

Research Article

**On the Existence of Infinitely Many Solutions for Nonlinear Klein–Gordon–Maxwell Systems Under Weak Nonlinearity Conditions**Mohsen Timoumi<sup>‡</sup>

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**Abstract**

In this paper, the existence of infinitely many nontrivial solutions to a class of nonlinear Klein–Gordon–Maxwell systems in  $\mathbb{R}^3$  is investigated, where the nonlinearity is either superquadratic or exhibits combined growth (i.e., both subquadratic and superquadratic behavior at infinity). The potential  $V(x)$  is allowed to be neither coercive nor uniformly positive. By employing variational methods, in particular the symmetric mountain pass theorem and the dual fountain theorem, new existence results are established under significantly weaker assumptions on the nonlinearity, without requiring the classical Ambrosetti–Rabinowitz condition. The obtained results extend and improve several known results in the literature and are applicable to a broader class of nonlinearities.

**Keywords:** variational methods; Klein–Gordon–Maxwell systems; infinitely many solutions; symmetric mountain pass theorem; dual fountain theorem.

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**1. Introduction**

This paper is concerned with the existence of infinitely many nontrivial solutions for the following nonlinear Klein–Gordon–Maxwell system:

$$(KGM) \quad \begin{cases} -\Delta u + V(x)u - (2\omega + \phi)\phi u = f(x, u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where  $\omega > 0$  is a constant,  $u, \phi : \mathbb{R}^3 \rightarrow \mathbb{R}$  are unknown functions,  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is a continuous potential, and  $f : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous nonlinear function. The system (KGM) is used to describe a nonlinear Klein–Gordon field interacting with an electromagnetic field and arises as a stationary solution of the Klein–Gordon–Maxwell equations when standing wave solutions of the form  $\psi(x, t) = u(x)e^{i\omega t}$  are considered, where  $u$  is in equilibrium with a purely electrostatic field  $E = -\nabla\phi(x)$ . In the past two decades, similar systems have been extensively studied by numerous researchers. One of the most frequently studied forms is the Klein–Gordon–Maxwell system with constant potential

$$\begin{cases} -\Delta u + [m_0^2 - (\omega + e\phi)^2]u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi + e^2u^2\phi = -\omega eu^2, & x \in \mathbb{R}^3, \end{cases} \quad (1)$$

where  $m_0$  and  $e$  are the mass and charge of the particle, respectively.

The existence of nontrivial solutions for (1) has been established by means of variational methods and critical point theory under various assumptions on  $m_0$ ,  $\omega$ , and  $p$ ; see [1, 3–7, 9, 10, 12]. More recently, attention has been directed toward the study of the system  $(\mathcal{KGM})$  with non-constant potentials and more general nonlinearities. For instance, Chen and Li [8] proved the existence of infinitely many solutions when  $V$  is radial, smooth, and bounded. Wang et al. [17] studied ground state solutions for  $(\mathcal{KGM})$  with periodic potential satisfying a coercivity-type condition  $V(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ . Moreover, various results on the existence of high energy or infinitely many solutions have been obtained under compactness-type assumptions on the potential, such as

$$(V_1) \quad \inf_{x \in \mathbb{R}^3} V(x) > 0;$$

$(V_2)$  there exists a constant  $r > 0$  such that

$$\lim_{|y| \rightarrow \infty} \text{meas} \{x \in \mathbb{R}^3 : |x - y| \leq r \text{ and } V(x) \leq M\} = 0, \quad \forall M > 0,$$

where  $\text{meas}$  denotes the Lebesgue measure in  $\mathbb{R}^3$ . These conditions were introduced by Bartsch et al. [2] to guarantee the compactness of the embedding of certain Sobolev spaces. The nature of the nonlinearity  $f(x, u)$  also plays a crucial role in the analysis. Under the Ambrosetti–Rabinowitz condition

$$(AR) \quad \exists \nu > 4 : \frac{1}{\nu} f(x, t)t \geq F(x, t) = \int_0^t f(x, s)ds,$$

He [13] used a variant of the fountain theorem to obtain infinitely many solutions of  $(\mathcal{KGM})$ . However, the  $(AR)$  condition can be restrictive and excludes many interesting nonlinearities. To overcome this limitation, subsequent works, including [11, 14, 15], replaced the conditions  $(AR)$  and  $(V_1)$  with the following weaker assumptions:

$$(V'_1) \quad \inf_{x \in \mathbb{R}^3} V(x) > -\infty,$$

$$\lim_{|t| \rightarrow \infty} \frac{f(x, t)}{|t|^4} = +\infty \text{ uniformly in } x, \quad f(x, t)t - 4F(x, t) \geq -\nu|t|^2, \quad \text{for some } \nu \geq 0.$$

Although these conditions are weaker than  $(AR)$ , they still exclude many nonlinearities exhibiting logarithmic or combined growth.

The aim of this paper is to establish the existence of infinitely many nontrivial solutions to the system  $(\mathcal{KGM})$  under minimal assumptions on both the potential  $V(x)$  and the nonlinearity  $f(x, u)$ . In contrast to many existing works, we do not assume that  $V(x)$  is coercive or strictly positive, nor do we impose the classical  $(AR)$  condition, which often restricts the class of admissible nonlinearities. Our framework accommodates a wide range of nonlinearities, including those that are superquadratic or involve combined subquadratic and superquadratic growth at infinity. The main tools used are variational methods, particularly the symmetric mountain pass theorem and the dual fountain theorem, which are applied under mild symmetry and growth conditions.

The remainder of the paper is organized as follows. Section 2 presents the necessary preliminaries and variational framework. Section 3 deals with the superquadratic case, while Section 4 is devoted to combined nonlinearities involving both subquadratic and superquadratic terms. The conclusion of the paper is given in Section 5.

## 2. Variational Framework and Preliminary Results

In this section, we present the variational framework and preliminary results that will be used to prove our main results. For  $1 \leq p < \infty$ , we set

$$L^p(\mathbb{R}^3) = \left\{ u : \mathbb{R}^3 \rightarrow \mathbb{R} : u \text{ is Lebesgue measurable and } \int_{\mathbb{R}^3} |u(x)|^p dx < \infty \right\}$$

equipped with the norm

$$\|u\|_p = \left( \int_{\mathbb{R}^3} |u(x)|^p dx \right)^{\frac{1}{p}}.$$

Let  $H^1(\mathbb{R}^3) = \{u \in L^2(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$  be the usual Sobolev space, endowed with the norm

$$\|u\|_{H^1(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^3} (|\nabla u(x)|^2 + u^2(x)) dx \right)^{\frac{1}{2}}. \quad (2)$$

Also, let  $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$  with the norm

$$\|u\|_{D^{1,2}} = \left( \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}}. \quad (3)$$

Throughout this section, we assume that the potential  $V$  satisfies the condition  $(V_1)$  instead of the condition  $(V'_1)$ , and we define the working space

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2(x) dx < \infty \right\}$$

with the inner product and norm defined by

$$\langle u, v \rangle = \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)u(x)v(x)] dx \quad \text{and} \quad \|u\| = \langle u, u \rangle^{\frac{1}{2}}. \quad (4)$$

Evidently,  $E$  is a Hilbert space. According to [4], we have the following:

**Lemma 2.1.** *Under assumptions  $(V_1)$  and  $(V_2)$ , the embedding  $E \hookrightarrow L^p(\mathbb{R}^3)$  is continuous for  $2 \leq p \leq 6$  and compact for  $2 \leq p < 6$ . Moreover, the embedding inequality*

$$\|u\|_p \leq \eta_p \|u\|, \quad \forall u \in E \quad (5)$$

holds for all  $p \in [2, 6]$ , where  $\eta_p > 0$  is a constant.

Due to the variational structure of the system  $(\mathcal{KGM})$ , its weak solutions  $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$  are critical points of the functional  $I$ , defined by

$$I(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - |\nabla \phi|^2 - (2\omega + \phi)\phi u^2] dx - \int_{\mathbb{R}^3} F(x, u) dx. \quad (6)$$

It is well known that  $I$  is continuously differentiable on  $E \times D^{1,2}(\mathbb{R}^3)$ , and its critical points are the solutions of the system  $(\mathcal{KGM})$ . The functional  $I$  is strongly indefinite, i.e., it is unbounded both from below and from above on infinite-dimensional subspaces. For this reason, the usual tools of critical point theory cannot be applied directly. To overcome this difficulty, we reduce the study of (6) to the analysis of a functional depending only on the variable  $u$ . For  $u \in E$ , consider the following bilinear form on  $D^{1,2}(\mathbb{R}^3)$ :

$$B(\phi, v) = \int_{\mathbb{R}^3} [\nabla \phi \nabla v + u^2 \phi v] dx. \quad (7)$$

Since  $E$  is continuously embedded into  $L^6(\mathbb{R}^3)$ , it follows that  $u^2 \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ . As  $1 < \frac{3}{2} < 3$ , interpolation yields  $u^2 \in L^{\frac{3}{2}}(\mathbb{R}^3)$ . Therefore, for all  $\phi, v \in D^{1,2}(\mathbb{R}^3)$ , we have

$$|B(\phi, v)| \leq \|\nabla\phi\|_2 \|v\|_2 + \|u^2\|_{\frac{3}{2}} \|\phi\|_6 \|v\|_6.$$

This shows that  $B(\phi, v)$  defines an inner product on  $D^{1,2}(\mathbb{R}^3)$  that is equivalent to the standard one.

