{5} + \frac{1}{r} < 1$, the cases considered in this section are $(2X, 5A, 5A)$, $(2X, 5A, 7A)$, and $(2X, 5A, 13Z)$ for all $X, Z \in \{A, B\}$.

Proposition 4.3.1. *The group $PSp_6(3)$ is not $(2A, 5A, 5A)$-generated, while it is $(2B, 5A, 5A)$-generated.*

Proof. For the non-generation part, we note that $\Delta_{PSp_6(3)}(2A, 5A, 5A) = 10 < 120 = |C_{PSp_6(3)}(5A)|$, and therefore the result follows from Lemma 2.5.

Table 4.3: The partial fusion maps into $PSp_6(3)$.

M_1-class	$2a$	$2b$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$3k$	$3l$
$\to PSp_6(3)$	$2A$	$2A$	$3B$	$3A$	$3D$	$3E$	$3C$	$3F$	$3D$	$3E$	$3F$	$3G$	$3C$	$3D$
	$3m$	$3n$	$3o$	$3p$	$3q$	$3r$	$5a$							
	$3B$	$3E$	$3A$	$3D$	$3C$	$3F$	$5A$							
M_2-class	$2a$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$3k$	$13a$	$13b$
$\to PSp_6(3)$	$2A$	$3B$	$3A$	$3C$	$3D$	$3F$	$3E$	$3G$	$3D$	$3E$	$3F$	$3G$	$13B$	$13A$
	$13c$	$13d$												
	$13B$	$13A$												
M_3-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$3k$
$\to PSp_6(3)$	$2A$	$2A$	$2B$	$3B$	$3A$	$3C$	$3D$	$3D$	$3E$	$3F$	$3G$	$3D$	$3G$	$3B$
	$3l$	$3m$	$3n$	$3o$	$3p$	$3q$	$3r$	$3s$	$3t$	$3u$	$3v$	$3w$	$3x$	
	$3C$	$3D$	$3F$	$3E$	$3E$	$3G$	$3A$	$3D$	$3C$	$3E$	$3F$	$3F$	$3G$	
M_4-class	$2a$	$2b$	$2c$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$	$3k$
$\to PSp_6(3)$	$2A$	$2B$	$2A$	$3A$	$3B$	$3A$	$3C$	$3D$	$3D$	$3C$	$3B$	$3F$	$3C$	$3E$
	$3l$	$3m$	$3n$	$5a$										
	$3E$	$3D$	$3F$	$5A$										
M_5-class	$2a$	$2b$	$2c$	$2d$	$3a$	$3b$	$3c$	$3d$	$3e$	$3f$	$3g$	$3h$	$3i$	$3j$
$\to PSp_6(3)$	$2A$	$2B$	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3C$	$3E$	$3F$	$3E$	$3F$	$3G$
M_6-class	$2a$	$3a$	$3b$	$3c$	$3d$	$7a$	$13a$	$13b$						
$\to PSp_6(3)$	$2B$	$3F$	$3E$	$3G$	$3G$	$7A$	$13B$	$13A$						
M_7-class	$2a$	$2b$	$2c$	$2d$	$2e$	$3a$	$3b$	$7a$						
$\to PSp_6(3)$	$2B$	$2A$	$2B$	$2B$	$2B$	$3C$	$3G$	$7A$						
M_8-class	$2a$	$2b$	$3a$	$3b$	$13a$	$13b$								
$\to PSp_6(3)$	$2A$	$2B$	$3D$	$3G$	$13A$	$13B$								
M_9-class	$2a$	$3a$	$7a$	$7b$	$7c$	$13a$	$13b$							
$\to PSp_6(3)$	$2B$	$3G$	$7A$	$7A$	$7A$	$13B$	$13A$							
M_{10}-class	$2a$	$3a$	$7a$	$7b$	$7c$	$13a$	$13b$							
$\to PSp_6(3)$	$2B$	$3G$	$7A$	$7A$	$7A$	$13B$	$13A$							
M_{11}-class	$2a$	$3a$	$5a$	$5b$										
$\to PSp_6(3)$	$2B$	$3G$	$5A$	$5A$										

For the generation part, we observe from Table 4.3 that M_4 and M_{11} are the only maximal subgroups that intersect both classes $2B$ and $5A$ nontrivially, with $h(5A, M_4) = 1$ and $h(5A, M_{11}) = 48$. Computations with GAP give $\Delta_{PSp_6(3)}(2B, 5A, 5A) = 1680$, $\Sigma(M_4) = 0$ and

$$\Sigma(M_{11}) = \Delta_{M_{11}}(2b, 5a, 5b) + \Delta_{M_{11}}(2b, 5b, 5b) = 5 + 0 = 5.$$

Since $h(5A, M_4) = 1$ and $h(5A, M_{11}) = 48$, we have

$$\Delta^*_{PSp_6(3)}(2B, 5A, 5A) \geq \Delta_{PSp_6(3)}(2B, 5A, 5A) - 1 \cdot \Sigma(M_4) - 48 \cdot \Sigma(M_{11}) = 1680 - 1(0) - 48(5) > 0.$$

Therefore, $(2B, 5A, 5A)$ generates $PSp_6(3)$. □

Proposition 4.3.2. *The group $PSp_6(3)$ is not $(2A, 5A, 7A)$-generated, while it is $(2B, 5A, 7A)$-generated.*

Proof. By Table 4.1, it holds that $\Delta_{PSp_6(3)}(2A, 5A, 7A) = 0$, and thus the non-generation part follows.

There is no maximal subgroup of $PSp_6(3)$ that fuse to the classes $2B, 5A$ and $7A$. Hence, no maximal subgroup contributes to the computation of $\Delta^*_{PSp_6(3)}(2B, 5A, 7A)$. It follows that

$$\Delta^*_{PSp_6(3)}(2B, 5A, 7A) = \Delta_{PSp_6(3)}(2B, 5A, 7A) = 1708,$$

and therefore the desired result follows. □

Proposition 4.3.3. *The group* $PSp_6(3)$ *is* $(2X, 5A, 13Y)$*-generated for all* $X, Y \in \{A, B\}$.

Proof. By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuse to the classes $2X, 5A$ and $13Y$ for all $X, Y \in \{A, B\}$. Thus, there is no contribution to the computation of $\Delta^*_{PSp_6(3)}(2X, 5A, 13Y)$ from any maximal subgroup. It follows that

$$\Delta^*_{PSp_6(3)}(2X, 5A, 13Y) = \Delta_{PSp_6(3)}(2X, 5A, 13Y) \;=\; \begin{cases} 52 & \text{when } X = A \text{ and } Y \in \{A, B\}, \\ 1586 & \text{when } X = B \text{ and } Y \in \{A, B\}. \end{cases}$$

Hence, $\Delta^*_{PSp_6(3)}(2X, 5A, 13Y) > 0$ for all $X, Y \in \{A, B\}$, and therefore the required result follows. □

4.4. $(2, 7, r)$-Generations

In this section, we discuss the following triples: $(2X, 7A, 7A)$ and $(2X, 7A, 13Y)$ for all $X, Y \in \{A, B\}$.

Proposition 4.4.1. *The group* $PSp_6(3)$ *is* $(2X, 7A, 7A)$*-generated for all* $X \in \{A, B\}$.

Proof. For the triple $(2A, 7A, 7A)$, there is only one maximal subgroup that fuses to the classes $2A$ and $7A$, namely, M_7. We obtain $\Delta_{PSp_6(3)}(2A, 7A, 7A) = 350$, $\Sigma(M_7) = 14$ and $h(7A, M_7) = 1$. Hence, we have

$$\Delta^*_{PSp_6(3)}(2A, 7A, 7A) \geq \Delta_{PSp_6(3)}(2A, 7A, 7A) - 1 \cdot \Sigma(M_7) = 350 - 1(14) > 0.$$

Therefore, $PSp_6(3)$ is generated by $(2A, 7A, 7A)$.

For the triple $(2B, 7A, 7A)$, there are four maximal subgroups, namely, M_6, M_7, M_9 and M_{10}, that fuse to the classes $2B$ and $7A$. By Remark 2.3, intersections may be ignored. We have $\Sigma(M_6) = 84$, $\Sigma(M_7) = 0$, $\Sigma(M_9) = 84$ and $\Sigma(M_{10}) = 84$. Now, with $\Delta_{PSp_6(3)}(2B, 7A, 7A) = 14560$, $h(7A, M_6) = h(7A, M_7) = 1$ and $h(7A, M_9) = h(7A, M_{10}) = 6$, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(2B, 7A, 7A) &\geq \Delta_{PSp_6(3)}(2B, 7A, 7A) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_7) - 6 \cdot \Sigma(M_9) - 6 \cdot \Sigma(M_{10}) \\ &= 14560 - 1(84) - 1(0) - 6(84) - 6(84) > 0, \end{aligned}$$

showing that $PSp_6(3)$ is generated by $(2B, 7A, 7A)$. □

Proposition 4.4.2. *The group* $PSp_6(3)$ *is* $(2X, 7A, 13Y)$*-generated for all* $X, Y \in \{A, B\}$.

Proof. Since there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $2A$, $7A$ and $13Y$ (see Table 4.3) for $Y \in \{A, B\}$, it follows that $\Delta^*_{PSp_6(3)}(2A, 7A, 13Y) = \Delta_{PSp_6(3)}(2A, 7A, 13Y) = 520$. Thus, $PSp_6(3)$ is $(2A, 7A, 13Y)$-generated for $Y \in \{A, B\}$.

For the triple $(2B, 7A, 13Y)$, with $Y \in \{A, B\}$, the maximal subgroups M_6, M_9 and M_{10} are the only ones that intersect all three classes $2B$, $7A$ and $13Y$ nontrivially. On the other hand, calculations via GAP give $\Sigma(M_6) = 78, \Sigma(M_9) = 13 + 13 + 13 = 39$ and $\Sigma(M_{10}) = 13 + 13 + 13 = 39$. Furthermore, $\Delta_{PSp_6(3)}(2B, 7A, 13Y) = 13520$ and $h(13Y, M_6) = h(13Y, M_9) = h(13Y, M_{10}) = 1$. Hence, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(2B, 7A, 13Y) &\geq \Delta_{PSp_6(3)}(2B, 7A, 13Y) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10}) \\ &= 13520 - 1(78) - 1(39) - 1(39) > 0. \end{aligned}$$

Therefore, $PSp_6(3)$ is generated by $(2B, 7A, 13Y)$ for $Y \in \{A, B\}$. □

4.5. $(2, 13, r)$-Generations

This section examines the generation of $(2X, 13Y, 13Z)$ for all $X, Y, Z \in \{A, B\}$.

Proposition 4.5.1. *The group $PSp_6(3)$ is $(2X, 13Y, 13Z)$-generated for all $X, Y, Z \in \{A, B\}$.*

Proof. We first deal with the triples involving the class $2A$. In this case, the maximal subgroups M_2 and M_8 are the only ones with the required fusions, and $\Sigma(M_2) = 117 + 117 = 234$, $\Sigma(M_8) = 26$, $h(13Z, M_2) = 2$ and $h(13Z, M_8) = 1$. From Table 4.1, we notice that $\Delta_{PSp_6(3)}(2A, 13Y, 13Z) = 754$. Thus,

$$\Delta^*_{PSp_6(3)}(2A, 13Y, 13Z) \geq \Delta_{PSp_6(3)}(2A, 13Y, 13Z) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_8) = 754 - 2(234) - 1(26) > 0.$$

Therefore, $PSp_6(3)$ is $(2A, 13Y, 13Z)$-generated for $Y, Z \in \{A, B\}$.

