

Research Article

Padé Approximant Related to Asymptotics for Somos' Quadratic Recurrence Constant

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Abstract

We study approximation formulas and inequalities related to the Somos' quadratic recurrence constant using Padé approximation techniques. We derive higher-order approximations and improve existing bounds, building on the prior literature.

Keywords: Somos' quadratic recurrence constant; approximation; generalized Euler constant.

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1. Introduction

Somos' quadratic recurrence constant is defined (see [9, 10]) by

$$\sigma = \sqrt{1\sqrt{2\sqrt{3\cdots}}} = \prod_{n=1}^{\infty} n^{1/2^n} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k}\right)^{1/2^k} = \exp\left\{\sum_{k=1}^{\infty} \frac{\ln k}{2^k}\right\} = 1.66168794\dots \tag{1}$$

or

$$\sigma = \exp\left\{-\int_0^1 \frac{1-x}{(2-x)\ln x} dx\right\} = \exp\left\{-\int_0^1 \int_0^1 \frac{x}{(2-xy)\ln(xy)} dx dy\right\}. \tag{2}$$

The constant σ arises in the study of the asymptotic behaviour of the sequence

$$g_0 = 1, \quad g_n = ng_{n-1}^2, \quad n \in \mathbb{N} := \{1, 2, 3, \dots\}, \tag{3}$$

with first few terms

$$g_0 = 1, \quad g_1 = 1, \quad g_2 = 2, \quad g_3 = 12, \quad g_4 = 576, \quad g_5 = 1658880, \quad \dots$$

This sequence behaves as follows (see [8, p. 446] and [6, 7]):

$$g_n \sim \frac{\sigma^{2^n}}{n} \left(1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \frac{10088}{n^7} - \frac{106918}{n^8} + \frac{1279220}{n^9} - \frac{17070418}{n^{10}} + \frac{251560472}{n^{11}} - \frac{4059954946}{n^{12}} + \dots\right)^{-1}. \tag{4}$$

The constant σ appears in important problems from pure and applied analysis, and it is the motivation of a large number of research papers (see, for example, [5–7, 9, 11–17, 20, 21]).

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Sondow and Hadjicostas [20] introduced and studied the generalized-Euler-constant function $\gamma(z)$, defined by

$$\gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left(\frac{1}{n} - \ln \frac{n+1}{n} \right), \quad (5)$$

where the series converges when $|z| \leq 1$. Pilehrood and Pilehrood [16] considered the function $z\gamma(z)$ (with $|z| \leq 1$). The function $\gamma(z)$ generalizes both Euler's constant $\gamma(1)$ and the alternating Euler constant $\ln \frac{4}{\pi} = \gamma(-1)$ (see [18, 19]).

It is known (see [20]) that

$$\gamma\left(\frac{1}{2}\right) = 2 \ln \frac{2}{\sigma}, \quad \text{equivalently,} \quad \sigma = 2 \exp \left\{ -\frac{1}{2} \gamma\left(\frac{1}{2}\right) \right\}. \quad (6)$$

Thus, the formula (5) is closely related to the Somos' quadratic recurrence constant σ . Define

$$\gamma_n(z) = \sum_{k=1}^n z^{k-1} \left(\frac{1}{k} - \ln \frac{k+1}{k} \right), \quad |z| \leq 1.$$

Mortici [14] proved that for $n \in \mathbb{N}$,

$$\frac{270(n+1)}{2^n(270n^3 + 1530n^2 + 1065n + 6293)} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{18}{2^n(18n^2 + 84n - 13)}. \quad (7)$$

Using the Maple software, we find, as $n \rightarrow \infty$,

$$2^n \left(\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) \right) = \frac{18}{18n^2 + 84n - 13} + O\left(\frac{1}{n^5}\right), \quad (8)$$

$$2^n \left(\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) \right) = \frac{270(n+1)}{270n^3 + 1530n^2 + 1065n + 6293} + O\left(\frac{1}{n^6}\right). \quad (9)$$