Now, for  $u \in E$ , consider the linear operator  $T_u : D^{1,2}(\mathbb{R}^3) \rightarrow \mathbb{R}$  defined by

$$T_u(v) = -\omega \int_{\mathbb{R}^3} u^2 v dx. \quad (8)$$

Since  $2 < \frac{12}{5} < 3 < 6$ , interpolation gives  $u \in L^{\frac{12}{5}}(\mathbb{R}^3)$ . Then, by Hölder's inequality, for all  $v \in D^{1,2}(\mathbb{R}^3)$

$$|T_u(v)| \leq \omega \|u\|_{\frac{12}{5}}^2 \|v\|_6,$$

which implies that  $T_u$  is continuous.

By the Lax–Milgram theorem, for each  $u \in E$ , there exists a unique  $\phi_u \in D^{1,2}(\mathbb{R}^3)$  such that

$$T_u(v) = B(\phi_u, v), \quad \forall v \in D^{1,2}(\mathbb{R}^3),$$

i.e.,

$$-\omega \int_{\mathbb{R}^3} u^2 v dx = \int_{\mathbb{R}^3} [\nabla\phi_u \nabla v + u^2 \phi_u v] dx, \quad \forall v \in D^{1,2}(\mathbb{R}^3)$$

which is equivalent to

$$-\omega u^2 = -\Delta\phi_u + u^2 \phi_u. \quad (9)$$

Thus, we define the mapping

$$\Phi : E \rightarrow D^{1,2}(\mathbb{R}^3) \quad (10)$$

such that, for every  $u \in E$ ,  $\Phi(u) = \phi_u$  is the unique solution of (9).

**Lemma 2.2** (see [4]). *The function  $\Phi$  is continuously differentiable on  $E$ , and for any  $u \in E$ , the following holds:*

(i)  $-\omega \leq \Phi(u) \leq 0$  on the set  $\{x \in \mathbb{R}^3 : u(x) \neq 0\}$ ;

(ii) there exist positive constants  $c_1, c_2$  such that

$$\|\Phi(u)\|_{D^{1,2}} \leq c_1 \|u\|^2 \quad \text{and} \quad \int_{\mathbb{R}^3} |\Phi(u)| u^2 dx \leq c_2 \|u\|^4.$$

Moreover, for all  $u \in E$ ,  $\Phi(u)$  solves (9), and hence

$$-\omega u^2 = -\Delta\Phi(u) + u^2 \Phi(u)$$

for which, taking the product with  $\Phi(u)$ , we have

$$-\int_{\mathbb{R}^3} \omega u^2 \Phi(u) dx = \int_{\mathbb{R}^3} |\nabla\Phi(u)|^2 dx + \int_{\mathbb{R}^3} u^2 \Phi(u)^2 dx. \quad (11)$$

Now, we define  $J(u) = I(u, \Phi(u))$  for  $u \in E$ . Then,  $J$  is continuously differentiable on  $E$ , and Equation (11) implies

$$\begin{aligned} J(u) &= \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - |\nabla \Phi(u)|^2 - (2\omega + \Phi(u))\Phi(u)u^2] dx - \int_{\mathbb{R}^3} F(x, u)dx \\ &= \frac{1}{2} \int_{\mathbb{R}^3} [|\nabla u|^2 + V(x)u^2 - \omega\Phi(u)u^2] dx - \int_{\mathbb{R}^3} F(x, u)dx. \end{aligned}$$

Let  $(u, \phi) \in E \times D^{1,2}(\mathbb{R}^3)$ . By Proposition 3.5 in [3],  $(u, \phi)$  is a critical point of  $I$  if and only if  $u$  is a critical point of  $J$  and  $\phi = \Phi(u)$ . Moreover, for any  $u, v \in E$ ,

$$J'(u)v = \int_{\mathbb{R}^3} [\nabla u \nabla v + V(x)uv] dx - \int_{\mathbb{R}^3} (2\omega + \Phi(u))\Phi(u)uv dx - \int_{\mathbb{R}^3} f(x, u)v dx.$$

**Definition 2.3.** Let  $E$  be an infinite-dimensional Banach space. We say that  $\psi \in C^1(E, \mathbb{R})$  satisfies the

(a) *Palais-Smale condition at level  $c$  (briefly, the  $(PS)_c$  condition) if any sequence  $(u_n) \subset E$  satisfying*

$$\psi(u_n) \rightarrow c \text{ and } \psi'(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty$$

*possesses a convergent subsequence;*

(b) *Cerami's condition at level  $c$  (briefly, the  $(C)_c$  condition) if any sequence  $(u_n) \subset E$  satisfying*

$$\psi(u_n) \rightarrow c \text{ and } \|\psi'(u_n)\| (\|u_n\| + 1) \rightarrow 0 \text{ as } n \rightarrow \infty$$

*possesses a convergent subsequence.*

**Theorem 2.4** (Symmetric Mountain Pass Theorem, [16]). Let  $E = Y \oplus Z$  be an infinite-dimensional space, where  $Y$  is finite-dimensional. If  $\psi \in C^1(E, \mathbb{R})$  satisfies the  $(PS)_c$  condition for all level  $c > 0$  and

(a) 
$$\psi(0) = 0, \psi(-u) = \psi(u), \forall u \in E;$$

(b) 
$$\psi|_{\partial B_\rho \cap Z} \geq \alpha \text{ for some } \rho, \alpha > 0;$$

(c) *for any finite-dimensional subspace  $\tilde{E} \subset E$ , there is a positive constant  $R = R(\tilde{E})$  such that  $\psi(u) \leq 0$  on  $\tilde{E} \setminus B_R$ ;*

*Then,  $\psi$  possesses an unbounded sequence of critical values.*

**Remark 2.5.** It is well known that a deformation lemma can be established under the  $(C)_c$  condition in place of the  $(PS)_c$  condition. Consequently, Theorem 2.4 remains valid when the  $(C)_c$  condition is used instead of the  $(PS)_c$  condition.

Now, let  $E$  be a Banach space with the norm  $\|\cdot\|$  and  $E = \overline{\bigoplus_{j \in \mathbb{N}} X_j}$ , where  $X_j$  is a finite-dimensional subspace of  $E$ . For each  $k \in \mathbb{N}$ , let  $Y_k = \bigoplus_{j=0}^k X_j$  and  $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$ . The functional  $\psi \in C^1(E, \mathbb{R})$  is said to satisfy the  $(PS)^*$  condition if any sequence  $(u_j)$  such that  $(\psi(u_j))$  is bounded,  $u_j \in Y_{k_j}$  for some  $k_j$  with  $k_j \rightarrow \infty$  and  $(\psi|_{Y_{k_j}})'(u_j) \rightarrow 0$  as  $j \rightarrow \infty$ , has a subsequence converging to a critical point of  $\psi$ .

**Theorem 2.6** (Dual Fountain Theorem, [18]). Suppose that the functional  $\psi \in C^1(E, \mathbb{R})$  is even and satisfies the  $(PS)^*$  condition. Assume that for each sufficiently large integer  $k$ , there exist  $0 < r_k < \rho_k$  such that

(a) 
$$a_k = \inf_{u \in Z_k, \|u\| = \rho_k} \psi(u) \geq 0;$$

(b) 
$$b_k = \max_{u \in Y_k, \|u\| = r_k} \psi(u) < 0;$$

(c) 
$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} \psi(u) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

*Then,  $\psi$  has a sequence of negative critical values converging to zero.*

### 3. Superquadratic Case

The goal of this section is to apply variational methods to obtain infinitely many high-energy solutions of the system  $(\mathcal{KGM})$ . The main difficulty arises from the fact that the associated functional is strongly indefinite and the classical Ambrosetti–Rabinowitz condition is not assumed. To overcome this difficulty, we work with a suitable decomposition of the energy space and verify the geometric structure required by Theorem 2.4. Roughly speaking, we show that the functional is positive on a suitable infinite-dimensional subspace and tends to  $-\infty$  along finite-dimensional subspaces. We then establish the compactness condition (Cerami condition) under weak growth assumptions on the nonlinearity.