Now, we deal with the triples involving the class $2B$. Here, the maximal subgroups involved are M_6, M_8, M_9 and M_{10}. By Remark 2.3, intersections may be ignored. The structure constants obtained are $\Delta_{PSp_6(3)}(2B, 13Y, 13Z) = 13520$, $\Sigma(M_6) = 78$ and $\Sigma(M_8) = \Sigma(M_9) = \Sigma(M_{10}) = 0$, with $h(13Z, M_6) = 1$. Hence, we have

$$\Delta^*_{PSp_6(3)}(2B, 13Y, 13Z) \geq \Delta_{PSp_6(3)}(2B, 13Y, 13Z) - 1 \cdot \Sigma(M_6) = 13520 - 1(78) > 0.$$

Therefore, $PSp_6(3)$ is $(2B, 13Y, 13Z)$-generated for $Y, Z \in \{A, B\}$. □

5. $(3, q, r)$-Generations

We only need to consider the cases $(3, 3, r)$, $(3, 5, r)$, $(3, 7, r)$ and $(3, 13, r)$.

5.1. $(3, 3, r)$-Generation

From the condition $\frac{1}{3} + \frac{1}{3} + \frac{1}{r} < 1$, it follows that r must be 5, 7 or 13. So, we consider the following triples: $(3X, 3Y, 5A)$, $(3X, 3Y, 7A)$ and $(3X, 3Y, 13Z)$ for all $X, Y \in \{A, B, C, D, E, F, G\}$ and $Z \in \{A, B\}$.

Proposition 5.1.1. *The group $PSp_6(3)$ is not $(3X, 3Y, 5A)$-generated for $X, Y \in \{A, B, C, D, E, F, G\}$, except for the triples $(3E, 3G, 5A)$, $(3F, 3G, 5A)$ and $(3G, 3G, 5A)$.*

Proof. First, we prove the non-generation part. By Tables 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6, we have

$$\Delta_{PSp_6(3)}(3X, 3Y, 5A) = 0$$

for $X, Y \in \{A, B, C, D, E, F, G\}$, except when $(X, Y) \in \{(C, D), (C, G), (E, E), (E, G), (F, F), (F, G), (G, G)\}$. Thus, $PSp_6(3)$ is not generated by any of these triples.

If $PSp_6(3)$ is $(3C, 3D, 5A)$-generated, then by Ree's theorem, we have $c_{3C} + c_{3D} + c_{5A} \leq 366$. But, from Table 5.7, we have $c_{3C} = c_{3D} = 148$ and $c_{5A} = 76$, from which we obtain $c_{3C} + c_{3D} + c_{5A} = 372 > 366$. Thus, according to Ree's theorem, $PSp_6(3)$ is not $(3C, 3D, 5A)$-generated.

Non-generation by the triples $(3C, 3G, 5A)$ and $(3X, 3X, 5A)$, with $X \in \{E, F\}$, follows from Lemma 2.5 because $\Delta_{PSp_6(3)}(3C, 3G, 5A) = 80 < 120 = |C_{PSp_6(3)}(5A)|$ and

$$\Delta_{PSp_6(3)}(3X, 3X, 5A) = 40 < 120 = |C_{PSp_6(3)}(5A)|, \quad X \in \{E, F\}.$$

In the rest of the proof, we discuss the generation part of the theorem. By Table 4.3, there is only one maximal subgroup of $PSp_6(3)$ that fuse to the classes $3E, 3G$ and $5A$, namely M_1. But, from Table 5.5, we have $\Delta_{PSp_6(3)}(3E, 3G, 5A) = 200$. We obtain $\Sigma(M_1) = 0 + 20 + 0 = 20$ and $h(5A, M_1) = 4$. Therefore,

$$\Delta^*_{PSp_6(3)}(3E, 3G, 5A) \geq \Delta_{PSp_6(3)}(3E, 3G, 5A) - 4 \cdot \Sigma(M_1) = 200 - 4(20) = 120 > 0.$$

Table 5.1: Structure constants of $PSp_6(3)$ for $(3A, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3A, 2A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 2B, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3A, rX)$	0	0	0	1	2	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3B, rX)$	0	0	0	0	0	1	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3C, rX)$	0	0	0	120	0	1	3	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3D, rX)$	0	0	120	0	2	2	0	6	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3E, rX)$	0	0	0	0	0	36	4	3	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3F, rX)$	0	0	0	0	36	0	6	4	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 3G, rX)$	0	0	0	0	0	0	0	0	4	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 5A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3A, 7A, rX)$	0	0	0	0	0	0	0	0	0	0	14	26	26
$\Delta_{PSp_6(3)}(3A, 13A, rX)$	0	0	0	0	0	0	0	0	0	0	28	39	26
$\Delta_{PSp_6(3)}(3A, 13B, rX)$	0	0	0	0	0	0	0	0	0	0	28	26	39
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Table 5.2: Structure constants of $PSp_6(3)$ for $(3B, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3B, 2A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 2B, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3A, rX)$	0	0	0	0	0	1	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3B, rX)$	0	0	1	0	2	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3C, rX)$	0	0	120	0	0	1	0	3	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3D, rX)$	0	0	0	120	2	2	6	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3E, rX)$	0	0	0	0	36	0	4	6	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3F, rX)$	0	0	0	0	0	36	3	4	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 3G, rX)$	0	0	0	0	0	0	0	0	4	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 5A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3B, 7A, rX)$	0	0	0	0	0	0	0	0	0	0	14	26	26
$\Delta_{PSp_6(3)}(3B, 13A, rX)$	0	0	0	0	0	0	0	0	0	0	28	39	26
$\Delta_{PSp_6(3)}(3B, 13B, rX)$	0	0	0	0	0	0	0	0	0	0	28	26	39
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Table 5.3: Structure constants of $PSp_6(3)$ for $(3C, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3C, 2A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 2B, rX)$	0	112	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3A, rX)$	0	0	0	120	0	1	3	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3B, rX)$	0	0	120	0	0	1	0	3	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3C, rX)$	0	0	0	0	1	74	6	6	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3D, rX)$	0	0	120	120	148	2	18	18	0	5	0	0	0
$\Delta_{PSp_6(3)}(3C, 3E, rX)$	0	0	2160	0	72	108	15	36	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3F, rX)$	0	0	0	2160	72	108	36	15	0	0	0	0	0
$\Delta_{PSp_6(3)}(3C, 3G, rX)$	0	0	0	0	0	0	0	0	6	80	35	26	26
$\Delta_{PSp_6(3)}(3C, 5A, rX)$	0	0	0	0	0	4374	0	0	648	15	0	104	104
$\Delta_{PSp_6(3)}(3C, 7A, rX)$	0	0	0	0	0	0	0	0	2430	0	735	1560	1560
$\Delta_{PSp_6(3)}(3C, 13A, rX)$	0	0	0	0	0	0	0	0	1944	960	1680	2106	2028
$\Delta_{PSp_6(3)}(3C, 13B, rX)$	0	0	0	0	0	0	0	0	1944	960	1680	2028	2106
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Similarly, we have $\Delta^*_{PSp_6(3)}(3F, 3G, 5A) \geq 120$. Hence, $(3X, 3G, 5A)$ generates $PSp_6(3)$ for $X \in \{E, F\}$.

Finally, by Table 4.3, there are only two maximal subgroups of $PSp_6(3)$ that fuse to the classes $3G$ and $5A$, namely M_1 and M_{11}. By Remark 2.3, intersections may be ignored. Calculations via GAP yield $\Delta_{PSp_6(3)}(3G, 3G, 5A) = 1920$, $\Sigma(M_1) = 0$ and $\Sigma(M_{11}) = 5$. We also have $h(5A, M_1) = 4$ and $h(5A, M_{11}) = 48$. Thus, we have

$$\Delta^*_{PSp_6(3)}(3G, 3G, 5A) \geq \Delta_{PSp_6(3)}(3G, 3G, 5A) - 4 \cdot \Sigma(M_1) - 48 \cdot \Sigma(M_{11}) = 1920 - 4(0) - 48(5) = 1680,$$

which implies that $PSp_6(3)$ is $(3G, 3G, 5A)$-generated. This completes the proof. □

Proposition 5.1.2. *The group $PSp_6(3)$ is not $(3X, 3Y, 7A)$-generated for $X, Y \in \{A, B, C, D, E, F, G\}$, except when $(X, Y) \in \{(C, G), (E, G), (F, G), (G, G)\}$.*

