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Research Article

Weakly Complemented Quantales and Localization

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Abstract

This paper develops a localization framework for coherent quantales based on weak polar elements, thereby extending localization techniques beyond the semiprime setting. By using multiplicatively closed subsets determined by weak polars, the corresponding localization quantale is constructed, and the properties of the canonical localization morphism are investigated. It is shown that this construction preserves and reflects important algebraic and spectral properties. Within this framework, weakly complemented quantales are introduced and studied, and characterizations are provided in terms of the radical frame, localization quantales, and the topology of the minimal m-prime spectrum. In particular, the weakly complemented property is shown to be equivalent to the compactness of the Zariski–Stone topology on the minimal spectrum and to the hyperarchimedean properties of the associated localization quantale. Moreover, generalized PP-quantales and weak fraction-dense quantales are investigated, and several equivalences connecting algebraic conditions, spectral topologies, and localization properties are established. The interplay between weak polar operators, spectral properties, and localization techniques provides a broader perspective on coherent quantales beyond the semiprime setting.

Keywords: coherent quantale; weak polar; localization quantale; weakly complemented quantale; generalized PP-quantale.

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1. Introduction

Quantales provide a natural framework for extending methods from lattice theory, topology, and commutative algebra to algebraic structures endowed with both an order and a multiplication.

Many structural results in quantale theory rely on the polar (annihilator) operator, which plays a central role in the study of complemented quantales and localization techniques [13]. However, this approach is essentially restricted to semiprime quantales. In the general case, the polar operator does not fully capture annihilator-type behavior, and classical constructions may lose structural precision or fail to extend naturally. To overcome this limitation, the notion of a *weak polar* was introduced as a generalization of annihilators in quantales [12]. Weak polars preserve important algebraic properties while remaining applicable beyond the semiprime setting. Moreover, they are closely related to the radical structure of a quantale and induce Boolean-type structures that extend classical results [10, 12].

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The aim of this paper is to systematically develop the theory of quantales based on weak polar operators. More precisely, we introduce the notion of a *weakly complemented quantale*, obtained by replacing the classical polar operator with the weak polar operator in the definition of a complemented quantale. In parallel, we construct a localization quantale associated with weak polar conditions, extending classical localization techniques from commutative algebra [2].

The main contributions of this paper are as follows:

- we define and characterize weakly complemented quantales in terms of the frame of radical elements;
- we construct a localization quantale based on weak polar operators and study its fundamental properties;
- we provide equivalent characterizations in terms of the minimal m-prime spectrum and its associated topologies;
- we establish connections with hyperarchimedean quantales and generalized PP-quantales, extending known results from the semiprime case [11, 13].

The paper is organized as follows. Section 2 recalls preliminaries on coherent quantales, including compact elements, residuation, and spectral constructions. Section 3 is devoted to weak polar elements and their properties. In Section 4, we develop the localization theory and introduce the corresponding localization quantale. Section 5 defines weakly complemented quantales and provides their main characterizations. Section 6 studies generalized PP-quantales and their relationship with weakly complemented and mp-quantales. Finally, Section 7 characterizes weak fraction-dense quantales, while Section 8 addresses functorial aspects of localization.

The results developed in this paper extend classical constructions in quantale theory to a broader setting and provide a unified framework for studying localization and spectral phenomena in non-semiprime quantales.

2. Preliminaries

The notions and results in quantale theory (respectively, frame theory) used in this paper can be found in the fundamental monographs [8, 21] (respectively, [16, 20]). Throughout this paper, the term *quantale* refers to a commutative and integral quantale. We also occasionally use the abbreviation "iff" for "if and only if".

Let A be a quantale. The set of compact elements of A is denoted by $K(A)$. The quantale A is said to be *algebraic* if every element is a join of compact elements. An algebraic quantale A is *coherent* if $1 \in K(A)$ and the set $K(A)$ is closed under multiplication.

Lemma 2.1. *For any $x, y \in A$, the following properties hold:*

(1) $xy \leq x \wedge y$;

(2) If $x \vee y = 1$, *then* $xy = x \wedge y$;

(3) If $x \vee y = 1$, *then* $x^n \vee y^n = 1$ *for any integer* $n \geq 1$.

According to [21], the *residuation* (*implication*) of A is defined by

$$x \to y = \bigvee \{a \in A \mid xa \leq y\},$$

and the *polar* (*annihilator*) of x is $x^{\perp_A} = x \to 0$. The algebraic structure $(A, \vee, \wedge, \cdot, \to, 0, 1)$ is a commutative residuated lattice. For the basic concepts and results of residuation theory, we refer to [9].

Lemma 2.2 (see [9]). *For any $x, y \in A$, the following properties hold:*

(1) $x \leq x^{\perp_A \perp_A}$; $x^{\perp_A} = x^{\perp_A \perp_A \perp_A}$;

(2) $(x \vee y)^{\perp_A} = x^{\perp_A} \wedge y^{\perp_A}$;

(3) $(x \vee x^{\perp_A})^{\perp_A} = 0$.

Let $B(A)$ denote the Boolean algebra of complemented elements of A. Moreover, for $x, y \in B(A)$, we have $x \wedge y = xy$. Hence, $B(A)$ is closed under multiplication. If $x \in B(A)$, then $x^{\perp_A}$ is the complement of x. In accordance with Lemma 15 of [7], we have $B(A) \subseteq K(A)$.

Lemma 2.3. *For all $x, y \in A$, the following properties hold:*

(1) $x \in B(A)$ if and only if $x \vee x^{\perp_A} = 1$;

(2) If $x \vee y = 1$ and $xy = 0$, then $x, y \in B(A)$;

(3) If $x, y \in B(A)$, then $x \wedge y = xy$.

An element $p < 1$ is *m-prime* if for all $x, y \in A$, $xy \leq p$ implies $x \leq p$ or $y \leq p$. When A is an algebraic quantale, $p < 1$ is m-prime if and only if for all $e, f \in K(A)$, $ef \leq p$ implies $e \leq p$ or $f \leq p$. The set of all m-prime elements is denoted by $Spec(A)$, and is called the m-prime spectrum of A. The *minimal m-prime spectrum* of A is defined as the set $Min(A)$ of minimal m-prime elements of A. For any $a \in A$, we denote $U(a) = \{p \in Min(A) \mid a \not\leq p\}$ and $V(a) = Min(A) \setminus U(a)$. Following [10], we equip $Min(A)$ with the following two topologies:

- the Zariski–Stone topology, having the family $\{U(c)\}_{c \in K(A)}$ as a basis;
- the flat topology, having the family $\{V(c)\}_{c \in K(A)}$ as a basis.

Thus, we obtain two topological spaces $Min_Z(A)$ and $Min_F(A)$, corresponding to $Min(A)$ endowed with the Zariski–Stone topology and the flat topology, respectively. By Corollary 8.6 of [10], we know that $Min_Z(A)$ is a zero-dimensional Hausdorff space and $Min_F(A)$ is a compact T_1-space.

Following [5], a topological space X is called a Boolean space if it is compact, Hausdorff, and totally disconnected. A topological space X is called a Stone space if it is compact, Hausdorff, and extremally disconnected.

For any $x \in A$, $\rho_A(x) = \bigwedge \{q \in Spec(A) \mid x \leq q\}$ is the *radical* of x. An element $x \in A$ is a radical element if $\rho_A(x) = x$. The quantale A is *semiprime* if $\rho_A(0) = 0$.

The operator $\rho_A(\cdot)$ satisfies standard properties, including monotonicity, idempotence, $x \leq \rho_A(x)$, $\rho_A(xy) = \rho_A(x \wedge y) = \rho_A(x) \wedge \rho_A(y)$, etc. The set $R(A)$ of all radical elements is a coherent frame in which the joins can be described as follows: for any $T \subseteq R(A)$, $\dot{\bigvee} T = \rho(\bigvee T)$. Particularly, $x \dot{\vee} y = \rho_A(x \vee y)$ for all $x, y \in R(A)$. By Lemma 8 of [7], for any coherent quantale A, it holds that $K(R(A)) = \rho(K(A))$.

Lemma 2.4. *If A is a coherent quantale, then $Min_Z(A) = Min_Z(R(A))$ and $Min_F(A) = Min_F(R(A))$.*

Lemma 2.5 (see [1, 17]). *Let A be a coherent quantale. For any $x \in A$, the following properties hold:*

(1) $\rho(x) = \bigvee \{d \in K(A) \mid d^n \leq x$ for some $n \in \mathbb{N}\}$;

(2) For any $d \in K(A)$, $d \leq \rho(x)$ if and only if $d^n \leq x$ for some $n \in \mathbb{N}$;

(3) A is semiprime if and only if $d^n = 0$ implies $d = 0$ for all $d \in K(A)$ and $n \in \mathbb{N}$.

3. Weak Polars

The weak polars in a quantale were introduced in [12] under the name weak annihilator elements. They are the quantale-theoretic abstraction of weak annihilator ideals in a commutative ring [19].

Let A be a quantale and $a \in A$. The *weak polar* of a is defined by

$$a^{\perp_w} = a^{\perp_{A,w}} = a \to \rho_A(0).$$

An element of the form $a^{\perp_w}$ is called a *weak polar* of A. Using residuation theory, one can prove that $a^{\perp_A} \leq a^{\perp_w}$. If A is semiprime, then $a^{\perp_A} = a^{\perp_w}$.

In what follows, we fix a coherent quantale A.