Consider the following assumptions on the nonlinearity  $f$ :

$(H_1)$  There exist constants  $a, b > 0$  and  $2 < p < 2^* = 6$  such that

$$|f(x, t)| \leq a|t| + b|t|^{p-1}, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R};$$

$(H_2)$  There exists a constant  $r > 0$  such that  $F(x, t) \geq 0$  for all  $x \in \mathbb{R}^3$  and  $|t| \geq r$ , and

$$\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^4} = +\infty \text{ for a.e. } x \in \mathbb{R}^3;$$

$(H_3)$   $F(x, -t) = F(x, t), \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R};$

$(H_4)$  There exist constants  $C \geq 0$  and  $\sigma > \frac{2^*}{2^*-2} = \frac{3}{2}$  such that

$$f(x, t)t - 4F(x, t) \geq 0, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}$$

and

$$|F(x, t)|^\sigma \leq C|t|^{2\sigma} [f(x, t)t - 4F(x, t)], \quad \forall x \in \mathbb{R}^3, |t| \geq r,$$

where  $r$  is the constant given in  $(H_2)$ ;

$(H'_4)$  There exist constants  $\mu \geq 4$  and  $\gamma > 0$  such that

$$\mu F(x, t) \leq f(x, t)t + \gamma t^2, \quad \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Now, we state the main results of this section.

**Theorem 3.1.** *Assume that the conditions  $(V'_1)$ ,  $(V_2)$ , and  $(H_1)$ – $(H_4)$  hold. Then, the system  $(\mathcal{KGM})$  has infinitely many nontrivial solutions.*

**Theorem 3.2.** *Assume that the conditions  $(V'_1)$ ,  $(V_2)$ ,  $(H_1)$ – $(H_3)$  and  $(H'_4)$  hold. Then, the system  $(\mathcal{KGM})$  has infinitely many nontrivial solutions.*

**Remark 3.3.** *Note that the (AR) condition is not required in Theorem 3.1. Indeed, consider*

$$F(x, t) = a(x) \left[ (4t^4 - 1) \ln \left( \frac{1}{2} + t^2 \right) - 2 \left( \frac{1}{2} + t^2 \right)^2 + 4t^2 + \frac{1}{2} - \ln 2 \right],$$

where  $a \in C(\mathbb{R}^3, \mathbb{R})$  such that  $0 < \inf_{x \in \mathbb{R}^3} a(x) \leq \sup_{x \in \mathbb{R}^3} a(x) < +\infty$ . It is clear that  $f(x, t) = \frac{\partial F}{\partial t}(x, t)$  satisfies  $(H_1)$ – $(H_3)$ . It remains to verify  $(H_4)$ . A direct computation yields

$$f(x, t)t - 4F(x, t) = a(x) \left[ (4t^4 - 1) \frac{2t^2}{2t^2 + 1} - 4t^2 + 4 \ln \left( \frac{1}{2} + t^2 \right) + 4 \ln 2 \right].$$

It is straightforward to show that  $f(x, t)t - 4F(x, t) \geq 0$  for all  $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$ . Moreover, we have

$$\left(\frac{F(x, t)}{t^2}\right)^\sigma [f(x, t)t - 4F(x, t)]^{-1} \cong_\infty (4a(x))^{\sigma-1} \frac{(t^2 \ln(\frac{1}{2} + t^2))^\sigma}{t^4}$$

which converges to 0 as  $|t| \rightarrow \infty$ , uniformly in  $x \in \mathbb{R}^3$  for  $\frac{3}{2} < \sigma < 2$ . Hence, there exist two positive constants  $r$  and  $C$  such that

$$\left(\frac{F(x, t)}{|t|^2}\right)^\sigma \leq C[f(x, t)t - 4F(x, t)], \forall x \in \mathbb{R}^3, |t| \geq r.$$

Therefore,  $(H_4)$  holds. However, it is easy to verify that the  $(AR)$  condition fails for any  $\nu > 4$ . By Theorem 3.1, the corresponding system  $(\mathcal{KGM})$  possesses infinitely many nontrivial solutions.

**Remark 3.4.** Theorem 3.1 generalizes and improves previous results in [11, 13]. In particular, the assumption  $(H_4)$  is strictly weaker than the  $(AR)$  condition.

In order to prove Theorems 3.1 and 3.2, we need some preparation first. We begin by noting that from  $(V'_1)$ , there exists a constant  $v_0 > 0$  such that  $\inf_{x \in \mathbb{R}^3} V(x) + v_0 > 0$ . Define

$$\bar{V}(x) = V(x) + v_0, \bar{f}(x, t) = f(x, t) + v_0 t.$$

Consider the equivalent Klein–Gordon–Maxwell system:

$$(\overline{\mathcal{KGM}}) \quad \begin{cases} -\Delta u + \bar{V}(x)u - (2\omega + \phi)\phi u = \bar{f}(x, u), & x \in \mathbb{R}^3, \\ \Delta \phi = (\omega + \phi)u^2, & x \in \mathbb{R}^3. \end{cases}$$

The assumptions  $(H_1) - (H_4)$  or  $(H_1) - (H_3)$  and  $(H'_4)$  remain valid for  $\bar{f}$ , and  $\bar{V}$  satisfies  $(V_1)$  and  $(V_2)$ . Thus, we may assume without loss of generality that  $V$  satisfies  $(V_1)$  in place of  $(V'_1)$ .

Let  $(e_j)_{j \in \mathbb{N}}$  be an orthonormal basis of the space  $E$ . Define

$$Y_m = \text{span} \{e_1, \dots, e_m\} \quad \text{and} \quad Z_m = \overline{\text{span} \{e_{m+1}, \dots\}}, \quad \text{where } m \in \mathbb{N}.$$

Then  $E = Y_m \oplus Z_m$ .

We proceed by establishing a sequence of lemmas that will allow us to apply critical point theory. The first step is to show that the functional is positive on a small sphere contained in a high-frequency subspace. Intuitively, functions in  $Z_m$  oscillate more as  $m$  increases, which weakens their  $L^q$  norms compared to the energy norm. As a result, the nonlinear term becomes negligible for small  $\|u\|$ , and the quadratic part dominates, making the functional positive.

**Lemma 3.5.** Assume that  $(V_1)$ ,  $(V_2)$  and  $(H_1)$  hold. Then, there exist constants  $m_0 \in \mathbb{N}$ ,  $\alpha > 0$ , and  $\rho > 0$  such that  $J|_{\partial B_\rho \cap Z_{m_0}} \geq \alpha$ .

**Proof.** By the condition  $(H_1)$ , we have

$$|F(x, t)| \leq \frac{a}{2} |t|^2 + \frac{b}{p} |t|^p, \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{12}$$

For any  $m \in \mathbb{N}$ , define

$$l_2(m) = \sup_{u \in Z_m \setminus \{0\}} \frac{\|u\|_2}{\|u\|} \quad \text{and} \quad l_p(m) = \sup_{u \in Z_m \setminus \{0\}} \frac{\|u\|_p}{\|u\|}. \tag{13}$$

It is clear that  $l_2(m + 1) \leq l_2(m)$ , so  $l_2(m) \rightarrow l \geq 0$  as  $m \rightarrow \infty$ . For each  $m \in \mathbb{N}$ , there exists  $u_m \in Z_m$  with  $\|u_m\| = 1$  such that

$$\|u_m\|_2 \geq \frac{1}{2} l_2(m).$$

Since  $u_m \in Z_m$ , we have  $u_m \rightarrow 0$  in  $E$ . By Lemma 2.1, it follows that  $u_m \rightarrow 0$  in  $L^2(\mathbb{R}^3)$ , which implies  $l = 0$ , and hence  $l_2(m) \rightarrow 0$  as  $m \rightarrow \infty$ . Similarly,  $l_p(m) \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore, we can choose a sufficiently large integer  $m_0$  such that

$$\|u\|_2^2 \leq \frac{1}{2a} \|u\|^2 \quad \text{and} \quad \|u\|_p^p \leq \frac{p}{4b} \|u\|^p, \quad \forall u \in Z_{m_0}. \quad (14)$$

Now, using (12), (14), and Lemma 2.2, for any  $u \in Z_{m_0}$ , we have

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|^2 + \frac{1}{2} \int_{\mathbb{R}^3} -\omega \Phi(u) u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx \\ &\geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^3} \left( \frac{a}{2} |u|^2 + \frac{b}{p} |u|^p \right) dx \\ &\geq \frac{1}{2} \|u\|^2 - \frac{a}{2} \|u\|_2^2 - \frac{b}{p} \|u\|_p^p \\ &\geq \frac{1}{2} \|u\|^2 - \frac{1}{4} \|u\|^2 - \frac{1}{4} \|u\|^p \\ &= \frac{1}{4} (\|u\|^2 - \|u\|^p). \end{aligned}$$

Now, take  $\rho = \frac{1}{2}$ . Then, for all  $u \in Z_{m_0}$  with  $\|u\| = \rho$ , we obtain

$$J(u) \geq \frac{1}{4} (\rho^2 - \rho^p) = \frac{1}{4} \left( \frac{1}{2} - \frac{1}{2^p} \right) = \frac{2^{p-2} - 1}{2^{p+2}} = \alpha > 0.$$

This completes the proof.  $\square$

To apply Theorem 2.4, we consider the decomposition  $E = Y \oplus Z$ , where  $Y = Y_{m_0}$  and  $Z = Z_{m_0}$ , with  $m_0$  as defined in Lemma 3.5.