Table 5.4: Structure constants of $PSp_6(3)$ for $(3D, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3D, 2A, rX)$	320	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 2B, rX)$	0	112	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3A, rX)$	0	0	120	0	2	2	0	6	0	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3B, rX)$	0	0	0	120	2	2	6	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3C, rX)$	0	0	120	120	148	2	18	18	0	5	0	0	0
$\Delta_{PSp_6(3)}(3D, 3D, rX)$	0	0	240	240	4	545	30	30	36	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3E, rX)$	0	0	0	4320	216	180	282	36	36	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3F, rX)$	0	0	4320	0	216	180	36	282	36	0	0	0	0
$\Delta_{PSp_6(3)}(3D, 3G, rX)$	0	0	0	0	0	3888	648	648	624	0	0	65	65
$\Delta_{PSp_6(3)}(3D, 5A, rX)$	0	0	0	0	8748	0	0	0	0	2075	896	260	260
$\Delta_{PSp_6(3)}(3D, 7A, rX)$	0	0	0	0	0	0	0	0	0	7680	5880	3120	3120
$\Delta_{PSp_6(3)}(3D, 13A, rX)$	0	0	0	0	0	0	0	0	4860	2400	3360	4095	4290
$\Delta_{PSp_6(3)}(3D, 13B, rX)$	0	0	0	0	0	0	0	0	4860	2400	3360	4290	4095
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Table 5.5: Structure constants of $PSp_6(3)$ for $(3E, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3E, 2A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3E, 2B, rX)$	0	0	0	0	0	0	729	0	0	0	14	13	13
$\Delta_{PSp_6(3)}(3E, 3A, rX)$	0	0	0	0	0	36	4	3	0	0	0	0	0
$\Delta_{PSp_6(3)}(3E, 3B, rX)$	0	0	0	0	36	0	4	6	0	0	0	0	0
$\Delta_{PSp_6(3)}(3E, 3C, rX)$	0	0	2160	0	72	108	15	36	0	0	0	0	0
$\Delta_{PSp_6(3)}(3E, 3D, rX)$	0	0	0	4320	216	180	282	36	36	0	0	0	0
$\Delta_{PSp_6(3)}(3E, 3E, rX)$	0	0	4320	2160	432	216	72	1026	36	40	28	13	13
$\Delta_{PSp_6(3)}(3E, 3F, rX)$	0	1008	2880	2880	180	1692	72	72	144	0	0	13	13
$\Delta_{PSp_6(3)}(3E, 3G, rX)$	0	0	0	0	0	3888	2592	648	828	200	266	247	247
$\Delta_{PSp_6(3)}(3E, 5A, rX)$	0	0	0	0	0	0	0	5832	1620	1440	1596	1950	1950
$\Delta_{PSp_6(3)}(3E, 7A, rX)$	0	24192	0	0	0	0	0	34992	18468	13680	17010	18720	18720
$\Delta_{PSp_6(3)}(3E, 13A, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20241	20358
$\Delta_{PSp_6(3)}(3E, 13B, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20358	20241
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Table 5.6: Structure constants of $PSp_6(3)$ for $(3F, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3F, 2A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(3F, 2B, rX)$	0	0	0	0	0	0	0	729	0	0	14	13	13
$\Delta_{PSp_6(3)}(3F, 3A, rX)$	0	0	0	0	36	0	6	4	0	0	0	0	0
$\Delta_{PSp_6(3)}(3F, 3B, rX)$	0	0	0	0	0	36	3	4	0	0	0	0	0
$\Delta_{PSp_6(3)}(3F, 3C, rX)$	0	0	0	2160	72	108	36	15	0	0	0	0	0
$\Delta_{PSp_6(3)}(3F, 3D, rX)$	0	0	4320	0	216	180	36	282	36	0	0	0	0
$\Delta_{PSp_6(3)}(3F, 3E, rX)$	0	1008	2880	2880	180	1692	72	72	144	0	0	13	13
$\Delta_{PSp_6(3)}(3F, 3F, rX)$	0	0	2160	4320	432	216	1026	72	36	40	28	13	13
$\Delta_{PSp_6(3)}(3F, 3G, rX)$	0	0	0	0	0	3888	648	2592	828	200	266	247	247
$\Delta_{PSp_6(3)}(3F, 5A, rX)$	0	0	0	0	0	0	5832	0	1620	1440	1596	1950	1950
$\Delta_{PSp_6(3)}(3F, 7A, rX)$	0	24192	0	0	0	0	34992	0	18468	13680	17010	18720	18720
$\Delta_{PSp_6(3)}(3F, 13A, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20241	20358
$\Delta_{PSp_6(3)}(3F, 13B, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20358	20241
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Proof. First, we consider non-generating cases. By Tables 5.1–5.6, we have $\Delta_{PSp_6(3)}(3X, 3Y, 7A) = 0$ for $X, Y \in \{A, B, C, D, E, F, G\}$, except when $(X, Y) \in \{(C, G), (E, E), (E, G), (F, F), (F, G), (G, G)\}$. Hence, $PSp_6(3)$ is not generated by any of these triples. Now, we consider the case $X \in \{E, F\}$. By Tables 5.5 and 5.6, we have $\Delta_{PSp_6(3)}(3X, 3X, 7A) = 28$. We observe from Table 4.3 that only the maximal subgroup M_6 has the required fusions. We calculate $\Sigma(M_6) = 28 = \Delta_{PSp_6(3)}(3X, 3X, 7A)$, and hence, by Lemma 2.6, $(3X, 3X, 7A)$ does not generate $PSp_6(3)$ for $X \in \{E, F\}$.

In the rest of the proof, we discuss the generating cases. By Table 4.3, there is only one maximal subgroup of $PSp_6(3)$ that fuse to the classes $3C, 3G$ and $7A$, namely M_7. From Table 5.3, we have $\Delta_{PSp_6(3)}(3C, 3G, 7A) = 35$. Also, we have $\Sigma(M_7) = 7$ and $h(7A, M_7) = 1$. Therefore,

$$\Delta^*_{PSp_6(3)}(3C, 3G, 7A) \geq \Delta_{PSp_6(3)}(3C, 3G, 7A) - 1 \cdot \Sigma(M_7) = 35 - 1(7) = 28,$$

showing that $(3C, 3G, 7A)$ generates $PSp_6(3)$.

Again, by Table 4.3, there is only one maximal subgroup of $PSp_6(3)$ that fuse to the classes $3E, 3G$ and $7A$, namely M_6. However, its contribution is 0 since $\Sigma(M_6) = 0$. So, there is no contribution to $\Delta^*_{PSp_6(3)}(3E, 3G, 7A)$ by any maximal subgroup. Hence, it follows that

$$\Delta^*_{PSp_6(3)}(3E, 3G, 7A) = \Delta_{PSp_6(3)}(3E, 3G, 7A) = 266.$$

Thus, $PSp_6(3)$ is a $(3E, 3G, 7A)$-generated group. Similarly, $\Delta^*_{PSp_6(3)}(3F, 3G, 7A) = 266$ and $PSp_6(3)$ is a $(3F, 3G, 7A)$-generated group.

Table 5.7: Cycle structures of the conjugacy classes of $PSp_6(3)$.

nX	Cycle Structure	c_i	d_i
$2A$	$1^{44}2^{160}$	204	160
$2B$	2^{182}	182	182
$3A$	$1^{121}3^{81}$	202	162
$3B$	$1^{121}3^{81}$	202	162
$3C$	$1^{40}3^{108}$	148	216
$3D$	$1^{40}3^{108}$	148	216
$3E$	$1^{13}3^{117}$	130	234
$3F$	$1^{13}3^{117}$	130	234
$3G$	$1^{4}3^{120}$	124	240
$4A$	$1^{40}2^{2}4^{80}$	120	244
$4B$	$1^{8}2^{18}4^{80}$	186	178
$4C$	$1^{4}2^{20}4^{80}$	104	260
$5A$	$1^{4}5^{72}$	76	288
$6A$	$1^{41}2^{40}3^{1}6^{40}$	122	242
$6B$	$1^{41}2^{40}3^{1}6^{40}$	122	242
$6C$	$1^{14}2^{13}3^{10}6^{49}$	86	278
$6D$	$1^{14}2^{13}3^{10}6^{49}$	86	278
$6E$	$1^{14}2^{13}3^{10}6^{49}$	86	278
$6F$	$1^{14}2^{13}3^{10}6^{49}$	86	278
$6G$	$1^{17}2^{52}3^{9}6^{36}$	114	250
$6H$	$1^{17}2^{52}3^{9}6^{36}$	114	250
$6I$	$1^{8}2^{16}3^{12}6^{48}$	84	280
$6J$	$1^{8}2^{16}3^{12}6^{48}$	84	280
$6K$	$1^{5}2^{4}3^{13}6^{52}$	74	290
$6L$	$1^{5}2^{4}3^{13}6^{52}$	74	290
$6M$	$2^{20}6^{54}$	74	290
$6N$	$1^{5}2^{4}3^{13}6^{52}$	74	290
$6O$	$1^{5}2^{4}3^{13}6^{52}$	74	290
$6P$	$2^{2}6^{60}$	62	302
$7A$	7^{52}	52	312
$8A$	$2^{2}4^{10}8^{40}$	52	312
$8B$	$1^{4}4^{10}8^{40}$	52	312
$9A$	$1^{13}3^{36}9^{27}$	76	288
$9B$	$1^{4}3^{39}9^{27}$	70	294
$9C$	$1^{4}3^{39}9^{27}$	70	294
$9D$	$1^{4}3^{39}9^{27}$	70	294
$9E$	$1^{4}3^{39}9^{27}$	70	294
$9F$	$1^{4}3^{39}9^{27}$	70	294
$9G$	$1^{1}3^{4}9^{39}$	44	320
$9H$	$1^{1}3^{4}9^{39}$	44	320

nX	Cycle Structure	c_i	d_i
$10A$	$1^4 5^8 10^{32}$	44	320
$12A$	$1^{13} 2^2 3^9 4^{26} 12^{18}$	68	296
$12B$	$1^{13} 2^2 3^9 4^{26} 12^{18}$	68	296
$12C$	$1^4 2^2 3^{12} 4^8 12^{24}$	50	314
$12D$	$1^1 2^{20} 3^1 4^{20} 12^{20}$	62	302
$12E$	$1^1 2^{20} 3^1 4^{20} 12^{20}$	62	302
$12F$	$1^5 2^6 3^1 4^{26} 6^4 12^{18}$	60	304
$12G$	$1^5 2^6 3^1 4^{26} 6^4 12^{18}$	60	304
$12H$	$1^4 2^2 3^{12} 4^8 12^{24}$	50	314
$12I$	$1^4 2^2 4^8 6^6 12^{24}$	44	320
$12J$	$1^4 2^3 3^2 4^8 6^5 12^{24}$	46	318
$12K$	$1^4 2^3 3^2 4^8 6^5 12^{24}$	46	318
$12L$	$1^4 2^3 3^2 4^8 6^5 12^{24}$	46	318
$12M$	$1^1 2^2 3^1 4^2 6^6 12^{26}$	38	326
$12N$	$1^1 2^2 3^1 4^2 6^6 12^{26}$	38	326
$13A$	13^{28}	28	336
$13B$	13^{28}	28	336
$14A$	14^{26}	26	338
$15A$	$1^1 3^1 5^{24} 15^{16}$	42	322
$15B$	$1^1 3^1 5^{24} 15^{16}$	42	322
$18A$	$1^5 2^4 3^4 6^{16} 9^3 18^{12}$	44	320
$18B$	$1^5 2^4 3^4 6^{16} 9^3 18^{12}$	44	320
$18C$	$1^2 2^1 3^5 6^{17} 9^3 18^{12}$	40	324
$18D$	$1^5 2^4 3^4 6^{16} 9^3 18^{12}$	40	324
$18E$	$1^5 2^4 3^4 6^{16} 9^3 18^{12}$	40	324
$18F$	$1^5 2^4 3^4 6^{16} 9^3 18^{12}$	40	324
$20A$	$2^2 5^8 20^{16}$	26	338
$24A$	$1^1 3^1 4^{10} 8^{10} 24^{10}$	32	332
$24B$	$1^1 3^1 4^{10} 8^{10} 24^{10}$	32	332
$30A$	$1^1 3^1 5^8 10^8 30^8$	26	338
$30B$	$1^1 3^1 5^8 10^8 30^8$	26	338
$36A$	$1^1 2^2 3^4 4^2 9^3 12^8 36^6$	26	338
$36B$	$1^1 2^2 3^4 4^2 9^3 12^8 36^6$	26	338

Now, from Table 5.8, it follows that $\Delta_{PSp_6(3)}(3G, 3G, 7A) = 3885$.

Calculations using GAP reveal the following distribution of pairs (x, y) satisfying $x, y \in 3G$, $xy = z$ for a fixed $z \in 7A$, and $\langle x, y\rangle \leq PSp_6(3)$:

- 7 pairs generate proper subgroups isomorphic to $PSL(3, 2)$ of order 168,
- 14 pairs generate proper subgroups isomorphic to 7:3 of order 21,
- 56 pairs generate proper subgroups isomorphic to $PSL(2, 27)$:3 of order 29484,
- 56 pairs generate proper subgroups isomorphic to $PSU(3, 3)$ of order 6048,
- 336 pairs generate proper subgroups isomorphic to $PSL(2, 13)$ of order 1092 and,
- 3416 pairs generate the entire of $PSp_6(3)$.