Lu and Song [13] improved Mortici's result (7) and obtained the following double inequality:

$$\frac{690n^2 + 3524n + 145}{6(2^n)(n+1)^2(115n^2 + 894n + 779)} < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{48n + 127}{3(2^n)(16n + 85)(n+1)^2} \quad (10)$$

for $n \in \mathbb{N}$. We find, as $n \rightarrow \infty$,

$$2^n \left(\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) \right) = \frac{48n + 127}{3(16n + 85)(n+1)^2} + O\left(\frac{1}{n^5}\right), \quad (11)$$

$$2^n \left(\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) \right) = \frac{690n^2 + 3524n + 145}{6(n+1)^2(115n^2 + 894n + 779)} + O\left(\frac{1}{n^6}\right). \quad (12)$$

You and Chen [21] improved Lu and Song's result (10) and obtained the following double inequality:

$$\begin{aligned} & \frac{20(1+n)(67661899 + 6374739n)}{3(2^n)(2550076487 + 2769781604n + 2723748210n^2 + 691902800n^3 + 42498260n^4)} \\ & < \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{10(342487 + 38928n)}{3(2^n)(2293579 + 5233860n + 1747170n^2 + 129760n^3)} \end{aligned} \quad (13)$$

for $n \in \mathbb{N}$.

We find, as $n \rightarrow \infty$,

$$2^n \left(\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right) \right) = \frac{10(342487 + 38928n)}{3(2293579 + 5233860n + 1747170n^2 + 129760n^3)} + O\left(\frac{1}{n^7}\right) \quad (14)$$

and

$$\begin{aligned} & 2^n \left(\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right) \right) \\ &= \frac{20(1+n)(67661899 + 6374739n)}{3(2550076487 + 2769781604n + 2723748210n^2 + 691902800n^3 + 42498260n^4)} + O\left(\frac{1}{n^8}\right). \end{aligned} \quad (15)$$

Remark 1.1. We introduce Padé approximant in the next section. It is known that a Padé approximant is the best approximation of a function by a rational function of a given order. The formulas (8) and (14) are Padé approximants, while the formulas (9), (11), (12) and (15) are not Padé approximants. Using the Padé approximation method, in the next section, we derive (8) and (14). We also present two approximations that are more accurate than that given by (15); see (35) and (37).

Based on the Padé approximation method, in this paper, we develop the approximation formulas (8) and (14) to produce a general result. More precisely, we determine the coefficients a_j and b_j such that

$$2^{n-1} \left(\gamma \left(\frac{1}{2} \right) - \gamma_{n-1} \left(\frac{1}{2} \right) \right) = \frac{a_2 n^{k-2} + a_3 n^{k-3} + \cdots + a_p}{n^k + b_1 n^{k-1} + \cdots + b_q} + O\left(\frac{1}{n^{2k+1}}\right) \quad (16)$$

as $n \rightarrow \infty$, where $k \geq 2$ is any given integer. Based on the obtained result, we present two new upper bounds for $\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right)$ (see (39) and (42)), improving the upper bound in (13).

Padé approximation use rational functions to mimic the behavior of series, often offering a wider range of validity and higher approximation efficiency than truncated Taylor series. In this paper, we exploit this feature to construct higher-order Padé approximation for Somos' quadratic recurrence constant, thereby obtaining more accurate asymptotic formulas and tighter bounds for inequalities.

The following lemma is required in our present investigation.

Lemma 1.2 (see [7]). As $n \rightarrow \infty$,

$$2^n \left(\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right) \right) \sim \sum_{j=0}^{\infty} \frac{c_j}{(n+1)^j}, \quad (17)$$

with the coefficients c_j given by the following recurrence relation:

$$c_0 = c_1 = 0, \quad c_2 = 1, \quad c_j = \frac{2(-1)^j}{j} + \sum_{k=2}^{j-1} a_k (-1)^{j-k} \binom{j-1}{j-k}, \quad j \geq 3. \quad (18)$$

The numerical values given in this paper are calculated via the computer program MAPLE 13.