Lemma 3.1. *If $a, b \in A$ and $\{a_i\}_{i \in I} \subseteq A$, then the following hold:*

(1) If $a \leq b$, then $b^{\perp_w} \leq a^{\perp_w}$;

(2) $a \leq a^{\perp_w \perp_w}$; $a^{\perp_w} = a^{\perp_w \perp_w \perp_w}$;

(3) $0^{\perp_w} = 1$; $1^{\perp_w} = \rho_A(0)$;

(4) $\rho_A(0) \leq a^{\perp_w}$;

(5) $(\bigvee_{i \in I} a_i)^{\perp_w} = \bigwedge_{i \in I} a_i^{\perp_w}$;

(6) $(a \vee a^{\perp_w})^{\perp_w} = \rho_A(0)$.

Proof. The properties (1)–(5) are verified using residuation rules. For the proof of (6), see Lemma 5.1(6) in [12]. □

Lemma 3.2 (see [12]). *For any $a \in A$, the following hold*

(1) $(\rho_A(a))^{\perp_w} = a^{\perp_w}$;

(2) If $a \in R(A)$, then $a^{\perp_{R(A)}} = a^{\perp_w}$;

(3) $a^{\perp_w} = (\rho_A(a))^{\perp_{R(A)}}$.

Lemma 3.3. *For all $x, y \in A$, the following hold*

(1) $xx^{\perp_w} \leq \rho_A(0)$;

(2) $xy \leq \rho_A(0)$ if and only if $x \wedge y \leq \rho_A(0)$;

(3) $x \leq y^{\perp_w}$ if and only if $x \wedge y \leq \rho_A(0)$;

(4) $xy \leq \rho_A(0)$ if and only if $x \leq y^{\perp_w}$.

Proof. (1) From residuation theory, we know that $a(a \to b) \leq b$ for all $a, b \in A$. Hence,

$$xx^{\perp_w} = x(x \to \rho_A(0)) \leq \rho_A(0).$$

(2) Since $xy \leq x \wedge y$, the inequality $x \wedge y \leq \rho_A(0)$ implies $xy \leq \rho_A(0)$. Now, assume that $xy \leq \rho_A(0)$. Let c be a compact element of A such that $c \leq x \wedge y$. Thus, $c \leq x$ and $c \leq y$, so $c^2 \leq xy \leq \rho_A(0)$. By Lemma 2.5(2), one can find an $n \in \mathbb{N}$ such that $c^{2n} = 0$. Using once more Lemma 2.5(2), we obtain $c \leq \rho_A(0)$. Therefore, we conclude that $x \wedge y \leq \rho_A(0)$.

(3) In accordance with (2), we have the following equivalences: $x \leq y^{\perp_w}$ iff $x \leq y \to \rho_A(0)$ iff $xy \leq \rho_A(0)$ iff $x \wedge y \leq \rho_A(0)$.

(4) This part follows from (2) and (3). □

We denote by $Pol_w(A)$ the set of all weak polars of A. By Lemma 3.1(2), we have

$$Pol_w(A) = \{a^{\perp_w} \mid a \in A\} = \{a \in A \mid a^{\perp_w \perp_w} = a\}.$$

Consider the set $Pol(R(A))$ of polars of the coherent frame $R(A)$:

$$Pol(R(A)) = \{a^{\perp_{R(A)}} \mid a \in R(A)\} = \{a \in R(A) \mid a^{\perp_{R(A)} \perp_{R(A)}} = a\}.$$

For any $\{x_i\}_{i \in I} \subseteq Pol(R(A))$, we set

$$\bigoplus_{i \in I} x_i = \left(\dot{\bigvee}_{i \in I} x_i\right)^{\perp_{R(A)} \perp_{R(A)}}.$$

We know from [3] that $(Pol(R(A)), \bigoplus, \bigwedge)$ is a complete Boolean algebra. Also, for any $\{a_i\}_{i \in I} \subseteq Pol_w(A)$, we set

$$\bigsqcup_{i \in I} a_i = \left(\bigvee_{i \in I} a_i\right)^{\perp_w \perp_w}.$$

Proposition 3.4 (see [12]). *The following properties hold:*

(1) $Pol_w(A) = Pol(R(A))$*;*

(2) $Pol_w(A)$ *is closed under the infinite operations* $\bigsqcup$ *and* $\bigwedge$*;*

(3) For any $\{a_i\}_{i \in I} \subseteq Pol_w(A)$*, it holds that* $\bigsqcup_{i \in I} a_i = \bigoplus_{i \in I} a_i$*.*

By virtue of Proposition 3.4, $(Pol_w(A), \bigsqcup, \bigwedge)$ is a complete Boolean algebra, identical to the complete Boolean algebra $Pol(R(A), \bigoplus, \bigwedge)$. For any element $a \in Pol_w(A)$, $a^{\perp_w}$ is its complement. We note that $\rho_A(0)$ (respectively, 1) is the bottom element (respectively, the top element). If A is semiprime, then $Pol_w(A) = Pol(A)$, and hence, $Pol(A)$ is a complete Boolean algebra.

The equality of the complete Boolean algebras $Pol_w(A)$ and $Pol(R(A))$ allows us to transfer some properties from $Pol(R(A))$ to $Pol_w(A)$, and conversely. The proof of the following proposition illustrates this method.

Proposition 3.5. *For all* $a, b \in A$*, the following equalities hold:*

(1) $(ab)^{\perp_w} = (a \wedge b)^{\perp_w}$*;*

(2) $a^{\perp_w \perp_w} \wedge b^{\perp_w \perp_w} = (ab)^{\perp_w \perp_w} = (a \wedge b)^{\perp_w \perp_w}$.

Proof. (1) We know that $\rho_A(ab) = \rho_A(a \wedge b)$. Therefore, using Lemma 3.2(3), the following equalities hold:

$$(ab)^{\perp_w} = (\rho_A(ab))^{\perp_w} = (\rho_A(a \wedge b))^{\perp_w} = (a \wedge b)^{\perp_w}.$$

(2) For all $x, y \in R(A)$, we have

$$x^{\perp_{R(A)} \perp_{R(A)}} \wedge y^{\perp_{R(A)} \perp_{R(A)}} = (x \wedge y)^{\perp_{R(A)} \perp_{R(A)}}.$$

Applying Lemma 3.2(3) for $x = \rho_A(a)$ and $y = \rho_A(b)$, and taking into account the previous equality, we obtain

$$\begin{aligned} a^{\perp_w \perp_w} \wedge b^{\perp_w \perp_w} &= (\rho_A(a))^{\perp_{R(A)} \perp_{R(A)}} \wedge (\rho_A(b))^{\perp_{R(A)} \perp_{R(A)}} = (\rho_A(a) \wedge \rho_A(b))^{\perp_{R(A)} \perp_{R(A)}} = (\rho_A(ab))^{\perp_{R(A)} \perp_{R(A)}} \\ &= (ab)^{\perp_w \perp_w}, \end{aligned}$$

which yields the first equality of (2). The second equality of (2) follows from (1). □

Proposition 3.6 (see [10]). *If $p \in Spec(A)$, then the following properties are equivalent:*

(1) $p \in Min(A)$;

(2) For any $c \in K(A)$, $c \leq p$ implies $c^{\perp_w} \not\leq p$;

(3) For any $c \in K(A)$, $c \leq p$ if and only if $c^{\perp_w} \not\leq p$.

4. Localization Quantales

Let A be a quantale and let $K(A)$ be its set of compact elements. A subset S of $K(A)$ is multiplicatively closed if $1 \in S$ and $x, y \in S$ implies $xy \in S$. We fix a multiplicatively closed subset S of a non-trivial coherent quantale A. Following [1, 18], for any $a \in A$, we set

$$a_S = \bigvee \{c \in K(A) \,|\, cs \leq a \text{ for some } s \in S\}.$$

We say that a_S is *the localization of a at S*. Consider the set $A_S = \{a_S \,|\, a \in A\}$ and the map $j_S : A \to A_S$ defined by $j_S(a) = a_S$, for any $a \in A$. The map j_S is called the localization map associated with the pair (A, S).

Lemma 4.1 (see [13]). *For all $a \in A$ and $d \in K(A)$, $d \leq a_S$ if and only if $ds \leq a$ for some $s \in S$.*

Lemma 4.2 (see [1, 18]). *For all $a, b \in A$ and $d \in K(A)$, the following hold:*

(1) If $a \leq b$, then $a_S \leq b_S$;

(2) $a \leq a_S$;

(3) $(a_S)_S = a_S$;

(4) $(a \wedge b)_S = a_S \wedge b_S$;

(5) $a_S = 1$ if and only if $s \leq a$ for some $s \in S$;

(6) $a \leq b_S$ if and only if $a_S \leq b_S$;

(7) $d_S = 0_S$ if and only if $ds = 0$ for some $s \in S$;

(8) $(a \to b)_S = a_S \to b_S$;

(9) $\rho_{A_S}(a_S) = (\rho_A(a))_S$.

According to Lemma 4.2(5), we have $a_S = 1$ for any $a \in S$. In particular, $1_S = 1$. The set A_S is endowed with a quantale structure with respect to the following operations:

- $\bigvee_{i \in I} (a_i)_S = (\bigvee_{i \in I} a_i)_S$ for any family $\{a_i\}_{i \in I} \subseteq A$;
- $a_S \wedge b_S = (a \wedge b)_S$ for all $a, b \in A$;
- $a_S \cdot_S b_S = (ab)_S$ for all $a, b \in A$.

Lemma 4.3 (see [1, 18]). *The following statements hold:*

(1) A_S *is a coherent quantale;*

(2) $K(A_S) = \{c_S \mid c \in K(A)\}$.