We now analyze the behavior of the functional on finite-dimensional subspaces. In this setting, all norms are equivalent; hence, large values of  $\|u\|$  imply that  $|u(x)|$  is large on a set of positive measure. Owing to the superquadratic growth condition in  $(H_2)$ , the nonlinear term dominates the quadratic part, forcing the functional to tend to  $-\infty$ .

**Lemma 3.6.** *If the conditions  $(V_1)$ ,  $(V_2)$ ,  $(H_1)$  and  $(H_2)$  hold, then for any finite-dimensional subspace  $\tilde{E} \subset E$ , there exists a constant  $R = R(\tilde{E}) > 0$  such that*

$$J(u) \leq 0, \quad \forall u \in \tilde{E}, \quad \|u\| \geq R. \quad (15)$$

**Proof.** To prove (15), it suffices to show that

$$J(u) \rightarrow -\infty \quad \text{as} \quad \|u\| \rightarrow \infty, \quad u \in \tilde{E}. \quad (16)$$

Assume, for the sake of contradiction, that there exists a sequence  $(u_n) \subset \tilde{E}$  such that  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$  and  $J(u_n) \geq -M$  for some constant  $M > 0$ , for all  $n \in \mathbb{N}$ . Define the normalized sequence  $v_n = \frac{u_n}{\|u_n\|}$ , so that  $\|v_n\| = 1$  for all  $n$ . Since  $\tilde{E}$  is finite-dimensional, the unit ball is compact; hence (up to a subsequence), we have  $v_n \rightarrow v$  strongly in  $E$ , which implies  $v_n(x) \rightarrow v(x)$  almost everywhere in  $\mathbb{R}^3$  and  $\|v\| = 1$ . Now, we define the sets

$$\Lambda_n(c, d) = \{x \in \mathbb{R}^3 : c \leq |u_n(x)| < d\}, \quad 0 \leq c < d,$$

and

$$\Lambda = \{x \in \mathbb{R}^3 : v(x) \neq 0\}.$$

For any  $x \in \Lambda$ , since  $|v_n(x)| \rightarrow |v(x)| > 0$  and  $\|v_n\| \rightarrow +\infty$ , it follows that  $|u_n(x)| = \|u_n\| |v_n(x)| \rightarrow \infty$ . Thus, for sufficiently large  $n$ , we have  $x \in \Lambda_n(r, \infty)$ , where  $r > 0$  is defined in the assumption  $(H_2)$ . Now, divide the inequality by  $\|u_n\|^2$ . Then, using (12), Lemmas 2.1 and 2.2, the assumption  $(H_2)$ , and Fatou’s lemma, we obtain

$$\begin{aligned}
 0 &= \lim_{n \rightarrow \infty} \frac{-M}{\|u_n\|^2} \leq \frac{J(u_n)}{\|u_n\|^2} \\
 &= \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + \frac{1}{2\|u_n\|^2} \int_{\mathbb{R}^3} -\omega\Phi(u_n)u_n^2 dx - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \right] \\
 &\leq \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + \frac{\omega}{2\|u_n\|^2} \int_{\mathbb{R}^3} |\Phi(u_n)| u_n^2 dx - \int_{\Lambda_n(0,r)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx - \int_{\Lambda_n(r,\infty)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \right] \\
 &\leq \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + \left(\frac{a}{2} + \frac{b}{p}r^{p-2}\right) \int_{\mathbb{R}^3} v_n^2 dx + \frac{\omega c_2}{2} \|u_n\|^2 - \int_{\Lambda_n(r,\infty)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \right] \\
 &\leq \left(\frac{1}{2} + \frac{a}{2} + \frac{b}{p}r^{p-2}\right) \eta_2^2 + \lim_{n \rightarrow \infty} \|u_n\|^2 \left[ \frac{\omega c_2}{2} - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^4} \chi_{\Lambda_n(r,\infty)} |v_n|^4 dx \right] \\
 &= \frac{1}{2} + \left(\frac{a}{2} + \frac{b}{p}r^{p-2}\right) \eta_2^2 + \lim_{n \rightarrow \infty} \|u_n\|^2 \left[ \frac{\omega c_2}{2} - \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^4} \chi_{\Lambda_n(r,\infty)} |v_n|^4 dx \right] \\
 &\leq \frac{1}{2} + \left(\frac{a}{2} + \frac{b}{p}r^{p-2}\right) \eta_2^2 + \lim_{n \rightarrow \infty} \|u_n\|^2 \left[ \frac{\omega c_2}{2} - \int_{\mathbb{R}^3} \liminf_{n \rightarrow \infty} \frac{F(x, u_n)}{|u_n|^4} \chi_{\Lambda_n(r,\infty)} |v_n|^4 dx \right] \\
 &= -\infty.
 \end{aligned} \tag{17}$$

This is a contradiction. Hence, our assumption is false, and we conclude that (16) holds, which proves the lemma. □

**Proof of Theorem 3.1.** By the assumption  $(H_3)$  and Lemmas 3.5 and 3.6, the functional  $J$  satisfies conditions (a), (b) and (c) of Theorem 2.4. It remains to verify the Cerami condition. The main difficulty is to recover compactness despite the lack of coercivity and the unbounded domain. The idea is to show that any Cerami sequence cannot escape to infinity. This is achieved by combining the superquadratic growth of  $F$  with a careful decomposition of the domain and estimates involving the Maxwell term.

**Lemma 3.7.** *Assume that the conditions  $(V_1)$ ,  $(V_2)$ ,  $(H_1)$ ,  $(H_2)$ , and  $(H_4)$  hold. Then,  $J$  satisfies the  $(C)_c$  condition for any level  $c > 0$ .*

**Proof.** Let  $(u_n) \subset E$  be a  $(C)_c$ -sequence, i.e.,  $J(u_n) \rightarrow c$  and  $\|J'(u_n)\| (1 + \|u_n\|) \rightarrow 0$  as  $n \rightarrow \infty$ . Assume, for the sake of contradiction, that  $(u_n)$  is unbounded. Then, up to a subsequence, we may assume  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Define  $v_n = \frac{u_n}{\|u_n\|}$ , so that  $\|v_n\| = 1$ . Passing to a subsequence, if necessary, we have  $v_n \rightharpoonup v$  in  $E$ , and by Lemma 2.1,  $v_n \rightarrow v$  in  $L^q(\mathbb{R}^3)$  for  $q = 2, p, 2\sigma' = \frac{2\sigma}{\sigma-1}$ , and  $v_n \rightarrow v$  a.e. on  $\mathbb{R}^3$ . We distinguish two cases depending on whether the weak limit of the normalized sequence is trivial or not.

If  $v \neq 0$ , then by Hölder’s inequality and following the argument used in Lemma 3.6, we have

$$0 = \lim_{n \rightarrow \infty} \frac{J(u_n)}{\|u_n\|^2} = \lim_{n \rightarrow \infty} \left[ \frac{1}{2} + \frac{1}{2\|u_n\|^2} \int_{\mathbb{R}^3} -\omega\Phi(u_n)v_n^2 dx - \int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \right] \leq -\infty,$$

which is a contradiction. Thus,  $(u_n)$  must be bounded.

If the limit is zero, a more delicate analysis using measure decomposition is required. As  $J(u_n) \rightarrow c$  and  $\|u_n\| \rightarrow \infty$ , it follows that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx = \limsup_{n \rightarrow \infty} \left[ -\frac{J(u_n)}{\|u_n\|^2} + \frac{1}{2} + \frac{1}{2\|u_n\|^2} \int_{\mathbb{R}^3} -\omega \Phi(u_n) u_n^2 dx \right] \geq \frac{1}{2}. \tag{18}$$

Using  $(H_2)$  and (12), for the constant  $r$  given in  $(H_2)$ , we have

$$\int_{\Lambda_n(0,r)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \leq \left( \frac{a}{2} + \frac{b}{p} r^{p-2} \right) \int_{\Lambda_n(0,r)} |v_n|^2 dx \leq \left( \frac{a}{2} + \frac{b}{p} r^{p-2} \right) \|v_n\|_2^2 \rightarrow 0. \tag{19}$$

Additionally, there exists a constant  $c_3 > 0$  such that for all  $n$ ,

$$\int_{\mathbb{R}^3} \left[ \frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right] dx = J(u_n) - \frac{1}{4} J'(u_n) u_n - \frac{1}{4} \int_{\mathbb{R}^3} \Phi(u_n)^2 u_n^2 dx \leq c_3.$$

Using Hölder’s inequality and the assumption  $(H_4)$ , we obtain

$$\begin{aligned} & \int_{\Lambda_n(r,\infty)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \\ & \leq \left[ \int_{\Lambda_n(r,\infty)} \left( \frac{F(x, u_n)}{|u_n|^2} \right)^\sigma dx \right]^{\frac{1}{\sigma}} \left( \int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'} dx \right)^{\frac{1}{\sigma'}} \\ & \leq (2C)^{\frac{1}{\sigma}} \left( \int_{\Lambda_n(r,\infty)} \left[ \frac{1}{4} f(x, u_n) u_n - F(x, u_n) \right] dx \right)^{\frac{1}{\sigma}} \left( \int_{\Lambda_n(r,\infty)} |v_n|^{2\sigma'} dx \right)^{\frac{1}{\sigma'}} \\ & \leq (2Cc_3)^{\frac{1}{\sigma}} \|v_n\|_{2\sigma'}^2 \rightarrow 0. \end{aligned} \tag{20}$$

Combining (19) and (20), we obtain

$$\int_{\mathbb{R}^3} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx = \int_{\Lambda_n(0,r)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx + \int_{\Lambda_n(r,\infty)} \frac{F(x, u_n)}{|u_n|^2} |v_n|^2 dx \rightarrow 0$$

which contradicts (18). Hence,  $(u_n)$  must be bounded.