2. Approximation to $\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right)$

For later use, we introduce Padé approximant (see [1–4]). Let f be a formal power series

$$f(t) = c_0 + c_1 t + c_2 t^2 + \cdots. \quad (19)$$

The Padé approximation of order (p, q) of the function f is the rational function, denoted by

$$[p/q]_f(t) = \frac{\sum_{j=0}^p a_j t^j}{1 + \sum_{j=1}^q b_j t^j}, \tag{20}$$

where $p \geq 0$ and $q \geq 1$ are two given integers, the coefficients a_j and b_j are given by (see [2–4])

$$\begin{cases} a_0 = c_0 \\ a_1 = c_0 b_1 + c_1 \\ a_2 = c_0 b_2 + c_1 b_1 + c_2 \\ \vdots \\ a_p = c_0 b_p + \dots + c_{p-1} b_1 + c_p \\ 0 = c_{p+1} + c_p b_1 + \dots + c_{p-q+1} b_q \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \dots + c_p b_q, \end{cases} \tag{21}$$

and the following equation holds:

$$[p/q]_f(t) - f(t) = O(t^{p+q+1}). \tag{22}$$

Thus, the first $p + q + 1$ coefficients of the series expansion of $[p/q]_f$ are identical to those of f . Moreover, we have (see [4])

$$[p/q]_f(t) = \frac{\begin{vmatrix} t^q f_{p-q}(t) & t^{q-1} f_{p-q+1}(t) & \cdots & f_p(t) \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} t^q & t^{q-1} & \cdots & 1 \\ c_{p-q+1} & c_{p-q+2} & \cdots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \cdots & c_{p+q} \end{vmatrix}}, \tag{23}$$

with $f_n(x) = c_0 + c_1 x + \dots + c_n x^n$, the n th partial sum of the series f (here, f_n is identically zero for $n < 0$). Let

$$f(n) = 2^{n-1} \left(\gamma \left(\frac{1}{2} \right) - \gamma_{n-1} \left(\frac{1}{2} \right) \right). \tag{24}$$

It follows from (17) that, as $n \rightarrow \infty$,

$$\begin{aligned} f(n) \sim \sum_{j=2}^{\infty} \frac{c_j}{n^j} &= \frac{1}{n^2} - \frac{8}{3n^3} + \frac{23}{2n^4} - \frac{332}{5n^5} + \frac{479}{n^6} - \frac{29024}{7n^7} + \frac{167491}{4n^8} - \frac{4349492}{9n^9} \\ &+ \frac{31374953}{5n^{10}} - \frac{995818760}{11n^{11}} + \dots \end{aligned} \tag{25}$$

with the coefficients c_j given by (18). In what follows, the function f is defined in (24).

Based on the Padé approximation method, we now give a derivation of formula (8). To this end, we consider

$$[2/2]_f(n) = \frac{\sum_{j=0}^2 a_j n^{-j}}{1 + \sum_{j=1}^2 b_j n^{-j}}.$$

Noting that

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = 1, \quad c_3 = -\frac{8}{3}, \quad c_4 = \frac{23}{2}, \quad c_5 = -\frac{332}{5}, \quad c_6 = 479 \quad (26)$$

holds, we have, by (21),

$$\begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = 1 \\ 0 = -\frac{8}{3} + b_1 \\ 0 = \frac{23}{2} - \frac{8}{3}b_1 + b_2, \end{cases}$$

that is,

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \quad b_1 = \frac{8}{3}, \quad b_2 = -\frac{79}{18}.$$

We thus obtain that

$$[2/2]_f(n) = \frac{\frac{1}{n^2}}{1 + \frac{8}{3n} - \frac{79}{18n^2}} = \frac{18}{18n^2 + 48n - 79}, \quad (27)$$

and we have, by (22),

$$2^{n-1} \left(\gamma \left(\frac{1}{2} \right) - \gamma_{n-1} \left(\frac{1}{2} \right) \right) - \frac{18}{18n^2 + 48n - 79} = O \left(\frac{1}{n^5} \right), \quad n \rightarrow \infty. \quad (28)$$

Replacing n by $n + 1$ in (28) yields (8).