It is clear from the above distribution (see the last bullet) that $PSp_6(3)$ is $(3G, 3G, 7A)$-generated. □

Table 5.8: Structure constants of $PSp_6(3)$ for $(3G, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(3G, 2A, rX)$	1920	0	0	0	0	0	0	0	81	0	7	13	13
$\Delta_{PSp_6(3)}(3G, 2B, rX)$	0	2016	0	0	0	0	0	0	648	240	168	156	156
$\Delta_{PSp_6(3)}(3G, 3A, rX)$	0	0	0	0	0	0	0	0	4	0	0	0	0
$\Delta_{PSp_6(3)}(3G, 3B, rX)$	0	0	0	0	0	0	0	0	4	0	0	0	0
$\Delta_{PSp_6(3)}(3G, 3C, rX)$	0	0	0	0	0	0	0	0	6	80	35	26	26
$\Delta_{PSp_6(3)}(3G, 3D, rX)$	0	0	0	0	0	3888	648	648	624	0	0	65	65
$\Delta_{PSp_6(3)}(3G, 3E, rX)$	0	0	0	0	0	3888	2592	648	828	200	266	247	247
$\Delta_{PSp_6(3)}(3G, 3F, rX)$	0	0	0	0	0	3888	648	2592	828	200	266	247	247
$\Delta_{PSp_6(3)}(3G, 3G, rX)$	51840	16128	51840	51840	1296	67392	14904	14904	15282	1920	3885	5265	5265
$\Delta_{PSp_6(3)}(3G, 5A, rX)$	0	48384	0	0	139968	0	29160	29160	15552	66960	51408	39312	39312
$\Delta_{PSp_6(3)}(3G, 7A, rX)$	311040	290304	0	0	524880	0	332424	332424	269730	440640	394065	336960	336960
$\Delta_{PSp_6(3)}(3G, 13A, rX)$	622080	290304	0	0	419904	524880	332424	332424	393660	362880	362880	373815	372762
$\Delta_{PSp_6(3)}(3G, 13B, rX)$	622080	290304	0	0	419904	524880	332424	332424	393660	362880	362880	372762	373815
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

Proposition 5.1.3. *The group* $PSp_6(3)$ *is not* $(3X, 3Y, 13Z)$*-generated for* $X, Y \in \{A, B, C, D, E, F, G\}$ *and* $Z \in \{A, B\}$*, except for the triples* $(3C, 3G, 13Z)$*,* $(3E, 3G, 13Z)$*,* $(3F, 3G, 13Z)$ *and* $(3G, 3G, 13Z)$*.*

Proof. Non-Generation Cases:
Calculations via GAP give $\Delta_{PSp_6(3)}(3X, 3Y, 13Z) = 0$, except for the triples $(3D, 3G, 13Z)$, $(3E, 3E, 13Z)$, $(3E, 3F, 13Z)$ and $(3F, 3F, 13Z)$.

For the triple $(3D, 3G, 13Z)$, we have $\Delta_{PSp_6(3)}(3D, 3G, 13Z) = 65$. Using GAP, we found that 13 pairs out of the 65 pairs generate subgroups of $PSp_6(3)$ that are isomorphic to $PSL(3, 3)$, while the remaining 52 pairs generate subgroups of $PSp_6(3)$ that are isomorphic to $3^6{:}PSL(3, 3)$. Thus, none of the 65 pairs $(x, y) \in 3D \times 3G$ with $xy = z$, generates the entire group $PSp_6(3)$, where z is a fixed element of $13Z$. Hence, $PSp_6(3)$ is not $(3D, 3G, 13Z)$-generated.

Now, we deal with $(3E, 3E, 13Z)$, $(3E, 3F, 13Z)$ and $(3F, 3F, 13Z)$ for $Z \in \{A, B\}$. In these cases, the maximal subgroups with the required fusions are M_2 and M_6 only. Calculations give $\Sigma(M_2) = 0$, $\Sigma(M_6) = 13$ and $\Delta_{PSp_6(3)}(3X, 3Y, 13Z) = 13$ for $X, Y \in \{E, F\}$. Since $\Sigma(M_6) = \Delta_{PSp_6(3)}(3X, 3Y, 13Z)$, Lemma 2.6 shows that $\Delta^*_{PSp_6(3)}(3X, 3Y, 13Z) = 0$, establishing the non-generation by the mentioned triples.

Generation Cases:
By Table 4.3, only M_2 has classes that fuse into classes $3C$, $3G$ and $13Z$ of $PSp_6(3)$. Since $\Sigma(M_2) = 0$, no maximal subgroup of $PSp_6(3)$ makes any contribution to the calculations of $\Delta^*_{PSp_6(3)}(3C, 3G, 13Z)$. It follows that $\Delta^*_{PSp_6(3)}(3C, 3G, 13Z) = \Delta_{PSp_6(3)}(3C, 3G, 13Z) = 26$. Hence, $PSp_6(3)$ is generated by $(3C, 3G, 13Z)$ for $Z \in \{A, B\}$.

Again, by Table 4.3, there are only two maximal subgroups of $PSp_6(3)$ that fuse to the classes $3E, 3G$ and $13Z$ for $Z \in \{A, B\}$, namely M_2 and M_6. We obtain $\Sigma(M_2) = 39$ and $\Sigma(M_6) = 0$. Since $\Delta_{PSp_6(3)}(3E, 3G, 13Z) = 247$, $h(13Z, M_2) = 2$ and $h(13Z, M_6) = 1$, we have

$$\Delta^*_{PSp_6(3)}(3E, 3G, 13Z) \geq \Delta_{PSp_6(3)}(3E, 3G, 13Z) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_6) = 247 - 2(39) - 1(0) = 169.$$

This implies that, for $Z \in \{A, B\}$, $(3E, 3G, 13Z)$ generates $PSp_6(3)$. Similarly, for $Z \in \{A, B\}$, $(3F, 3G, 13Z)$ generates $PSp_6(3)$.

For a fixed $z \in 13Z$, with $Z \in \{A, B\}$, there are 5265 pairs $(x, y) \in 3G \times 3G$ such that $xy = z$. Now, investigations through GAP reveal the following distribution of these pairs:

- 26 pairs generate proper subgroups of the form $13{:}3$ of order 39,
- 26 pairs generate proper subgroups of the form $PSL(2, 13)$ of order 1092,
- 39 pairs generate proper subgroups of the form $PSL(3, 3)$ of order 5616,
- 52 pairs generate proper subgroups of the form $PSL(2, 27){:}3$ of order 29484,

- 208 pairs generate proper subgroups of the form $(3^6{:}13){:}3$ of order 28431,
- 208 pairs generate proper subgroups of the form $(3^3{:}13){:}3$ of order 1053,
- 1560 pairs generate proper subgroups of the form $3^6{:}PSL(3,3)$ of order 4094064, and the remaining,
- 3146 pairs generate the whole of $PSp_6(3)$.

We see in the last bullet that 3146 pairs $(x,y) \in 3G \times 3G$, with $xy = z \in 13Z$, generate $PSp_6(3)$. Therefore, we conclude that $PSp_6(3)$ is $(3G, 3G, 13Z)$-generated for $Z \in \{A, B\}$. □

5.2. $(3, 5, r)$-Generation

In this section, we deal with the following triples: $(3X, 5A, 5A), (3X, 5A, 7A)$ and $(3X, 5A, 13Y)$ for all $X \in \{A, B, C, D, E, F, G\}$ and $Y \in \{A, B\}$.

Proposition 5.2.1. *The group* $PSp_6(3)$ *is*

(i) not $(3X, 5A, 5A)$*-generated for every* $X \in \{A, B, C\}$*,*

(ii) $(3X, 5A, 5A)$*-generated for every* $X \in \{D, E, F, G\}$*.*

Proof. (i) From Tables 5.1 and 5.2, we have $\Delta_{PSp_6(3)}(3X, 5A, 5A) = 0$ for $X \in \{A, B\}$. Furthermore, $\Delta_{PSp_6(3)}(3C, 5A, 5A) = 15 < 120 = |C_{PSp_6(3)}(5A)|$. Thus, non-generation by the triples $(3A, 5A, 5A)$, $(3B, 5A, 5A)$ and $(3C, 5A, 5A)$ immediately follows.

(ii) By Table 4.3, there are only two maximal subgroups of $PSp_6(3)$ that fuse to the classes $3D$ and $5A$, namely M_1 and M_4. We obtain $\Sigma_{M_1}(3x, 5a, 5a) = 80 + 375 + 0 + 0 = 455, \Sigma_{M_4}(3y, 5a, 5a) = 0 + 0 + 75 = 75$ for all $x \in \{c, g, l, p\}$ and $y \in \{e, f, m\}$, and also, $h(5A, M_1) = 4$ and $h(5A, M_4) = 1$. By Table 5.4, it holds that $\Delta_{PSp_6(3)}(3D, 5A, 5A) = 2075$. Thus, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3D, 5A, 5A) &\geq \Delta_{PSp_6(3)}(3D, 5A, 5A) - 4 \cdot \Sigma_{M_1}(3x, 5a, 5a) - 1 \cdot \Sigma_{M_4}(3y, 5a, 5a) \\ &= 2075 - 4(455) - 1(75) > 0. \end{aligned}$$

This implies that $(3D, 5A, 5A)$ generates $PSp_6(3)$.

By Table 4.3, the only maximal subgroups of $PSp_6(3)$ with non-trivial intersection with the classes $3E$ and $5A$ are M_1 and M_4. We obtain $\Sigma_{M_1}(3x, 5a, 5a) = 60+150+0 = 210$ and $\Sigma_{M_4}(3y, 5a, 5a) = 0+0 = 0$ for all $x \in \{d, h, n\}$ and $y \in \{k, l\}$, and also, $h(5A, M_1) = 4$ and $h(5A, M_4) = 1$. Since $\Delta_{PSp_6(3)}(3E, 5A, 5A) = 1440$, it follows that

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3E, 5A, 5A) &\geq \Delta_{PSp_6(3)}(3E, 5A, 5A) - 4 \cdot \Sigma_{M_1}(3x, 5a, 5a) - 1 \cdot \Sigma_{M_4}(3y, 5a, 5a) \\ &= 1440 - 4(210) - 1(0) > 0. \end{aligned}$$

This implies that $(3E, 5A, 5A)$ generates $PSp_6(3)$. Similarly, $(3F, 5A, 5A)$ generates $PSp_6(3)$.

By Table 4.3, there are only two maximal subgroups of $PSp_6(3)$ that fuse to the classes $3G$ and $5A$, namely, M_1 and M_{11}. We obtain $\Delta_{PSp_6(3)}(3G, 5A, 5A) = 66960$, $\Sigma(M_1) = 2700$, $\Sigma(M_{11}) = 5 + 5 = 10$ (no proper subgroup of $M_{11} \cong A_5$ contains elements of orders 3 and 5), $h(5A, M_1) = 4$ and $h(5A, M_{11}) = 48$. Therefore, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3G, 5A, 5A) &\geq \Delta_{PSp_6(3)}(3G, 5A, 5A) - 4 \cdot \Sigma(M_1) - 48 \cdot \Sigma(M_{11}) \\ &= 66960 - 4(2700) - 48(10) > 0, \end{aligned}$$

which shows that $PSp_6(3)$ is $(3G, 5A, 5A)$-generated. □

Proposition 5.2.2. *The group* $PSp_6(3)$ *is*

(i) not $(3X, 5A, 7A)$*-generated for every* $X \in \{A, B, C\}$*,*

(ii) $(3X, 5A, 7A)$*-generated for every* $X \in \{D, E, F, G\}$*.*

Proof. (i) The result follows from the fact that $\Delta_{PSp_6(3)}(3X, 5A, 7A) = 0$ for $X \in \{A, B, C\}$.