The coherent quantale A_S is called *the localization quantale of* A *at* S. Hence, $j_S : A \to A_S$ is a coherent quantale morphism.

Proposition 4.4 (see [1]). $Spec(A_S) = \{p \in Spec(A) \mid (p] \cap S = \emptyset\}$.

Now, we present a new proof of the following known proposition.

Proposition 4.5 (see [13]). *For any* $a \in A$*, it holds that* $(a^{\perp_{A,w}})_S = (a_S)^{\perp_{A_S,w}}$.

Proof. Let a be an element of A. Recall that the weak polar of a is defined by $a^{\perp_{A,w}} = a \to \rho_A(0)$, while in the localization quantale A_S, we have $(a_S)^{\perp_{A_S,w}} = a_S \to \rho_{A_S}(0_S)$. By Lemma 4.2(8), localization preserves residuation, and hence, we have

$$(a^{\perp_{A,w}})_S = (a \to \rho_A(0))_S = a_S \to (\rho_A(0))_S. \quad (1)$$

Now, by Lemma 4.2(9), the radical operator preserves localization, and hence, $(\rho_A(0))_S = \rho_{A_S}(0_S)$. Using this equality in (1), we obtain $(a^{\perp_{A,w}})_S = a_S \to \rho_{A_S}(0_S) = (a_S)^{\perp_{A_S,w}}$. □

We note that, for convenience, instead of $(a^{\perp_{A,w}})_S = (a_S)^{\perp_{A_S,w}}$, we sometimes use the abbreviated form $(a^{\perp_w})_S = (a_S)^{\perp_w}$.

Example 4.6 (see [13]). *We set* $S(A) = \{c \in K(A) \mid c^{\perp_A} = 0\}$*. Then,* $S(A)$ *is a multiplicatively closed subset of* A*, and hence, we can consider the localization quantale* $Q(A) = A_{S(A)}$ *of* A *at* $S(A)$*. When* A *is a coherent frame,* $Q(A)$ *coincides with the coherent frame* qA*, defined in [6]. Moreover,* $Q(A)$ *is a quantale-theoretic version of the notion of the total ring of quotients from commutative algebra. We assume that* A *is a non-trivial quantale. Therefore, from Lemma 3.10 of [14], it follows that* $Q(A)$ *is a non-trivial quantale.*

We note that the construction of the coherent quantale $Q(A)$ relies on the polar operator $(\cdot)^{\perp_A}$. Now, we define a new localization quantale $\tilde{Q}(A)$ by replacing $(\cdot)^{\perp_A}$ with the weak polar operator $(\cdot)^{\perp_w}$. We set $T(A) = \{c \in K(A) \mid c^{\perp_w} = \rho_A(0)\}$. According to Lemma 3.1(4), for each $a \in A$, the following equivalence holds: $a^{\perp_w} = \rho_A(0)$ iff $a^{\perp_w} \leq \rho_A(0)$. Hence, we have $T(A) = \{c \in K(A) \mid c^{\perp_w} \leq \rho_A(0)\}$.

Lemma 4.7. *The set* $T(A)$ *is a multiplicatively closed subset of* A.

Proof. Since $1^{\perp_w} = \rho_A(0)$, we have $1 \in T(A)$. Assume that $c, d \in T(A)$. Then c and d are compact elements of A such that $c^{\perp_w} \leq \rho_A(0)$ and $d^{\perp_w} \leq \rho_A(0)$. We have to prove that $(cd)^{\perp_w} \leq \rho_A(0)$. Let e be a compact element of A such that $e \leq (cd)^{\perp_w}$. Thus, $e \leq (cd) \to \rho_A(0)$, and hence, we have $ecd \leq \rho_A(0)$. From Lemma 3.3, it follows that $ec \leq d^{\perp_w} \leq \rho_A(0)$. Hence, using Lemma 3.3(4), we obtain $e \leq c^{\perp_w} \leq \rho_A(0)$. Thus, $e \leq \rho_A(0)$ for all $e \in K(A)$ such that $e \leq (cd)^{\perp_w}$. Since A is coherent, we obtain $(cd)^{\perp_w} \leq \rho_A(0)$. Then, $cd \in T(A)$. Therefore, we conclude that $T(A)$ is multiplicatively closed. □

The localization quantale $A_{T(A)}$ of A at $T(A)$ is denoted by $\tilde{Q}(A)$. If A is semiprime, then $T(A) = S(A)$ and $\tilde{Q}(A) = Q(A)$.

Lemma 4.8. $\tilde{Q}(A)$ *is a non-trivial quantale.*

Proof. Suppose that $\tilde{Q}(A) = A_{T(A)}$ is a trivial quantale, i.e. $0_{T(A)} = 1$. By Lemma 4.2(5), there exists $t \in T(A)$ such that $1t \leq 0$. Hence, $t = 0$, which implies that $t^{\perp_w} = 0^{\perp_w} = 1$. On the other hand, for $t \in T(A)$, we have $\rho_A(0) = t^{\perp_w} = 1$. According to Lemma 2.5(2), $1 \in K(A)$ and $1 \leq \rho_A(0)$ imply that there exists a natural number n such that $1 = 1^n = 0$, contradicting the assumption that A is non-trivial. Thus, $\tilde{Q}(A)$ is a non-trivial quantale. □

Lemma 4.9. *For any $c \in K(A)$, the following equivalences hold:*

$$c_{T(A)} = 1 \Leftrightarrow c^{\perp_w} \leq \rho_A(0) \Leftrightarrow c^{\perp_w} = \rho_A(0).$$

Proof. Assume that $c_{T(A)} = 1$. By Lemma 4.2(5), there exists $s \in T(A)$ such that $s \leq c$. Hence, using Lemma 3.2(1), we obtain $c^{\perp_w} \leq s^{\perp_w} \leq \rho_A(0)$. Conversely, suppose that $c^{\perp_w} \leq \rho_A(0)$. Then, $c \in T(A)$. Hence, using Lemma 4.2(5), we have $c_{T(A)} = 1$. The second equivalence follows from Lemma 3.1(4). □

Lemma 4.10. *For any $a \in A$, the following equivalence holds:*

$$a_{T(A)} \leq \rho_{A_{T(A)}}(0_{T(A)}) \Leftrightarrow a \leq \rho_A(0).$$

Proof. By Lemma 4.2(9), we have $\rho_{A_{T(A)}}(0_{T(A)}) = (\rho_A(0))_{T(A)}$. Assume that $a_{T(A)} \leq \rho_{A_{T(A)}}(0_{T(A)})$. Then, $a_{T(A)} \leq (\rho_A(0))_{T(A)}$. Let c be a compact element of A such that $c \leq a$. Thus, using Lemma 4.2(2), we obtain $c \leq a \leq a_{T(A)} \leq (\rho_A(0))_{T(A)}$. By Lemma 4.1, we conclude that there exists $s \in T(A)$ such that $cs \leq \rho_A(0)$. By Lemma 3.3(4), we obtain $c \leq s \to \rho_A(0) = s^{\perp_w} \leq \rho_A(0)$. Since A is an algebraic quantale, we conclude that $a \leq \rho_A(0)$.

In order to establish the converse implication, suppose that $a \leq \rho_A(0)$. Then, using the properties (1) and (9) from Lemma 4.2, we obtain $a_{T(A)} \leq (\rho_A(0))_{T(A)} = \rho_{A_{T(A)}}(0_{T(A)})$. □

Lemma 4.11. *For any $a \in A$, it holds that $(a^{\perp_w})_{T(A)} = a^{\perp_w}$.*

Proof. Assume that $c \in K(A)$ and $c \leq (a^{\perp_w})_{T(A)}$. By Lemma 4.1, there exists $s \in T(A)$ such that $cs \leq a^{\perp_w}$. Then, $csa \leq \rho_A(0)$, and hence, we have $ca \leq s^{\perp_w} = \rho_A(0)$. From $ca \leq \rho_A(0)$, we obtain $c \leq a \to \rho_A(0) = a^{\perp_w}$. Therefore, it follows that $(a^{\perp_w})_{T(A)} \leq a^{\perp_w}$. The converse inequality follows from Lemma 4.2(2). □

Theorem 4.12. $Min(A) = Min(\tilde{Q}(A))$.

Proof. We set $T = T(A)$. Then, $\tilde{Q}(A) = A_T$. Recall from Proposition 4.4 that

$$Spec(A_T) = \{p \in Spec(A) \mid (p] \cap T = \emptyset\}.$$

First, we prove that $Min(A) \subseteq Min(A_T)$. Assume that $p \in Min(A)$. Suppose that $(p] \cap T \neq \emptyset$. Then, there exists $c \in T$ such that $c \leq p$. Applying Proposition 3.6(3), from $c \leq p$, we obtain $c^{\perp_{A,w}} \not\leq p$. But, $c \in T$ implies $c^{\perp_{A,w}} = \rho_A(0)$, and hence, $\rho_A(0) \not\leq p$. This last property contradicts $p \in Min(A)$, and therefore, it holds that $(p] \cap T = \emptyset$. Consequently, $p \in Spec(A_T)$, and hence, $p \in A_T$. To prove that $p \in Min(A_T)$, let x be a compact element of A_T such that $x \leq p$. By Lemma 4.3(2), there exists $c \in K(A)$ such that $x = c_T$. We know that $p_T = p$. Hence, using the properties (1) and (2) from Lemma 4.2, we obtain $c \leq c_T \leq p_T = p$. Recall from Proposition 4.5 that $(c^{\perp_{A,w}})_T = (c_T)^{\perp_{A_T,w}}$. In accordance with Lemma 4.2(2), we have $c^{\perp_{A,w}} \leq (c^{\perp_{A,w}})_T$. But, $c^{\perp_{A,w}} \not\leq p$, and hence, we have $(c_T)^{\perp_{A_T,w}} \not\leq p$. We proved that $x^{\perp_{A_T,w}} \not\leq p$ for all $x \in K(A_T)$ such that $x \leq p$. Now, Proposition 3.6(3) gives $p \in Min(A_T)$. Therefore, we conclude that $Min(A) \subseteq Min(A_T)$.