We now give a derivation of formula (14). To this end, we consider

$$[3/3]_f(x) = \frac{\sum_{j=0}^3 a_j x^{-j}}{1 + \sum_{j=1}^3 b_j x^{-j}}.$$

Noting that (26) holds, we have, by (21),

$$\begin{cases} a_0 = 0 \\ a_1 = 0 \\ a_2 = 1 \\ a_3 = b_1 - \frac{8}{3} \\ 0 = \frac{23}{2} - \frac{8}{3}b_1 + b_2 \\ 0 = -\frac{332}{5} + \frac{23}{2}b_1 - \frac{8}{3}b_2 + b_3 \\ 0 = 479 - \frac{332}{5}b_1 + \frac{23}{2}b_2 - \frac{8}{3}b_3, \end{cases}$$

that is,

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 1, \quad a_3 = \frac{303559}{38928}, \quad b_1 = \frac{135789}{12976}, \quad b_2 = \frac{13305}{811}, \quad b_3 = -\frac{1322871}{129760}.$$

We thus obtain that

$$\begin{aligned}
 [3/3]_f(n) &= \frac{\frac{1}{n^2} + \frac{303559}{38928n^3}}{1 + \frac{135789}{12976n} + \frac{13305}{811n^2} - \frac{1322871}{129760n^3}} \\
 &= \frac{10(38928n + 303559)}{3(129760n^3 + 1357890n^2 + 2128800n - 1322871)}, \tag{29}
 \end{aligned}$$

and we have, by (22),

$$2^{n-1} \left(\gamma \left(\frac{1}{2} \right) - \gamma_{n-1} \left(\frac{1}{2} \right) \right) - \frac{10(38928n + 303559)}{3(129760n^3 + 1357890n^2 + 2128800n - 1322871)} = O \left(\frac{1}{n^7} \right). \tag{30}$$

Replacing n by $n + 1$ in (30) yields (14).

From the Padé approximation method and the expansion (25), we now present the following general result.

Theorem 2.1. *The Padé approximation of order (p, q) of the asymptotic formula (25) of the function*

$$f(n) = 2^{n-1} \left(\gamma \left(\frac{1}{2} \right) - \gamma_{n-1} \left(\frac{1}{2} \right) \right)$$

(at the point $n = \infty$) is the following rational function:

$$[p/q]_f(n) = \frac{\sum_{j=2}^p a_j n^{-j}}{1 + \sum_{j=1}^q b_j n^{-j}} = n^{q-p} \left(\frac{a_2 n^{p-2} + a_3 n^{p-3} + \dots + a_p}{n^q + b_1 n^{q-1} + \dots + b_q} \right), \tag{31}$$

where $p \geq 2$ and $q \geq 1$ are any given integers, the coefficients a_j and b_j are given by

$$\begin{cases} a_2 = c_0 b_2 + c_1 b_1 + c_2 \\ \vdots \\ a_p = c_0 b_p + \dots + c_{p-1} b_1 + c_p \\ 0 = c_{p+1} + c_p b_1 + \dots + c_{p-q+1} b_q \\ \vdots \\ 0 = c_{p+q} + c_{p+q-1} b_1 + \dots + c_p b_q, \end{cases} \tag{32}$$

and c_j is given in (18), and the following holds:

$$f(n) - [p/q]_f(n) = O \left(\frac{1}{n^{p+q+1}} \right), \quad x \rightarrow \infty. \tag{33}$$

Moreover, we have

$$[p/q]_f(n) = \frac{\begin{vmatrix} \frac{1}{n^q} f_{p-q}(n) & \frac{1}{n^{q-1}} f_{p-q+1}(n) & \dots & f_p(n) \\ c_{p-q+1} & c_{p-q+2} & \dots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \dots & c_{p+q} \end{vmatrix}}{\begin{vmatrix} \frac{1}{n^q} & \frac{1}{n^{q-1}} & \dots & 1 \\ c_{p-q+1} & c_{p-q+2} & \dots & c_{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_p & c_{p+1} & \dots & c_{p+q} \end{vmatrix}}, \tag{34}$$

with $f_m(n) = \sum_{j=0}^m \frac{c_j}{n^j}$, the m th partial sum of the asymptotic series (25).

Remark 2.2. Setting $(p, q) = (k, k)$ in (33), we obtain (16).