(ii) There is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $3X$, $5A$ and $7A$ for $X \in \{D, E, F, G\}$. Thus, there is no contribution to the calculations of $\Delta^*_{PSp_6(3)}(3X, 5A, 7A)$ from any maximal subgroup. It follows that $\Delta^*_{PSp_6(3)}(3X, 5A, 7A) = \Delta_{PSp_6(3)}(3X, 5A, 7A)$. We calculate

$$\begin{aligned}
\Delta^*_{PSp_6(3)}(3D, 5A, 7A) &= \Delta_{PSp_6(3)}(3D, 5A, 7A) = 896,\\
\Delta^*_{PSp_6(3)}(3X, 5A, 7A) &= \Delta_{PSp_6(3)}(3X, 5A, 7A) = 1596, \text{ for } X \in \{E, F\}\\
\Delta^*_{PSp_6(3)}(3G, 5A, 7A) &= \Delta_{PSp_6(3)}(3G, 5A, 7A) = 51408,
\end{aligned}$$

showing that $PSp_6(3)$ is $(3X, 5A, 7A)$-generated for every $X \in \{D, E, F, G\}$. □

Proposition 5.2.3. *The group* $PSp_6(3)$ *is*

(i) not $(3X, 5A, 13Z)$*-generated for* $X, Z \in \{A, B\}$*.*

(ii) $(3X, 5A, 13Z)$*-generated for* $X \in \{C, D, E, F, G\}$ *and* $Z \in \{A, B\}$*.*

Proof. (i) We note that $\Delta_{PSp_6(3)}(3X, 5A, 13Z) = 0$ for $X, Z \in \{A, B\}$, and hence, the desired result follows.

(ii) By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $3X$, $5A$ and $13Z$ for $X \in \{C, D, E, F, G\}$ and $Z \in \{A, B\}$. Hence, there is no contribution to the calculations of $\Delta^*_{PSp_6(3)}(3X, 5A, 13Z)$ from any maximal subgroup. So, $\Delta^*_{PSp_6(3)}(3X, 5A, 13Z) = \Delta_{PSp_6(3)}(3X, 5A, 13Z)$. Here, we have $\Delta^*_{PSp_6(3)}(3C, 5A, 13Z) = \Delta_{PSp_6(3)}(3C, 5A, 13Z) = 104$,

$$\begin{aligned}
\Delta^*_{PSp_6(3)}(3D, 5A, 13Z) &= \Delta_{PSp_6(3)}(3D, 5A, 13Z) = 260,\\
\Delta^*_{PSp_6(3)}(3X, 5A, 13Z) &= \Delta_{PSp_6(3)}(3X, 5A, 13Z) = 1950, \text{ for } X \in \{E, F\},\\
\Delta^*_{PSp_6(3)}(3G, 5A, 13Z) &= \Delta_{PSp_6(3)}(3G, 5A, 13Z) = 39312.
\end{aligned}$$

Since $\Delta^*_{PSp_6(3)}(3X, 5A, 13Z) > 0$ for all $X \in \{C, D, E, F, G\}$ and $Z \in \{A, B\}$, we conclude that $PSp_6(3)$ is $(3X, 5A, 13Z)$-generated. □

5.3. $(3, 7, r)$-Generation

We only need to deal with the following triples: $(3X, 7A, 7A)$ and $(3X, 7A, 13Z)$ for all $Z \in \{A, B\}$ and $X \in \{A, B, C, D, E, F, G\}$.

Proposition 5.3.1. *The group* $PSp_6(3)$ *is generated by* $(3X, 7A, 7A)$ *for every* $X \in \{A, B, C, D, E, F, G\}$*.*

Proof. By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $3X$ and $7A$ for $X \in \{A, B, D\}$. Thus, there is no contribution to $\Delta^*_{PSp_6(3)}(3X, 7A, 7A)$ from any maximal subgroup. Hence, we have

$$\Delta^*_{PSp_6(3)}(3X, 7A, 7A) = \Delta_{PSp_6(3)}(3X, 7A, 7A) = \begin{cases} 14 & \text{when } X \in \{A, B\},\\ 5880 & \text{when } X = D. \end{cases}$$

Therefore, $PSp_6(3)$ is generated by $(3X, 7A, 7A)$ for $X \in \{A, B, D\}$.

Only the maximal subgroup M_7 has classes that fuse to the classes $3C$ and $7A$ of $PSp_6(3)$. We obtain $\Sigma(M_7) = 7$, $h(7A, M_7) = 1$ and $\Delta_{PSp_6(3)}(3C, 7A, 7A) = 735$. Thus, we have

$$\Delta^*_{PSp_6(3)}(3C, 7A, 7A) \geq \Delta_{PSp_6(3)}(3C, 7A, 7A) - 1 \cdot \Sigma(M_7) = 735 - 1(7) > 0,$$

which proves generation by the triple $(3C, 7A, 7A)$.

Only the maximal subgroup M_6 has classes that fuse to the classes $3X$ and $7A$ of $PSp_6(3)$ for $X \in \{E, F\}$. Here, calculations yield $\Sigma(M_6) = 70$, $h(7A, M_6) = 1$ and $\Delta_{PSp_6(3)}(3X, 7A, 7A) = 17010$. Now,

$$\Delta^*_{PSp_6(3)}(3X, 7A, 7A) \geq \Delta_{PSp_6(3)}(3X, 7A, 7A) - 1 \cdot \Sigma(M_6) = 17010 - 1(70) > 0.$$

This shows that $PSp_6(3)$ is generated by the triples $(3E, 7A, 7A)$ and $(3F, 7A, 7A)$.

From Table 4.3, we note that the only maximal subgroups of $PSp_6(3)$ that fuse to the classes $3G$ and $7A$ are M_6, M_7, M_9 and M_{10}. By Remark 2.3, intersections may be ignored. Calculations with GAP give $\Sigma(M_6) = 0$, $\Sigma(M_7) = 189$, $\Sigma(M_9) = 28$ and $\Sigma(M_{10}) = 28$. The corresponding values of h are $h(7A, M_6) = h(7A, M_7) = 1$ and $h(7A, M_9) = h(7A, M_{10}) = 6$. By Table 5.8, we have that $\Delta_{PSp_6(3)}(3G, 7A, 7A) = 394065$. Hence,

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3G, 7A, 7A) &\geq \Delta_{PSp_6(3)}(3G, 7A, 7A) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_7) - 6 \cdot \Sigma(M_9) - 6 \cdot \Sigma(M_{10}) \\ &= 394065 - 1(0) - 1(189) - 6(28) - 6(28) > 0, \end{aligned}$$

which yields the desired result. □

Proposition 5.3.2. *The group $PSp_6(3)$ is generated by $(3X, 7A, 13Z)$ for all $X \in \{A, B, C, D, E, F, G\}$ and $Z \in \{A, B\}$.*

Proof. By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $3X, 7A$ and $13Z$ for $X \in \{A, B, C, D\}$ and $Z \in \{A, B\}$. Thus, there is no contribution to $\Delta^*_{PSp_6(3)}(3X, 7A, 13Z)$ from any maximal subgroup. Consequently, for $X, Z \in \{A, B\}$, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3X, 7A, 13Z) &= \Delta_{PSp_6(3)}(3X, 7A, 13Z) &&= 26, \\ \Delta^*_{PSp_6(3)}(3C, 7A, 13Z) &= \Delta_{PSp_6(3)}(3C, 7A, 13Z) &&= 1560, \\ \Delta^*_{PSp_6(3)}(3D, 7A, 13Z) &= \Delta_{PSp_6(3)}(3D, 7A, 13Z) &&= 3120, \end{aligned}$$

and therefore, generation follows.

By Table 4.3, there is only one maximal subgroup of $PSp_6(3)$ that fuses to the classes $3E$, $7A$ and $13Z$ for $Z \in \{A, B\}$, namely M_6. Here, we calculate $\Sigma(M_6) = 78$ and $h(13Z, M_6) = 1$. From Table 5.5, we have $\Delta_{PSp_6(3)}(3E, 7A, 13Z) = 18720$. Therefore,

$$\Delta^*_{PSp_6(3)}(3E, 7A, 13Z) \geq \Delta_{PSp_6(3)}(3E, 7A, 13Z) - 1 \cdot \Sigma(M_6) = 18720 - 1(78) = 18642.$$

Similarly, $\Delta^*_{PSp_6(3)}(3F, 7A, 13Z) \geq 18642$. Thus, $PSp_6(3)$ is generated by $(3X, 7A, 13Z)$ for $X \in \{E, F\}$ and $Z \in \{A, B\}$.

For the triple $(3G, 7A, 13Z)$, with $Z \in \{A, B\}$, only three maximal subgroups, namely M_6, M_9 and M_{10}, have the required fusions. By Remark 2.3, intersections may be ignored. Calculations via GAP yield $\Sigma(M_6) = 0$, $\Sigma(M_9) = 26$, $\Sigma(M_{10}) = 26$ and $\Delta_{PSp_6(3)}(3G, 7A, 13Z) = 336960$ for $Z \in \{A, B\}$. With $h(13Z, M_6) = h(13Z, M_9) = h(13Z, M_{10}) = 1$, we have

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3G, 7A, 13A) &\geq \Delta_{PSp_6(3)}(3G, 7A, 13A) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10}) \\ &= 336960 - 1(0) - 1(26) - 1(26) > 0. \end{aligned}$$

Therefore, $(3G, 7A, 13Z)$ generates $PSp_6(3)$ for $Z \in \{A, B\}$. □

5.4. $(3, 13, r)$-Generation

We only need to consider the triple $(3X, 13Y, 13Z)$ for all $X \in \{A, B, C, D, E, F, G\}$ and $Y, Z \in \{A, B\}$.

Proposition 5.4.1. *The group $PSp_6(3)$ is generated by $(3X, 13Y, 13Z)$ for all $X \in \{A, B, C, D, E, F, G\}$ and $Y, Z \in \{A, B\}$.*

Proof. Case 1: Consider $(3X, 13Y, 13Z)$ with $X, Y, Z \in \{A, B\}$.
There is only one maximal subgroup of $PSp_6(3)$ that fuses to the classes $3A$ and $13Y$ for $Y \in \{A, B\}$, namely M_2. Here, we have $h(13Z, M_2) = 2$. We calculate $\Sigma_{M_2}(3b, 13y, 13y) = 13$ and $\Sigma_{M_2}(3b, 13y, 13z) = 0$ for all $y, z \in \{a, b, c, d\}$ with $y \neq z$.