Therefore, there exists a subsequence (still denoted by  $(u_n)$ ) such that  $u_n \rightharpoonup u$  in  $E$ , and  $u_n \rightarrow u$  in  $L^s(\mathbb{R}^3)$  for  $s \in [2, 6[$ . Observe that

$$\begin{aligned} \|u_n - u\|^2 &= (J'(u_n) - J'(u))(u_n - u) + \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \\ &+ \int_{\mathbb{R}^3} [(2\omega + \Phi(u_n)) \Phi(u_n) u_n - (2\omega + \Phi(u)) \Phi(u) u] (u_n - u) dx. \end{aligned} \tag{21}$$

By the assumption  $(H_1)$  and Hölder’s inequality, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^3} (f(x, u_n) - f(x, u))(u_n - u) dx \right| \\ & \leq \int_{\mathbb{R}^3} [a(|u_n| + |u|) + b(|u_n|^{p-1} + |u|^{p-1})] |u_n - u| dx \\ & \leq a \int_{\mathbb{R}^3} (|u_n| + |u|) |u_n - u| dx + b \int_{\mathbb{R}^3} (|u_n|^{p-1} + |u|^{p-1}) |u_n - u| dx \\ & \leq a (\|u_n\|_2 + \|u\|_2) \|u_n - u\|_2 + b \left( \|u_n\|_p^{p-1} + \|u\|_p^{p-1} \right) \|u_n - u\|_p \rightarrow 0. \end{aligned} \tag{22}$$

Similarly, using Hölder’s and Sobolev’s inequalities and Lemma 2.1, for positive constants  $c_4$  and  $c_5$ , we obtain

$$\begin{aligned}
 & \left| \int_{\mathbb{R}^3} (2\omega + \Phi(u_n))\Phi(u_n) u_n(u_n - u)dx \right| \\
 & \leq \left| \int_{\mathbb{R}^3} 2\omega\Phi(u_n)u_n(u_n - u)dx \right| + \left| \int_{\mathbb{R}^3} \Phi(u_n)^2u_n(u_n - u)dx \right| \\
 & \leq 2\omega \|\Phi(u_n)u_n\|_2 \|u_n - u\|_2 + \|\Phi(u_n)^2u_n\|_2 \|u_n - u\|_2 \tag{23} \\
 & \leq 2\omega \|\Phi(u_n)\|_6 \|u_n\|_3 \|u_n - u\|_2 + \|\Phi(u_n)\|_6^2 \|u_n\|_6 \|u_n - u\|_2 \\
 & \leq c_4 [\|\Phi(u_n)\|_{D^{1,2}} \|u_n\|_3 + \|\Phi(u_n)\|_{D^{1,2}}^2 \|u_n\|_6] \|u_n - u\|_2 \\
 & \leq c_5 [\|u_n\|^2 \|u_n\|_3 + \|u_n\|^4 \|u_n\|_6] \|u_n - u\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Likewise, we have

$$\left| \int_{\mathbb{R}^3} (2\omega + \Phi(u)) \Phi(u)u(u_n - u)dx \right| \leq c_5 [\|u\|^2 \|u\|_3 + \|u\|^4 \|u\|_6] \|u_n - u\|_2 \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{24}$$

It is clear that

$$(J'(u_n) - J'(u))(u_n - u) \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{25}$$

Combining (21)–(25), we obtain  $\|u_n - u\| \rightarrow 0$  as  $n \rightarrow \infty$ , this completes the proof of Lemma 3.7  $\square$

Consequently, Theorem 2.4, together with Remark 2.5, implies that the functional  $J$  possesses an unbounded sequence of critical points. Therefore, the system  $(\mathcal{KGM})$  admits infinitely many solutions. This completes the proof of Theorem 3.1.  $\square$

**Proof of Theorem 3.2.** Under the assumption  $(H'_4)$ , the compactness argument becomes simpler. This condition provides a direct control of the nonlinear term in terms of the energy, allowing us to bound Cerami sequences without relying on the finer estimates used in Lemma 3.7.

**Lemma 3.8.** *Assume that the conditions  $(V_1)$ ,  $(V_2)$ , and  $(H'_4)$  are satisfied. Then, for every positive constant  $c$ , the functional  $J$  satisfies the  $(C)_c$  condition.*

**Proof.** Let  $c > 0$ . Let  $(u_n) \subset E$  be a  $(C)_c$ -sequence, that is,  $J(u_n) \rightarrow c$  and  $\|J'(u_n)\| (1 + \|u_n\|) \rightarrow 0$  as  $n \rightarrow \infty$ . Assume, for the sake of contradiction, that  $(u_n)$  is unbounded. Then, up to a subsequence, we may assume that  $\|u_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Define  $v_n = \frac{u_n}{\|u_n\|}$ , so that  $\|v_n\| = 1$  for all  $n$ . By the assumption  $(H'_4)$ , there exists a constant  $c_6$  such that for all  $n$ ,

$$\begin{aligned}
 c_6 & \geq J(u_n) - \frac{1}{\mu} J'(u_n)u_n \\
 & = \frac{\mu - 2}{2\mu} \|u_n\|^2 + \frac{\mu - 4}{2\mu} \int_{\mathbb{R}^3} -\omega\Phi(u_n)u_n dx + \frac{1}{\mu} \int_{\mathbb{R}^3} \Phi(u_n)^2u_n^2 dx + \int_{\mathbb{R}^3} \left[ \frac{1}{\mu} f(x, u_n)u_n - F(x, u_n) \right] dx \\
 & \geq \frac{\mu - 2}{2\mu} \|u_n\|^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{\mu} f(x, u_n)u_n - F(x, u_n) \right] dx \\
 & \geq \frac{\mu - 2}{2\mu} \|u_n\|^2 - \frac{\gamma}{\mu} \|u_n\|_2^2.
 \end{aligned}$$

It follows that

$$\limsup_{n \rightarrow \infty} \|v_n\|_2^2 \geq \frac{\mu - 2}{2\gamma}. \quad (26)$$

Since  $\|v_n\| = 1$ , passing to a subsequence (if necessary), we have  $v_n \rightharpoonup v$  in  $E$ . Then, by Lemma 2.1,  $v_n \rightarrow v$  in  $L^2(\mathbb{R}^3)$ , which, together with (26), implies that  $v \neq 0$ . Arguing similarly to (17), a contradiction is obtained. Therefore,  $(u_n)$  must be bounded. Following the same reasoning as in the proof of Lemma 3.7, we conclude that  $(u_n)$  possesses a convergent subsequence. Hence,  $J$  satisfies the  $(C)_c$  condition. This completes the proof of Lemma 3.8.  $\square$

As in the proof of Theorem 3.1, we deduce that the functional  $J$  possesses an unbounded sequence of critical points, and thus, the proof of Theorem 3.2 is complete.  $\square$

## 4. Combined Nonlinearities

In this section, nonlinearities composed of both subquadratic and superquadratic terms are considered. This mixed growth renders the variational structure more delicate, since neither purely coercive nor purely superlinear techniques apply directly. The approach is based on the dual fountain theorem, which is particularly suited for obtaining sequences of small-energy solutions. Its geometric assumptions are verified on increasing finite-dimensional subspaces, and a suitable compactness condition adapted to this mixed growth setting is established.

Assume that  $f(x, t)$  is of the form  $f(x, t) = g(x, t) + h(x, t)$ , where  $g, h : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$  are continuous functions, satisfying the following conditions:

(H<sub>5</sub>) There exist constants  $\gamma, \sigma$  satisfying  $1 < \gamma < 2$  and  $1 < \sigma < 2$ , and functions  $c_0, a \in L^{\frac{2}{2-\gamma}}(\mathbb{R}^3, \mathbb{R}^+)$  and  $b \in L^{\frac{2}{2-\sigma}}(\mathbb{R}^3, \mathbb{R}^+)$  such that

$$c_0(x) |t|^\gamma \leq g(x, t)t, \quad |g(x, t)| \leq a(x) |t|^{\gamma-1} + b(x) |t|^{\sigma-1}, \quad a.e. \ x \in \mathbb{R}^3, \ \forall t \in \mathbb{R};$$

(H<sub>6</sub>)  $H(x, t) = \int_0^t h(x, s)ds \geq 0$  and there exists  $2 < \mu < 4$ ,  $c \in L^2(\mathbb{R}^3, \mathbb{R}^+)$  and  $d \in L^\infty(\mathbb{R}^3, \mathbb{R}^+)$  such that

$$|h(x, t)| \leq c(x) + d(x) |t|^{\mu-1}, \quad a.e. \ x \in \mathbb{R}^3, \ \forall t \in \mathbb{R};$$

(H<sub>7</sub>) There exist  $\rho \geq 4$ ,  $1 < \delta < 2$  and  $\theta \in C(\mathbb{R}^3, \mathbb{R}^+) \cap L^{\frac{2}{2-\delta}}(\mathbb{R}^3, \mathbb{R}^+)$  such that

$$\rho H(x, t) - h(x, t)t \leq \theta(x) |t|^\delta, \quad a.e. \ x \in \mathbb{R}^3, \ \forall t \in \mathbb{R};$$

(H<sub>8</sub>)  $F(x, t) = \int_0^t f(x, s)ds$  is even in  $t$ .