Remark 2.3. Using (34), we can also derive (27) and (29). Indeed, we have

$$[2/2]_f(n) = \frac{\begin{vmatrix} \frac{1}{n^2}f_0(n) & \frac{1}{n}f_1(n) & f_2(n) \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}}{\begin{vmatrix} \frac{1}{n^2} & \frac{1}{n} & 1 \\ c_1 & c_2 & c_3 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 & \frac{1}{n^2} \\ 0 & 1 & -\frac{8}{3} \\ 1 & -\frac{8}{3} & \frac{23}{2} \end{vmatrix}}{\begin{vmatrix} \frac{1}{n^2} & \frac{1}{n} & 1 \\ 0 & 1 & -\frac{8}{3} \\ 1 & -\frac{8}{3} & \frac{23}{2} \end{vmatrix}} = \frac{18}{18n^2 + 48n - 79}$$

and

$$[3/3]_f(n) = \frac{\begin{vmatrix} \frac{1}{n^3}f_0(n) & \frac{1}{n^2}f_1(n) & \frac{1}{n}f_2(n) & f_3(n) \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_3 & c_4 & c_5 & c_6 \end{vmatrix}}{\begin{vmatrix} \frac{1}{n^3} & \frac{1}{n^2} & \frac{1}{n} & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_3 & c_4 & c_5 & c_6 \end{vmatrix}} = \frac{\begin{vmatrix} 0 & 0 & \frac{1}{n^3} & \frac{1}{n^2} - \frac{8}{3n^3} \\ 0 & 1 & -\frac{8}{3} & \frac{23}{2} \\ 1 & -\frac{8}{3} & \frac{23}{2} & -\frac{332}{5} \\ -\frac{8}{3} & \frac{23}{2} & -\frac{332}{5} & 479 \end{vmatrix}}{\begin{vmatrix} \frac{1}{n^3} & \frac{1}{n^2} & \frac{1}{n} & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_2 & c_3 & c_4 & c_5 \\ c_3 & c_4 & c_5 & c_6 \end{vmatrix}} = \frac{303559 + 38928n}{1620n^3} \\ = \frac{129760n^3 + 1357890n^2 + 2128800n - 1322871}{5400n^3} \\ = \frac{10(38928n + 303559)}{3(129760n^3 + 1357890n^2 + 2128800n - 1322871)}.$$

Setting $(p, q) = (4, 4)$, we obtain by Theorem 2.1 that, as $n \rightarrow \infty$,

$$2^n \left(\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right) \right) = U(n) + O \left(\frac{1}{n^9} \right), \quad (35)$$

where

$$U(n) = \frac{5(129824698710n^2 + 2591031580396n + 10908261571089)}{3(216374497850n^4 + 5328133623960n^3 + 38176633328670n^2 + 86922599176700n + 44894793967479)}. \quad (36)$$

Setting $(p, q) = (5, 5)$, we obtain by Theorem 2.1 that, as $n \rightarrow \infty$,

$$2^n \left(\gamma \left(\frac{1}{2} \right) - \gamma_n \left(\frac{1}{2} \right) \right) = V(n) + O \left(\frac{1}{n^{11}} \right), \quad (37)$$

with

$$V(n) = \frac{7V_1(n)}{15V_2(n)}, \quad (38)$$

where

$$V_1(n) = 1309818323703984683040n^3 + 24465697736539620142710n^2 \\ + 25987062928275708344760n - 569488232827853386340501$$

and

$$V_2(n) = 611248551061859518752n^5 + 13047321746550114783270n^4 \\ + 39890795686784252183960n^3 - 270065513551849474801650n^2 \\ - 631459388683754150359560n + 275572963466622181424373.$$

Clearly, among approximation formulas (8), (9), (11), (12), (14), (15), (35) and (37), the formula (37) would be the best one.

3. A New Upper Bound for $\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right)$

The formula (35) motivates us to establish the following result.

Theorem 3.1. For $n \in \mathbb{N}$,

$$\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{U(n)}{2^n}, \quad (39)$$

with $U(n)$ given in (36).