By Table 5.1, we have $\Delta_{PSp_6(3)}(3X, 13Y, 13Y) = 39$ for $X, Y \in \{A, B\}$, and hence,

$$\Delta^*_{PSp_6(3)}(3X, 13Y, 13Y) \geq \Delta_{PSp_6(3)}(3X, 13Y, 13Y) - 2 \cdot \Sigma_{M_2}(3b, 13y, 13y) = 39 - 2(13) = 13 > 0.$$

Also, by Table 5.1, we have $\Delta_{PSp_6(3)}(3X, 13Y, 13Z) = 26$ for $X, Y, Z \in \{A, B\}$ with $Y \neq Z$, and hence,

$$\Delta^*_{PSp_6(3)}(3X, 13Y, 13Z) \geq \Delta_{PSp_6(3)}(3X, 13Y, 13Z) - 2 \cdot \Sigma_{M_2}(3b, 13y, 13z) = 26 - 2(0) = 26 > 0.$$

Consequently, we conclude that $PSp_6(3)$ is generated by $(3X, 13Y, 13Z)$ for all $X, Y, Z \in \{A, B\}$.

Case 2: Consider $(3C, 13Y, 13Z)$ with $Y, Z \in \{A, B\}$.
By Table 4.3, there is only one maximal subgroup of $PSp_6(3)$ that fuses to the classes $3C$ and $13Y$ for each $Y \in \{A, B\}$, namely M_2, and $h(13Z, M_2) = 2$. We obtain $\Sigma_{M_2}(3a, 13y, 13y) = 78$ and $\Sigma_{M_2}(3a, 13y, 13z) = 0$ for all $y, z \in \{a, b, c, d\}$ with $y \neq z$. By Table 5.3, we have $\Delta_{PSp_6(3)}(3C, 13Y, 13Y) = 2106$ for $Y \in \{A, B\}$, and hence,

$$\Delta^*_{PSp_6(3)}(3C, 13Y, 13Y) \geq \Delta_{PSp_6(3)}(3C, 13Y, 13Y) - 2 \cdot \Sigma_{M_2}(3a, 13y, 13y) = 2106 - 2(78) = 1950 > 0.$$

Also, by Table 5.3, we have $\Delta_{PSp_6(3)}(3C, 13Y, 13Z) = 2028$ for $Y, Z \in \{A, B\}$ with $Y \neq Z$, and thus,

$$\Delta^*_{PSp_6(3)}(3C, 13Y, 13Z) \geq \Delta_{PSp_6(3)}(3C, 13Y, 13Z) - 2 \cdot \Sigma_{M_2}(3a, 13y, 13z) = 2028 - 2(0) = 2028 > 0.$$

Therefore, we conclude that $PSp_6(3)$ is generated by $(3C, 13Y, 13Z)$ for all $Y, Z \in \{A, B\}$.

Case 3: Consider $(3D, 13Y, 13Z)$ with $Y, Z \in \{A, B\}$.
For this case, there are two maximal subgroups of $PSp_6(3)$ that fuse to the classes $3D$ and $13Y$ for $Y \in \{A, B\}$, namely M_2 and M_8, with $h(13Z, M_2) = 2$ and $h(13Z, M_8) = 1$. By Remark 2.3, the intersection may be ignored. Calculations give $\Sigma_{M_2}(3x, 13y, 13y) = 156$, $\Sigma_{M_2}(3x, 13y, 13z) = 351$, $\Sigma_{M_8}(3a, 13u, 13u) = 13$ and $\Sigma_{M_8}(3a, 13u, 13v) = 26$ for all $x \in \{d, h\}$, $y, z \in \{a, b, c, d\}$ and $u, v \in \{a, b\}$ with $y \neq z$ and $u \neq v$. By Table 5.4, we have $\Delta_{PSp_6(3)}(3D, 13Y, 13Y) = 4095$ for $Y \in \{A, B\}$, and hence,

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3D, 13Y, 13Y) &\geq \Delta_{PSp_6(3)}(3D, 13Y, 13Y) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13y) - 1 \cdot \Sigma_{M_8}(3a, 13u, 13u) \\ &= 4095 - 2(156) - 1(13) > 0. \end{aligned}$$

Also, by Table 5.4, we have $\Delta_{PSp_6(3)}(3D, 13Y, 13Z) = 4290$ for $Y, Z \in \{A, B\}$ with $Y \neq Z$, and thus,

$$\begin{aligned} \Delta^*_{PSp_6(3)}(3D, 13Y, 13Z) &\geq \Delta_{PSp_6(3)}(3D, 13Y, 13Z) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13z) - 1 \cdot \Sigma_{M_8}(3a, 13u, 13v) \\ &= 4290 - 2(351) - 1(26) > 0. \end{aligned}$$

Therefore, $PSp_6(3)$ is generated by $(3D, 13Y, 13Z)$ for all $Y, Z \in \{A, B\}$.

Case 4: Consider $(3X, 13Y, 13Z)$ with $X \in \{E, F\}$ and $Y, Z \in \{A, B\}$.
By Table 4.3, only M_2 and M_6 have classes that fuse to the classes $3E$ (respectively $3F$) and $13Y$ for all $Y \in \{A, B\}$. By Remark 2.3, the intersection may be ignored. Computations with GAP give $\Sigma_{M_2}(3x, 13y, 13y) = 234$, $\Sigma_{M_2}(3x, 13y, 13z) = 351$, $\Sigma_{M_6}(3b, 13u, 13u) = 91$, $\Sigma_{M_6}(3b, 13u, 13v) = 78$, $h(13Z, M_2) = 2$ and $h(13Z, M_6) = 1$ for all $x \in \{f, i\}$, $y, z \in \{a, b, c, d\}$ and $u, v \in \{a, b\}$ with $y \neq z$ and $u \neq v$. By Tables 5.5 and 5.6, we have $\Delta_{PSp_6(3)}(3X, 13Y, 13Y) = 20241$ for $X \in \{E, F\}$ and $Y \in \{A, B\}$, and hence,

$$\begin{aligned}\Delta^*_{PSp_6(3)}(3X, 13Y, 13Y) &\geq \Delta_{PSp_6(3)}(3X, 13Y, 13Y) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13y) - 1 \cdot \Sigma_{M_6}(3b, 13u, 13u)\\ &= 20241 - 2(234) - 1(91) > 0.\end{aligned}$$

Also, by Tables 5.5 and 5.6, we have $\Delta_{PSp_6(3)}(3X, 13Y, 13Z) = 20358$ for $X \in \{E, F\}$, $Y, Z \in \{A, B\}$ and $Y \neq Z$. Hence, we have

$$\begin{aligned}\Delta^*_{PSp_6(3)}(3X, 13Y, 13Z) &\geq \Delta_{PSp_6(3)}(3X, 13Y, 13Z) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13z) - 1 \cdot \Sigma_{M_6}(3b, 13u, 13v)\\ &= 20358 - 2(351) - 1(78) > 0.\end{aligned}$$

Therefore, $PSp_6(3)$ is generated by $(3X, 13Y, 13Z)$ for $X \in \{E, F\}$ and $Y, Z \in \{A, B\}$.

Before considering the remaining two cases, which involve the class $3G$, we note that there are five maximal subgroups of $PSp_6(3)$ that fuse to the classes $3G$ and $13Y$ for $Y \in \{A, B\}$, namely M_2, M_6, M_8, M_9 and M_{10}. Also, $h(13Z, M_2) = 2$ and $h(13Z, M_i) = 1$ for $i \in \{6, 8, 9, 10\}$. By Remark 2.3, all the intersections will be ignored.

Case 5: Consider $(3G, 13Y, 13Z)$ with $Y, Z \in \{A, B\}$ and $Y = Z$.
Here, computations with GAP reveal that $\Delta_{PSp_6(3)}(3G, 13Y, 13Y) = 373815$, $\Sigma_{M_2}(3x, 13y, 13y) = 6318$, $\Sigma_{M_6}(3t, 13u, 13u) = 0$, $\Sigma_{M_8}(3a, 13u, 13u) = 117$, $\Sigma_{M_9}(3a, 13u, 13u) = 26$ and $\Sigma_{M_{10}}(3a, 13u, 13u) = 26$ for $x \in \{g, k\}$, $y \in \{a, b, c, d\}$, $t \in \{c, d\}$ and $u \in \{a, b\}$. Now, we have

$$\begin{aligned}\Delta^*_{PSp_6(3)}(3G, 13Y, 13Y) &\geq \Delta_{PSp_6(3)}(3G, 13Y, 13Y) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13y) - 1 \cdot \Sigma_{M_6}(3t, 13u, 13u)\\ &\quad -1 \cdot \Sigma_{M_8}(3a, 13u, 13u) - 1 \cdot \Sigma_{M_9}(3a, 13u, 13u) - 1 \cdot \Sigma_{M_{10}}(3a, 13u, 13u)\\ &= 373815 - 2(6318) - 1(0) - 1(117) - 1(26) - 1(26) > 0.\end{aligned}$$

This shows that the triples $(3G, 13A, 13A)$ and $(3G, 13B, 13B)$ generate $PSp_6(3)$.

Case 6: Consider $(3G, 13Y, 13Z)$ with $Y, Z \in \{A, B\}$ and $Y \neq Z$.
In the present case, we have $\Sigma_{M_2}(3x, 13y, 13z) = 2106$, $\Sigma_{M_6}(3t, 13u, 13v) = 0$, $\Sigma_{M_8}(3a, 13u, 13v) = 78$, $\Sigma_{M_9}(3a, 13u, 13v) = \Sigma_{M_{10}}(3a, 13u, 13v) = 0$ and $\Delta_{PSp_6(3)}(3G, 13A, 13B) = \Delta_{PSp_6(3)}(3G, 13B, 13A) = 372762$. Therefore, we obtain

$$\begin{aligned}\Delta^*_{PSp_6(3)}(3G, 13Y, 13Z) &\geq \Delta_{PSp_6(3)}(3G, 13Y, 13Z) - 2 \cdot \Sigma_{M_2}(3x, 13y, 13z) - 1 \cdot \Sigma_{M_6}(3t, 13u, 13v)\\ &\quad -1 \cdot \Sigma_{M_8}(3a, 13u, 13v) - 1 \cdot \Sigma_{M_9}(3a, 13u, 13v) - 1 \cdot \Sigma_{M_{10}}(3a, 13u, 13v)\\ &= 372762 - 2(2106) - 1(0) - 1(78) - 1(0) - 1(0) > 0,\end{aligned}$$

which implies the desired result. □

6. $(5, q, r)$-Generations of $PSp_6(3)$

We only need to handle the cases of $(5, 5, r)$-, $(5, 7, r)$-, and $(5, 13, r)$-generations.

6.1. $(5, 5, r)$-Generations

We only need to discuss the triples $(5A, 5A, 5A), (5A, 5A, 7A)$ and $(5A, 5A, 13X)$ for $X \in \{A, B\}$.

Proposition 6.1.1. *The group* $PSp_6(3)$ *is generated by* $(5A, 5A, 5A)$.