To prove the converse inclusion $Min(A_T) \subseteq Min(A)$, assume that $p \in Min(A_T)$. Then, p is an m-prime element of A. We have to show that p is a minimal m-prime element of A. Let c be a compact element of A such that $c \leq p$. By Lemma 4.3(2), c_T is a compact element of A_T. Due to Lemma 4.2(3), from $p \in A_T$ it follows that $p_T = p$. Hence, using Lemma 4.2(1), we have $c_T \leq p_T = p$. According to Proposition 3.6(3), $c_T \leq p$ implies $(c_T)^{\perp_{A_T,w}} \not\leq p$. Taking into account Proposition 4.5 and Lemma 4.11, we have $(c_T)^{\perp_{A_T,w}} = (c^{\perp_{A,w}})_T = c^{\perp_{A,w}}$, and hence, $c^{\perp_{A,w}} \not\leq p$. From Proposition 3.6, it follows that $p \in Min(A)$. In conclusion, we have $Min(A_T) \subseteq Min(A)$. □

Corollary 4.13. *If A is semiprime, then $Min(A) = Min(Q(A))$.*

Proposition 4.14. *If A is a coherent quantale, then the following statements are equivalent:*

(1) A is semiprime;

(2) $Q(A)$ is semiprime.

Proof. $(1) \Rightarrow (2)$. Assume that A is semiprime. We set $S = S(A)$. Then, $Q(A) = A_S$. First, we prove that $0_S = 0$. Let c be a compact element of A such that $c \leq 0_S$. Using Lemma 4.1, we obtain $cs = 0$ for some $s \in S$. Thus, $c \leq s^{\perp_A} = 0$, and hence, $c = 0$. Since A is coherent, it follows that $0_S \leq 0$, and therefore, $0_S = 0$. Using Lemma 4.2(9), we obtain $\rho_{A_S}(0_S) = (\rho_A(0))_S = 0_S$. Consequently, $Q(A)$ is semiprime.

$(2) \Rightarrow (1)$. Assume that $Q(A)$ is semiprime, i.e. $\rho_{A_S}(0_S) = 0_S$. Then, by Lemma 4.2(9), we have $\rho_{A_S}(0_S) = (\rho_A(0))_S$, and hence, $(\rho_A(0))_S = 0_S$. Now, let $c \in K(A)$ such that $c \leq \rho_A(0)$. Hence, using Lemma 4.2(1), we have $c_S \leq (\rho_A(0))_S = 0_S$. By Lemma 4.1, there exists $s \in S$ such that $cs \leq 0$. Since $s^{\perp_A} = 0$, we obtain $c \leq s^{\perp_A} = 0$. Since A is algebraic, we have $\rho_A(0) = 0$. Thus, A is semiprime. □

Corollary 4.15. *If A is a semiprime coherent quantale, then $\tilde{Q}(A)$ is semiprime.*

Proof. If A is a semiprime coherent quantale, then $T(A) = S(A)$, and hence, $\tilde{Q}(A) = Q(A)$. From Proposition 4.14, it follows that $Q(A)$ is semiprime. Therefore, $\tilde{Q}(A)$ is semiprime. □

We say that the localization map $j_T : A \to A_T$ *reflects* $\rho_A(0)$ if $j_T(\rho_A(0)) = 0_T$ implies $\rho_A(0) = 0$.

Proposition 4.16. *If A is a coherent quantale, then the following statements are equivalent:*

(1) A is semiprime;

(2) $\tilde{Q}(A)$ is semiprime and j_T reflects $\rho_A(0)$.

Proof. $(1) \Rightarrow (2)$. Assume that A is semiprime. By Corollary 4.15, $\tilde{Q}(A)$ is semiprime. Since $\rho_A(0) = 0$, it follows that j_T reflects $\rho_A(0)$.

$(2) \Rightarrow (1)$. Assume that $\tilde{Q}(A)$ is semiprime and j_T reflects $\rho_A(0)$. Since $\tilde{Q}(A)$ is semiprime, we have $\rho_{A_T}(0_T) = 0_T$. Using the identity $\rho_{A_T}(0_T) = (\rho_A(0))_T$, we obtain $j_T(\rho_A(0)) = 0_T$. But, j_T reflects $\rho_A(0)$, and hence, we have $\rho_A(0) = 0$. Therefore, A is semiprime. □

5. Weakly Complemented Quantales

Following Definition 3.5 of [6], a frame L is *complemented* if, for each $c \in K(L)$, there exists $d \in K(L)$ such that $c \wedge d = 0$ and $(c \vee d)^{\perp_L} = 0$.

The complemented quantales were introduced in [13] as a quantale generalization of complemented frames. Namely, a quantale A is *complemented* if, for any $c \in K(A)$, there exists $d \in K(A)$ such that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. Replacing the polar operator in this definition with the weak polar operator yields the concept of a weakly complemented quantale. Then, a quantale A is *weakly complemented* if, for any $c \in K(A)$, there exists $d \in K(A)$ such that $cd \leq \rho_A(0)$ and $(c \vee d)^{\perp_w} = \rho_A(0)$. If A is semiprime, then the notions of weakly complemented quantales and complemented quantales coincide.

Remark 5.1. *Assume that $c, d \in K(A)$. By Lemma 3.1(4), we have $\rho_A(0) \leq (c \vee d)^{\perp_w}$. Therefore, A is weakly complemented if and only if, for any $c \in K(A)$, there exists $d \in K(A)$ such that $cd \leq \rho_A(0)$ and $(c \vee d)^{\perp_w} \leq \rho_A(0)$.*

We fix a coherent quantale A. Recall from [7] that $R(A)$ is a coherent frame.

Theorem 5.2. *The following statements are equivalent:*

(1) A is a weakly complemented quantale;

(2) $R(A)$ is a complemented frame.

Proof. $(1) \Rightarrow (2)$. Assume that A is weakly complemented. Let x be a compact element of $R(A)$. From Lemma 2.10 of [7], it follows that $K(R(A)) = \rho_A(K(A))$. Hence, there exists $c \in K(A)$ such that $x = \rho_A(c)$. By hypothesis, there exists $d \in K(A)$ such that $cd \leq \rho_A(0)$ and $(c \vee d)^{\perp_w} = \rho_A(0)$. We set $y = \rho_A(d)$. Thus, $y \in K(R(A))$ and, from $cd \leq \rho_A(0)$, we obtain

$$x \wedge y = \rho_A(cd) \leq \rho_A(\rho_A(0)) = \rho_A(0). \tag{2}$$

Since the bottom element of $R(A)$ is $\rho_A(0)$, inequality (2) gives $x \wedge y = \rho_A(0)$. Recall that $x \dot{\vee} y = \rho_A(x \vee y)$. By Lemma 3.2(3), the following equalities hold:

$$(x \dot{\vee} y)^{\perp_{R(A)}} = (\rho_A(c \vee d))^{\perp_{R(A)}} = (c \vee d)^{\perp_w} = \rho_A(0).$$

Therefore, $R(A)$ is a complemented frame.

$(2) \Rightarrow (1)$. Suppose that $R(A)$ is complemented and $c \in K(A)$. Then, $x = \rho_A(c)$ is a compact element of $R(A)$. By hypothesis, there exists $y \in K(R(A))$ such that $x \wedge y = \rho_A(0)$ and $(x \dot{\vee} y)^{\perp_{R(A)}} = \rho_A(0)$. We know that y has the form $y = \rho_A(d)$ for some $d \in K(A)$. We observe that $\rho_A(cd) = \rho_A(c) \wedge \rho_A(d) = x \wedge y = \rho_A(0)$. Since $cd \leq \rho_A(cd)$, we have $cd \leq \rho_A(0)$. In accordance with Lemma 3.2(3), the following equalities hold:

$$(c \vee d)^{\perp_w} = (\rho_A(c \vee d))^{\perp_{R(A)}} = (\rho_A(c) \dot{\vee} \rho_A(d))^{\perp_{R(A)}} = \rho_A(0).$$

Consequently, it follows that A is a weakly complemented quantale. □

Corollary 5.3 (see [13]). *If A is semiprime, then the following statements are equivalent:*

(1) A is a complemented quantale;

(2) $R(A)$ is a complemented frame.

We recall from [4] the following characterizations of complemented frames:

Lemma 5.4. *If L is a coherent frame, then the following statements are equivalent:*

(1) L is a complemented frame;

(2) $Min_Z(L) = Min_F(L)$;

(3) $Min_Z(L)$ is a compact space;

(4) $Min_Z(L)$ is a Boolean space.

Recall from [7] that a quantale A is *hyperarchimedean* if, for any $c \in K(A)$, there exists an integer $n \geq 1$ such that $c^n \in B(A)$.