**Theorem 4.1.** *Assume that the conditions  $(V_1)$ ,  $(V_2)$ , and  $(H_5)$ – $(H_8)$  are satisfied. Then, the system  $(\mathcal{KGM})$  possesses infinitely many small energy solutions.*

**Remark 4.2.** *Let  $F(x, t) = G(x, t) + H(x, t)$ , where*

$$G(x, t) = \left( \frac{1}{1 + |x|^2} \right) |t|^{\frac{4}{3}} + \left( \frac{1}{1 + |x|^2} \right)^{\frac{1}{2}} |t|^{\frac{5}{3}},$$

$$H(x, t) = \left( \frac{1}{1 + |x|^{\frac{3}{2}}} \right)^3 \left[ |t|^{\frac{4}{3}} \ln(1 + |t|^{\frac{1}{3}}) + |t|^{\frac{5}{3}} \right].$$

*By a standard computation, it can be verified that conditions  $(H_5)$ – $(H_8)$  are satisfied. Hence, the corresponding system  $(\mathcal{KGM})$  admits infinitely many solutions.*

**Proof of Theorem 4.1.** The argument is developed through a sequence of lemmas. The first step is to establish a compactness property for the nonlinear term. Since the nonlinearity exhibits mixed growth (subquadratic and superquadratic), standard compactness arguments do not apply directly. The following lemma ensures that weak convergence in  $E$  is sufficient to pass to the limit in the nonlinear term, which will be crucial in verifying a Palais–Smale type condition later.

**Lemma 4.3.** *Assume that  $(V_1)$ ,  $(V_2)$ ,  $(H_5)$ , and  $(H_6)$  are satisfied. If  $u_n \rightharpoonup u$  in  $E$ , then*

$$f(\cdot, u_n) \rightarrow f(\cdot, u) \text{ in } L^2(\mathbb{R}^3). \quad (27)$$

**Proof.** We argue by contradiction. By Lemma 2.1, there exists a subsequence  $u_{n_j}$  such that

$$u_{n_j} \rightarrow u \text{ in both } L^2(\mathbb{R}^3) \text{ and } L^{2(\mu-1)}(\mathbb{R}^3) \text{ and } u_{n_j} \rightarrow u \text{ a.e. in } \mathbb{R}^3 \text{ as } j \rightarrow \infty \quad (28)$$

and

$$\int_{\mathbb{R}^3} |f(x, u_{n_j}(x)) - f(x, u(x))|^2 dx \geq \epsilon_0, \quad \forall j \in \mathbb{N}, \quad (29)$$

for some positive constant  $\epsilon_0$ . By (28) and passing to a further subsequence (if necessary), we can assume that

$$\sum_{j=1}^{\infty} \|u_{n_j} - u\|_2 < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} \|u_{n_j} - u\|_{2(\mu-1)} < \infty.$$

Define  $w(x) = \sum_{j=1}^{\infty} |u_{n_j}(x) - u(x)|$ . Then  $w \in L^2(\mathbb{R}^3) \cap L^{2(\mu-1)}(\mathbb{R}^3)$ . By  $(H_5)$  and  $(H_6)$ , for all  $j \in \mathbb{N}$  and  $x \in \mathbb{R}^3$ , we have

$$\begin{aligned} |f(x, u_{n_j}) - f(x, u)|^2 &\leq \left[ |f(x, u_{n_j})| + |f(x, u)| \right]^2 \\ &\leq \left[ |g(x, u_{n_j})| + |h(x, u_{n_j})| + |g(x, u)| + |h(x, u)| \right]^2 \\ &\leq \left[ a |u_{n_j}|^{\gamma-1} + b |u_{n_j}|^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} + 2c + d |u_{n_j}|^{\mu-1} + d |u|^{\mu-1} \right]^2 \\ &\leq \left[ a (|u_{n_j} - u| + |u|)^{\gamma-1} + b (|u_{n_j} - u| + |u|)^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} \right. \\ &\quad \left. + 2c + d (|u_{n_j} - u| + |u|)^{\mu-1} + d |u|^{\mu-1} \right]^2 \\ &\leq \left[ a (w + |u|)^{\gamma-1} + b (w + |u|)^{\sigma-1} + a |u|^{\gamma-1} + b |u|^{\sigma-1} \right. \\ &\quad \left. + 2c + d (w + |u|)^{\mu-1} + d |u|^{\mu-1} \right]^2 \\ &\leq c_7 \left[ a^2 w^{2(\gamma-1)} + a^2 |u|^{2(\gamma-1)} + b^2 w^{2(\sigma-1)} + b^2 |u|^{2(\sigma-1)} \right. \\ &\quad \left. + c^2 + d^2 w^{2(\mu-1)} + d^2 |u|^{2(\mu-1)} \right] = k(x) \end{aligned} \quad (30)$$

where  $c_7$  is a positive constant. It is easy to see that  $k \in L^1(\mathbb{R}^3)$ . Therefore, by (28), (30), and the Lebesgue dominated convergence theorem, we conclude that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{R}^3} |f(x, u_{n_j}(x)) - f(x, u(x))|^2 dx = 0,$$

which contradicts (29). Hence, (27) holds, completing the proof of Lemma 4.3.  $\square$

It is well known that under the assumptions of Theorem 4.1,  $J \in C^1(E, \mathbb{R})$ . Moreover, the derivative  $J' : E \rightarrow E^*$  is compact, and the critical points of  $J$  in  $E$  are classical solutions of the system  $(\mathcal{KGM})$ .

We prove Theorem 4.1 by applying Theorem 2.6. Let  $(e_j)$  be a complete orthonormal basis of  $E$ , and define  $X_j = \mathbb{R}e_j$ . Then  $Y_k$  and  $Z_k$  can be defined as in Section 2. By the assumption  $(H_8)$ , the functional  $J \in C^1(E, \mathbb{R})$  is even. We next verify that all the conditions of Theorem 2.6 are satisfied.

To apply the dual fountain theorem, we need a suitable compactness condition. In this setting, we verify a variant of the Palais-Smale condition adapted to sequences lying in increasing finite-dimensional subspaces. The key point is to show that such sequences are precompact despite the lack of global coercivity.

**Lemma 4.4.** *Assume that  $(V_1)$ ,  $(V_2)$ ,  $(H_5)$ , and  $(H_7)$  are satisfied. Then,  $J$  satisfies the  $(PS)^*$  condition.*

**Proof.** Let  $(u_j)$  be a  $(PS)^*$ -sequence, that is,  $(J(u_j))$  is bounded,  $u_j \in Y_{k_j}$  for some  $k_j$  with  $k_j \rightarrow \infty$  and  $(J_{|Y_{k_j}})'(u_j) \rightarrow 0$  as  $j \rightarrow \infty$ . We will show that  $(u_j)$  is bounded in  $E$ . By (5),  $(H_5)$ , and  $(H_7)$ , there exists a positive constant  $M$  such that

$$\begin{aligned} \rho M + M \|u_j\| &\geq \rho J(u_j) - J'(u_j)u_j \\ &= \left(\frac{\rho}{2} - 1\right) \|u_j\|^2 + \frac{\rho - 4}{2} \int_{\mathbb{R}^3} -\omega \Phi(u_j)u_j^2 dx + \int_{\mathbb{R}^3} \Phi(u_j)^2 u_j^2 dx + \int_{\mathbb{R}^3} [f(x, u_j)u_j - \rho F(x, u_j)] dx \\ &\geq \left(\frac{\rho}{2} - 1\right) \|u_j\|^2 + \int_{\mathbb{R}^3} [g(x, u_j)u_j - \rho G(x, u_j)] dx + \int_{\mathbb{R}^3} [h(x, u_j)u_j - \rho H(x, u_j)] dx \\ &\geq \left(\frac{\rho}{2} - 1\right) \|u_j\|^2 - \int_{\mathbb{R}^3} [a(x) |u_j|^\gamma + b(x) |u_j|^\sigma] dx \\ &\quad - \rho \int_{\mathbb{R}^3} \left[ \frac{a(x)}{\gamma} |u_j|^\gamma + \frac{b(x)}{\sigma} |u_j|^\sigma \right] dx - \int_{\mathbb{R}^3} \theta(x) |u_j|^\delta dx \\ &\geq \left(\frac{\rho}{2} - 1\right) \|u_j\|^2 - \left(1 + \frac{\rho}{\gamma}\right) \|a\|_{\frac{2}{2-\gamma}} \|u_j\|_2^\gamma - \left(1 + \frac{\rho}{\sigma}\right) \|b\|_{\frac{2}{2-\sigma}} \|u_j\|_2^\sigma - \|\theta\|_{\frac{2}{2-\delta}} \|u_j\|_2^\delta \\ &\geq \left(\frac{\rho}{2} - 1\right) \|u_j\|^2 - \left(1 + \frac{\rho}{\gamma}\right) \eta_2^\gamma \|a\|_{\frac{2}{2-\gamma}} \|u_j\|^\gamma - \left(1 + \frac{\rho}{\sigma}\right) \eta_2^\sigma \|b\|_{\frac{2}{2-\sigma}} \|u_j\|^\sigma - \|\theta\|_{\frac{2}{2-\delta}} \eta_2^\delta \|u_j\|^\delta. \end{aligned}$$