Proof. By Table 4.3, there are only three maximal subgroups of $PSp_6(3)$ that fuse to the class $5A$, namely M_1, M_4 and M_{11}. Calculations yield $\Sigma_{M_1}(5a, 5a, 5a) = 7203$, $\Sigma_{M_4}(5a, 5a, 5a) = 343$ and

$$\begin{aligned}\Sigma_{M_{11}}(5x, 5y, 5z) &= \Delta_{M_{11}}(5a, 5a, 5a) + \Delta_{M_{11}}(5b, 5b, 5b) + \Delta_{M_{11}}(5a, 5a, 5b) + \Delta_{M_{11}}(5a, 5b, 5b)\\ &= 5 + 1 + 1 + 5 = 12\end{aligned}$$

for all $x, y, z \in \{a, b\}$. By Remark 2.3, intersections of M_1, M_4 and M_{11} may be ignored. We also have $h(5A, M_1) = 4$, $h(5A, M_4) = 1$ and $h(5A, M_{11}) = 48$. From Table 6.1, we have the structure constant $\Delta_{PSp_6(3)}(5A, 5A, 5A) = 69063$. Therefore,

$$\begin{aligned}\Delta^*_{PSp_6(3)}(5A, 5A, 5A) &\geq \Delta_{PSp_6(3)}(5A, 5A, 5A) - 4 \cdot \Sigma_{M_1}(5a, 5a, 5a) - 1 \cdot \Sigma_{M_4}(5a, 5a, 5a) - 48 \cdot \Sigma_{M_{11}}(5x, 5y, 5z)\\ &= 69063 - 4(7203) - 1(343) - 48(12) > 0,\end{aligned}$$

which implies that $(5A, 5A, 5A)$ generates $PSp_6(3)$. □

Table 6.1: Structure constants of $PSp_6(3)$ for $(5A, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(5A, 2A, rX)$	0	0	0	0	0	0	0	0	0	10	0	52	52
$\Delta_{PSp_6(3)}(5A, 2B, rX)$	0	6048	0	0	0	0	0	0	1944	1680	1708	1586	1586
$\Delta_{PSp_6(3)}(5A, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(5A, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta_{PSp_6(3)}(5A, 3C, rX)$	0	0	0	0	0	4374	0	0	648	15	0	104	104
$\Delta_{PSp_6(3)}(5A, 3D, rX)$	0	0	0	0	8748	0	0	0	0	2075	896	260	260
$\Delta_{PSp_6(3)}(5A, 3E, rX)$	0	0	0	0	0	0	0	5832	1620	1440	1596	1950	1950
$\Delta_{PSp_6(3)}(5A, 3F, rX)$	0	0	0	0	0	0	5832	0	1620	1440	1596	1950	1950
$\Delta_{PSp_6(3)}(5A, 3G, rX)$	0	48384	0	0	139968	0	29160	29160	15552	66960	51408	39312	39312
$\Delta_{PSp_6(3)}(5A, 5A, rX)$	51840	338688	0	0	26244	1815210	209952	209952	542376	69063	158648	301444	301444
$\Delta_{PSp_6(3)}(5A, 7A, rX)$	0	2951424	0	0	0	6718464	1994544	1994544	3569184	1359840	1997408	2728	2728
$\Delta_{PSp_6(3)}(5A, 13A, rX)$	2488	2951	0	0	1679	2099	2624	2624	2939	2782	2938	960 2992	960 2992
$\Delta_{PSp_6(3)}(5A, 13B, rX)$	320 2488	424 2951	0	0	616 1679	520 2099	400 2624	400 2624	328 2939	560 2782	880 2938	912 2992	912 2992
$\lvert C_{PSp_6(3)}(rX)\rvert$	320 622 080	424 241 92	1259 7120	1259 7120	616 209 952	520 104 976	400 17496	400 17496	328 972	560 120	880 14	912 13	912 13

Proposition 6.1.2. *The group* $PSp_6(3)$ *is generated by* $(5A, 5A, 7A)$.

Proof. By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $5A$ and $7A$. Thus, there is no contribution to $\Delta^*_{PSp_6(3)}(5A, 5A, 7A)$ from any maximal subgroup. It follows that $\Delta^*_{PSp_6(3)}(5A, 5A, 7A) = \Delta_{PSp_6(3)}(5A, 5A, 7A) = 158648$. Hence, $PSp_6(3)$ is generated by $(5A, 5A, 7A)$. □

Proposition 6.1.3. *The group* $PSp_6(3)$ *is generated by* $(5A, 5A, 13Z)$ *for* $Z \in \{A, B\}$.

Proof. By Table 4.3, there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $5A$ and $13Z$ for $Z \in \{A, B\}$. It follows that $\Delta^*_{PSp_6(3)}(5A, 5A, 13Z) = \Delta_{PSp_6(3)}(5A, 5A, 13Z) = 301444$. Hence, $PSp_6(3)$ is generated by $(5A, 5A, 13Z)$ for $Z \in \{A, B\}$. □

6.2. $(5, 7, r)$- and $(5, 13, r)$-Generations

We only need to deal with the following triples: $(5A, 7A, 7A), (5A, 7A, 13X)$ and $(5A, 13X, 13Y)$ for all $X, Y \in \{A, B\}$.

Proposition 6.2.1. *The group* $PSp_6(3)$ *is* $(5A, 7A, 7A)$*-,* $(5A, 7A, 13X)$*- and* $(5A, 13X, 13Y)$*-generated for all* $X, Y \in \{A, B\}$.

Proof. We observe from Table 4.3 that there is no maximal subgroup of $PSp_6(3)$ that fuses to the classes $5A$ and $7A$. Also, there is no maximal subgroup that fuses to the classes $5A$ and $13X$ for $X \in \{A, B\}$. Thus, there is no contribution to the calculations of $\Delta^*_{PSp_6(3)}(5A, qK, rL)$, with $qK, rL \in \{7A, 13X\}$, from any maximal subgroup for $X, Y \in \{A, B\}$. Therefore, we have

$$\begin{aligned}
\Delta^*_{PSp_6(3)}(5A, 7A, 7A) &= \Delta_{PSp_6(3)}(5A, 7A, 7A) = 1997408,\\
\Delta^*_{PSp_6(3)}(5A, 7A, 13X) &= \Delta_{PSp_6(3)}(5A, 7A, 13X) = 2728960,\\
\Delta^*_{PSp_6(3)}(5A, 13X, 13Y) &= \Delta_{PSp_6(3)}(5A, 13X, 13Y) = 2992912.
\end{aligned}$$

Since $\Delta^*_{PSp_6(3)}(5A, qK, rL) > 0$, for the involved triples, the desired result follows. □

7. $(7, q, r)$- and $(13, q, r)$-Generations

We only need to consider the triples $(7, 7, r)$, $(7, 13, r)$ and $(13, 13, 13)$.

7.1. $(7, 7, r)$-Generations

Here, we only need to consider $r \in \{7, 13\}$ because $p \le q \le r$.

Proposition 7.1.1. *The group* $PSp_6(3)$ *is generated by* $(7A, 7A, 7A)$.

Proof. There are only four maximal subgroups of $PSp_6(3)$ that fuse to the class $7A$, namely M_6, M_7, M_9 and M_{10}. Calculations with GAP show that $\Delta_{PSp_6(3)}(7A, 7A, 7A) = 20005445$ (see Table 7.1), $\Sigma_{M_6}(7a, 7a, 7a) = 369$, $\Sigma_{M_7}(7a, 7a, 7a) = 343$ and $\Sigma_{M_9}(7x, 7y, 7z) = \Sigma_{M_{10}}(7x, 7y, 7z) = 28 + 15 = 43$ for all $x, y, z \in \{a, b, c\}$ and $h(7A, M_6) = h(7A, M_7) = 1$. Also, we have $h(7A, M_9) = h(7A, M_{10}) = 6$. By Remark 2.3, intersections may be ignored. Therefore,

$$\begin{aligned}
\Delta^*_{PSp_6(3)}(7A, 7A, 7A) &\ge \Delta_{PSp_6(3)}(7A, 7A, 7A) - 1 \cdot \Sigma_{M_6}(7a, 7a, 7a) - 1 \cdot \Sigma_{M_7}(7a, 7a, 7a)\\
&\quad -6 \cdot \Sigma_{M_9}(7x, 7y, 7z) - 6 \cdot \Sigma_{M_{10}}(7x, 7y, 7z)\\
&= 20005445 - 1(369) - 1(343) - 6(43) - 6(43) = 20004217.
\end{aligned}$$

This implies that $(7A, 7A, 7A)$ generates $PSp_6(3)$. □

Proposition 7.1.2. *The group* $PSp_6(3)$ *is generated by* $(7A, 7A, 13Z)$ *for each* $Z \in \{A, B\}$.

Proof. We have only three maximal subgroups that fuse to the classes $7A$ and $13Z$ for $Z \in \{A, B\}$, which are M_6, M_9 and M_{10}. By Remark 2.3, intersections may be ignored. Computations via GAP give $\Delta_{PSp_6(3)}(7A, 7A, 13Z) = 23395840$, $\Sigma_{M_6}(7a, 7a, 13z) = 468$ and $\Sigma_{M_9}(7x, 7y, 13z) = \Sigma_{M_{10}}(7x, 7y, 13z) = 39$ for all $z \in \{a, b\}$ and $x, y \in \{a, b, c\}$, where $x \ne y$ and $h(13Z, M_6) = h(13Z, M_9) = h(13Z, M_{10}) = 1$. Thus,

$$\begin{aligned}
\Delta^*_{PSp_6(3)}(7A, 7A, 13Z) &\ge \Delta_{PSp_6(3)}(7A, 7A, 13Z) - 1 \cdot \Sigma_{M_6}(7a, 7a, 13z) - 1 \cdot \Sigma_{M_9}(7x, 7y, 13z)\\
&\quad -1 \cdot \Sigma_{M_{10}}(7x, 7y, 13z)\\
&= 23395840 - 1(468) - 1(39) - 1(39) = 23395294.
\end{aligned}$$

Hence, $PSp_6(3)$ is generated by $(7A, 7A, 13Z)$ for $Z \in \{A, B\}$. □

Table 7.1: Structure constants of $PSp_6(3)$ for $(7A, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(7A, 2A, rX)$	0	0	0	0	0	0	0	0	486	0	350	520	520
$\Delta_{PSp_6(3)}(7A, 2B, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\Delta_{PSp_6(3)}(7A, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	14	26	26
$\Delta_{PSp_6(3)}(7A, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	14	26	26
$\Delta_{PSp_6(3)}(7A, 3C, rX)$	0	0	0	0	0	0	0	0	2430	0	735	1560	1560
$\Delta_{PSp_6(3)}(7A, 3D, rX)$	0	0	0	0	0	0	0	0	0	7680	5880	3120	3120
$\Delta_{PSp_6(3)}(7A, 3E, rX)$	0	24192	0	0	0	0	0	34992	18468	13680	17010	18720	18720
$\Delta_{PSp_6(3)}(7A, 3F, rX)$	0	24192	0	0	0	0	34992	0	18468	13680	17010	18720	18720
$\Delta_{PSp_6(3)}(7A, 3G, rX)$	311040	290304	0	0	524880	0	332424	332424	269 730	440 640	394 065	336 960	336 960
$\Delta_{PSp_6(3)}(7A, 5A, rX)$	0	2951 424	0	0	0	6718 464	1994 544	1994 544	3569 184	1359 840	1997 408	2728 960	2728 960
$\Delta_{PSp_6(3)}(7A, 7A, rX)$	1555 2000	2515 9680	1259 7120	1259 7120	1102 2480	4408 9920	2125 7640	2125 7640	2735 9370	1712 0640	2000 5445	2339 5840	2339 5840
$\Delta_{PSp_6(3)}(7A, 13A, rX)$	2488 3200	2515 9680	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 0400	2519 5520	2519 2960	2519 2960
$\Delta_{PSp_6(3)}(7A, 13B, rX)$	2488 3200	2515 9680	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 4240	2519 0400	2519 5520	2519 2960	2519 2960
$\lvert C_{PSp_6(3)}(rX)\rvert$	622 080	241 92	1259 7120	1259 7120	209 952	104 976	17496	17496	972	120	14	13	13

7.2. (7, 13, 13)- and (13, 13, 13)-Generations

Here, we discuss generations by $(7A, 13X, 13Y)$ and $(13X, 13Y, 13Z)$ for $X, Y, Z \in \{A, B\}$.