Lemma 5.5. *Any hyperarchimedean coherent quantale is weakly complemented.*

Proof. Let A be a hyperarchimedean coherent quantale. Assume that c is a compact element of A. Then, $c^n \in B(A)$ for some integer $n \geq 1$. Hence, there exists $d \in K(A)$ such that $c^n d = 0$ and $c^n \vee d = 1$. Consequently, we have $(cd)^n = c^n d^n = 0$ and $cd \in K(A)$. By Lemma 2.5(2), it holds that $cd \leq \rho_A(0)$. But, $c^n \vee d = 1$ and $c^n \leq c$ imply $c \vee d = 1$, and hence, $(c \vee d)^{\perp_w} = 1^{\perp_w} = \rho_A(0)$. Therefore, A is weakly complemented. □

Theorem 5.6. *If A is a coherent quantale, then the following statements are equivalent:*

(1) A is a weakly complemented quantale;

(2) $Min_Z(A) = Min_F(A)$;

(3) $Min_Z(A)$ is a compact space;

(4) $Min_Z(A)$ is a Boolean space;

(5) $\tilde{Q}(A)$ is a hyperarchimedean quantale;

(6) $\tilde{Q}(A)$ is a weakly complemented quantale.

Proof. $(1) \Leftrightarrow (2)$. We know that $Min_Z(A) = Min_Z(R(A))$ and $Min_F(A) = Min_F(R(A))$. By Theorem 5.2 and Lemma 5.4, the following properties are equivalent:

- A is a weakly complemented quantale;
- $R(A)$ is a complemented frame;
- $Min_Z(R(A)) = Min_F(R(A))$;
- $Min_Z(A) = Min_F(A)$.

$(1) \Leftrightarrow (3) \Leftrightarrow (4)$. These equivalences follow from Theorem 5.2 and Lemma 5.4.

$(1) \Rightarrow (5)$. Suppose that A is weakly complemented. We set $T = T(A)$. Then, $\tilde{Q}(A) = A_T$. Recall from Lemma 4.3(1) that A_T is a coherent quantale. To show that A_T is hyperarchimedean, consider a compact element x of A_T. By Lemma 4.3(2), there exists $c \in K(A)$ such that $x = c_T$. Since A is weakly complemented, there exists $d \in K(A)$ such that $cd \leq \rho_A(0)$ and $(c \vee d)^{\perp_w} = \rho_A(0)$. According to Lemma 4.3(2), $y = d_T$ is a compact element of A_T. By virtue of Lemma 4.2(1), we have

$$x \cdot_T y = c_T \cdot_T d_T = (cd)_T \leq \rho_{A_T}(0_T).$$

Since A_T is a coherent quantale (cf. Lemma 4.3) and $x, y \in K(A_T)$, Lemma 2.5(2) (applied to A_T) implies that there exists an integer $n \geq 1$ such that $x^n \cdot_T y^n = 0_T$. Using Lemma 4.9, from $(c \vee d)^{\perp_w} = \rho_A(0)$, we infer $(c \vee d)_T = 1$. Thus, the following equalities hold:

$$x \vee_T y = c_T \vee_T d_T = (c \vee d)_T = 1.$$

Since A_T is a coherent quantale, Lemma 2.1(3) applies to A_T. Thus, we obtain $x^n \vee_T y^n = 1$. From Lemma 2.3, it follows that $x^n \in B(A_T)$, and hence, $\tilde{Q}(A) = A_T$ is hyperarchimedean.

$(5) \Rightarrow (6)$. This implication follows from Lemma 5.5.

$(6) \Rightarrow (1)$. Assume that $\tilde{Q}(A) = A_T$ is a weakly complemented quantale. Let c be a compact element of A. By Lemma 4.3(2), c_T is a compact element of A_T. By hypothesis, there exists $y \in K(A_T)$ such that $c_T \cdot_T y \leq \rho_{A_T}(0_T)$ and $(c_T \vee_T y)^{\perp_{A_T,w}} = \rho_{A_T}(0_T)$. By Lemma 4.3(2), there exists $d \in K(A)$ such that $y = d_T$. Hence, $(cd)_T = c_T \cdot_T d_T \leq \rho_{A_T}(0_T)$ and

$$((c \vee d)_T)^{\perp_{A_T,w}} = (c_T \vee_T d_T)^{\perp_{A_T,w}} = \rho_{A_T}(0_T).$$

On the other hand, by Lemma 4.2(9) and Proposition 4.5, we have $\rho_{A_T}(0_T) = (\rho_A(0))_T$ and

$$((c \vee d)_T)^{\perp_{A_T,w}} = ((c \vee d)^{\perp_{A,w}})_T.$$

Thus, we obtain $(cd)_T \leq (\rho_A(0))_T$ and $((c \vee d)^{\perp_{A,w}})_T \leq (\rho_A(0))_T$. Now, from Lemma 4.10, it follows that $cd \leq \rho_A(0)$ and $(c \vee d)^{\perp_{A,w}} \leq \rho_A(0)$. Therefore, A is weakly complemented. $\square$

Corollary 5.7 (see [13]). *If A is a semiprime coherent quantale, then the following statements are equivalent:*

(1) A is a complemented quantale;

(2) $Min_Z(A) = Min_F(A)$;

(3) $Min_Z(A)$ is a compact space;

(4) $Min_Z(A)$ is a Boolean space;

(5) $Q(A)$ is a hyperarchimedean quantale;

(6) $Q(A)$ is a complemented quantale.

6. Generalized PP-Quantales

Let A be an arbitrary quantale. Recall from [11] that A is a PP-*quantale* if $c^{\perp_A} \in B(A)$ for any $c \in K(A)$. We remark that A is a PP-quantale if and only if $c^{\perp_A} \vee c^{\perp_A \perp_A} = 1$ for any $c \in K(A)$. Replacing the polar operator in this equivalent definition of a PP-quantale with the weak polar operator yields the notion of the generalized PP-quantale: A is called a *generalized* PP-*quantale* (abbreviated GPP-*quantale*) if the equality $c^{\perp_w} \vee c^{\perp_w \perp_w} = 1$ holds for any $c \in K(A)$.

Lemma 6.1. *If $a \in A$, then $a^{\perp_w} = (a^n)^{\perp_w}$ for any integer $n \geq 1$.*

Proof. The required conclusion follows from Proposition 3.5(1). $\square$

Lemma 6.2. *Any hyperarchimedean coherent quantale is a GPP-quantale.*

Proof. Let A be a hyperarchimedean quantale and $c \in K(A)$. Then, $c^n \in B(A)$ for some integer $n \geq 1$, and hence, $c^n \vee (c^n)^{\perp_A} = 1$. We set $y = c^n$. Then, $y \vee y^{\perp_A} = 1$. On the other hand, we have $y \leq y^{\perp_w \perp_w}$ and $y^{\perp_A} \leq y^{\perp_w}$. Hence, $y^{\perp_w} \vee y^{\perp_w \perp_w} = 1$. According to Lemma 6.1, we have $y^{\perp_w} = c^{\perp_w}$ and $y^{\perp_w \perp_w} = c^{\perp_w \perp_w}$. Therefore, $c^{\perp_w} \vee c^{\perp_w \perp_w} = 1$, and hence, A is a GPP-quantale. □

Theorem 6.3. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a GPP-quantale;

(2) $R(A)$ is a PP-frame.

Proof. $(1) \Rightarrow (2)$. Assume that A is a GPP-quantale. Let x be a compact element of the frame $R(A)$. Then, $x = \rho_A(c)$ for some $c \in K(A)$. By hypothesis, we have $c^{\perp_w} \vee c^{\perp_w \perp_w} = 1$. According to Lemma 3.2(3), we have

$$x^{\perp_{R(A)}} = (\rho_A(c))^{\perp_{R(A)}} = c^{\perp_w}.$$

Thus, $c^{\perp_w} \in R(A)$. Now, applying Lemma 3.2(2), we obtain

$$x^{\perp_{R(A)} \perp_{R(A)}} = (c^{\perp_w})^{\perp_{R(A)}} = c^{\perp_w \perp_w}.$$

Consequently, we have

$$x^{\perp_{R(A)}} \dot{\vee} x^{\perp_{R(A)} \perp_{R(A)}} = c^{\perp_w} \dot{\vee} c^{\perp_w \perp_w} = \rho_A(c^{\perp_w} \vee c^{\perp_w \perp_w}) \geq c^{\perp_w} \vee c^{\perp_w \perp_w}.$$

Since A is a GPP-quantale, we have $c^{\perp_w} \vee c^{\perp_w \perp_w} = 1$, and hence, $x^{\perp_{R(A)}} \dot{\vee} x^{\perp_{R(A)} \perp_{R(A)}} = 1$. Therefore, we conclude that $R(A)$ is a PP-frame.

$(2) \Rightarrow (1)$. This implication is proved in a similar manner by applying Lemma 3.2. □

Recall from [10] that A is an *mp-quantale* if, for each $p \in Spec(A)$, there exists a unique $q \in Min(A)$ such that $q \leq p$.

Theorem 6.4 (see [11]). *If A is a semiprime coherent quantale, then the following properties are equivalent:*

(1) A is a PP-quantale;

(2) A is an mp-quantale and $Min_Z(A)$ is a compact topological space.

Corollary 6.5. *If A is a semiprime coherent quantale, then the following properties are equivalent:*

(1) A is a PP-quantale;

(2) A is a complemented mp-quantale.

Proof. By Corollary 5.7, A is a complemented quantale if and only if $Min_Z(A)$ is a compact space. Therefore, by applying Theorem 6.4, we obtain the desired equivalence. □

Corollary 6.6. *If L is a coherent frame, then the following properties are equivalent:*

(1) L is a PP-frame;

(2) L is a complemented mp-frame.