Since  $\rho > 2$  and  $\gamma, \sigma, \delta < 2$ , it follows that  $(u_j)$  is bounded in  $E$ . By the reflexivity of  $E$ , we can extract a subsequence (still denoted by  $(u_j)$ ) such that  $u_j \rightharpoonup u$  in  $E$  and  $u_j \rightarrow u$  in  $L^s(\mathbb{R}^3)$  for all  $s \in [2, 6)$ . Now,

$$\begin{aligned} \|u_j - u\|^2 &= (J'(u_j) - J'(u))(u_j - u) + \int_{\mathbb{R}^3} (f(x, u_j) - f(x, u))(u_j - u) dx \\ &\quad + \int_{\mathbb{R}^3} [(2\omega + \Phi(u_j)) \Phi(u_j)u_j - (2\omega + \Phi(u)) \Phi(u)u] (u_j - u) dx. \end{aligned} \tag{31}$$

By Hölder’s inequality, (5), and Lemma 4.3, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (f(x, u_j) - f(x, u))(u_j - u) dx \right| &\leq \|f(\cdot, u_j) - f(\cdot, u)\|_2 \|u_j - u\|_2 \\ &\leq \eta_2 \|f(\cdot, u_j) - f(\cdot, u)\|_2 \|u_j - u\| \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned} \tag{32}$$

Also, by Hölder’s and Sobolev’s inequalities, and Lemma 2.2, we obtain

$$\begin{aligned}
 \left| \int_{\mathbb{R}^3} (2\omega + \Phi(u_j)) \Phi(u_j) u_j (u_j - u) dx \right| &\leq \left| \int_{\mathbb{R}^3} 2\omega \Phi(u_j) u_j (u_j - u) dx \right| + \left| \int_{\mathbb{R}^3} \Phi(u_j)^2 u_j (u_j - u) dx \right| \\
 &\leq 2\omega \|\Phi(u_j) u_j\|_2 \|u_j - u\|_2 + \|\Phi(u_j)^2 u_j\|_2 \|u_j - u\|_2 \\
 &\leq 2\omega \|\Phi(u_j)\|_6 \|u_j\|_3 \|u_j - u\|_2 + \|\Phi(u_j)\|_6^2 \|u_j\|_6 \|u_j - u\|_2 \\
 &\leq c_8 [\|\Phi(u_j)\|_{D^{1,2}} \|u_j\|_3 + \|\Phi(u_j)\|_{D^{1,2}}^2 \|u_j\|_6] \|u_j - u\|_2 \\
 &\leq c_9 [\|u_j\|^2 \|u_j\|_3 + \|u_j\|^4 \|u_j\|_6] \|u_j - u\|_2 \rightarrow 0 \text{ as } j \rightarrow \infty
 \end{aligned} \tag{33}$$

where  $c_8$  and  $c_9$  are positive constants. Similarly,

$$\left| \int_{\mathbb{R}^3} (2\omega + \Phi(u)) \Phi(u) u (u_j - u) dx \right| \leq c_9 [\|u\|^2 \|u\|_3 + \|u\|^4 \|u\|_6] \|u_j - u\|_2 \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{34}$$

It is also clear that

$$(J'(u_j) - J'(u)) (u_j - u) \rightarrow 0 \text{ as } j \rightarrow \infty. \tag{35}$$

Combining (31)–(35), we obtain  $\|u_j - u\| \rightarrow 0$  as  $j \rightarrow \infty$ . This completes the proof of Lemma 4.4.  $\square$

Next, we show that the functional is positive on high-frequency components with small norm. This reflects the fact that the linear part dominates the nonlinear terms in this regime.

**Lemma 4.5.** *Assume that  $(V_1)$ ,  $(V_2)$ ,  $(H_5)$ , and  $(H_6)$  are satisfied. Then, for any sufficiently large  $k \in \mathbb{N}$ , there exists  $\rho_k > 0$  such that*

$$a_k = \inf_{u \in Z_k, \|u\| = \rho_k} J(u) \geq 0. \tag{36}$$

**Proof.** For any  $k \in \mathbb{N}$ , define

$$l_2(k) = \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_2}{\|u\|} \text{ and } l_p(k) = \sup_{u \in Z_k \setminus \{0\}} \frac{\|u\|_p}{\|u\|}.$$

As in the proof of Lemma 3.5, we have  $l_2(k) \rightarrow 0$  and  $l_p(k) \rightarrow 0$  as  $k \rightarrow \infty$ . By the mean-value theorem, along with  $(H_5)$ ,  $(H_6)$ , and (5), we have, for any  $u \in Z_k$ ,

$$\begin{aligned}
 J(u) &= \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^3} F(x, u) dx + \int_{\mathbb{R}^3} -\omega \Phi(u) u^2 dx \\
 &\geq \frac{1}{2} \|u\|^2 - \int_{\mathbb{R}^3} \left[ \frac{1}{\gamma} a |u|^\gamma + \frac{1}{\sigma} b |u|^\sigma \right] dx - \int_{\mathbb{R}^3} \left[ c |u| + \frac{1}{\mu} d |u|^\mu \right] dx \\
 &\geq \frac{1}{2} \|u\|^2 - \frac{1}{\gamma} \|a\|_{\frac{2}{2-\gamma}} \|u\|_2^\gamma - \frac{1}{\sigma} \|b\|_{\frac{2}{2-\sigma}} \|u\|_2^\sigma - \|c\|_2 \|u\|_2 - \frac{1}{\mu} \|d\|_\infty \|u\|_2^\mu \\
 &\geq \frac{1}{2} \|u\|^2 - \frac{1}{\gamma} l_2^\gamma(k) \|a\|_{\frac{2}{2-\gamma}} \|u\|^\gamma - \frac{1}{\sigma} l_2^\sigma(k) \|b\|_{\frac{2}{2-\sigma}} \|u\|^\sigma - l_2(k) \|c\|_2 \|u\| - \frac{1}{\mu} \eta_\mu^\mu \|d\|_\infty \|u\|^\mu.
 \end{aligned} \tag{37}$$

In view of (37), and using the facts that  $\mu > 2$  and  $\gamma, \sigma > 1$ , we obtain

$$J(u) \geq \frac{1}{4} \|u\|^2 - \left( \frac{1}{\gamma} l_2^\gamma(k) \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^\sigma(k) \|b\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \right) \|u\| \tag{38}$$

for

$$\|u\| \leq \inf \left\{ 1, \left( \frac{\mu}{4 \|d\|_\infty \eta_\mu^\mu} \right)^{\frac{1}{\mu-2}} \right\}.$$

Let

$$\rho_k = 8 \left( \frac{1}{\gamma} l_2^\gamma(k) \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^\sigma(k) \|b\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \right).$$

It is easy to see that  $\rho_k \rightarrow 0$  as  $k \rightarrow \infty$ . Thus, for sufficiently large integers  $k$ , (38) implies that

$$a_k \geq \frac{1}{8} \rho_k^2 > 0.$$

This completes the proof of Lemma 4.5. □

Next, we show that the minimum energy on small balls in  $Z_k$  tends to zero as  $k \rightarrow \infty$ . This reflects the fact that the influence of the nonlinear terms vanishes asymptotically on high-frequency components.