Proposition 7.2.1. *The group* $PSp_6(3)$ *is generated by* $(7A, 13Y, 13Z)$ *for* $Y, Z \in \{A, B\}$.

Proof. As in the proof of Proposition 7.1.2, only the maximal subgroups M_6, M_9 and M_{10} of $PSp_6(3)$ have classes that fuse to the classes $7A$ and $13Y$ for $Y \in \{A, B\}$. By Remark 2.3, intersections may be ignored. Here, we obtain $\Sigma_{M_6}(7a, 13y, 13z) = 468$, $\Sigma_{M_9}(7x, 13y, 13z) = 13$ and $\Sigma_{M_{10}}(7x, 13y, 13z) = 13$ for all $y, z \in \{a, b\}$ and $x \in \{a, b, c\}$. Also, we have $h(13Y, M_6) = 1$, $h(13Y, M_9) = h(13Y, M_{10}) = 1$ and $\Delta_{PSp_6(3)}(7A, 13Y, 13Z) = 25192960$ for all $Y, Z \in \{A, B\}$; see Table 7.1. Thus,

$$\begin{aligned}\Delta^*_{PSp_6(3)}(7A, 13Y, 13Z) &\geq \Delta_{PSp_6(3)}(7A, 13Y, 13Z) - 1 \cdot \Sigma_{M_6}(7a, 13y, 13z) - 1 \cdot \Sigma_{M_9}(7x, 13y, 13z)\\ &\quad -1 \cdot \Sigma_{M_{10}}(7x, 13y, 13z)\\ &= 25192960 - 1(468) - 1(13) - 1(13) = 25192466.\end{aligned}$$

Therefore, $PSp_6(3)$ is $(7A, 13Y, 13Z)$-generated for all $Y, Z \in \{A, B\}$. □

Proposition 7.2.2. *The group* $PSp_6(3)$ *is generated by* $(13X, 13Y, 13Z)$ *for* $X, Y, Z \in \{A, B\}$.

Proof. By Table 4.3, there are only five maximal subgroups of $PSp_6(3)$ that fuse to the class $13X$ for each $X \in \{A, B\}$, which are M_2, M_6, M_8, M_9 and M_{10}. By Table 3.1, $h(13X, M_2) = 2$ and $h(13X, M_i) = 1$ for $i \in \{6, 8, 9, 10\}$. By Remark 2.3, the intersections of these maximal subgroups may be ignored.

Case 1: Consider $(13X, 13X, 13X)$ for $X \in \{A, B\}$.
In this case, we have $\Delta_{PSp_6(3)}(13X, 13X, 13X) = 27218986$ for $X \in \{A, B\}$; see Tables 7.2 and 7.3 Contributions from the involved maximal subgroups are $\Sigma(M_2) = 9477 + 9477 + 48843 = 67797$, $\Sigma(M_6) = 522$, $\Sigma(M_8) = 106$, $\Sigma(M_9) = 15$ and $\Sigma(M_{10}) = 15$. Thus,

$$\begin{aligned}\Delta^*_{PSp_6(3)}(13X, 13X, 13X) &\geq \Delta_{PSp_6(3)}(13X, 13X, 13X) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_8)\\ &\quad -1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10})\\ &= 27218986 - 2(67797) - 1(522) - 1(106) - 1(15) - 1(15) = 27082734.\end{aligned}$$

Table 7.2: Structure constants of $PSp_6(3)$ for $(13A, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(13A, 2A, rX)$	0	0	0	0	0	0	0	0	972	480	560	754	754
$\Delta_{PSp_6(3)}(13A, 2B, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\Delta_{PSp_6(3)}(13A, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	28	39	26
$\Delta_{PSp_6(3)}(13A, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	28	39	26
$\Delta_{PSp_6(3)}(13A, 3C, rX)$	0	0	0	0	0	0	0	0	1944	960	1680	2106	2028
$\Delta_{PSp_6(3)}(13A, 3D, rX)$	0	0	0	0	0	0	0	0	4860	2400	3360	4095	4290
$\Delta_{PSp_6(3)}(13A, 3E, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20241	20358
$\Delta_{PSp_6(3)}(13A, 3F, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20241	20358
$\Delta_{PSp_6(3)}(13A, 3G, rX)$	622	290	0	0	419	524	332	332	393	362	362	373	372
	080	304			904	880	424	424	660	880	880	815	762
$\Delta_{PSp_6(3)}(13A, 5A, rX)$	2488	2951	0	0	1679	2099	2624	2624	2939	2782	2938	2992	2992
	320	424			616	520	400	400	328	560	880	912	912
$\Delta_{PSp_6(3)}(13A, 7A, rX)$	2488	2515	2519	2519	2519	2519	2519	2519	2519	2519	2519	2519	2519
	3200	9680	4240	4240	4240	4240	4240	4240	4240	0400	5520	2960	2960
$\Delta_{PSp_6(3)}(13A, 13A, rX)$	3608	2515	3779	3779	3401	3306	2724	2724	2794	2762	2713	2721	2723
	0640	9680	1360	1360	2224	7440	1272	1272	9860	6880	0880	8986	8669
$\Delta_{PSp_6(3)}(13A, 13B, rX)$	3608	2515	2519	2519	3275	3464	2739	2739	2787	2762	2713	2723	2723
	0640	9680	4240	4240	2512	2080	8736	8736	1128	6880	0880	8669	8669
$\lvert C_{PSp_6(3)}(rX)\rvert$	622	241	1259	1259	209	104	17496	17496	972	120	14	13	13
	080	92	7120	7120	952	976							

Table 7.3: Structure constants of $PSp_6(3)$ for $(13B, qX, rY)$.

rX	$2A$	$2B$	$3A$	$3B$	$3C$	$3D$	$3E$	$3F$	$3G$	$5A$	$7A$	$13A$	$13B$
$\Delta_{PSp_6(3)}(13B, 2A, rX)$	0	0	0	0	0	0	0	0	972	480	560	754	754
$\Delta_{PSp_6(3)}(13B, 2B, rX)$	0	24192	0	0	0	0	17496	17496	11664	14640	14560	13520	13520
$\Delta_{PSp_6(3)}(13B, 3A, rX)$	0	0	0	0	0	0	0	0	0	0	28	26	39
$\Delta_{PSp_6(3)}(13B, 3B, rX)$	0	0	0	0	0	0	0	0	0	0	28	26	39
$\Delta_{PSp_6(3)}(13B, 3C, rX)$	0	0	0	0	0	0	0	0	1944	960	1680	2028	2106
$\Delta_{PSp_6(3)}(13B, 3D, rX)$	0	0	0	0	0	0	0	0	4860	2400	3360	4290	4095
$\Delta_{PSp_6(3)}(13B, 3E, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20358	20241
$\Delta_{PSp_6(3)}(13B, 3F, rX)$	0	24192	0	0	0	0	17496	17496	18468	18000	20160	20358	20241
$\Delta_{PSp_6(3)}(13B, 3G, rX)$	622	290	0	0	419	524	332	332	393	362	362	372	373
	080	304			904	880	424	424	660	880	880	762	815
$\Delta_{PSp_6(3)}(13B, 5A, rX)$	2488	2951	0	0	1679	2099	2624	2624	2939	2782	2938	2992	2992
	320	424			616	520	400	400	328	560	880	912	912
$\Delta_{PSp_6(3)}(13B, 7A, rX)$	2488	2515	2519	2519	2519	2519	2519	2519	2519	2519	2519	2519	2519
	3200	9680	4240	4240	4240	4240	4240	4240	4240	0400	5520	2960	2960
$\Delta_{PSp_6(3)}(13B, 13A, rX)$	3608	2515	2519	2519	3275	3464	2739	2739	2787	2762	2713	2723	2723
	0640	9680	4240	4240	2512	2080	8736	8736	1128	6880	0880	8669	8669
$\Delta_{PSp_6(3)}(13B, 13B, rX)$	3608	2515	3779	3779	3401	3306	2724	2724	2794	2762	2713	2723	2721
	0640	9680	1360	1360	2224	7440	1272	1272	9860	6880	0880	8669	8986
$\lvert C_{PSp_6(3)}(rX)\rvert$	622	241	1259	1259	209	104	17496	17496	972	120	14	13	13
	080	92	7120	7120	952	976							

Case 2: Consider $(13A, 13A, 13B)$.
In this case, we have $\Delta_{PSp_6(3)}(13A, 13A, 13B) = 27238669$. Contributions from the maximal subgroups are $\Sigma(M_2) = 9477 + 20160 + 20160 = 49797$, $\Sigma(M_6) = 549$, $\Sigma(M_8) = 133$, $\Sigma(M_9) = 3$ and $\Sigma(M_{10}) = 3$. Therefore, we have

$$\begin{aligned}\Delta^*_{PSp_6(3)}(13A, 13A, 13B) &\geq \Delta_{PSp_6(3)}(13A, 13A, 13B) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_6) - 1 \cdot \Sigma(M_8) \\ &\quad -1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10}) \\ &= 27238669 - 2(49797) - 1(549) - 1(133) - 1(3) - 1(3) = 27138387.\end{aligned}$$

Case 3: Consider $(13A, 13B, 13B)$.
As in the preceding cases, we find $\Sigma(M_2) = 29160 + 29160 + 29160 + 9477 = 96957$, $\Sigma(M_6) = 549$, $\Sigma(M_8) = 133$, $\Sigma(M_9) = 3$ and $\Sigma(M_{10}) = 3$. Thus,

$$\begin{aligned}\Delta^*_{PSp_6(3)}(13A, 13B, 13B) &\geq \Delta_{PSp_6(3)}(13A, 13B, 13B) - 2 \cdot \Sigma(M_2) - 1 \cdot \Sigma(M_6) \ - 1 \cdot \Sigma(M_8) \\ &\quad -1 \cdot \Sigma(M_9) - 1 \cdot \Sigma(M_{10}) \\ &= 27238669 - 2(96957) - 1(549) - 1(133) - 1(3) - 1(3) = 27044067.\end{aligned}$$

In all three cases, $\Delta^*_{PSp_6(3)}(13X, 13Y, 13Z)$ is greater than zero, which means that $PSp_6(3)$ is generated by the given triples. □

Gathering all the propositions from Sections 4 to 7 proves Theorem 1.1.

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