Proof. Since any coherent frame is a particular case of a semiprime coherent quantale, it suffices to apply Corollary 6.5. □

Theorem 6.7 (see [11]). *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is an mp-quantale;

(2) $R(A)$ is an mp-frame.

Theorem 6.8. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a GPP-quantale;

(2) A is a weakly complemented mp-quantale.

Proof. Recall that $R(A)$ is a coherent frame. Applying Theorems 5.2, 6.3 and 6.7, together with Corollary 6.6, we obtain the equivalence of the following properties:

- A is a GPP-quantale;
- $R(A)$ is a PP-frame;
- $R(A)$ is a complemented mp-frame;
- A is a weakly complemented mp-quantale.

$\square$

We remark that Theorem 6.8 is a quantale-theoretic generalization of Corollary 6.6. Its proof illustrates how the radical functor $R(\cdot)$ can be used to transfer certain results from coherent frames to coherent quantales.

Theorem 6.9. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a weakly complemented quantale;

(2) $\tilde{Q}(A)$ is a GPP-quantale.

Proof. $(1) \Rightarrow (2)$. Assume that A is weakly complemented. By Theorem 5.6, $\tilde{Q}(A)$ is hyperarchimedean. Hence, from Lemma 6.2, it follows that $\tilde{Q}(A)$ is a GPP-quantale.

$(2) \Rightarrow (1)$. Suppose that $\tilde{Q}(A)$ is a GPP-quantale. By virtue of Theorem 6.8, $\tilde{Q}(A)$ is weakly complemented. Therefore, by Theorem 5.6, we infer that A is weakly complemented. $\square$

Theorem 6.10. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a GPP-quantale;

(2) A is an mp-quantale and $Min_Z(A)$ is compact;

(3) A is an mp-quantale and $\tilde{Q}(A)$ is hyperarchimedean.

Proof. By Theorem 5.6, A is weakly complemented iff $Min_Z(A)$ is compact iff $\tilde{Q}(A)$ is hyperarchimedean. Therefore, by Theorem 6.8, the following statements are equivalent:

- A is a GPP-quantale;
- A is a weakly complemented mp-quantale;
- A is an mp-quantale and $Min_Z(A)$ is compact;
- A is an mp-quantale and $\tilde{Q}(A)$ is hyperarchimedean. $\square$

Lemma 6.11. *Any complemented coherent quantale is semiprime.*

Proof. Let A be a complemented coherent quantale and $c \in K(A)$. Then, there exists $d \in K(A)$ such that $cd = 0$ and $(c \vee d)^{\perp_A} = 0$. Assume that $c^2 = 0$. Hence $c \leq c^{\perp_A}$. From $cd = 0$, it follows that $c \leq d^{\perp_A}$, and therefore, $c \leq c^{\perp_A} \wedge d^{\perp_A} = (c \vee d)^{\perp_A} = 0$. We proved that $c^2 = 0$ implies $c = 0$. By induction, it can be proved that for each integer $n \geq 1$, $c^n = 0$ implies $c = 0$. Consequently, from Lemma 2.5(3), it follows that A is semiprime. □

Lemma 6.12. *Any semiprime and hyperarchimedean coherent quantale is a PP-quantale.*

Proof. The required conclusion follows from Corollary 6.1 of [13]. □

Lemma 6.13. *Any coherent PP-quantale is complemented.*

Proof. The required conclusion follows from Lemma 6.2 of [13]. □

Theorem 6.14. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a complemented quantale;

(2) $Q(A)$ is a semiprime hyperarchimedean quantale;

(3) $Q(A)$ is a PP-quantale;

(4) $Q(A)$ is a complemented quantale.

Proof. $(1) \Rightarrow (2)$. Suppose that A is complemented. By Lemma 6.11 and Proposition 4.14, $Q(A)$ is semiprime. From the implication $(1) \Rightarrow (5)$ of Corollary 5.7, it follows that $Q(A)$ is a hyperarchimedean quantale.

$(2) \Rightarrow (3)$. Assume that $Q(A)$ is semiprime and hyperarchimedean. By Lemma 6.12, $Q(A)$ is a PP-quantale.

$(3) \Rightarrow (4)$. This implication follows from Lemma 6.13.

$(4) \Rightarrow (1)$. Assume that $Q(A)$ is complemented. By virtue of Lemma 6.11, $Q(A)$ is semiprime. Since $Q(A)$ is semiprime, Proposition 4.14 implies that A is semiprime. Therefore, by Corollary 5.7, A is a complemented quantale. □

Proposition 6.15. *Any complemented coherent quantale is weakly complemented.*

Proof. Let A be a complemented coherent quantale. By Lemma 6.11, A is semiprime. Hence, $\rho_A(0) = 0$. Therefore, the weak polars and the polars of A coincide. Consequently, A is weakly complemented. □

Corollary 6.16. *Any coherent PP-quantale is a GPP-quantale.*

Proof. Let A be a coherent PP-quantale. By Lemma 6.13, A is a complemented coherent quantale. From Lemma 6.11, it follows that A is semiprime. Hence, the weak polars and the polars of A coincide. Therefore, we conclude that A is a GPP-quantale. □

Example 6.17. *Let $A = \{0, a, b, 1\}$ be the four-element chain ordered by $0 < a < b < 1$. We define a multiplication on A by the following table:*

$\cdot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	b	b
1	0	a	b	1

Thus, A is an integral commutative quantale. Since A is finite, we have $K(A) = A$. Hence, A is a coherent quantale. A simple computation shows that $Spec(A) = \{a, b\}$ and $Min(A) = \{a\}$. Thus, $\rho_A(0) = a \wedge b = a \neq 0$, and hence, A is not a semiprime quantale. According to Lemma 6.11, A is not a complemented quantale. Since $Min_Z(A)$ is a compact space, by Theorem 5.6, A is weakly complemented. Therefore, the converse of Proposition 6.15 does not hold.

Now, we compute the weak polars of A as follows:
$0^{\perp_w} = 1$;
$a^{\perp_w} = a \to \rho_A(0) = a \to a = 1$;
$b^{\perp_w} = b \to \rho_A(0) = b \to a = a$;
$1^{\perp_w} = \rho_A(0) = a$.
Then, $T(A) = \{x \in K(A) \mid x^{\perp_w} = \rho_A(0)\} = \{b, 1\}$. Since $b \in T(A)$, we have $b_{T(A)} = 1$. According to Lemma 4.2(7), $b \in T(A)$ and $ab = 0$ imply $a_{T(A)} = 0_{T(A)}$. Hence, $\tilde{Q}(A) = \{0_{T(A)}, 1\}$, which is a semiprime quantale. Since A is not semiprime, we conclude that the converse of Corollary 4.15 does not hold.

Moreover, the quantale A is a GPP-quantale. Indeed, A is an mp-quantale, since $Min(A) = \{a\}$, and we have shown above that A is weakly complemented. Therefore, by Theorem 6.8, A is a GPP-quantale. It follows from the definition of polars that $a^{\perp_A} = b$. Since $b \notin B(A)$, A is not a PP-quantale.

Example 6.17 shows that the generalized PP-quantales properly extend the class of PP-quantales.

7. Weak Fraction-Dense Quantales

Fraction-dense frames were introduced in [6] as an abstraction of fraction-dense f-rings [15]. The notions of weak fraction-dense and fraction-dense quantales were introduced in [13] as quantale-theoretic generalizations of fraction-dense frames.

A quantale A is called

- *fraction-dense* if, for each $a \in A$, there exists $c \in K(A)$ such that $a^{\perp_A} = c^{\perp_A}$;
- *weak fraction-dense* if, for each $a \in A$, there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w}$.

For semiprime quantales, these two notions coincide.

Theorem 7.1 (see [13]). *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a weak fraction-dense quantale;

(2) $R(A)$ is a fraction-dense frame.

Theorem 7.2. *If L is a coherent frame, then the following properties are equivalent:*

(1) L is a fraction-dense frame;

(2) For any $a \in L$, there exists $c \in K(L)$ such that $a \wedge c = 0$ and $(a \vee c)^{\perp_L} = 0$;

(3) $Min_Z(L)$ is a Stone space.

Proof. (1) $\Leftrightarrow$ (3). This equivalence follows from Corollary 3.7 of [6].

(2) $\Leftrightarrow$ (3). This equivalence follows from Theorem 4.14 of [5]. □

We now generalize Theorem 7.2 to the setting of quantales.

Theorem 7.3. *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a weak fraction-dense quantale;

(2) For any $a \in A$, there exists $c \in K(A)$ such that $ac \leq \rho_A(0)$ and $(a \vee c)^{\perp_w} = \rho_A(0)$;

(3) $Min_Z(A)$ is a Stone space.

Proof. (1) $\Rightarrow$ (2). Assume that A is a weak fraction-dense quantale. First, we recall that $\dot{\vee}$ is the join in the frame $R(A)$. To verify condition (2), let a be an element of A. Then, we have $\rho_A(a) \in R(A)$. According to Theorem 7.1, $R(A)$ is a fraction-dense frame. Hence, there exists $x \in K(R(A))$ such that $\rho_A(a) \wedge x = \rho_A(0)$ and $(\rho_A(a)\dot{\vee}x)^{\perp_{R(A)}} = \rho_A(0)$. Since $K(R(A)) = \rho_A(K(A))$, there exists $c \in K(A)$ such that $x = \rho_A(c)$. Thus, we have $\rho_A(ac) = \rho_A(a) \wedge \rho_A(c) = \rho_A(a) \wedge x = \rho_A(0)$. From $\rho_A(ac) = \rho_A(0)$ and $ac \leq \rho_A(ac)$, we obtain $ac \leq \rho_A(0)$. Using Lemma 3.2(3), we have

$$(a \vee c)^{\perp_w} = (\rho_A(a \vee c))^{\perp_{R(A)}} = (\rho_A(a)\dot{\vee}\rho_A(c))^{\perp_{R(A)}} = (\rho_A(a)\dot{\vee}x)^{\perp_{R(A)}} = \rho_A(0).$$

Therefore, we conclude that $ac \leq \rho_A(0)$ and $(a \vee c)^{\perp_w} = \rho_A(0)$.