Proof. Define the sequence (x_n) by

$$x_n = \gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) - \frac{U(n)}{2^n}. \quad (40)$$

It is well known that for $-1 < t \leq 1$ and $m \in \mathbb{N}$,

$$\sum_{j=1}^{2m} \frac{(-1)^{j-1}}{j} t^j < \ln(1+t) < \sum_{j=1}^{2m-1} \frac{(-1)^{j-1}}{j} t^j,$$

which implies that for $x \geq 1$ and $m \in \mathbb{N}$,

$$\sum_{j=2}^{2m+1} \frac{(-1)^j}{jx^j} < \frac{1}{x} - \ln\left(1 + \frac{1}{x}\right) < \sum_{j=2}^{2m} \frac{(-1)^j}{jx^j}. \quad (41)$$

Using the second inequality in (41), we find that, for $n \geq 1$,

$$2^n(x_n - x_{n+1}) = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n+1}\right) + \frac{U(n+1)}{2} - U(n) \\ < \frac{1}{2(n+1)^2} - \frac{1}{3(n+1)^3} + \frac{1}{4(n+1)^4} - \frac{1}{5(n+1)^5} + \frac{1}{6(n+1)^6} \\ - \frac{1}{7(n+1)^7} + \frac{1}{8(n+1)^8} + \frac{U(n+1)}{2} - U(n) \\ = -\frac{P_7(n-1)}{840(n+1)^8 P_8(n)},$$

where

$$\begin{aligned}
 P_7(n) = & 24711973487640913713901821571939 + 90724400754532917054970033932436n \\
 & + 138288155884347116547377871757788n^2 \\
 & + 115379752789296732118806897058168n^3 \\
 & + 57410897939712946316515803624070n^4 \\
 & + 17101136105388780278068083630840n^5 \\
 & + 2827627799263034799228544425820n^6 \\
 & + 200325884207787240715973591200n^7
 \end{aligned}$$

and

$$\begin{aligned}
 P_8(n) = & (216374497850n^4 + 6193631615360n^3 + 55459281187650n^2 \\
 & + 180125764697320n + 175538534594659)(216374497850n^4 + 5328133623960n^3 \\
 & + 38176633328670n^2 + 86922599176700n + 44894793967479)
 \end{aligned}$$

Hence, the sequence (x_n) is strictly increasing for $n \geq 1$, and we have

$$x_n < \lim_{m \rightarrow \infty} x_m = 0, \quad n \geq 1.$$

Therefore, the inequality (39) holds for all $n \in \mathbb{N}$. □

Remark 3.2. *The upper bound in (39) is better than the upper bound in (13).*

Remark 3.3. *Following the same method as used in the proof of Theorem 3.1, we can prove that, for $n \in \mathbb{N}$,*

$$\gamma\left(\frac{1}{2}\right) - \gamma_n\left(\frac{1}{2}\right) < \frac{V(n)}{2^n}, \quad (42)$$

where $V(n)$ is given in (38).

In view of (8), (14), (39) and (42), we propose the following conjecture.

Conjecture 3.4. *For $n \geq 2$,*

$$\begin{aligned}
 f(n) &= 2^{n-1} \left(\gamma\left(\frac{1}{2}\right) - \gamma_{n-1}\left(\frac{1}{2}\right) \right) \\
 &< [p/q]_f(n) = \frac{\sum_{j=2}^p a_j n^{-j}}{1 + \sum_{j=1}^q b_j n^{-j}} = n^{q-p} \left(\frac{a_2 n^{p-2} + a_3 n^{p-3} + \dots + a_p}{n^q + b_1 n^{q-1} + \dots + b_q} \right),
 \end{aligned}$$

where $p \geq 2$ and $q \geq 1$ are any given integers, whereas the coefficients a_j and b_j are given by (32).

4. Concluding Remarks and Observations

Based on the Padé approximation method, we obtain the approximation formula for Somos' quadratic recurrence constant (see Theorem 2.1). Based on the obtained result, we establish an inequality for Somos' quadratic recurrence constant (see Theorem 3.1). The upper bound in (39) is better than the upper bound in (13). In view of (8) and (14), (39) and (42), we propose Conjecture 3.4.

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