**Lemma 4.6.** *Assume that  $(V_1)$ ,  $(V_2)$ ,  $(H_5)$ , and  $(H_6)$  are satisfied. Then*

$$d_k = \inf_{u \in Z_k, \|u\| \leq \rho_k} J(u) \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{39}$$

**Proof.** By (38), for any  $u \in Z_k$ , we have

$$J(u) \geq - \left( \frac{1}{\gamma} l_2^\gamma(k) \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^\sigma(k) \|b\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \right) \|u\|. \tag{40}$$

Therefore, for  $\|u\| \leq \rho_k$ , it follows that

$$0 \geq d_k \geq - \left( \frac{1}{\gamma} l_2^\gamma(k) \|a\|_{\frac{2}{2-\gamma}} + \frac{1}{\sigma} l_2^\sigma(k) \|b\|_{\frac{2}{2-\sigma}} + l_2(k) \|c\|_2 \right) \rho_k. \tag{41}$$

Since  $l_2(k), \rho_k \rightarrow 0$  as  $k \rightarrow \infty$ , we conclude that  $d_k \rightarrow 0$  as  $k \rightarrow \infty$ . □

We now establish the second geometric condition of the dual fountain theorem. On finite-dimensional subspaces  $Y_k$ , all norms are equivalent, and the subquadratic part of the nonlinearity dominates near the origin. This forces the functional to become negative on small spheres, creating the “descending geometry” required by the theorem.

**Lemma 4.7.** *Assume that  $(V_1)$ ,  $(V_2)$ ,  $(H_5)$ , and  $(H_6)$  are satisfied. Then, for every integer  $k$ , there exists  $0 < r_k < \rho_k$  such that*

$$b_k = \inf_{u \in Y_k, \|u\|=r_k} J(u) < 0, \forall k \in \mathbb{N}. \tag{42}$$

**Proof.** We first claim that there exists  $\epsilon > 0$  such that

$$\text{meas} \left( \{x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \epsilon \|u\|^\gamma\} \right) \geq \epsilon, \forall u \in Y_k \setminus \{0\}. \tag{43}$$

If not, there exists a sequence  $(u_n) \subset Y_k$  with  $\|u_n\| = 1$  such that

$$\text{meas} \left( \left\{ x \in \mathbb{R}^3 : c_0(x) |u_n(x)|^\gamma \geq \frac{1}{n} \right\} \right) \leq \frac{1}{n}. \tag{44}$$

Since  $\dim Y_k < \infty$ , it follows from the compactness of the unit sphere in  $Y_k$  that there exists a subsequence, still denoted by  $(u_n)$ , that converges to some  $u \in Y_k$ . Hence,  $\|u\| = 1$ . Because all norms are equivalent in the finite-dimensional space  $Y_k$ , we have  $u_n \rightarrow u$  in  $L^2(\mathbb{R}^3)$ . By Hölder’s inequality, we obtain

$$\int_{\mathbb{R}^3} c_0(x) |u_n - u|^\gamma dx \leq \|c_0\|_{\frac{2}{2-\gamma}} \left( \int_{\mathbb{R}^3} |u_n - u|^2 dx \right)^{\frac{\gamma}{2}} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{45}$$

Thus, there exists  $\epsilon_0 > 0$  such that

$$\text{meas} \left( \{x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \epsilon_0\} \right) \geq \epsilon_0. \tag{46}$$

In fact, if not, then for every  $n \in \mathbb{N}$ ,

$$\text{meas} \left( \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \frac{1}{n} \right\} \right) \leq \frac{1}{n}.$$

Let  $n \in \mathbb{N}$ . Then, for every integer  $m \geq n$ ,

$$\text{meas} \left( \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \frac{1}{n} \right\} \right) \leq \text{meas} \left( \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \frac{1}{m} \right\} \right) \leq \frac{1}{m}$$

which implies

$$\text{meas} \left( \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \frac{1}{n} \right\} \right) = 0.$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}^3} c_0(x) |u|^{\gamma+2} dx &= \int_{\{x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \leq \frac{1}{n}\}} c_0(x) |u|^{\gamma+2} dx \\ &\leq \frac{1}{n} \int_{\mathbb{R}^3} |u|^2 dx \leq \frac{\eta_2^2}{n} \|u\|^2 = \frac{\eta_2^2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

which implies  $u = 0$ , contradicting the fact that  $\|u\| = 1$ . Therefore, (46) holds. Define

$$\Omega_0 = \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \epsilon_0 \right\} \text{ and } \Omega_n = \left\{ x \in \mathbb{R}^3 : c_0(x) |u_n(x)|^\gamma \leq \frac{1}{n} \right\}.$$

From (44) and (46), it follows that

$$\text{meas} \left( \Omega_0 \cap \Omega_n \right) = \text{meas} \left( \Omega_0 \setminus \left( \Omega_n^c \cap \Omega_0 \right) \right) \geq \text{meas}(\Omega_0) - \text{meas} \left( \Omega_n^c \cap \Omega_0 \right) \geq \epsilon_0 - \frac{1}{n}, \forall n \in \mathbb{N}.$$

Choose  $n$  large enough such that  $\epsilon_0 - \frac{1}{n} \geq \frac{1}{2}\epsilon_0$  and  $\frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n} \geq \frac{\epsilon_0}{2^\gamma}$ . Then, we have

$$\int_{\mathbb{R}^3} c_0(x) |u_n - u|^\gamma dx \geq \int_{\Omega_0 \cap \Omega_n} c_0(x) |u_n - u|^\gamma dx \geq \left( \frac{\epsilon_0}{2^{\gamma-1}} - \frac{1}{n} \right) \text{meas} \left( \Omega_0 \cap \Omega_n \right) \geq \frac{\epsilon_0^2}{2^{\gamma+1}},$$

for every large integer  $n$ , which contradicts (45). Hence, (43) is proved.

Now, for the  $\epsilon$  given in (43), define

$$\Omega_u = \left\{ x \in \mathbb{R}^3 : c_0(x) |u(x)|^\gamma \geq \epsilon \|u\|^\gamma \right\}, \forall u \in Y_k \setminus \{0\}. \tag{47}$$

By (43), we obtain

$$\text{meas}(\Omega_u) \geq \epsilon, \forall u \in Y_k \setminus \{0\}. \tag{48}$$

For any  $u \in Y_k$  with  $\|u\| < 1$ , by the mean-value theorem, assumptions  $(H_5)$ ,  $(H_6)$ , and using (47) and (48), we have

$$\begin{aligned} J(u) &= \frac{1}{2} \|u\|^2 + \int_{\mathbb{R}^3} -\omega \Phi(u) u^2 dx - \int_{\mathbb{R}^3} F(x, u) dx \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\omega c_2}{2} \|u\|^4 - \frac{1}{\gamma} \int_{\mathbb{R}^3} c_0(x) |u|^\gamma dx - \int_{\mathbb{R}^3} H(x, u) dx \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\omega c_2}{2} \|u\|^4 - \frac{1}{\gamma} \int_{\Omega_u} c_0(x) |u|^\gamma dx \\ &\leq \frac{1}{2} \|u\|^2 + \frac{\omega c_2}{2} \|u\|^4 - \frac{\epsilon}{\gamma} \|u\|^\gamma \text{meas}(\Omega_u) \\ &\leq \frac{1 + \omega c_2}{2} \|u\|^2 - \frac{\epsilon^2}{\gamma} \|u\|^\gamma. \end{aligned}$$

Choose

$$0 < r_k < \inf \left\{ 1, \rho_k, \left( \frac{\epsilon^2}{\gamma(1\omega c_2)} \right)^{\frac{1}{2-\gamma}} \right\}.$$

A direct computation shows that

$$b_k = \inf_{u \in Y_k, \|u\|=r_k} J(u) \leq \frac{1}{2}r_k^2 - r_k^{2-\gamma}r_k^\gamma = -\frac{1}{2}r_k^2 < 0.$$

This completes the proof of Lemma 4.7.  $\square$

By Lemmas 4.4, 4.5, 4.6, and 4.7, all the conditions of Theorem 2.6 are satisfied. Therefore, by Theorem 2.6,  $J$  has infinitely many nontrivial critical points. That is, the system  $(\mathcal{KGM})$  possesses infinitely many solutions.  $\square$

To conclude this section, it is observed that the functional exhibits a dual geometric structure: it is positive on small spheres in  $Z_k$  and negative on small spheres in  $Y_k$ , while a suitable compactness condition is satisfied. This configuration fits precisely within the framework of the dual fountain theorem.

## 5. Conclusion

In this work, the existence of infinitely many nontrivial solutions for a class of nonlinear Klein–Gordon–Maxwell systems in  $\mathbb{R}^3$  has been investigated under weak assumptions on both the potential  $V(x)$  and the nonlinearity  $f(x, u)$ . By employing variational methods, in particular the symmetric mountain pass theorem and the dual fountain theorem, multiplicity results have been established without relying on the classical Ambrosetti–Rabinowitz condition. The obtained results extend several known contributions in the literature by allowing the potential to be non-coercive and not necessarily uniformly positive, and by considering nonlinearities exhibiting either superquadratic growth or combined subquadratic-superquadratic behavior.

Several directions for future research arise naturally from this study. One possible extension is the investigation of Klein–Gordon–Maxwell systems involving critical Sobolev exponents, where the lack of compactness becomes more delicate. Another interesting direction is the study of systems with sign-changing or unbounded potentials under even weaker compactness assumptions. Furthermore, exploring the qualitative properties of the solutions obtained here, such as symmetry, regularity, and concentration behavior, would provide deeper insight into the structure of solutions.

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