(2) $\Rightarrow$ (3). Let x be an element of $R(A)$. We can find an element $a \in A$ such that $x = \rho_A(a)$. Using condition (2), there exists $c \in K(A)$ such that $ac \leq \rho_A(0)$ and $(a \vee c)^{\perp_w} = \rho_A(0)$. From $ac \leq \rho_A(0)$, we obtain $x \wedge \rho_A(c) = \rho_A(ac) \leq \rho_A(\rho_A(0))$. Since the radical operator $\rho_A(\cdot)$ is idempotent, we have $\rho_A(\rho_A(0)) = \rho_A(0)$, and hence, $x \wedge \rho_A(c) \leq \rho_A(0)$. By Lemma 3.2(3), we have

$$(x\dot{\vee}\rho_A(c))^{\perp_{R(A)}} = (\rho_A(a)\dot{\vee}\rho_A(c))^{\perp_{R(A)}} = (\rho_A(a \vee c))^{\perp_{R(A)}} = (a \vee c)^{\perp_w} = \rho_A(0).$$

Hence, the coherent frame $R(A)$ satisfies condition (2) of Theorem 7.2, and therefore, $Min_Z(R(A))$ is a Stone space. By Lemma 2.4, we have $Min_Z(R(A)) = Min_Z(A)$, and hence, $Min_Z(A)$ is a Stone space.

(3) $\Rightarrow$ (1). Suppose that $Min_Z(A)$ is a Stone space. Then, $Min_Z(R(A))$ is also a Stone space (by Lemma 2.4). Therefore, from Theorem 7.2, it follows that $R(A)$ is a fraction-dense frame. Thus, by Theorem 7.1, A is a weak fraction-dense quantale. □

Remark 7.4. *Theorem 5.2 in [13] contains a statement that requires correction, namely, the assertion that a coherent quantale A is weak fraction-dense if and only if $Min_Z(A)$ is a Boolean space. This statement can be corrected by replacing "Boolean space" with "Stone space" in both the statement and its proof. As shown above, Theorem 7.3 corrects and completes the corresponding result in [13].*

A quantale A is said to be a *strongly PP-quantale* (or *strongly projectable*) if, for any $a \in A$, it holds that $a^{\perp_A} \vee a^{\perp_A \perp_A} = 1$. Starting from this concept and using the weak polars, we can formulate the following notion: the quantale A is a *strongly GPP-quantale* if, for any $a \in A$, it holds that $a^{\perp_w} \vee a^{\perp_w \perp_w} = 1$. If A is a semiprime quantale, then A is a strongly PP-quantale if and only if it is a strongly GPP-quantale.

Lemma 7.5 (see [13]). *If A is a coherent quantale, then the following properties are equivalent:*

(1) A is a weak fraction-dense quantale;

(2) For any $a \in A$, there exists $c \in K(A)$ such that $a^{\perp_w} = c^{\perp_w \perp_w}$.

Theorem 7.6. *Let A be a coherent quantale. Then, A is weak fraction-dense if and only if $\tilde{Q}(A)$ is a strongly GPP-quantale.*

Proof. Assume that A is a weak fraction-dense quantale. We set $T = T(A)$. Then, $\tilde{Q}(A) = A_{T(A)}$. Let x be an element of A_T. Then, $x = a_T$ for some $a \in A$. By Lemma 7.5, there exist $c, d \in K(A)$ such that $a^{\perp_{A,w}} = c^{\perp_{A,w}\perp_{A,w}}$ and $a^{\perp_{A,w}\perp_{A,w}} = c^{\perp_{A,w}\perp_{A,w}\perp_{A,w}} = c^{\perp_{A,w}} = d^{\perp_{A,w}\perp_{A,w}}$. We observe that

$$d^{\perp_{A,w}} = d^{\perp_{A,w}\perp_{A,w}\perp_{A,w}} = a^{\perp_{A,w}\perp_{A,w}\perp_{A,w}} = a^{\perp_{A,w}},$$

and hence,

$$(c \vee d)^{\perp_{A,w}} = c^{\perp_{A,w}} \wedge d^{\perp_{A,w}} = d^{\perp_{A,w}\perp_{A,w}} \wedge d^{\perp_{A,w}} = \rho_A(0).$$

By using Lemma 4.9, from $(c \vee d)^{\perp_{A,w}} = \rho_A(0)$, we obtain $(c \vee d)_T = 1$. From Proposition 4.5, it follows that

$$(a_T)^{\perp_{A_T,w}} \vee_T (a_T)^{\perp_{A_T,w}\perp_{A_T,w}} = (a^{\perp_{A,w}})_T \vee_T (a^{\perp_{A,w}\perp_{A,w}})_T = (c^{\perp_{A,w}\perp_{A,w}} \vee d^{\perp_{A,w}\perp_{A,w}})_T.$$

By Lemma 3.1(2), we have $c \leq c^{\perp_{A,w}\perp_{A,w}}$ and $d \leq d^{\perp_{A,w}\perp_{A,w}}$. Thus, using Lemma 4.2(1), we obtain

$$1 = (c \vee d)_T \leq (c^{\perp_{A,w}\perp_{A,w}} \vee d^{\perp_{A,w}\perp_{A,w}})_T = (a_T)^{\perp_{A_T,w}} \vee_T (a_T)^{\perp_{A_T,w}\perp_{A_T,w}}.$$

Since 1 is the top element of A_T, it follows that $x^{\perp_{A_T,w}} \vee_T x^{\perp_{A_T,w}\perp_{A_T,w}} = 1$. Therefore, $\tilde{Q}(A)$ is a strongly GPP-quantale.

Conversely, suppose that $\tilde{Q}(A)$ is a strongly GPP-quantale. We verify the property by which a weak fraction-dense quantale is defined. Let a be an element of A. We have to prove that there exists $c \in K(A)$ such that $a^{\perp_{A,w}} = c^{\perp_{A,w}}$. Since $\tilde{Q}(A)$ is a strongly GPP-quantale, for the element $a_T \in A_T$, we have

$$(a_T)^{\perp_{A_T,w}} \vee_T (a_T)^{\perp_{A_T,w}\perp_{A_T,w}} = 1.$$

Since $1 \in K(A_T)$ and A_T is coherent, there exist compact elements $u, v \in K(A_T)$ such that $u \vee_T v = 1$ and

$$u \leq (a_T)^{\perp_{A_T,w}}; \ v \leq (a_T)^{\perp_{A_T,w}\perp_{A_T,w}}.$$

Since $K(A_T) = \{c_T \mid c \in K(A)\}$, we may write $u = b_T$ and $v = c_T$ with $b, c \in K(A)$. Hence, we obtain $b_T \vee_T c_T = 1$ and

$$b_T \leq (a_T)^{\perp_{A_T,w}}; \ c_T \leq (a_T)^{\perp_{A_T,w}\perp_{A_T,w}}.$$

Using parts (1) and (2) of Lemma 3.1, from $c_T \leq (a_T)^{\perp_{A_T,w}\perp_{A_T,w}}$, we obtain $(a_T)^{\perp_{A_T,w}} \leq (c_T)^{\perp_{A_T,w}}$. Now, we prove the converse inequality. Since $b_T \leq (a_T)^{\perp_{A_T,w}}$, we have $b_T \wedge a_T \leq \rho_{A_T}(0_T)$. Thus, it follows that $a_T \leq (b_T)^{\perp_{A_T,w}}$. On the other hand, since $b_T \vee_T c_T = 1$, we have

$$(b_T)^{\perp_{A_T,w}} \wedge (c_T)^{\perp_{A_T,w}} = (b_T \vee_T c_T)^{\perp_{A_T,w}} = 1^{\perp_{A_T,w}} = \rho_{A_T}(0_T).$$

But, $a_T \leq (b_T)^{\perp_{A_T,w}}$, and hence, we obtain

$$a_T \wedge (c_T)^{\perp_{A_T,w}} \leq (b_T)^{\perp_{A_T,w}} \wedge (c_T)^{\perp_{A_T,w}} = \rho_{A_T}(0_T).$$

Therefore, we obtain the converse inequality $(c_T)^{\perp_{A_T,w}} \leq (a_T)^{\perp_{A_T,w}}$. Together with the previous inequality, this gives $(a_T)^{\perp_{A_T,w}} = (c_T)^{\perp_{A_T,w}}$. We now transfer this equality from A_T to A. By Proposition 4.5, we have

$$(a_T)^{\perp_{A_T,w}} = (a^{\perp_{A,w}})_T; \ (c_T)^{\perp_{A_T,w}} = (c^{\perp_{A,w}})_T.$$

Hence, we obtain $(a_T)^{\perp_{A_T,w}} = (c^{\perp_{A,w}})_T$. According to Lemma 4.11, the following equalities hold:

$$a^{\perp_{A,w}} = (a_T)^{\perp_{A_T,w}} = (c^{\perp_{A,w}})_T = c^{\perp_{A,w}}.$$

We have shown that $a^{\perp_{A,w}} = c^{\perp_{A,w}}$ for some $c \in K(A)$. Therefore, A is weak fraction-dense. $\square$

8. Localization and Quantale Morphisms

Let A and B be two quantales. In accordance with [21], a map $u : A \to B$ is called a *quantale morphism* if it preserves arbitrary joins and multiplication. Particularly, any quantale morphism preserves the bottom element 0. A quantale morphism $u : A \to B$ is *integral* if it preserves the top element 1. For coherent quantales A and B, an integral quantale morphism $u\colon A \to B$ is called *coherent* if $u(c) \in K(B)$ for every $c \in K(A)$. The category of coherent quantales and coherent quantale morphisms is denoted by $CohQuant$.

Lemma 8.1. *If $u : A \to B$ is a coherent quantale morphism and S is a multiplicatively closed subset of A, then $u(S)$ is a multiplicatively closed subset of B.*

Assume now that $u : A \to B$ is a coherent quantale morphism and S (respectively, T) is a multiplicatively closed subset of A (respectively, B). For any $c \in K(A)$, we write $u(c)_T$ instead of $(u(c))_T$.

Now, we study the functoriality of the construction $\tilde{Q}(\cdot)$. To this end, we recall a general extension theorem from [14].

Theorem 8.2 (see [14]). *Let $u : A \to B$ be a coherent quantale morphism, and let S (respectively, T) be a multiplicatively closed subset of A (respectively, B). If $u(S) \subseteq T$, then there exists a unique coherent quantale morphism $u_{ST} : A_S \to B_T$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "j_S"'] & B \arrow[d, "j_T"] \\ A_S \arrow[r, "u_{ST}"'] & B_T \end{tikzcd}$$

Corollary 8.3 (cf. Theorem 7.1 of [13]). *Let $u : A \to B$ be a coherent quantale morphism such that $u(T(A)) \subseteq T(B)$. Then, there exists a unique coherent quantale morphism $\tilde{Q}(u) : \tilde{Q}(A) \to \tilde{Q}(B)$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "j_{T(A)}"'] & B \arrow[d, "j_{T(B)}"] \\ \tilde{Q}(A) \arrow[r, "\tilde{Q}(u)"'] & \tilde{Q}(B) \end{tikzcd}$$

Proof. Since $u(T(A)) \subseteq T(B)$, Theorem 8.2 applies to the multiplicatively closed sets $S = T(A)$ and $T = T(B)$. Therefore, there exists a unique coherent quantale morphism $\tilde{Q}(u) : \tilde{Q}(A) \to \tilde{Q}(B)$ such that $\tilde{Q}(u) \circ j_{T(A)} = j_{T(B)} \circ u$. $\square$

Let $u : A \to B$ be a coherent quantale morphism and S (respectively, T) be a multiplicatively closed subset of A (respectively, B). Assume that $u(S) \subseteq T$. By definition, $u_{ST}(a_S) = (u(a))_T$ for any $a \in A$. For simplicity, we write $u(a)_T$ instead of $(u(a))_T$. Hence, $u_{ST}(a_S) = u(a)_T$ for any $a \in A$.

Remark 8.4. *Let $u : A \to B$ be a coherent quantale morphism. Since $\tilde{Q}(A) = A_{T(A)}$ and $\tilde{Q}(B) = A_{T(B)}$, we have $\tilde{Q}(u)(a_{T(A)}) = u(a)_{T(B)}$ for any $a \in A$.*

A coherent quantale morphism $u : A \to B$ is said to be a *$\tilde{Q}$-morphism* if $u(T(A)) \subseteq T(B)$. It is clear that the composition of two $\tilde{Q}$-morphisms is a $\tilde{Q}$-morphism. Thus, one can consider the category $\tilde{Q}CohQuant$ of coherent quantales and their $\tilde{Q}$-morphisms. Let $u : A \to B$ and $v : B \to C$ be two coherent quantale morphisms. Using Corollary 8.3, it is easy to prove that $\tilde{Q}(v \circ u) = \tilde{Q}(v) \circ \tilde{Q}(u)$. Therefore, the assignments $A \mapsto \tilde{Q}(A)$ and $u \mapsto \tilde{Q}(u)$ provide a covariant functor $\tilde{Q} : \tilde{Q}CohQuant \to CohQuant$.

Lemma 8.5. *If A is a coherent quantale, then $S(A) \subseteq T(A)$.*

Proof. Assume that $c \in S(A)$. Then, $c \in K(A)$ and $c^{\perp_A} = 0$. To show that $c^{\perp_w} \leq \rho_A(0)$, consider an element $d \in K(A)$ such that $d \leq c^{\perp_w}$. Then, $cd \leq \rho_A(0)$, and hence, $c^n d^n = 0$ for some integer $n \geq 1$. Thus, $d^n c^{n-1} \leq c^{\perp_A} = 0$, and therefore, $d^n c^{n-1} = 0$. Repeating this argument inductively, we obtain $d^n = 0$, and hence, $d \leq \rho_A(0)$. We have shown that $c^{\perp_w} \leq \rho_A(0)$, and therefore, $S(A) \subseteq T(A)$. □

Proposition 8.6. *If A is a coherent quantale, then there exists a unique coherent quantale morphism $\epsilon_A : Q(A) \to \tilde{Q}(A)$ such that the following diagram is commutative:*

$$\begin{tikzcd} A \arrow[r, "j_{S(A)}"] \arrow[dr, "j_{T(A)}"'] & Q(A) \arrow[d, "\varepsilon_A"] \\ & \tilde{Q}(A) \end{tikzcd}$$

Proof. By Lemma 8.5, we have $S(A) \subseteq T(A)$. Thus, applying Theorem 8.2 to the identity quantale morphism $id : A \to A$, we find a unique coherent quantale morphism $\epsilon_A : Q(A) \to \tilde{Q}(A)$ such that $\epsilon_A \circ j_{S(A)} = j_{T(A)}$. □

According to Remark 8.4, $\epsilon_A(a_{S(A)}) = a_{T(A)}$ for any $a \in A$.

Consider now the coherent quantale morphism $\rho_A : A \to R(A)$ defined by the assignment $a \mapsto \rho_A(a)$. For convenience, we write ρ instead of ρ_A.

Let S be a multiplicatively closed subset of A. By Lemma 8.1, $\rho(S)$ is a multiplicatively closed subset of the frame $R(A)$. By Theorem 8.2, there exists a coherent quantale morphism $\rho_{S\rho(S)} : A_S \to (R(A))_{\rho(S)}$, defined by $\rho_{S\rho(S)}(a_S) = (\rho_A(a))_{\rho(S)}$ for any $a \in A$. We write ρ_S instead of $\rho_{S\rho(S)}$.

Consider the map $\Phi : R(A_S) \to (R(A))_{\rho(S)}$ obtained by restricting $\rho_{S\rho(S)}$ to $R(A_S)$. Thus, Φ is a coherent frame morphism.

Theorem 8.7 (see [14]). *The map Φ is a frame isomorphism.*

Proposition 8.8. *If A is a coherent quantale, then the following properties hold:*

(1) $\rho(T(A)) = S(R(A))$;

(2) $R(\tilde{Q}(A))$ and $Q(R(A))$ are isomorphic frames.

Proof. (1). Since $T(A) = \{c \in K(A) \mid c^{\perp_w} = \rho_A(0)\}$, it follows that

$$\rho(T(A)) = \{\rho_A(c) \mid c \in T(A)\} = \{\rho_A(c) \mid c \in K(A), c^{\perp_w} = \rho(0)\}.$$

Recall that $K(R(A)) = \{\rho_A(c) \mid c \in K(A)\}$. From Lemma 3.2(3), it follows that $c^{\perp_w} = (\rho_A(c))^{\perp_{R(A)}}$ for every $c \in K(A)$. Hence, the following equalities hold:

$$\begin{aligned} S(R(A)) &= \{x \in K(R(A)) \mid x^{\perp_{R(A)}} = \rho_A(0)\} = \{\rho_A(c) \mid c \in K(A), (\rho_A(c))^{\perp_{R(A)}} = \rho_A(0)\} \\ &= \{\rho_A(c) \mid c \in K(A), c^{\perp_w} = \rho_A(0)\} = \rho(T(A)). \end{aligned}$$

(2). By part (1) and Theorem 8.7, we have

$$R(\tilde{Q}(A)) = R(A_{T(A)}) \cong (R(A))_{\rho(T(A))} = (R(A))_{S(R(A))} = Q(R(A)).$$

Therefore, the frames $R(\tilde{Q}(A))$ and $Q(R(A))$ are isomorphic. □

Corollary 8.9 (see [14]). *If A is a semiprime quantale, then $R(Q(A))$ and $Q(R(A))$ are isomorphic frames.*

Remark 8.10. *Proposition 8.8 and Corollary 8.9 can be used to transfer certain results from coherent frames to coherent quantales. To illustrate this method, we provide a second proof of Theorem 6.9. First, we recall from Proposition 5.1 of [6] that a coherent frame L is complemented if and only if $Q(L)$ is a PP-frame. Then, using Theorems 5.2 and 6.3, and Proposition 8.8(2), we obtain the equivalence of the following properties:*

- *A is a weakly complemented quantale;*
- *$R(A)$ is a complemented frame;*
- *$Q(R(A))$ is a PP-frame;*
- *$R(\tilde{Q}(A))$ is a PP-frame;*
- *$\tilde{Q}(A)$ is a GPP-quantale.*

Therefore, A is a weakly complemented quantale if and only if $\tilde{Q}(A)$ is a GPP-quantale.

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